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Finitely Additive Phenomena

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1. Introduction

In 1976 Dorothy Maharam's milestone paper [26] started with the following sentence: *Some years ago S. Bochner remarked to the author that finitely additive measures are more interesting, and perhaps more important, than countably additive ones. Certainly there has been increasing interest in them shown by mathematicians and statisticians; and they lead quickly to problems that seem hard to answer.*

The aim of these notes is to prove the assertion still to be true more than twenty years later.

The first question that an analyst (if unfamiliar with Measure Theory) would probably ask is: *why should one bother with finitely additive measures, when everything is so nice and smooth and already done with countably additive ones?*

Let me just mention three reasons that any mathematician would hopefully judge good enough to keep reading these notes:

- the vector space $ba(\Omega, \Sigma)$ of bounded finitely additive measure is the topological dual of the space $L^\infty(m)$ (with m countably additive);
- in Stochastic Integration, the classical Doléans measure, that fully describes a process, and that allows to reasonably define a stochastic integral in Banach spaces, is in general only finitely additive;

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- finitely additive probabilities are the conceptual foundations for *subjective probability*, as in the work of say De Finetti, Dubins, Savage.

By *finitely additive phenomenon* I mean a difference in the behaviour of finitely additive measures versus countably additive ones. In these notes I will for instance show that:

– $0 - 0$ and $\varepsilon - \delta$ absolute continuity fail to be equivalent;

– non atomic finitely additive measures do not necessarily enjoy the Darboux property;

– the Lyapounoff theorem does not hold any more;

– convergence in measure and almost everywhere do not compare;

– L^p spaces may be non-complete;

– the Radon-Nikodym theorem fails to be true.

There are of course several more topics concerning finitely additive measures that will be neglected in these notes, like the pathology in the existence of liftings, various decomposition results, the existence of controls and Rybakov controls. On the other side I have emphasized the existence of countably additive restrictions, an argument that is hardly treated elsewhere.
In other words I have made choices about the finitely additive phenomena that I wanted to describe.
My selection has been generally driven by my feeling of competence and my taste, but also by the bounds of the reasonable lenghts of these notes and more than anything else by the determined purpose of keeping things simple: this survey is in fact aimed at the youngest in the audience, those who know almost nothing of finitely additive measures, but just the standard amount of classical Measure Theory and very little of Functional Analysis. Therefore use it just as a first seight into a flavoured, interesting topic in advanced Measure Theory, but not as an updating on a research field.
In this very perspective I have reported the results in the most elementary cases, $\mathbb{R}$ or $\mathbb{R}^n$, and only mentioned the extension to infinite

dimensional cases at the end of each of the five sections in which the material has been arranged:

1. The Stone extension;
2. Non atomicity and other regularity assumptions;
3. The range of a finitely additive measure;
4. Countably additive restrictions;
5. Integration and related pathologies.

The list of references that I have included is only functional to these notes, for I have definitively given up the project of including a more complete bibliography after having tried for a few days, only to conclude that the literature is too vaste and I know too little!
These notes also include 14 critical exercises, that the reader intrigued by this field of research may use as a first training; this in obeyence with the general philosophy, that in this particular case would recite: *to start thinking finitely additive, you've got to first flirt with finitely additive objects!*

2. A big brother - the Stone extension

The material of this first lecture, and much more on the topic, can be found in the books [8] and [33]; I am just reporting what is needed for the purposes of this minicourse.
Although this concept may be familiar for most of the readers, in the aim to make these notes self-contained, I am going to give the definition of a Booelean algebra and some related definitions.

DEFINITION 2.1. *A* Boolean algebra *is a non empty set* $\mathbf{B}$ *equipped with two binary operations* $\vee$ *and* $\wedge$ *and a unary operation* ^{c} *satisfying the following identities*

(2.1.1) $a \wedge b = b \wedge a, a \vee b = b \vee a$ *for all* $a, b \in \mathbf{B}$*;*

(2.1.2) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *and* $a \vee (b \vee c) = (a \vee b) \vee c$ *for all* $a, b, c \in \mathbf{B}$

(2.1.3) $(a \vee b) \wedge b = b, (a \wedge b) \vee b = b$ *for all* $a, b \in \mathbf{B}$*;*

(2.1.4) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ *and* $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *for all* $a, b, c \in \mathbf{B}$*;*

(2.1.5) $(a \vee a^c) \wedge b = b$, *and* $(a \wedge a^c) \vee b = b$ *for all* $a, b \in \mathbf{B}$.

EXERCISE 2.2: Prove that for each $a, b \in \mathbf{B}$, $a \wedge a^c = b \wedge b^c$ and $a \vee a^c = b \vee b^c$

In view of the previous Exercise, there are two elements 0 and 1 in $\mathbf{B}$ such that $a \wedge a^c = 1$ and $a \vee a^c = 0$ for every $a \in \mathbf{B}$.
We introduce a partial ordering on $\mathbf{B}$ in the following way: For $a, b \in \mathbf{B}$ $a \leq b$ if $a \vee b = b$ or equivalently $a \wedge b = a$. The relation $\leq$ is a partial order, namely it is reflexive, antisymmetric and transitive.

DEFINITION 2.3. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two Boolean algebras, and* h *a map from* $\mathbf{A}$ *to* $\mathbf{B}$. h *is an* isomorphism *if it is one-to-one, onto and such that*

(2.2.1) $h(a \wedge b) = h(a) \wedge h(b)$, $h(a \vee b) = h(a) \vee h(b)$, *and* $h(a^c) = h(a)^c$ *for all* $a, b \in \mathbf{A}$.

We shall now introduce the notion of a filter.

DEFINITION 2.4. *A subset* $\mathbf{J}$ *of a Boolean algebra* $\mathbf{B}$ *is a* proper filter *if*

(2.3.1) $0 \notin \mathbf{J}$*;*

(2.3.2) $a \wedge b \in \mathbf{J}$ *for all* $a, b \in \mathbf{J}$ *;*

(2.3.3) if $b \in \mathbf{J}$, $a \in \mathbf{B}$ and $a \geq b$, then $a \in \mathbf{J}$.

$\mathbf{J}$ *is a* maximal filter in $\mathbf{B}$ *if there is no proper filter in* $\mathbf{B}$ *that properly contains* $\mathbf{J}$.

EXERCISE 2.5:

(2.1) Let $\{\mathbf{F}_i, i \in I\}$ be a family of filters; then $\bigcap_{i \in I} \mathbf{F}_i$ is a filter;

(2.2) If H is a subset of $\mathbf{B}$ such that $\bigwedge_{i=1}^{n} a_i \neq 0$ for each finite collection $a_1, \ldots, a_n$ in H, then there exists a filter containing H.

(2.3) If $\mathbf{F}$ is a filter in $\mathbf{B}$, and $a \in \mathbf{B}$ is such that $a \wedge f \neq 0$, for every $f \in \mathbf{F}$, then there is a filter $\mathbf{J}$ containing $\mathbf{F}$ and a.

(2.4) Every filter is contained in a maximal filter (use Zorn's Lemma).

If Σ is an algebra of subsets of a set Ω then Σ is a Boolean algebra with respect to union, intersection and complementation; it is natural to ask whether the converse is true, namely is any Boolean algebra in fact an algebra of sets? The following theorem provides a partially affirmative answer.

THEOREM 2.6. *(Stone Representation Theorem). Let* $\mathbf{B}$ *be a Boolean algebra. Then there exists a compact totally disconnected space* S *such that* $\mathbf{B}$ *is isomorphic to the algebra* $\mathcal{G}$ *of clopen subsets of* S.

Proof. Let S be the set of all maximal filters in $\mathbf{B}$. For each $a \in \mathbf{B}$ let $h(a)$ be the set of all filters in S containing a. Then the following properties hold:

(2.1.i) $h(0) = \emptyset$;

(2.1.ii) $h(a \vee b) = h(a) \cup h(b)$ for $a, b \in \mathbf{B}$;

(2.1.iii) $h(a \wedge b) = h(a) \cap h(b)$ for $a, b \in \mathbf{B}$;

(2.1.iv) $h(a^c) = S \setminus h(a)$ for $a \in \mathbf{B}$;

(2.1.v) $h(a) \neq h(b)$ if $a \neq b$.

Consider the family $\mathcal{G} = \{h(a), a \in \mathbf{B}\}$. By (2.1.iii) $\mathcal{G}$ is closed under finite intersections, therefore the collection τ of unions of sets from $\mathcal{G}$ is a topology on S, for which $\mathcal{G}$ is a base.

CLAIM 1: (S, τ) is a Hausdorff space.

Proof. Let $\mathbf{J}_1$ and $\mathbf{J}_2$ be two distinct elements in S; then there exists $a \in \mathbf{B}$ such that $a \in \mathbf{J}_1$ and $a^c \in \mathbf{J}_2$. Thus $\mathbf{J}_1 \in h(a)$ and $\mathbf{J}_2 \in h(a^c)$. Since both $h(a)$ and $h(a^c)$ belong to $\mathcal{G}$ they are open, and from (2.1.iii) $h(a) \cap h(a^c) = \emptyset$. □

Claim 2: (S, τ) is compact.

Proof. Since every closed set in S is an intersection of sets from $\mathcal{G}$, it suffices to show that $\mathcal{G}$ has the finite intersection property. Let $\{h(a_i), i \in I\}$ be a family in $\mathcal{G}$; for each finite subset Γ of I $\bigwedge_{i \in \Gamma} a_i \neq \emptyset$. From (2.2) and (2.4) there exists a maximal filter $\mathbf{J} \supset \{h(a_i), i \in I\}$, for which clearly one has $\mathbf{J} \in \bigcap_{i \in I} h(a_i)$. □

Claim 3: $\mathcal{G}$ is the collection of clopen sets of S.

Proof. From (2.1.iv) $\mathcal{G}$ is closed under complementation, therefore each element in $\mathcal{G}$ is clopen. Conversely, let U be a clopen subset of S: then U is a union of elements from $\mathcal{G}$, and since U is closed, and S is compact, this union admits a finite subcover. As $\mathcal{G}$ is closed under finite unions, by (2.1.ii) $U \in \mathcal{G}$. □

Claim 4: (S, τ) is totally disconnected.

Proof. this is an obvious consequence of Claim 3 and the fact that $\mathcal{G}$ is a base for τ. □

Claim 5: B and $\mathcal{G}$ are isomorphic.

Proof. Indeed the map h from $\mathbf{B}$ to $\mathcal{G}$ is the requested isomorphism: property (2.2.1) follows immediately from (2.1.i), (2.1.ii) and (2.1.iii), and it is obvious that h is onto; finally, from (2.1.v) h is also one-to-one. □

We will call the space (S, τ) the *Stone space* associated to the Boolean algebra $\mathbf{B}$.

Exercise 2.7: Prove the properties (2.1.i) - (2.1.v).

Proposition 2.8. *Let* $\mathbf{B}$ *be a Boolean algebra,* S *its Stone space and* $\mathcal{G}$ *the algebra of its clopen sets. Then for every infinite family* $\{A_i, i \in I\}$ *of pairwise disjoint non empty elements of* $\mathcal{G}$*,* $\bigcup_{i \in I} A_i \notin \mathcal{G}$.

Proof. Assume by contradiction that $A = \bigcup_{i \in I} A_i \in \mathcal{G}$. Then A is clopen in S, and therefore compact; thus the open cover $\{A_i, i \in I\}$ should admit a finite subcover, namely $\bigcup_{i \in I} A_i = \bigcup_{j=1}^{k} A_j$ thus contradicting the assumption that I is infinite and the A_i's are pairwise disjoint and non empty. $\square$

EXERCISE 2.9: Let $\{A_n, n \in \mathbb{N}\}$ be a countable family of non empty elements of $\mathcal{G}$, (not necessarily pairwise disjoint!). Could $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{G}$?

Let now consider a measurable space (Ω, Σ), where Σ is an algebra of sets, and let $m : \Sigma \to \mathbb{R}_0^+$ be a finitely additive measure, namely a set function such that

(f.a.m.1) $m(\emptyset) = 0$;

(f.a.m.2) $m(A \cup B) = m(A) + m(B)$ whenever $A, B \in \Sigma$ and $A \cap B = \emptyset$.

