Rend. Istit. Mat. Univ. Trieste
Vol. XXXIII, 113–125 (2001)

Extensions of Asymmetric Norms to Linear Spaces

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SUMMARY. - *Let M be a subset of a (real) linear space that is closed with respect to the sum of vectors and the product by nonnegative scalars. An asymmetric seminorm on M is a nonnegative and subbaditive positively homogeneous function q defined on M. Moreover, q is an asymmetric norm if in addition for every non zero element x such that $-x$ belongs to M, $q(x)$ or $q(-x)$ are different from zero. Consider the linear expansion X of M. In this paper we characterize when (M,q) can be extended to an asymmetric normed linear space (X,q^*), i.e. when there exists an asymmetric norm q^* on X such that $q^*|_M = q$. As an application we study these extensions in the case of subsets of normed lattices.*

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AMS Subject Classification: 46B20, 54E50, 54H99
Keywords: asymmetric norm, semilinear space, extension
The authors thank the support of the Spanish Ministry of Science and Technology, grant BFM2000-1111.

1. Introduction

Let R be the set of real numbers, and let R^+ be the set of nonnegative real numbers. An *algebraically closed* space M (ac-space for short) is a subset of a (real) linear space X which is closed with respect to the sum on X and with respect to the product by non negative scalars, i.e.

$$x + y \in M, \qquad for\ every\ \ x, y \in M$$

and

$$ax \in M \qquad for\ every\ \ x \in M\ \ and\ \ a \in R^+.$$

In particular, $0 \in M$. An easy example of an ac-space is the positive cone C_n of the finite dimensional space R^n for $n \in N$, where N is the set of natural numbers. For instance,

$$C_2 = \{(x_1, x_2) \in R^2 : x_1 \geq 0, x_2 \geq 0\}$$

defines an ac-space. An *asymmetric seminorm* on an ac-space M is a function $q : M \to R^+$ such that for all $x, y \in M$ and $a \in R^+$:

1) $q(ax) = aq(x)$.
2) $q(x + y) \leq q(x) + q(y)$.

We say that the couple (M, q) is an *asymmetric seminormed ac-space.* Moreover, if the function q satisfies the following property,

3) for every $x \in M$ such that $-x \in M$, then $q(x) = q(-x) = 0$ if and only if $x = 0$,

it is called an *asymmetric norm* on M. In this case, we say that the couple (M, q) is an *asymmetric normed ac-space.* This definition is the reasonable restriction to ac-spaces of the notion of an asymmetric norm on a linear space (a quasi-norm in [1] and [2]). If X is a linear space, an *asymmetric seminorm* on X is a function $q : X \to R^+$ satisfying the conditions above 1) and 2). In this case, the third condition for q to be an *asymmetric norm* on X is the separation axiom

3') $q(x) = q(-x) = 0$ if and only if $x = 0$.

The function $q^{-1} : X \to R^+$ defined by $q^{-1}(x) := q(-x)$ is also an asymmetric (semi) norm. The function $q^s(.) : X \to R^+$ given by the formula $q^s(x) := max\{q(x), q^{-1}(x)\}$ is a (semi) norm on X.

The aim of the present paper is to obtain conditions under which it is possible to extend an asymmetric norm defined on an ac-space M to the corresponding linear span $span\{M\}$. Our motivation is that a great part of the asymmetric linear spaces that appear in applied contexts are in fact extensions of asymmetric norms defined on ac-spaces (see [7] and [5]). For example, the natural definition of the dual of an asymmetric normed linear space X (the linear and upper semicontinuous functions $f : X \to R$) gives an asymmetric normed ac-space. In Section 2 we characterize those asymmetric seminorms defined on an ac-space M that can be extended at least to an asymmetric seminorm on $span\{M\}$. However, note that in general such an extension does not lead to an asymmetric norm on $span\{M\}$, since we cannot assure that the third axiom of the definition above -the separation axiom- is satisfied. For example, the asymmetric seminorm q_2 defined on C_2 as $q_2((x_1, x_2)) = x_1$ can be extended to the function $\overline{q}_2$,

$$\overline{q}_2((x_1, x_2)) = x_1 \quad \text{if} \quad x_1 > 0,$$

and $\overline{q}_2((x_1, x_2)) = 0$ in the other case. It is clear that $\overline{q}_2$ does not satisfy the third axiom of the definition of an asymmetric norm, although q_2 is an asymmetric norm on C_2.