We introduce an equivalence relation $\sim$ in the following way: $A \sim B$ if $m(A \Delta B) = 0$. Then the quotient space $\Sigma_{|\sim}$ becomes a Boolean algebra; let S be its associated Stone space, $\mathcal{G}$ the algebra of clopen sets, and h the isomorphism of Theorem 2.1.
We define a set function $\widetilde{m} : \mathcal{G} \to \mathbb{R}_0^+$ by setting

$$\widetilde{m}(G) = m(h^{-1}(G)), \quad G \in \mathcal{G},$$

where we have abused the notation, since $h^{-1}(G)$ is an equivalence class in Σ; since every representative of it has the same m measure, by the very definition of $\sim$, $\widetilde{m}$ is well defined.
Observe that, since h is an isomorphism of algebras, $\widetilde{m}$ becomes a finitely additive measure, and, thanks to Proposition 1.1, it is in fact countably additive on $\mathcal{G}$; thus, by means of a Carathéodory Extension procedure, it can be uniquely extended to a countably additive measure on the σ-algebra $\mathcal{G}_\sigma$ generated by $\mathcal{G}$; we will keep the notation $\widetilde{m}$ for this extension, and will call it the *Stone extension* of the finitely additive measure m.
Indeed, $\widetilde{m}$ will often play the role of a big brother of m, rescueing

it from many of the troubles in which m is set by the lack of the countable additivity.

Notes on Section 2

In many cases, the set function m that one considers ranges in spaces much more general than the half line $[0, \infty)$; according to relationships (f.a.m.1) and (f.a.m.2) all what we need to define a finitely additive measure is a 0 element and a sum; thus a semigroup with a neutral element would be enough. However, to carry out a technique of this kind, and therefore to have a countable additivity notion, we need a topology on the range space! But even in this case, remember that Carathéodory Theorem is a scalar one (for it is based on inequalities), and therefore one needs an extension theorem in the more general setting. A good old one is due to Maurice Sion:

THEOREM 2.10. *[34] Let Γ be a commutative, complete topological group, and let $m : \mathcal{G} \to \Gamma$ be a countably additive measure, such that for each monotone sequence $(A_n)_n$ in $\mathcal{G}$, $m(A_n)$ admits a limit in Γ; then m can be uniquely extended as a countably additive measure to the whole $\mathcal{G}_\sigma$.*

A different approach can be followed in the case that the target space has an order together with the suitable algebraic operations; in this case the σ-convergence can be replaced by order convergence. Clearly the countable additivity is consequently redefined. A recent result in this direction is due to Antonio Boccuto :

THEOREM 2.11. *[10] Let Γ be a σ-complete commutative lattice group, and let $m : \mathcal{G} \to \Gamma$ be a positive countably additive measure; then m can be uniquely extended as a countably additive measure to the whole $\mathcal{G}_\sigma$.*

In Boccuto's paper there are also many references concerning the existence of σ-additive extensions of vector or group-valued measures.

3. Chemical phenomena of finitely additive measures - Strong non atomicity and agglutination

In Grado 1995, Professor Bhaskara Rao lectured on *Some important Theorems in Measure Theory*: the first section of the course concerned the celebrated Liapounoff's Theorem. I shall report the statement of the Theorem without its proof, referring the reader to those notes [7] to find it.
However, younger readers should be aware that Liapounoff's Theorem is really an important result for many different fields in Analysis; people in Measure Theory should have read a proof of it somewhen in their life...

Definition 3.1. *Let* (Ω, Σ, m) *be a measure space, i.e.* Σ *is a* σ*-algebra, and* $m : \Sigma \to R_0^+$ *a countably additive measure.* m *is said to be* non atomic *provided, for each* $A \in \Sigma$ *with* $m(A) > 0$ *there exists* $B \in \Sigma$, $B \subset A$ *such that* $0 < m(B) < m(A)$.

Theorem 3.2. *(Liapounoff's Theorem) Let* (Ω, Σ) *be a measurable space, and let* $m_i : \Sigma \to I\!R_0^+$, $i = 1, \ldots, n$ *be non atomic countably additive measures.*
Then the range of the vector measure $m = (m_1, \ldots, m_n)$ *is compact and convex.*

Exercise 3.3: In [7] the author omits the condition $B \subset A$ in the definition of non atomicity. Find an example of a measure that would satisfy that condition of non atomicity, and not the one in Definition 3.1, and such that Theorem 3.2 also fails to be true.

Definition 3.4. *Let* (Ω, Σ, m) *be as in Definition 3.1;* m *is said to be* semiconvex *if for every* $A \in \Sigma$ *there is* $B \in \Sigma$, $B \subset A$ *such that* $m(B) = \frac{1}{2} m(A)$.

Definition 3.5. *Let* (Ω, Σ, m) *be as in Definition 3.1;* m *is said to be* strongly non atomic *if for every* $\varepsilon > 0$ *there exists a decomposition of* Ω *into finitely many* Σ*-measurable sets each of* m*-measure underneath* ε.

Proposition 3.6. *Let* (Ω, Σ) *be a measurable space, and let* $m : \Sigma \to R_0^+$ *be a countably additive measure. Then the following are equivalent:*

(3.6.1) m is non atomic;

(3.6.2) m is semiconvex;

(3.6.3) m is strongly non atomic.

Proof. To show that (3.6.1) $\Longrightarrow$ (3.6.2) we shall use Liapounoff's Theorem with $n = 1$ hereditarily on $\Sigma_B := \{C \cap B : C \in \Sigma\}$ for any $B \in \Sigma$. Since the range of m restricted to Σ_B is, according to Liapounoff's Theorem, the interval $[0, m(B)]$, there is a set in Σ_B whose measure is the midpoint of the interval.
We next show that (3.6.2) $\Longrightarrow$(3.6.3): let $\varepsilon > 0$ be fixed, and let k be such that $\frac{1}{2^k} m(\Omega) \leq \varepsilon$. Let $\Omega_0 := \Omega$, and $\Omega_j \subset \Omega_{j-1}$ be such that $m(\Omega_j) = \frac{1}{2} m(\Omega_{j-1})$. Then clearly $m(\Omega_k) = \frac{1}{2^k} m(\Omega) \leq \varepsilon$. Put $\Omega^{(1)} = \Omega \setminus \Omega_k$; if $m(\Omega^{(1)}) \leq \varepsilon$ we are done, otherwise reapply the same reasoning to $\Omega^{(1)}$ and keep repeating the construction until $m(\Omega^{(n)}) \leq \varepsilon$.
Finally, to prove that (3.6.3) $\Longrightarrow$ (3.6.1), assume by contradiction that A is an atom of m, with $m(A) > 0$. Let $\varepsilon < m(A)$, and let $\{\Omega_1, \ldots, \Omega_n)$ be an ε-small decomposition of Ω. Then $m(A \cap \Omega_i) < \varepsilon$ for $i = 1, \ldots, n$ and since A is an atom, necessarily $m(A \cap \Omega_i) = 0$. But then, $A = \bigcup_{i=1}^{n} (A \cap \Omega_i)$ would measure 0 itself, which is a contradiction. □

We now turn our attention to the finitely additive case. In the remaining of this section (Ω, Σ) will be a measurable space, and $m : \Sigma \to R_0^+$ will be a finitely additive measure.
We shall consider the validity of Liapounoff's Theorem in this setting in a further section, but we shall need a version in the case $n = 1$ right now, in order to investigate which ones of the implications of Proposition 3.6 will remain true.

THEOREM 3.7. *[26] If m is strongly non atomic, then the range of m is the interval* $[0, m(\Omega)]$.

Proof. Note first that if m is strongly non atomic, and $B \in \Sigma$, then the restriction on Σ_B is still strongly non atomic.

We shall show that for each $\alpha \in]0, m(\Omega)[$ there is $A \in \Sigma$ such that $m(A) = \alpha$.
Assume by contradiction that this is false for some α fixed. Put $B_1 = \emptyset$ and $C_1 = \Omega$: we shall construct recursively two sequences, $(B_n)_n$ increasing and $(C_n)_n$ decreasing, with $B_n \subset C_n$ and

$$\alpha - \frac{1}{n} < m(B_n) < \alpha < m(C_n) < \alpha + \frac{1}{n}. \tag{1}$$

Suppose we have done this for $n \leq k$ for some $k > 1$.
Let

$$0 < \varepsilon < \min\left\{\alpha - m(B_k), m(C_k) - \alpha, \frac{1}{k(k+1)}\right\}$$

and by the strong non atomicity express $C_k \backslash B_k$ as the union of say r pairwise disjoint sets $D_1, \dots, D_r$ with $m(D_j) < \varepsilon$.
Put

$$B_{k+1} = B_k \cup [\bigcup_{i=1}^{t} D_i], \qquad C_{k+1} = B_{k+1} \cup D_{t+1}$$

with $t = max\{j \text{ such that } m(B_{k+1} \cup [\bigcup_{i=1}^{j} D_i]) < \alpha\}$.
Then $B_k \subset B_{k+1}$, $C_{k+1} \subset C_k$, $B_{k+1} \subset C_{k+1}$ and (1) holds. Now define $A = \bigcup_n B_n$. Then, for each n, $B_n \subset A \subset C_n$. Therefore $\alpha - \frac{1}{n} < m(A) < \alpha + \frac{1}{n}$ whence $m(A) = \alpha$. $\square$

We should now be able to investigate what of Proposition 3.6 remains true.

PROPOSITION 3.8. *Consider the properties:*

(3.8.1) m is non atomic;

(3.8.2) m is semiconvex;

(3.8.3) m is strongly non atomic.

Then (3.8.2) $\Longleftrightarrow$ (3.8.3) $\Longrightarrow$(3.8.1).

Proof. The implications (3.8.2) $\Rightarrow$ (3.8.3) $\Rightarrow$ (3.8.1) are proven in the same way as in Proposition 3.1, while the implication (3.8.3) $\Rightarrow$ (3.8.2) can be proven analogously to the implication (3.6.1) $\Rightarrow$ (3.6.2) by means of Theorem 3.2. $\square$

What about the remaining implication? We shall show that it is in general false in the finitely additive setting. Obviously Definition 3.1 can be given exactly in the same way in the finitely additive case; therefore if $m : \Sigma \to R_0^+$ is a finitely additive measure that *is not* non atomic, it will admit an atom. In the finitely additive case this atom being a singleton or not makes a difference.
We first state the formal definitions.

DEFINITION 3.9. *A Σ-measurable set A with $m(A) > 0$ is an* atom *of a finitely additive measure $m : \Sigma \to R_0^+$ if m assumes on the Σ-measurable subsets of A only the two values 0 or $m(A)$.*
If there is $\omega \in \Omega$ such that $\{\omega\} \in \Sigma$ and $m(\{\omega\}) > 0$ then $\{\omega\}$ is an atom of m; in this case we shall say that m is concentrated in ω.
If A is an atom of m but m is not concentrated in any point of A, we will say that m is agglutinated on A.

THEOREM 3.10. *[1] Let $m_1, m_2 : \Sigma \to I\!R_0^+$ be two finitely additive measures, with m_1 agglutinated on some subset and m_2 strongly non atomic. If $m_2(A) > 0$ for every atom A of m_1, then $m := m_1 + m_2$ is non atomic but not strongly non atomic.*

Proof. Let $A \in \Sigma$ with $m(A) > 0$ be fixed.
If A is not an atom of m_1, then it is obvious that it is not an atom m. If A is an atom of m_1 instead, then $m_2(A) > 0$ and according to Proposition 3.8 there is a set $B \in \Sigma, B \subset A$ such that $m_2(B) = \dfrac{1}{2} m_2(A)$. Hence

$$0 < m_2(B) \leq m_1(B) + m_2(B) = m(B) < m_1(B) + m_2(A) \leq m(A),$$

namely $0 < m(B) < m(A)$ and so A is not an atom of m. This shows that m is non atomic.
In order to show that m is not strongly non atomic, let A be an atom of m_1 and let $\varepsilon = m_1(A)$.
Let $\{E_1, \ldots, E_k\}$ be any finite decomposition of Ω into Σ-measurable

sets. Since $m_1(A) = \sum_{j=1}^{k} m_1(E_j \cap A)$ and A is an atom of m_i there exists only one index i for which $m_1(E_i \cap A) = m_1(A) = \varepsilon$, while for every index $j \neq i$ $m_1(E_j \cap A) = 0$.
This yields that $m(E_i) \geq m_1(E_i) \geq m_1(E_i \cap A) = \varepsilon$ namely m does not admit decompositions that are ε-small. □

EXERCISE 3.11: Prove that the finitely additive measure m_1 in Theorem 3.10 cannot be countably additive.

Notes on Section 3

Often in the literature the subadditive case has to be taken into consideration.

DEFINITION 3.12. *We shall call* $m : \Sigma \to R_0^+$ *a* monotone submeasure *provided*

(s.a.1) $m(\ \emptyset\ ) = 0$;

(s.a.2) if $A \cap B = \emptyset$ *then* $m(A \cup B) \leq m(A) + m(B)$;

(s.a.3) if $A \subseteq B$ *then* $m(A) \leq m(B)$.

Moreover we shall say that a monotone submeasure is semiconvex *provided for every* $A \in \Sigma$ *there exists* $B \subset A$ $B \in \Sigma$ *such that* $m(B) = m(A \setminus B) = \frac{1}{2}m(A)$.

DEFINITION 3.13. *On* Σ *the equality*

(3.13.1) $d(A, B) := m(A \Delta B)$

defines the Fréchet-Nikodym pseudometric *determined by the finitely additive measure* m.

EXERCISE 3.14: Prove that d is a pseudometric on Σ.

In [12] the following extension of Theorem 3.7 is given:

PROPOSITION 3.15. *[12] Let* $m : \Sigma \to R_0^+$ *be a semiconvex monotone submeasure. Then* $m(\Sigma)$ *is the closed interval* $[0, m(\Omega)]$.

Proof. Our proof will proceed by steps:
CLAIM 1: *For any $A \in \Sigma$ and for any dyadic rational $p \in [0,1]$ there is $A_p \in \Sigma_A$ such that $m(A_p) = pm(A)$; furthermore, if $p < q$ then $A_p \subset A_q$.*

Proof. Let A be fixed; from the semiconvexity there exists $A_{\frac{1}{2}} := B$ such that
$$m(A_{\frac{1}{2}}) = m(A \setminus A_{\frac{1}{2}}) = \frac{1}{2} m(A)$$
and there exist $A_{\frac{1}{4}} := (A_{\frac{1}{2}})_{\frac{1}{2}}$ and $(A \setminus A_{\frac{1}{2}})_{\frac{1}{2}}$ such that
$$\begin{aligned} m(A_{\frac{1}{4}}) &= m(A_{\frac{1}{2}} \setminus A_{\frac{1}{4}}) = m[(A \setminus A_{\frac{1}{2}})_{\frac{1}{2}}] \\ &= m[(A \setminus A_{\frac{1}{2}}) \setminus (A \setminus A_{\frac{1}{2}})_{\frac{1}{2}}] \\ &= \frac{1}{4} m(A). \end{aligned}$$
Thus putting $A_{\frac{3}{4}} := A_{\frac{1}{2}} \cup (A \setminus A_{\frac{1}{2}})_{\frac{1}{2}}$ we find $m(A_{\frac{3}{4}}) \leq \frac{3}{4} m(A)$, but since
$$m(A) \leq m(A_{\frac{3}{4}}) + m(A \setminus A_{\frac{3}{4}}) \leq \frac{3}{4} m(A) + \frac{1}{4} m(A)$$
it has to be $m(A_{\frac{3}{4}}) = \frac{3}{4} m(A)$. Thus proceeding, at the n-th step we shall find 2^n pairwise disjoint subsets of A, on each of which m is $\frac{1}{2^n} m(A)$, and 2^n subsets, $A_{\frac{r}{2^n}}$ for $r = 1, \ldots, 2^n$ obtained as unions of those disjoint 2^n sets; in this way if $p < q$ necessarily $A_p \subset A_q$. $\square$

CLAIM *2: For any $A \in \Sigma$ and for any $t \in [0,1]$ there is $A_t \in \Sigma_A$ such that $m(A_t) = tm(A)$. Furthermore if $t < s$ then $A_t \subset A_s$.*

Proof. Without loss of generality, let $t \in]0,1[$ be fixed, and let $(p_n)n$ be an increasing sequence of dyadic numbers such that $p_n \uparrow t$. By Claim 1 there is an increasing sequence of subsets of A, say $(A_n)_n$ such that $m(A_n) = p_n m(A)$; then obviously $\lim_n m(A_n) = tm(A)$.
If m were a countably additive measure, our proof was done! Being in the subadditive case, we shall work a little more to achieve the

result.
From the inclusion $A_n \subset \bigcup_n A_n$ we have that

$$tm(A) = \lim_n m(A_n) \leq m(\bigcup_n A_n).$$

Let now $q_n \downarrow t$ be a decreasing sequence of dyadic numbers, and repeat the same argument to find sets A'_n such that $m(A'_n) = q_n m(A)$. Then eventually one finds that

$$tm(A) = \lim_n m(A'_n) \geq m(\bigcap_n A'_n).$$

Since $p_n \leq q_k$ for any pair n, k, it also is $\bigcup_n A_n \subset \bigcap_n A'_n$ and thus on both these sets m assumes the value $tm(A)$. Put then $A_t = \bigcup_n A_n$. Note that it is essential that Σ is a σ-algebra. It is also clear that when $t < s$ it is $A_t \subset A_s$. □

CLAIM *3: (Σ, d) is arcwise connected.*

Proof. Let A and B be two sets in Σ with $A \neq B$. For every $t \in [0,1]$ consider the two sets $(B \backslash A)_t$ and $(A \backslash B)_{1-t}$ whose existence is established in Claim 2.
Define the map $\varphi : [0,1] \to \Sigma$ by setting

$$\varphi(t) = (A \backslash B)_{1-t} \cup (A \cap B) \cup (B \backslash A)_t;$$

then easily $\varphi(0) = A$ while $\varphi(1) = B$. Moreover, if $t < s$ then

$$\varphi(s) \backslash \varphi(t) = (B \backslash A)_s \backslash (B \backslash A)_t$$

and

$$\varphi(t) \backslash \varphi(s) = (A \backslash B)_{1-t} \backslash (A \backslash B)_{1-s}.$$

Then

$$d(\varphi(s), \varphi(t)) = (s-t)[m(B \backslash A) + m(A \backslash B)] \leq (s-t)m(\Omega)$$

and so φ is continuous, which concludes the proof of Claim 3. □

Claim 4: m is uniformly continuous with respect to d.

Proof. Indeed for every $\varepsilon > 0$ if A, B are such that $d(A,B) \leq \varepsilon$ then

$$\begin{aligned} |m(A) - m(B)| &= |m(A \backslash B) - m(B \backslash A)| \\ &\leq m(A \backslash B) + m(B \backslash A) \\ &= d(A,B) \leq \varepsilon. \end{aligned}$$

which proves the assertion. □

Finally, we conclude that the range of m, being the continuous image of an arcwise connected pseudometric space is a connected subset of the real line, and since m is bounded it is therefore an interval. The endpoints are the images of $\emptyset$ and the whole space Ω respectively, and this concludes the proof. □

4. An evanescent leaf - The range of a finitely additive measure

In Section 3 we have mentioned Liapounoff's Theorem, and have established that by using the correct extension of the non atomicity, the same result is true for non negative scalar finitely additive measures.
In this section we will face the vector case; the signed case will be trated as a byproduct of this case.

Definition 4.1. *Let $m : \Sigma \to I\!R^n$, $m = (m_1, \dots, m_n)$ be a finitely additive measure (namely, each component is a finitely additive measure); we shall say that it is* non negative *provided each component m_i is non negative, and we shall say that m is* semiconvex *if for every $A \in \Sigma$ there exists $B \in \Sigma$ $B \subset A$ such that $m(B) = \frac{1}{2}m(A)$.*

Lemma 4.2. *[13] Let $m : \Sigma \to I\!R^n$ be non negative and semiconvex. Then for every $E \in \Sigma$ there exists a family $(E_t)_{t\in[0,1]}$ of Σ-measurable subsets of E such that:*

(4.2.1) $E_0 = \emptyset$, $E_1 = E$

(4.2.2) if $s < t$ then $E_s \subset E_t$;

(4.2.3) $m(E_t) = tm(E)$, *for every* $t \in [0,1]$.

Proof. It is completely analogous to that of Proposition 3.3. First, by means of the semiconvexity, one determines a family $(E_p)_{p \in Q(2)}$, for every dyadic rational p, satisfying (4.2.1), (4.2.2), (4.2.3). Then for every $t \in [0,1]$, choosing two sequences $(p_n)_n$ and $(q_n)_n$ in $\mathbf{Q}$ (2) with $p_n \uparrow t$ and $q_n \downarrow t$, it will follow that

$$m(\bigcup_n E_{p_n}) = m(\bigcap_n E_{q_n}) = tm(E)$$

and setting for instance $E_t = \bigcap_n E_{q_n}$, then $(E_t)_{t \in [0,1]}$ is the required family. $\square$

Then, analogously to the proof of Proposition 3.6 the range $R(m)$ is arcwise connected in this case too.
We shall prove that it is indeed convex.

Definition 4.3. *Let* $\mu, m : \Sigma \to R_0^+$ *be two finitely additive measures. We shall say that* μ *is* m-continuous, *and write* $\mu \ll m$, *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that whenever* $E \in \Sigma$ *is such that* $m(E) < \delta$ *then* $\mu(E) < \varepsilon$.

Remark 4.4. *Remember that in the countably additive case, and if* Σ *is a* σ*-algebra, the previous definition is equivalent to the* $0 - 0$ *absolute continuity of a measure with respect to another. This equivalence fails to be true in the finitely additive case, as we shall see at the end of this section.*

Definition 4.5. *For a given non-negative finitely additive measure* $m : \Sigma \to \mathbb{R}^n$ *the symbol* $\overline{m}$ *will denote the* variation *of* m, *namely the finitely additive measure* $\overline{m} : \Sigma \to \mathbb{R}_0^+$ *defined as* $\overline{m} := \sum_{i=1}^n m_i$.

Theorem 4.6. *[13] Let* $m : \Sigma \to \mathbb{R}^n$ *be a non negative semiconvex finitely additive measure, and let* $E, F \in \Sigma$. *Then there exists a family* $(C_t)_{t \in [0,1]}$ *in* Σ *with:*

(4.6.i) $C_0 = E, \ C_1 = F$;

(4.6.ii) $m(C_t) = tm(E) + (1-t)m(F)$;

(4.6.iii) if $\mu : \Sigma \to I\!R_0^+$ is $\overline{m}$-continuous, then $t \mapsto \mu(C_t)$ is a continuous function.

Proof. Define $C_t = (F \setminus E)_t \cup (E \cap F) \cup (F \setminus E)_{(1-t)}$, where the lower index have the same meaning of Lemma 4.2. Then we only need to show the third condition. Let $x, y \in [0,1]$; clearly $|\mu(C_x) - \mu(C_y)| \leq \mu(C_x \Delta C_y)$. Since easily $\overline{m}(C_x \Delta C_y) \leq |x-y| \sum_i m_i(E \Delta F)$ for every fixed $\varepsilon > 0$ it is enough to choose x and y close enough as to have $\overline{m}(C_x \Delta C_y) < \delta$. $\square$

As a consequence, a non negative semiconvex vector finitely additive measure has convex range.
In Section 3 we have shown that strong non atomicity and semiconvexity are equivalent. It is therefore natural to ask whether the same equivalence holds in the vector case. It is immediate that, when m is semiconvex, each component is semiconvex and therefore strongly non atomic. Our next result establishes that the converse is also true.