This motivates the study of extensions satisfying the separation axiom. In Section 3 we characterize when this condition is also satisfied, under the assumption that such an extension exists. Section 4 is devoted to the application of these results to the particular case of the increasing asymmetric seminorms that appear in several interesting applied frameworks.

Each asymmetric (semi)norm q on a linear space X defines the quasi-(pseudo)metric $d_q(x, y) = q(y - x)$, $x, y \in X$. A fundamental system of neighborhoods of 0 for the topology induced by d_q is given by the sets

$$V(0, \epsilon) := \{x \in X : q(x) < \epsilon\}, \qquad \epsilon > 0.$$

In the same way, the sets

$$V(y,\epsilon) := y + V(0,\epsilon), \qquad \epsilon > 0$$

define a fundamental system of neighborhoods of y for every $y \in X$. A sequence $(x_n)_{n=1}^{\infty}$ of elements of X converges to a point $x \in X$ if $lim_n q(x_n - x) = 0$. Our basic references about quasi-metrics, asymmetric norms and applications are [3], [9], [2], [1] and [7]. For general questions related to norms and seminorms on linear spaces and normed lattices see [8] and [6].

2. Extensions of asymmetric seminorms defined on ac-spaces

Let M be an ac-space and let $X = span\{M\}$. In this section we develop a constructive technique to obtain extensions of an asymmetric seminorm q from M to X. Two basic functions are needed in order to construct the extension. The first one is q. The second function that is needed is another asymmetric seminorm p_0 on M. It is clear that the inversion map $i(x) = -x$ defines a linear isomorphism $i : X \to X$ such that $i(M) = -M = \{-x \in X : x \in M\}$ and then $-M$ is also an ac-space. Thus we can use p_0 in order to define an asymmetric seminorm p on $-M$ as $p(x) := p_0(-x)$ for every $x \in -M$. The following definition gives the canonical construction of an asymmetric seminorm from q and p. Note that each element $x \in X$ can be decomposed as a sum $x = x_1 + x_2$, where $x_1 \in M$ and $x_2 \in -M$.

Definition 2.1. *Let q and p be asymmetric seminorms on the ac-spaces M and $-M$, respectively. We define the function $q^*_{q,p}$ induced by the couple (q,p) by mean of the expression*

$$q^*_{q,p}(x) = inf\{q(x_1) + p(x_2) : x_1 \in M, x_2 \in -M, x = x_1 + x_2\}$$

for every $x \in X$.

It is easy to prove that $q^*_{q,p}$ defines an asymmetric seminorm on X.

Definition 2.2. *Let q be an asymmetric seminorm on the ac-space M. We say that the asymmetric seminorm q^* defined on X is an extension of q if the restriction of q^* to M coincides with q, i.e. $q^*|_M = q$.*

The asymmetric seminorm $q^*_{q,p}$ is closely related to the possible extensions of q to X. For instance, consider the positive cone C_+ of a Köthe function space $(E, \| \|, <)$. A Köthe function space is a Banach lattice of functions with its natural order (see [6]). If (Ω, Σ, μ) is a complete σ-finite measure, a Banach space E consisting of equivalence classes, modulo equality almos everywhere of locally integrable real valued functions is called a Köthe function space if the following conditions hold.

1) If $|f(\omega)| \leq |g(\omega)|$ a.e. on Ω, with f measurable and $g \in E$, then $f \in E$ and $\|f\| \leq \|g\|$.