THEOREM 4.7. *[13] Let $m : \Sigma \to I\!R^n$, $m = (m_1, \ldots, m_n)$ be a non-negative finitely additive measure such that each component is strongly non atomic. Then m is semiconvex.*

Without loss of generality, we assume that $n > 1$. We divide our proof in two steps:

CLAIM 1: *The assertion is true if $m_i \ll m_{i-1}, i = 2, \ldots, n$.*

Proof. By induction, we assume that $m' := (m_1, \ldots, m_{n-1})$ is semiconvex. Let $E \in \Sigma$: then there exists $E' \in \Sigma, E' \subset E$ such that $m'(E') = \frac{1}{2}m'(E)$. If $m_n(E') = \frac{1}{2}m'(E)$ we are finished; otherwise assume, for instance, that $m_n(E') < \frac{1}{2}m_n(E)$ and apply Theorem 4.6 to the sets E' and $E \setminus E'$. Since $m_n \ll \overline{m'}$, there are $t \in [0,1]$ and $C_t \in \Sigma, C_t \subset E$ such that, from (4.6.ii) $m'(C_t) = (1-t)m'(E') + tm'(E \setminus E')$ and from (4.6.iii) $m_n(C_t) = \frac{1}{2}m_n(E)$. But one easily checks that $m'(C_t) = \frac{1}{2}m'(E)$. $\square$

CLAIM 2: *m is semiconvex.*

Proof. For each $j = 1, \dots, n$ set $m'_j := \sum_{i=j}^{n} m_i$. Then the finitely additive measure $m' := (m'_1, \dots, m'_n)$ is semiconvex by the first Claim, since $m'_j \ll m'_{j-1}$. Let $T = (t_{ij})$ be the matrix with entries:

$$t_{ij} = \begin{cases} 1 & \text{if } j = i \\ -1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Then $m = Tm'$, whence obviously m is semiconvex too. □

EXERCISE 4.8: Let $m : \Sigma \to I\!R^2$ be a non negative, semiconvex finitely additive measure. Define the two functions $g, G \colon [0, m_1(\Omega)] \to I\!R$ as

$$\begin{aligned} g(x) &= inf\{m_2(E) : E \in \Sigma, m_1(E) = x\}, \\ G(x) &= sup\{m_2(E) : E \in \Sigma, m_1(E) = x\}. \end{aligned}$$

Prove that

- g is convex and monotone non decreasing;
- $G(x) = m_2(\Omega) - g[m_1(\Omega) - x]$ (and therefore G is concave and non decreasing).

Therefore the *finitely additive phenomenon* in the vector case is the lack of the compactness of the range; more precisely the following example shows that the range is not necessarily closed (its boundedness is a trivial consequence of the finiteness of m).
In order to state the counterexample, we shall need an extension result for finitely additive measures; we just report the statement of the result, referring to [8] for its proof and related results.

THEOREM 4.9. *([8, Corollary 3.3.4]) Let $\mathcal{C}$ be a field of subsets of Ω, $m : \mathcal{C} \to I\!R_0^+$ a bounded finitely additive measure on it, $\mathcal{F}$ a field on Ω containing $\mathcal{C}$. Then there exists a bounded finitely additive measure $\overline{m} : \mathcal{F} \to I\!R_0^+$ which extends m from $\mathcal{C}$ to $\mathcal{F}$.*

REMARK 4.10. *Therefore every* countably additive measure *on a σ-algebra can be extended in a finitely additive fashion to the whole power set 2^{Ω}; note also that if the measure is non atomic (and therefore, being countably additive, strongly non atomic), then every finitely additive extension will be strongly non atomic.*

EXAMPLE 4.11. *[12] Let $\Omega = [0,1[$, and let m_L be any extension to 2^{Ω} of the Lebesgue measure on it. Denote by $\mathcal{D}$ the family of subsets D of Ω such that the following limit exists:*

$$d(D) = lim_{n\to+\infty}\frac{m_L\left(D\cap\left[\frac{2^n-1}{2^n},1\right[\right)}{\left(\frac{1}{2}\right)^n},$$

Then $\mathcal{D}$ is a weak Dynkin system, namely:

1) $\Omega \in \mathcal{D}$;

2) if $D_1, D_2 \in \mathcal{D}$ and $D_1 \cap D_2 = \emptyset$, then $D_1 \cup D_2 \in \mathcal{D}$;

3) if $D_1, D_2 \in \mathcal{D}$ and $D_1 \subset D_2$ then $D_2 \setminus D_1 \in \mathcal{D}$.

Moreover $\mathcal{D}$ contains every subinterval of $[0,1[$; furthermore, if $D\in\mathcal{D}$ and D' is such that $m_L(D\Delta D') = 0$ then $D' \in \mathcal{D}$ and $d(D) = d(D')$. Consider now the intervals

$$A_n := \left[\frac{2^n-1}{2^n}, \frac{2^{n+1}-1}{2^{n+1}}\right[$$

with $n = 0, 1, \ldots$.
For each n and each k let us divide every A_n into 2^k contiguous subintervals, of the same lenght, and let us denote them with $A^{(k)}_{n,1}, \ldots, A^{(k)}_{n,2^k}$.
Now, for every k and $i \le 2^k$ set $B_{i,k} := \bigcup_{n=0}^{\infty} A^{(k)}_{n,i}$.
Clearly, for each $i \le 2^k$ there is a suitable $r \le 2^{k-1}$ such that $B_{i,k} \subset B_{r,k-1}$; moreover $B_{i,k} \cap B_{j,k} = \emptyset$ if $i \ne j$. Since for each m

$$m_L\left(B_{i,k}\cap\left[\frac{2^m-1}{2^m},1\right[\right) = \frac{1}{2^m}\frac{1}{2^k}$$

necessarily $B_{i,k} \in \mathcal{D}$ *for each* k *and* $i \leq 2^k$*; furthermore*

$$d(B_{i,k}) = \left(\frac{1}{2}\right)^k$$

for every $i \leq 2^k$. *It is clear that* $B_{i,k} \cap B_{j,h}$ *is either empty or coincide with one of them. Finally the intersection of a set* $B_{i,k}$ *with an interval is again a set of* $\mathcal{D}$.
Consider the family $\mathcal{F}$*' formed by the subintervals (possibly one-point) of* Ω, *the sets* $B_{i,k}$ *and* $\mathcal{F}$ *the family formed by the elements of* $\mathcal{F}$*' and the finite intersections of elements of* $\mathcal{F}$*'; hence* $\mathcal{F} \subset \mathcal{D}$.
Let Σ *be the family of all sets that are* m_L *equivalent to some set in* $\mathcal{F}$*; then* Σ *is closed under intersection and it is contained in* $\mathcal{D}$. *As in [5] the weak Dynkin system generated by* Σ *is an algebra which is contained in* $\mathcal{D}$ *since* $\Sigma \subset \mathcal{D}$. *Then, d is defined on an algebra and therefore, according to Theorem 4.9, it admits some finitely additive extension, say* δ, *to the whole* 2^Ω.
Thanks to the $B_{i,k}$*'s, it is clear that d is strongly non atomic, and from Remark 4.4.* δ *is strongly non atomic as well.*
Now observe that if $E \subset \Omega$ *is such that* $m_L(E) = 1$ *then necessarily* $\delta(E) = 1$, *while* δ *vanishes on the intervals of the form* $[0, 1-\varepsilon[$.
Therefore, the range of the 2-dimensional non negative semiconvex finitely additive measure $(m_L, \delta) : 2^\Omega \to I\!R^2$ *is easily seen to be the square*

$$(]0,1[\times]0,1[) \cup ([0,1[\times\{0\}) \cup (]0,1] \times \{1\}) .$$

Remark 4.12. *(1) In the previous example* $\delta \ll m_L$ *in the* $0-0$ *sense but not in the* $\varepsilon - \delta$ *sense.*

(2) If one considers the signed finitely additive measure $m_L - \delta$, *then* $R(m_L - \delta) =]-1, 1[$: *hence, even in the scalar case, if we allow negative values, the Liapounoff property does not hold anymore.*

Notes on Section 4

The Liapounoff Theorem does not hold anymore for *countably additive measures* when the target space is infinite dimensional.
Liapounoff himself gave an example in this sense: let $\Omega = [0, 1]$, Σ

be the Borel σ-algebra on Ω and μ be the Lebesgue measure; let $E = L^1(\mu)$. Then $m : \Sigma \to E$ defined as $m(A) = 1_A$ is countably additive, has no atoms, but its range is neither convex nor compact (not even in the weak topology).
We refer to [19] for more details. Only, as a curiosity, we mention that it is more difficult to get a counterexample where only one of the two properties lacks, and refer to the paper [4] for some results where conditions are investigated ensuring that one of the two properties can be derived from the other.
Because of the lack of compactness, in the literature, results concerning the range of a countably additive measure ranging on an infinite dimensional Banach space, generally concern the strong or the weak closure of the range or of its convex hull.
A *positive finitely additive phenomena* in this more general setting is that usually the same properties are recovered in the infinite dimensional case: in fact it can be shown ([13, Theorem 3.2]) that if $m : \Sigma \to E$ is a finitely additive measure and E is a Banach space, then m admits a Stone extension $\widetilde{m} : \mathcal{G} \to E$ and that $R(m)$ is strongly dense in $R(\widetilde{m})$, while the weak closure of $R(m)$ is contained in that of $R(\widetilde{m})$. Hence the compactness results that hold in the countably additive case can be inherited in the finitely additive case. We refer to ([13] Section 3) for more details on this topic.
For the sake of completeness we mention that the range of a finitely additive measure with values in a topological group has been investigated also, and refer to the papers [27] and [25] for results in this sense.
Another question that might tease the reader's curiosity might be: *what is known about $R(m)$ if m has atoms?* The answer to this question when m is countably additive is given for dimension $n = 1$ in [2] and in the case $n > 1$ partly investigated in [21]. In the scalar case the result that I know is the following:

THEOREM 4.13. *[2] If $m : \Sigma \to R_0^+$ is a measure, then for $R(m)$ the following alternatives hold:*

- *$R(m)$ is finite;*

- *$R(m) = [0, m(\Omega)]$;*

- *$R(m)$ is the finite union of closed sets;*

- *$R(m)$ is a Cantor set.*

Also in the same paper some investigation is carried out in the finitely additive case.
In the case $n > 1$ in [21] it is proven that the distance between $R(m)$ and its convex hull depends on the dimension. More precisely, define for a set $A \subset I\!R^n$ the *maximum dent size* as

$$D(A) = sup\{d(x, A), x \in coA\}$$

and assume that $m : \Sigma \to I\!R^n$ is a non-negative measure; let α_m be defined as

$$\alpha_m = max\{a : m_i \text{ has an atom } A, \text{ with } m_i(A) = a\}$$

(namely every coordinate of m has atoms of mass not greater than α_m).

THEOREM 4.14. *[21]*

$$D(R(m)) \leq \alpha_m \frac{n}{2}.$$

5. Moving up and down - Extensions and restrictions

In Section 4 we have stated an extension result (Theorem 4.3) and noted as a consequence that the following fact is true:
Every countably additive non atomic measure $m : \Sigma \to R_0^+$ on an algebra admits finitely additive strongly non atomic extensions to the power set 2^Ω.

This is another *positive finitely additive phenomenon*, in the sense that the same property does not hold in the countably additive case, because of Ulam's Theorem:

THEOREM 5.1. *[35] Let us assume the continuum hypothesis, and suppose that $|\Omega| > \aleph_0$. Then if $m : 2^\Omega \to I\!R_0^+$ is non concentrated and countably additive, then $m = 0$.*

Therefore, it is impossible to extend a countably additive measure to the whole power set. It is natural to ask why one should be interested in extending to the whole power set. One reasonable argument is in integration theory: every function is measurable with respect to the power set, thus it might be desirable to have the set function defined on it as well, in order to get rid of difficult measurability troubles! On the other side, one has to keep in mind that the use of such extensions never extend the countably additive case...
An obvious remark is that Theorem 4.6 holds for a vector measure, namely if $m : \Sigma \to I\!R^n$ is a countably additive measure on an algebra Σ then it admits finitely additive extensions to the whole 2^Ω.
The rest of this section will be devoted to the converse question, that is:
Does a finitely additive measure $m : 2^\Omega \to I\!R^n$ *admit a countably additive restriction on some subfamily of* 2^Ω?
Raised in this way the question has an obvious answer: it is enough to choose $\Sigma = \{ \emptyset ,\Omega\}$ to obtain a positive answer. Positive as much as useless: it is clear that something more has to be requested in order to have a more significant restriction. In other words we want a countably additive restriction that preserves something *nice* of the original m.
For instance, having already noted that the strong non atomicity is immediately inherited by any extension, a property that one would like to preserve in restricting to Σ could be this one.

EXERCISE 5.2: Let $m : 2^\Omega \to I\!R_0^+$ be a strongly non atomic measure. Prove that for every $\varepsilon > 0$ there exists a σ-algebra Σ_ε such that m is countably additive on Σ_ε and there is a Σ_ε-measurable decomposition of Ω, say $(\Omega_1, \dots, \Omega_k)$ such that $m(\Omega_j) \leq \varepsilon$ for $j = 1, \dots, k$.

We are not going into further details about this problem, for we are going in fact to answer a more general question, that is:
Given a finitely additive strongly non atomic non negative measure $m : 2^\Omega \to I\!R^n$ *does a subalgebra* $\Sigma \subset 2^\Omega$ *exists such that:*

i) $m_{|\Sigma}$ *is countably additive;*

ii) m *is strongly non atomic;*

iii) $m(\Sigma) = R(m)$?

EXERCISE 5.3: Conditions ii) and iii) above are independent.

5.1) Produce a countably additive *atomic* probability $m : \Sigma \to R_0^+$ such that $m(\Sigma) = [0, 1]$, (thus iii) $\not\Rightarrow$ ii));

5.2) Produce a finitely additive semiconvex measure $m : 2^\Omega \to I\!R^2$ whose range is the parallelogram with vertices in $(0, 0), (1, 1)$, $\left(\frac{1}{3}, \frac{1}{2}\right), \left(\frac{2}{3}, \frac{1}{2}\right)$;

5.3) Show in this second case that there is a filtering family $(E_t)_{t\in[0,1]}$ satisfying (4.2.1), (4.2.2), (4.2.3) (with $E = \Omega$) and such that m is countably additive on the σ-algebra Σ generated by the family (thus ii) $\not\Rightarrow$ iii), since $m(\Sigma)$ is only the diagonal of the parallelogram).

We shall see that, surprisingly, the answer to the question depends upon the cardinality of the set Ω. Let us face the case $|\Omega| = \aleph_0$ first.

EXERCISE 5.4: If Ω is a set, Σ a σ-algebra on Ω and $m : \Sigma \to R_0^+$ a non atomic measure on it, then there exists a non empty set $E \in \Sigma$ such that $m(E) = 0$.

THEOREM 5.5. *If $\Omega = I\!N$, Σ is a σ-algebra on Ω and $m : \Sigma \to R_0^+$ is a non atomic measure, then $m = 0$.*

Proof. Let $(I\!N, \Sigma_0, m_0)$ be the completion of $(I\!N, \Sigma, m)$, and set

$$K = \{k \in I\!N, \{k\} \in \Sigma_0\}.$$

Clearly $K \in \Sigma_0$ and from the strong non atomicity of m_0, $\{k\} \in \Sigma_0$ iff $\{k\}$ is a subset of a m-null set in Σ. Therefore $m_0(\{k\}) = 0$ for every $k \in K$, and by the countable additivity $m_0(K) = 0$.
Let $I\!N^* = I\!N \setminus K$, and $\Sigma^* = \{A \setminus K, A \in \Sigma_0\}$. Then $(I\!N^*, \Sigma^*, m_0)$ is a measure space, with m_0 non atomic. Let $B \in \Sigma^*$, with $m_0(B) = 0$. Thus there exists $A \in \Sigma_0$ such that $B = A \setminus K$; therefore $m_0(A) = 0$ and so $\{a\} \in \Sigma_0$ $\forall\, a \in A$. Consequently $A \subset K$ whence $B = \emptyset$. From Exercise 5.4 $m_0 = 0$ and then m is null on Σ. $\square$

Corollary 5.6. *Let $m : 2^{\mathbb{N}} \to \mathbb{R}_0^+$ be a strongly non atomic non trivial finitely additive measure. Then no algebra Σ exists on $\mathbb{N}$ such that $m_{|_\Sigma}$ is a non atomic countably additive measure.*

It is clear now that the interesting case becomes that of an uncountable set Ω. Throughout the rest of this section we shall assume that $|\Omega| = c$.
We first establish an affirmative answer to the question, and will later comment about the result.

Theorem 5.7. *[9] Let $m : 2^\Omega \to \mathbb{R}^n$ be any bounded finitely additive measure. Then there exists an algebra Σ such that i) $m_{|_\Sigma}$ is countably additive, and ii) $m(\Sigma) = R(m)$. Furthermore, iii) if m is strongly non atomic, then Σ can be chosen in such a way that $m_{|_\Sigma}$ is strongly non atomic too.*

Proof. It is easy to show that without loss of generality the theorem can be proved for a non negative m, with $m_i(\Omega) = 1$ for every component m_i of m.
Also, by decomposing Ω into at least c many pairwise disjoint sets, each of cardinality c, and by making use of the countable chain condition (which is verified for the range of m is finitely dimensional), that is
(CCC) Given a family $\{A_t\}_t$ of pairwise disjoint sets, then $m(A_t) \neq 0$ for at most countably many indices t,
we see that there exists a set $X \subset \Omega$ with $|X| = c$, $m(X) = 0$ and $|\Omega \setminus X| = |\Omega|$.
Put $Y = \Omega \setminus X$. Since $|R(m)| \leq c$, there is a family $\mathcal{A} \subset 2^Y$ with $|\mathcal{A}| = c$ and $m(\mathcal{A}) = R(m)$.
If m is strongly non atomic, replace $\mathcal{A}$ with the family $\mathcal{A} \cup \{Y_t, t \in [0,1]\}$ where the sets Y_t are those whose existence has been established in Lemma 4.2.
Now, let $\mathcal{F}$ be the algebra on Y generated by $\mathcal{A}$. Then $|\mathcal{F}| = c$. Let S be the Stone space of $\mathcal{F}$, and let $\varphi : \mathcal{F} \to \mathcal{G}$ denote the Stone isomorphism. Set now $\widetilde{m}(C) = m(\varphi^{-1}(C)), C \in \mathcal{G}$ and keep the notation $\widetilde{m}$ to denote the extension of this countably additive set function to the whole $\mathcal{G}_\sigma$: since $|\mathcal{F}| = |\mathcal{G}| = c$, then $|\mathcal{G}_\sigma| = c$.
Now, applying the Axiom of Choice, there exists a set $T \subset S$ such

that for each non empty $G \in \mathcal{G}_\sigma$, $T \cap G$ is a singleton. Thus $|T| = c$. Set now $\widetilde{m_T} : \mathcal{G}_\sigma \cap T \to I\!R^n$ by putting

$$\widetilde{m_T}(B) = \widetilde{m}(B \cap T).$$

Since $T \cap B = \emptyset$ implies $B = \emptyset$, $\widetilde{m_T}$ is well defined; furthermore it is countably additive and $\widetilde{m_T}(S) = (1, \dots, 1)$. Finally for every $H \in \mathcal{F}$ we find

$$m(H) = \widetilde{m}(\varphi(H)) = \widetilde{m_T}(\varphi(H) \cap T). \tag{2}$$

Since $|T| = |X| = c$, by means of any bijection we can identify X with T and set therefore $\Omega = Y \cup T$ with Y and T disjoint and $m(T) = 0$. Let us define Σ as

$$\Sigma = \{H \cup (\varphi(H) \cap T), H \in \mathcal{F}\}.$$

$\mathcal{F}$ being a field, Σ is a field too; indeed Σ is the requested field, as we are going to show.
We first prove that i) holds: indeed, if $(F_k)_k$ is a sequence in Σ decreasing to $\emptyset$, then there are H_k in $\mathcal{F}$ such that $F_k = H_k \cup (\varphi(H_k) \cap T)$ for every k. As $F_k \cap T = \varphi(H_k) \cap T$, it follows that $\varphi(H_k) \cap T \downarrow \emptyset$, whence $\widetilde{m_T}(\varphi(H_k) \cap T) \to 0$, that is $m(H_k) \to 0$ from (1). But $m(F_k) = m(H_k)$ since $m(T) = 0$, whence i) follows.
We now prove ii). For every $x \in R(m)$ there is $H \in \mathcal{F}$ with $m(H) = x$. Then $F = H \cup (\varphi(H) \cap T) \in \Sigma$ and $m(F) = m(H) = x$, whence $m(\Sigma) = R(m)$.

Finally, if m is strongly non atomic, for each $\varepsilon > 0$ choose $k \in I\!N$ such that $\frac{1}{k} < \varepsilon$. We can find in $\mathcal{F}$ sets $H_1, \dots, H_k$ such that $m_i(H_j) = \frac{1}{k}$ for each $i = 1, \dots, n$ and for every $j = 1, \dots, k$, (for instance take $H_1 = Y_{\frac{1}{k}}$, $H_2 = Y_{\frac{2}{k}} \setminus Y_{\frac{1}{k}}$ and so on). Then clearly, for $F_j = H_j \cup (\varphi(H_j) \cap T)$, we have

$$m(F_j) = m(H_j) = \left(\frac{1}{k}, \dots, \frac{1}{k}\right)$$

which proves iii). $\square$

Remark 5.8. *The complete solution of the question went along a path having Theorem 5.7 has one of the last steps.*
The problem was first investigated and solved in [14] for a positive scalar finitely additive measure. The scalar techniques though could not apply in higher dimension, the trouble being that sketched in Exercise 5.3.

In [28] the case of $n = 2$ was solved by means of ad hoc techniques. These included in particular the *Hereditarily Overlapping Boundary Property* (HOBP), which proved to be a property of interest by itself, in particular for application to Radon-Nikodym Theorems: it is stated and proven in Theorem 5.9 below.
Again the two-dimensional techniques could not be carried out in higher dimensions, and eventually Theorem 5.7 appeared. Finally, a new constructive proof was given in [17], solving the problem in the general case when $m : 2^\Omega \to G$ where G is an abelian topological group.

To state and prove the (HOBP) we shall need some notation.

Let $m : \Sigma \to I\!R^2$ be a non negative semiconvex finitely additive measure. For the sake of simplicity, we will assume $m_i(A) = 1$, $i = 1, 2$. Let $A \in \Sigma$, and define $R_A = \{m(B) : B \in \Sigma \cap A\}$.
In Exercise 4.8 we have defined the maps g, G. When we consider only pairs in R_A we shall use the notation g_A, G_A, and keep the notation g, G when $A = \Omega$.
Clearly, for every $A \in \Sigma$, g_A, G_A have the same properties of g, G established in Exercise 4.8.
Finally we set:

$$\begin{aligned}
\partial_A^+ &= \{(x, G_A(x)) : x \in [0, m_1(A)]\} \cup \{(0, y) : y \in [0, G_A(0)]\}, \\
\partial_A^- &= \{(x, g_A(x)) : x \in [0, m_1(A)]\} \cup \\
&\quad\ \{(m_1(A), y) : y \in [g_A(m_1(A)), m_2(A)]\}.
\end{aligned}$$

Then it is easy to prove that $\partial R_A = \partial_A^- \cup \partial_A^+$.

Theorem 5.9. *[28] (Hereditarily Overlapping Boundary Property) Let $m : 2^\Omega \to I\!R^2$ be a non negative strongly non atomic finitely additive measure with closed range. Then the following assertions hold:*

(5.9.1) for every set $A \subset \Omega$ such that $m(A) \in \partial^-$, $\partial_A^- = \partial^-$ (in the sense that for $t \in [0, m_1(A)]$, $g_A(t) = g(t)$, and, if $m_1(A) = 1$ every point $(0, y)$ with $g_A(1) \le y \le m_2(A)$ belongs to ∂_A^-);

(5.9.2) for every set $A \subset \Omega$ such that $m(A) \in \partial^+$, $\partial_A^+ = \partial^+$ in the same sense.

Proof. We will prove only (5.9.1) since (5.9.2) can be proven analogously.
Let $P \in \partial^-$; since $R(m)$ is closed, there is $A \subset \Omega$ such that $m(A) = P$.
We will consider two different cases:
Assume first that $m_2(A) = g(m_1(A))$: set $B = A^c$. We shall denote by g'_+, g'_-, G'_+, G'_- the right and left hand side derivatives of g and G (whose existence is a consequence of Exercise 4.8).
By the same Exercise we have that

$$G'_+(t) = g'_-(1-t), \; G'_-(t) = g'_+(1-t).$$

Let $\alpha = g'_-(m_1(A))$, $\beta = g'_+(m_1(A))$. Then the properties of g imply that $0 \le \alpha \le \beta \le +\infty$. Moreover necessarily $\alpha < +\infty$.
Assume now that $\beta < +\infty$. By the convexity of g for each $0 < x < m_1(A)$ it is $g'_+(x) \le \alpha$.
Consider the two parallelograms

$$\begin{aligned} Q_A &= \{(x,y) : y \in [0, m_2(A)], y \le \alpha x \le y + \alpha m_1(A) - m_2(A)\} \\ Q_B &= \{(x,y) : x \in [0, m_1(B)], \beta x \le y \le \beta x + m_2(B) - \beta m_1(B)\}. \end{aligned}$$

Then $R_A \subset Q_A$ and $R_B \subset Q_B$. (Indeed the lines $y = \beta x$ and $y = \beta x + m_2(B) - \beta m_1(B)$ support R_B at $(0,0)$ and $m(B)$ respectively, and analogously the lines $y = \alpha x$ and $y = \alpha x - \alpha m_1(A) + m_2(A)$ support R_A at $(0,0)$ and $m(A)$).
Therefore, if $\alpha < \beta$ then $R_A \cap R_B = (0,0)$, while, if $\alpha = \beta$ the intersection is a segment of the line $y = \alpha x$.