2) For every $\sigma \in \Sigma$ with $\mu(\sigma) < \infty$, the characteristic function χ_σ of σ belongs to E.

An easy example of such a space is a (real) Hilbert space of integrable functions $L_2(\nu)$, where ν is a finite measure.

If E is a Köthe function space, it is easy to see that the function $r(x) := \|x \vee 0\|$ define an asymmetric norm. In fact, the definition of r is given by the evaluation of the norm of the positive part of the function. This construction provides a broad class of examples of asymmetric normed linear spaces of the type (E, r). The reader can find information about related examples in [2] and [1].

It is easy to see that the positive cone (C_+, r) is an asymmetric normed ac-space. Now consider the trivial seminorm $p_1(x) = 0$ defined on $-C_+$. A direct calculation shows that $q^*_{r,p_1}|_{C_+} = q$. Another extension of r to E is the norm $\| \|$. It is also easy to prove that $\| \|$ is equivalent to q^*_{r,p_2}, where $p_2(x) := \|x \wedge 0\| = \|x\|$ for every $x \in -C_+$. Moreover, $q^*_{r,p_2}|_{C_+} = r$.

The example above shows that we can find different extensions of an asymmetric seminorm defined on an ac-space M to the linear space X. In fact, the asymmetric normed linear spaces (E, q^*_{r,p_1}) and (E, q^*_{r,p_2}) are absolutely different from a topological point of view. (E, q^*_{r,p_2}) is a Hausdorff space (it is in fact a Banach space). However, it can be easily proved that q^*_{r,p_1} does not define a Hausdorff topology on E ([4]). Anyway, the existence of such an extension cannot be assured in general. The following theorem characterizes the asymmetric seminorms defined on ac-spaces M which can be extended to $span\{M\}$, in terms of their moduli of asymmetry.

DEFINITION 2.3. *Let q be an asymmetric seminorm on the ac-space M. We define the modulus of asymmetry of q as the real function $\Phi_q : M \to R$ given by the formula*

$$\Phi_q(x) := sup\{q(y) - q(y+x) : y \in M\}$$

for every $x \in M$.

Note that $\Phi_q(x) = q(-x)$ if q is a norm on X.

THEOREM 2.4. *Let q be an asymmetric seminorm on the ac-space M. Then:*

1) There exists an extension of q to X if and only if there is an asymmetric seminorm p on $-M$ such that

$$\Phi_q(x) \leq p(-x) \qquad for\ every\ \ x \in M.$$

2) Such an extension can be obtained as the asymmetric seminorm $q^*_{q,p}$ induced by the couple (q,p).

Proof. The proof is a direct consequence of the properties of the asymmetric seminorm $q^*_{q,p}$. It is defined on the whole linear space $span\{M\}$. Then we just need to show that its restriction to M is exactly q. It is clear that $q^*_{q,p}(x) \leq q(x)$ for every $x \in M$, since

$$inf\{q(x_1) + p(x_2) : x_1 \in M, x_2 \in -M, x = x_1 + x_2\} \leq$$

$$\leq q(x) + p(0) = q(x).$$

On the other hand, consider an element $x \in M$, an $\epsilon > 0$ and a decomposition $x = x_1 + x_2$, where $x_1 \in M$ and $x_2 \in -M$, that satisfies

$$q(x_1) + p(x_2) < q^*_{q,p}(x) + \epsilon.$$

Then we obtain the following inequalities using the condition given in 1) for Φ_q.

$$q^*_{q,p}(x) + \epsilon > q(x_1) + p(x_2) = q(x - x_2) + p(x_2) \geq$$
$$\geq q(x - x_2) + sup\{q(y) - q(y - x_2) : y \in M\} \geq$$
$$q(x - x_2) + q(x) - q(x - x_2) = q(x).$$

Thus, $q^*_{q,p}(x) = q(x)$ for every $x \in M$, since the former inequalities hold for each $\epsilon > 0$.