Let now $C \subset \Omega$ be such that $m_1(C) < m_1(A)$ and $C \in \partial^-$. Then, $m_2(C) = g(m_1(C))$. From the convexity of g the function $f = \alpha x - g(x)$ satisfies $f'_\pm = \alpha - g'_\pm \ge 0$, for $x \le m_1(A)$. Hence f is non decreasing, whence

$$0 \le \alpha m_1(C) - m_2(C) \le \alpha m_1(A) - m_2(A). \tag{3}$$

Set now

$$\sigma := \frac{m_2(A) - m_2(C)}{m_1(A) - m_1(C)}.$$

From (3) two possible cases may happen:
If $\sigma < \alpha$ then $[m(C) - Q_B] \cap R_A = \{m(C)\}$. In this case then $m(C) - m(C \cap B) = m(C)$, namely $m(C \cap B) = 0$. Therefore $m(C) = m(C \cap A) \in R_A$.
If $\sigma = \alpha$, g is linear between $m(C)$ and $m(A)$. Put

$$x_0 = \min\{x : \frac{m_2(A) - g(x)}{m_1(A) - x} = \alpha\}.$$

Then $x_0 \leq m_1(C) < m_1(A)$; let $C_0 \subset \Omega$ be such that $m(C_0) = (x_0, g(x_0))$. Then, analogously to the previous case, $[m(C_0) - Q_B] \cap R_A = \{m(C_0)\}$, which again yields that $m(C_0) = m(C_0 \cap A) \in R_A$. By the convexity of R_A the whole segment joining $m(C_0)$ with $m(A)$ lies in R_A, and since $m(C)$ belongs to this segment, we have obtained in this second case that $m(C) \in R_A$.
In the case that $\beta = +\infty$ the same reasoning will apply with

$$Q_B = \{(0, y), 0 \leq y \leq m_2(B)\},$$

since in this case necessarily $m_1(A) = 1$.
It only remains to discuss the case of $m_1(A)=1$ and $m_2(A)>g(m_1(A))$. Let C be a set such that $m_1(C) = 1$ and $m_2(C) = g(1)$; then from the previous case with $\beta = +\infty$ we can suppose that $C \subset A$; again a convexity argument shows that the segment joining $m(C)$ with $m(A)$ belongs to R_A, and the proof is now complete. □

Notes on Section 5

The extension result established with Theorem 5.7 only gives an algebra as a solution of the question. It could be desirable however to have a σ-algebra on which m is countably additive *and* preserves the range and the strong non atomicity.
Clearly, Example 4.11 shows that this should not be expected in general: indeed, if m admits a countably additive restriction on a σ-algebra preserving the range and the strong non atomicity, then

because of Liapounoff's result its range should be closed.
It would be nice if this last condition could be also sufficient to get the desired restriction; unfortunately, one should not expect this to be the case. Indeed, while there is just one countably additive extension from an algebra to the generated σ-algebra, there could be many different finitely additive extensions. A concrete example of this phenomenon is given in [14]. Also the situation is faced for not necessarily strongly non atomic finitely additive measures in [18], under different points of view.
So, even when $R(m)$ is closed, what could enable us to say that our m precisely coincides with the countably additive extension to Σ_σ of $m_{|\Sigma}$ whose existence we have established in Theorem 5.7 ? In [14] some results in this sense were given in the case of $n = 1$.
In [28] the problem of the existence of a σ-algebra was given an approximate answer, namely there always is an increasing sequence of σ-algebras on which m is countably additive and strongly non atomic, and such that the sequence of subranges approximates $R(m)$ in the Lebesgue measure of $\mathbb{R}^n$. Compare with Exercise 5.2.

6. Can we cure it? - Pathologies in Integration Theory

Integration is the field were most of the finitely additive pathologies occurr.
Throughout this section $m : \Sigma \to \mathbb{R}$ will be a bounded finitely additive measure, with Σ a σ-algebra on a set Ω, and we shall assume for the sake of simplicity that

$$\begin{aligned} m^+(\Omega) &= sup\{m(A) : A \in \Sigma\} < +\infty \\ m^-(\Omega) &= -inf\{m(A) : A \in \Sigma\} < +\infty. \end{aligned}$$

The finitely additive measures defined on Σ by

$$\begin{aligned} m^+(A) &:= sup\{m(E) : E \in \Sigma_A\}, \\ m^-(A) &:= -inf\{m(E) : E \in \Sigma_A\}. \end{aligned}$$

are called the *positive* and *negative variations* of m: one easily obtains the *Jordan decomposition* for m

$$m = m^+ - m^-,$$

while the *total variation* of m is defined as $|m| := m^+ + m^-$.

DEFINITION 6.1. *Let $m : \Sigma \to R_0^+$ be a finitely additive measure; following [20] and [8] we shall adopt the notation*

$$m^*(A) = inf\{m(E) : E \supset A\}.$$

*Given $m : \Sigma \to I\!R$, we will say that A is m-*null *provided $|m|^*(A) = 0$, and a map $f : \Omega \to I\!R$ is said to be m-*null if $\{|f| > \varepsilon\}$ is m-null for every $\varepsilon > 0$.

When m is finitely but not countably additive, m-null functions need not to vanish m-a.e., as the following example shows.

EXAMPLE 6.2. *Let $\Omega = I\!N$ and let Σ be the σ-algebra generated by the field of finite and cofinite subsets of Ω. Let μ be any finitely additive extension to Σ of the (already only finitely additive) set function defined as*

$$\mu(A) = \begin{cases} 0, & \textit{if } A \textit{ is finite} \\ 1, & \textit{if } A \textit{ is cofinite} \end{cases}$$

Let the function f be defined on Ω by $f(k) = \dfrac{1}{k}$. Then f is μ-null, since $\{|f| > \varepsilon\}$ is finite for each $\varepsilon > 0$, but $|\mu|^*(\{f \neq 0\}) = 1$.

The first pathology occurring in the finitely additive setting concerns the convergence of sequences of functions.

EXERCISE 6.3: Let $m : \Sigma \to R_0^+$ be a finitely non countably additive measure:

(12.1) Find a sequence $(f_n)_n$ converging to 0 m-a.e. but such that $m(\{|f_n| > \alpha\}) \not\to 0$ for some $\alpha > 0$.

(22.2) Find a sequence $(f_n)_n$ such that $f_n \overset{m}{\to} 0$ but no subsequence of $(f_n)_n$ converges to 0 m-a.e.

(The definition of convergence m-a.e. and convergence in m-measure are meant in exactly the same sense as in the countably additive setting.)

Precisely because of this phenomenon, the equality m-a.e. has to be replaced by the more suitable equivalence relation:

$$f \cong g \ [m] \qquad \text{if } f - g \text{ is } m\text{-null} \tag{4}$$

and the convergence in m-measure has to be replaced by the hazy convergence:

DEFINITION 6.4. *A sequence $(f_n)_n$* converges hazily*to f if*

$$lim_{n\to+\infty}|m|^*(\{|f_n - f| > \varepsilon\}) = 0,$$

for every $\varepsilon > 0$.

REMARK 6.5. *If f_n and f are Σ-measurable (in the usual sense, for (Ω, Σ) is a measurable space) then $(f_n)_n$ hazily converges to f iff $(f_n)_n$ $|m|$-converges to f. In fact, when $f_n - f$ is Σ-measurable, it is clear that*

$$|m|^*(\{|f_n - f| > \varepsilon\}) = |m|(\{|f_n - f| > \varepsilon\}).$$

From now on our functions will be Σ-measurable, unless differently stated.
The first problem that one has to face in order to define integration is the measurability of f: to explain this we begin by observing that one of the main engines of a theory of integration are convergence results, namely establishing which topologies make the integral a *continuous operator*.

In the countably additive case, the integral of f is usually defined by means of the limit of the sequence of the integrals of a sequence of simple functions converging to f m-a.e. As already noticed this convergence is not the suitable one in the finitely additive setting.
We begin by defining the right form of measurability.

DEFINITION 6.6. *A function f is said to be* totally measurable *if there exists a sequence of Σ-measurable simple functions that converges to f in $|m|$-measure.*

DEFINITION 6.7. *A function f is said to be* m-integrable *if there exists a sequence of simple functions $(f_n)_n$ such that*

(6.7.1) $(f_n)_n$ *converges in* $|m|$*-measure to* f*;*

(6.7.2) $\lim_{n,k\to+\infty} \int |f_n - f_k| d|m| = 0.$

By (6.7.2) the limit $\lim_{n\to+\infty} \int_E f_n dm$ *exists for every* $E \in \Sigma$*, and a routine argument proves its independence from the sequence* $(f_n)_n$*. Therefore we define*

$$\int_E f dm := \lim_{n\to+\infty} \int_E f_n dm, \quad E \in \Sigma.$$

For the properties of this integral we refer the reader to [20] Chapter III or to [8] Chapter 4.
Another approach is that of the monotone integral

DEFINITION 6.8. *Let* $m : \Sigma \to R_0^+$ *be a finitely additive measure, and let* $f : \Omega \to I\!R_0^+$ *be a* Σ*-measurable function. Define the distribution function of* f *as*

$$\Phi(t) = m(\{f > t\}), \quad t \in I\!R_0^+.$$

f *is said to be* integrable in the monotone sense *(or briefly* $(\widehat{\cdot})$ *integrable)* if Φ *is Lebesgue integrable on* $I\!R_0^+$*. In this case we set*

$$\widehat{\int}_E f dm := \int_0^{+\infty} \Phi^E(t) dt.$$

This definition extends to the case of $f : \Omega \to I\!R$ *by the decomposition* $f = f^+ - f^-$ *as well as to the case of* $m : \Sigma \to I\!R$ *by means of the Jordan decomposition.*

Apparently Definition 6.8 applies to a class larger than that of totally measurable functions. Indeed there might be Σ-measurable functions that are not totally measurable. The next Exercise enlightens what happens in the case of the monotone integral.

EXERCISE 6.9: Let $f : \Omega \to I\!R_0^+$, $m : \Sigma \to R_0^+$.

(6.9.1) If f is bounded, then it is totally measurable, m-integrable and $(\widehat{\cdot})$-integrable; furthermore

$$\int f dm = \widehat{\int} f dm;$$

(6.9.2) if $(f_n)_n$ is a sequence of totally measurable functions and $f_n \overset{m}{\to} f$ then f is totally measurable;

(6.9.3) if f is $\widehat{(\cdot)}$-integrable, then $f \wedge n \overset{m}{\to} f$ (use Markov inequality).

As a consequence if f is $\widehat{(\cdot)}$- integrable, then f is totally measurable. The remaining of the comparison is based upon the Vitali Theorem for both integrals:

THEOREM 6.10. *[20, 32] Let $(f_n)_n$ be a sequence of m-integrable (resp. $\widehat{(\cdot)}$-integrable) functions such that*

(6.10.1) $(f_n)_n$ m-converges to f;

(6.10.2) the sequence of finitely additive measures defined by

$$\left(\int_\bullet f_n dm_n\right) \; (resp. (\widehat{\int_\bullet} f_n dm)_n)$$

is m-continuous (in the sense of Definition 4.3) uniformly with respect to $n \in \mathbb{N}$.

Then f is m-integrable (resp. $\widehat{(\cdot)}$-integrable) and

$$\int_\bullet f dm = \lim_{n \to +\infty} \int_\bullet f_n dm.$$

(resp. $\widehat{\int_\bullet} f dm = \lim_{n \to +\infty} \widehat{\int_\bullet} f_n dm$*).*

COROLLARY 6.11. *Let f be Σ-measurable. Then f is $\widehat{(\cdot)}$-integrable iff f is m-integrable. Moreover in this case*

$$\int_\bullet f dm = \widehat{\int_\bullet} f dm.$$

The second engine of an Integration Theory is the Radon-Nikodym Theorem, which ensures that in some spaces the integral is an invertible operator. And here sits another finitely additive phenomenon!

DEFINITION 6.12. *Let $m : \Sigma \to \mathbb{R}$ be a finitely additive measure; a pair of Σ-measurable sets (P, N) is a* Hahn decomposition *if: $P \cap N$ is a m-null set, $P \cup N = \Omega$ and*

$$m^+(P) = m^+(\Omega), \; m^-(N) = m^-(\Omega).$$

Consequently, if $A \in \Sigma \cap P$ then $m(A) = m^+(A) \geq 0$ whence $m^-(A) = 0$, namely P is a positive set. *(Analogously N is a* negative set*).*

Differently from what happens with countably additive measures, the assumption $m_2 \ll m_1$ is not enough to ensure that a Radon-Nikodym density exists, namely that m_2 is in fact the indefinite integral of some m_1-integrable function, as the following Example shows.

EXAMPLE 6.13. *Consider the finitely additive measures m_l and δ of Example 4.11. Let $m_1 = m_L$ and $m_2 = m_L + \delta$; then $m_1 \ll m_2$, but $\dfrac{dm_1}{dm_2}$ does not exist. In fact, if we assume by contradiction that a density f exists, then the sets $P = \left\{ f > \dfrac{1}{2} \right\}$ and $N = P^c$ would be a Hahn decomposition for $m_1 - \dfrac{1}{2} m_2 = \dfrac{1}{2}(m_L - \delta)$ and therefore for $m_L - \delta$. But then $(m_L - \delta)(P) = 1$ and $(m_L - \delta)(N) = -1$, thus contradicting Remark 4.12.*

Indeed there is a strong link between the existence of Hahn decompositions and that of the Radon-Nikodym density, as stated in the following result, that is the finitely additive version of the original Theorem as due to Greco [22].

THEOREM 6.14. *[22] Let $m_1, m_2 : \Sigma \to \mathbb{R}_0^+$ be two finitely additive measures. The following are equivalent:*

(6.14.1) there exists a map f such that

$$m_2(E) = \int_E f dm_1 \quad \forall E \in \Sigma;$$

(6.14.2) there exists a family of sets $\{A_r\}_{r \in I\!R_0^+}$ in Σ such that (A_r, A_r^c) is a Hahn decomposition for $m_2 - rm_1$ and $\lim_{n \to +\infty} m_2(A_r) = 0$.

Proof. To show that (6.14.1) $\Rightarrow$ (6.14.2) observe that if f is the Radon-Nikodym density, then the sets $A_r = \{f \geq r\}$ and A_r^c form a Hahn decomposition for $m_2 - rm_1$; in fact for $E \in \Sigma \cap A_r$

$$m_2(E) = \int_E f dm_1 \geq rm_1(E) \tag{5}$$

and analogously $m_2(E) \leq rm_1(E)$ for every $E \in \Sigma \cap A_r^c$. Now, from (5), $m_1(A_r) \leq \frac{1}{r} m_2(A_r) \leq \frac{1}{r} m_2(\Omega)$ and therefore $\lim_{r \to +\infty} m_1(A_r) = 0$. But from (6.14.1) $m_2 \ll m_1$ and so $\lim_{r \to +\infty} m_2(A_r) = 0$.
We shall now prove the converse inclusion.
Consider the family of sets $\{B_r\}_{r \in \mathbf{Q}(2)}$ defined inductively by

$$B_0 = \Omega$$

$$B_n = A_n \cap B_{n-1} \quad \text{for} \quad n \in I\!N$$

and

$$B_{(2k+1)/2^{n+1}} = \left(A_{(2k+1)/2^{n+1}} \cap B_{(k-1)/2^n}\right) \cup B_{k/2^n} \quad \text{for} \quad n, k \in I\!N.$$

Then the family is decreasing with respect to increasing r; therefore

$$\lim_{r \to +\infty} m_2(B_r) = inf\{m_2(B_r), r \in \mathbf{Q}(2)\},$$

and since by definition $m_2(B_n) \leq m_2(A_n) = 0$, we have shown that

$$\lim_{r \to +\infty} m_2(B_r) = 0.$$

Also (B_r, B_r^c) is a Hahn decomposition for $m_2 - rm_1$. The proof of this fact follows from the following steps,

– if $r > s$ then $A_r \cap B_s$ is a positive set for $m_2 - rm_1$;

– if $r > s > t$ then $(A_r \cap B_s) \cup B_t$ is a positive set for $m_2 - tm_1$;

(we are sparing the reader of the lenghty details).
Now the map defined as

$$f(\omega) = sup\{r \in \mathbf{Q}(2) : \omega \in B_r\}$$

is the Radon-Nikodym derivative. To show this (again we shall just sketch the main passages and skip the details) consider the sequence of simple functions defined by $f_n := \frac{1}{2^n} \sum_{k=1}^{n2^n} 1_{B_{k/2^n}}$. Then

$$(f \wedge n) - \left(f \wedge \frac{1}{2^n} \right) \leq f_n \leq f$$

for every $n \in I\!N$ and therefore it is a defining sequence. Now for $E \in \Sigma$

$$\begin{aligned} \int_E f_n dm_1 &= \frac{1}{2^n} \sum_{k=1}^{n2^n} m_1(E \cap B_{k/2^n}) \\ &= \sum_{k=1}^{n2^n-1} \frac{k}{2^n} \left[m_1(E \cap B_{k/2^n}) - m_1(E \cap B_{(k+1)/2^n}) \right] . \end{aligned} \tag{6}$$

Then, since B_r is a positive set for $m_2 - rm_1$

$$\begin{aligned} \int_E f_n dm_1 &\leq \sum_{k=1}^{n2^n-1} \left[m_2(E \cap B_{k/2^n}) - m_2(E \cap B_{(k+1)/2^n}) \right] = \\ &= m_2(E \cap B_{1/2^n}) \pm m_2(E \cap B_n) \leq m_2(E) \end{aligned}$$

and letting $n \to \infty$ we find $\int_E f dm_1 \leq m_2(E)$.
Conversely, writing (6) as

$$\int_E f_n dm_1 = \sum_{k=1}^{n2^n-1} \left[\frac{k+1}{2^n} \left(m_1(E \cap B_{k/2^n}) - m_1(E \cap B_{(k+1)/2^n}) \right) \right]$$

$$+ n m_1(E \cap B_n) - \frac{1}{2^n} \sum_{k=1}^{n2^n-1} \left[m_1(E \cap B_{k/2^n}) - m_1(E \cap B_{(k+1)/2^n}) \right]$$

and using the fact that B_r^c is a negative set for $m_2 - rm_1$ and that $B_{(k+1)/2^n} \subset B_{k/2^n}$ we obtain

$$\begin{aligned}
\int_E f_n dm_1 &\geq m_2(E \cap B_{1/2^n}) - m_2(E \cap B_n) + nm_1(E \cap B_n) \\
&- \frac{1}{2^n}\left[m_1(E \cap B_{1/2^n}) - m_1(E \cap B_n)\right] \\
&\geq m_2(E \cap B_{1/2^n}) - m_2(E \cap B_n) \\
&- \frac{1}{2^n}\left[m_1(E \cap B_{1/2^n}) - m_1(E \cap B_n)\right]
\end{aligned}$$

which in turn implies that $\int_E f dm_1 \geq m_2(E)$ □

EXERCISE 6.15: Let $m_1, m_2 : \Sigma \to I\!R_0^+$ be two strongly non atomic finitely additive measures, $m = (m_1, m_2)$ and let (A_r, A_r^c) be a Hahn decomposition of $m_2 - rm_1$; then $m(A_r) \in \partial R(m)$.

PROPOSITION 6.16. *[15] Let $m_1, m_2 : \Sigma \to I\!R_0^+$ be two strongly non atomic measures, with $m_2 \ll m_1$, and let $m := (m_1, m_2)$. If $R(m)$ is closed, then there exists a Radon-Nikodym density $\dfrac{dm_2}{dm_1}$.*

Proof. From Theorem 4.6 and Theorem 4.7 $R(m)$ is compact and convex. We will show that (6.14.2) is satisfied. Let $\alpha = g'_+(0)$ and fix

$$r \in \left[\alpha, \frac{m_2(\Omega)}{m_1(\Omega)}\right].$$

By Exercise 4.8, g is convex; hence there is a point $x_r \in [0, m_1(\Omega)]$ such that $g(x_r) = rx_r$, and there is $t_r \in [0, x_r]$ such that the line $y(x) = g(t_r) + r(x - t_r)$ supports $R(m)$ at $(t_r, g(t_r))$. By the closedness of $R(m)$ there exists $A_r \in \Sigma$ such that $m(A_r) = (t_r, g(t_r))$, so that $m(A_r) \in \partial R(m)$.
From Theorem 5.9, $g_{A_r}(s) = g(s)$ for all $s \in [0, t_r]$ and hence $(g_{A_r})'_-(t_r) = g'_-(t_r) \leq r$. By the central simmetry of the subrange R_{A_r} then $(G_{A_r})'_+(0) \leq r$. Therefore we have that R_{A_r} is contained in the cone delimited by the lines $y(x) = \alpha x$ and $y(x) = rx$, while $R_{A_r^c}$ is above the line $y(x) = rx$. Hence for $B \in \Sigma \cap A_r$ $m_2(B) \geq rm_1(B)$,

while for $B \in \Sigma \cap A_r^c$ $m_2(B) \le r m_1(B)$.
Let $\beta = G'_+(0)$, and define the family $\{F_r\}_{r \in I\!R_0^+}$ by putting:

$$F_r = \begin{cases} \Omega & \text{if } r \in [0, \alpha] \\ A_r & \text{if } r \in \left]\alpha, \dfrac{m_2(\Omega)}{m_1(\Omega)}\right] \\ A_r^c & \text{if } r \in \left]\dfrac{m_2(\Omega)}{m_1(\Omega)}, \beta\right] \\ \emptyset & \text{if } r > \beta. \end{cases} \tag{7}$$

Note that if $\beta < +\infty$ then $m_2(F_r) = 0$ for $r > \beta$, while, if $\beta = +\infty$ since $m_2(F_r) = m_2(A_r^c)$ for r large, and $m_2(A_r^c) = G(m_1(A_r^c)$ we have that $\lim_{r \to +\infty} m_2(A_r^c) = 0$, since $m_2 \ll m_1$ and $m_1(A_r^c) \to 0$. The assertion follows then from Theorem 6.14. □

It would be nice if the above Proposition could be reverted, namely if the closedness of $R(m)$ would charachterize the pairs $m_2 \ll m_1$ admitting a density. Unfortunately, this is not the case, as the following example shows.

Example 6.17. *[15] Let $\Omega, \Sigma, m_1 = m_L + \delta$ be as in Example 4.11. Then m_1 is strongly non atomic, and, as one is easily convinced, it enjoys the property*
(σ) the ideal of m_1-null sets is stable under countable unions;
Also remember that $m_1([0,t[) = t$ for $t < 1$ and $m_1(\Omega) = 2$. Let $f : [0,1[\to I\!R_0^+$ be defined as $f(x) = 2(1-x)$, and consider the finitely additive measure

$$m_2(A) = \int_A f dm_1.$$

Then clearly f is a density. Note that, from Corollary 6.11,

$$\begin{aligned} m_2(\Omega) &= \int_0^1 f dm_1 = \int_0^2 m_1(\{f > t\}) dt \\ &= \int_0^2 m_1(\left[0, 1 - \frac{t}{2}\right]) dt = \int_0^2 \left(1 - \frac{t}{2}\right) dt = 1 \end{aligned}$$

while, if $A = [0, y[$ $y < 1$ then

$$m_1\left(A\cap\left[0,1-\frac{t}{2}\right[\right)=m_1\left([0,y[\cap\left[0,1-\frac{t}{2}\right[\right)=y\wedge\left(1-\frac{t}{2}\right)$$

whence

$$\begin{aligned} m_2(A) &= \int_0^2 m_1(A\cap\{f>t\})dt \\ &= \int_0^{2(1-y)} y dt + \int_{2(1-y)}^2 \left(1-\frac{t}{2}\right)dt \\ &= 2y-y^2. \end{aligned}$$

Hence $(1,1)\in\overline{R(m)}$.
Suppose that $m_2(E)=0$ *: then* $\int_0^2 m_1\left(E\cap\left[0,1-\frac{t}{2}\right]\right)dt=0$. *Necessarily* $m_1([0,s[\cap E)=0$ *for every* $s\in[0,1[$.
From (σ) *it follows that* $m_1(E)=0$. *Therefore* $m_2(E)=1$ *implies that* $m_1(E)=2$, *that is* $(1,1)\notin R(m)$.

In [15] the following charachterization is proven

Definition 6.18. *Let* K *be a convex subset of* $I\!R^n$; $P\in\partial K$ *is an* exposed point for K *if every hyperplane* H *supporting* K *at* P *is such that* $H\cap K=\{P\}$. *The set of exposed points of* K *is denoted by* $ExpK$.

Theorem 6.19. *[15] Let* $m_1,m_2:\Sigma\to I\!R_0^+$ *be two strongly non atomic finitely additive measures, with* $m_2\ll m_1$, *and let* $m=(m_1,m_2)$. *Then the following are equivalent:*

(6.19.1) there exists $\dfrac{dm_2}{dm_1}$;

(6.19.2) for every $P\in Exp\overline{R(m)}$ *there is* $A\in\Sigma$ *such that* $m(A)=P$.

The charachterization of Theorem 6.19 without the assumption that m_i is strongly non atomic, and with condition (6.19.2) replaced by

$$\forall P \in Exp\ \overline{co}R(m)\ \exists A \in \Sigma \qquad \text{such that} \quad m(A) = P;$$

has been proven in [6] . The proofs of both results are rather long and difficult and therefore we shall not reproduce them in these notes.
No matter how short the author would like it to be, this exposition would be too deficient if it would neglect another finitely additive phenomenon which can occurr in connection with Integration Theory: *the fact that $L^p(m)$ is not necessarily complete.* To convince the reader of this, we shall report a result from [3] : this is a very recent and complete update of the book [8] relative to this topic.

DEFINITION 6.20. *Let $m : \Sigma \to R_0^+$ be a finitely additive measure. In (4) we have defined the equivalence relation $f \cong g\ [m]$. We shall denote by $\mathcal{L}^1(\Omega, \Sigma, m)$ (or more shortly $\mathcal{L}^1(m)$) the vector space of equivalence classes $[f]$ of m-integrable functions, normed by the usual norm*

$$\|\ [f]\ \|_1 = \int_\Omega |f| dm.$$

DEFINITION 6.21. *Let $m : \Sigma \to R_0^+$ be a finitely additive measure, h the Stone isomorphism introduced in Section 2. For any simple function $f : \Omega \to I\!R$, say $f = \sum_{i=1}^n c_i 1_{A_i}$, we set $\overline{f} : S \to I\!R$ for the simple function $\overline{f} = \sum_{i=1}^n c_i 1_{h(A_i)}$. Note that $\int f dm = \int \overline{f} d\widetilde{m}$.*
If f is an m-integrable function, and $(f_n)_n$ is a defining sequence, then $\lim_{n,k} \int_S |\overline{f_n} - \overline{f_k}| d\widetilde{m} = 0$. Since $\widetilde{m}$ is countably additive, $L^1(\widetilde{m})$ is complete; hence there exists $\overline{f} \in L^1(\widetilde{m})$ such that $\overline{f_n} \xrightarrow{L^1} \overline{f}$. We shall call $\overline{f}$ the Stone extension *of f.*
Note that for every $A \in \Sigma$,

$$\int_A f dm = \int_{h(A)} \overline{f} d\widetilde{m}. \tag{8}$$

THEOREM 6.22. *[3] Let $m : \Sigma \to R_0^+$ be a finitely additive measure. Then $\mathcal{L}^1(\widetilde{m})$ is the completion of $\mathcal{L}^1(m)$.*

THEOREM 6.23. *[3] Let $m : \Sigma \to R_0^+$ be a finitely additive measure. If $\mathcal{L}^1(m)$ is complete, then for every $\lambda \ll m$ there exists an m-integrable function f such that*

$$\lambda(A) = \int_A f dm \quad A \in \Sigma.$$

Proof. By the previous result, if $\mathcal{L}^1(m)$ is complete, then $\mathcal{L}^1(m) = \mathcal{L}^1(\widetilde{m})$. Let $\lambda \ll m$; then $\partial R(m,\lambda)$, does not contain vertical lines, and since $R(\widetilde{m},\widetilde{\lambda}) = \overline{R(m,\lambda)}$ the same is true for $R(\widetilde{m},\widetilde{\lambda})$, namely $\widetilde{\lambda} \ll \widetilde{m}$. From the classical Radon-Nikodym Theorem then there is $\varphi \in \mathcal{L}^1(\widetilde{m})$ such that

$$\widetilde{\lambda} = \int \varphi d\widetilde{m}.$$

But the embedding $\vartheta : \mathcal{L}^1(m) \hookrightarrow \mathcal{L}^1(\widetilde{m})$ defined by $\vartheta([f]) = [\overline{f}]$ is by assumption onto. Hence there exists $[f] \in \mathcal{L}^1(m)$ such that $\vartheta([f]) = \varphi$. From (5), for $A \in \Sigma$,

$$\int_A f dm = \int_{h(A)} \varphi d\widetilde{m} = \widetilde{\lambda}(h(A)) \overset{def}{=} \lambda(A).$$

$\square$

From this Theorem, the finitely additive measure $m_l + \delta$ in Example 6.13 provides an example of a case of incomplete $\mathcal{L}^1(m)$.

REMARK 6.24. *We want to stress the fact that Theorem 6.23 in [3] is actually a necessary and sufficient condition.*

Notes on Section 6

We start with the obvious remark that, given two vector spaces E and F paired by some product, it is clear how to integrate simple E-valued (resp. F-valued) functions with respect to finitely additive measures ranging on F (resp. E).
When a topology is available, a limit process can be used to integrate larger classes of functions, approximating via simple functions.
The easiest situation of this type is that of a real Banach space X and the scalar field $\mathbb{R}$. Therefore one finds in the literature, besides the integral of a vector function (in the several senses it is commonly defined: Bochner, Pettis, Gelfand, ...) also the integral of *scalar* functions with respect to *vector* finitely additive measure.

This last integration is of particular interest in Stochastic Integration. Clearly in this setting one can also define a monotone integral. The comparison between the m-integral and the monotone integral with a vector m started in [11].
Notice first that in this case the distribution functions $\Phi^E(t)$ range on the Banach space X; then the monotone integral can be defined in many different ways, according to the type of vector integration one takes for Φ^E. The first attempt ([11]) of taking Bochner integrability proved to be too restrictive, and also the Pettis approach turned out to be unsuitable. The right approach was recently found to be the Mc Shane-Fremlin vector integration which sits in between the Bochner and the Pettis integral ([24]).
For what concerns the Radon-Nikodym Theorem, let us remind the reader that even in the countably additive case in the infinite dimensional setting the simple absolute continuity can be not enough. There is however a sharp difference between finite and countable additivity under this respect. It is known that functional analysts overcome the difficulty in the countably additive case by looking *at the space* X, and assuming the (RNP). By virtue of Example 6.13 this approach would be senseless in the finitely additive setting: indeed not even $I\!R$ would enjoy a (FARNP)!
For this reason in the finitely additive setting authors focus *on the properties of the pair* (m_1, m_2). This very fact makes the list of references concerning Radon-Nikodym Theorems in the finitely additive setting considerably larger than in the countably additive one. A complete list of references would be rather long: we bound ourselves to mentioning the papers [23, 16] and [31] for the integral with respect to a scalar m, and to the notes [30] and [29] for what concerns integration with respect to X-valued m.
Finally we wish to quote the fact that there are also extensions to the case X LCTVS or X nuclear space.
On the other side, it is known that in the scalar case, the following approximate Radon-Nikodym Theorem always hold:

THEOREM 6.25. *([20],IV.9.14) Let* $m : \Sigma \to R_0^+$ *be a non-negative finitely additive measure, and let* $\lambda : \Sigma \to I\!R$ *be a finitely additive measure with* $\lambda \ll m$. *Then* $\forall\, \varepsilon > 0$ *there exists a* m*-measurable simple function* f_ε *such that for the set function*

$$\mu(E) = \int_E f_\varepsilon dm$$

the inequality $var(\lambda - \mu)(\Omega) < \varepsilon$ *holds.*

Therefore the following Definition in the vector case makes sense:

DEFINITION 6.26. *Let* X *be a Banach space,* $m : \Sigma \to X$ *be a finitely additive measure; we shall say that* m *is of* bounded variation *if the scalar finitely additive measure*

$$|m|(E) := sup\{\sum_{i=1}^{n} \|m(E_i)\|, \{E_1, \dots, E_n\} \in \mathcal{D}(E), n \in I\!N\}$$

is finite on Ω *(and therefore on the whole of* Σ*), where* $\mathcal{D}(E)$ *represents the class of finite* Σ*-measurable decompositions of* E*.*
We shall say that the space X *satisfies the* Finitely Additive Radon-Nikodym Property *(briefly* (FARNP)*) if for every measurable space* (Ω, Σ) *and for every finitely additive measure* $m : \Sigma \to X$ *of bounded variation, for each* $\varepsilon > 0$ *there exists a simple* Σ*-measurable function* $f_\varepsilon : \Omega \to X$ *such that* $|m - \lambda|(\Omega) < \varepsilon$ *where* $\lambda(\cdot)$ *is defined as*

$$\lambda(E) = \int_E f_\varepsilon d|m|$$

for every $E \in \Sigma$*.*

Then the following result holds

THEOREM 6.27. X *has* (FARNP) *if and only if* X *has* (RNP)*.*

Proof. Assume first that X has (RNP); let m be a finitely additive measure of bounded variation, and let $\varepsilon > 0$ be fixed. Since m is of bounded variation, it is well known ([13]) that it admits a Stone extension, namely there exists a countably additive measure $\widetilde{m}$ on $\mathcal{G}_\sigma$, the Baire σ-algebra of the Stone space associated to $\Sigma_{|N(|m|)}$. Also it is known that $\widetilde{m}$ is of bounded variation and that

$$|\widetilde{m}| = \widetilde{|m|}.$$

By (RNP) there exists a Bochner integrable function $f : S \to X$ such that

$$\widetilde{m}(E) = \int_E f d|\widetilde{m}|$$

for $E \in \mathcal{G}$. A density argument shows that there exists then a $\mathcal{G}$-simple function $g : S \to X$ such that

$$\int_S \|g - f\| d|\widetilde{m}| \leq \varepsilon.$$

Let $g = \sum x_i 1_{G_i}$; since $G_i \in \mathcal{G}$ there are pairwise disjoint sets $A_1, \ldots, A_n \in \Sigma$ such that $h([A_i]) = G_i$. Put $\gamma = \sum x_i 1_{A_i}$, and let $\lambda : \Sigma \to X$ be defined as

$$\lambda = \int \gamma d|m|.$$

Then easily $\widetilde{\lambda} = \displaystyle\int g d|\widetilde{m}|$, and since

1. $|\widetilde{m} - \widetilde{\lambda}| = \int \|f - g\| d|\widetilde{m}|$;
2. $\widetilde{m - \lambda} = \widetilde{m} - \widetilde{\lambda}$;
3. $|m - \lambda|(\Omega) = |\widetilde{m - \lambda}|(S)$.

we find that $|m - \lambda|(\Omega) \leq \varepsilon$.
Conversely, assume that X has the (FARNP); let $m : \Sigma \to X$ be a (countably additive) measure of bounded variation. For each $\varepsilon > 0$ let f_ε be any simple X-valued function such that

$$var(m - \int f_\varepsilon d|m|) \leq \varepsilon. \tag{9}$$

Consider $\varepsilon_k = 2^{-k}$ and put $f_k = f_{\varepsilon_k}$. We want to show that $(f_k)_k$ is Cauchy in $L^1(|m|)$. Indeed, for each $n, k \in I\!N$

$$\|f_n - f_k\|_1 = \int_\Omega \|f_n - f_k\|_X d|m| = var(\lambda_n - \lambda_k),$$

where $\lambda_j = \int f_j d|m|$.
Since the total variation is a norm on $bvca(\Omega, \Sigma)$,

$$var(\lambda_n - \lambda_k) \leq var(\lambda_n - m) + var(m - \lambda_k)$$

whence $\|f_n - f_k\|_1 \leq \varepsilon_n + \varepsilon_k$.
Since $|m|$ is countably additive, $L^1(|m|)$ is complete. Hence there exists $f \in L^1(|m|)$ such that $f_n \overset{L^1(|m|)}{\to} f$. By definition, $(f_n)_n$ is a defining sequence for f, namely

$$\int f_n d|m| \to \int f d|m|$$

From (8) then $var(m - \int f d|m|) = 0$ namely, being $var(\cdot)$ a norm,

$$m = \int f d|m|$$

$\square$

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