For the converse, consider an extension q^* of q to $span\{M\}$. Then for every $x, y \in M$,

$$q(x + y) + q^*(-x) = q^*(x + y) + q^*(-x) \geq q^*(y) = q(y),$$

since $x+y \in M$. Now let us define on $-M$ the asymmetric seminorm $p = q^*|_{-M}$ and fix $x \in M$. We obtain for every $y \in M$ the inequality

$$p(-x) \geq q(y) - q(x + y).$$

Then

$$p(-x) \geq \Phi_q(x) \qquad for\ every\ \ x \in M.$$

2) is a direct consequence of the constructive procedure used in the proof of 1). □

The next example shows that it is possible to find asymmetric seminorms defined on ac-spaces that cannot be extended to the corresponding linear span. According to Theorem 2.4 we just need to show that there is not any seminorm satisfying the required property. In fact, it is enough to find an element $x \in M$ such that $\Phi_q(x) = \infty$.

EXAMPLE 2.5. *Consider the positive cone S_+ of the lattice R_0^N whose elements are the sequences of real numbers $(x_n)_{n=1}^{\infty}$ that are non zero only for a finite set of indexes, with the usual order. S_+ is obviously an ac-space. Let us define the asymmetric norm q_+ on S_+ as follows. Consider the canonical basis of R_0^N, $\{e_n : n \in N\}$. Then for every $\overline{x} = (x_n)_{n=1}^{\infty}$, if there is no $\lambda \in R^+$ such that $\overline{x} = \lambda e_n$ for any $n \in N$, we define*

$$q_+(\overline{x}) := \sum_{n=1}^{\infty} x_n,$$

and $q_+(\lambda e_n) := \lambda n$ *in the other case.*

It is easy to prove that q_+ is an asymmetric norm on S_+. However, the element e_1 satisfies that $\Phi_{q_+}(e_1) = \infty$ since

$$\Phi_{q_+}(e_1) = sup\{q(\overline{y}) - q(e_1 + \overline{y}) : \overline{y} \in S_+\} \geq$$

$$sup\{q(e_n) - q(e_1 + e_n) : n \in N\} =$$

$$= sup\{n - 2 : n \in N\} = \infty.$$

Then there is no asymmetric seminorm p on $-M$ satisfying $p(-e_1) \geq \Phi_{q_+}(e_1)$. Moreover, note that this conclusion does not depend on the separation properties of the space (C_+, q_+). It is easy to see that $q_+(\overline{x}) = 0$ implies $\overline{x} = 0$ in the former example. However, an easy change of the definition of q_+ would lead to an asymmetric seminorm which does not satisfy this separation property but does not admit an extension yet. The conditions required for the characterization of extensions that are asymmetric norms are different that the ones that assures the existence of the extension. The next section is devoted to study these conditions.

3. Extensions that satisfy the separation axiom

Definition 3.1. *Two asymmetric norms q and p given on the ac-spaces M and $-M$ respectively, define a compatible couple (q,p) if the extension $q^*_{q,p}$ exists and satisfies that $q^*_{q,p}|_M = q$ and $q^*_{q,p}|_{-M} = p$.*

Note that any extension $q^*_{q,p}$ of an asymmetric seminorm q can be obtained by mean of a compatible couple. It is enough to replace the seminorm p by $p_0 = q^*_{q,p}|_{-M}$. A direct computation shows that $q^*_{q,p} = q^*_{q,p_0}$. Thus we can use compatible couples without loss of generality.

Definition 3.2. *Consider an asymmetric seminormed ac-space (M,q) that admits an extension by mean of the compatible couple (q,p). We define the set $\overline{M}_{q,p}$ as the closure of M on the seminormed space*

$(span\{M\},(q^*_{q,p})^s)$. *Moreover, we say that the ac-space* M *is closed if* $M=\overline{M}_{q,p}$.

For each element $y\in\overline{M}_{q,p}$ there exists a sequence $(x_n)_{n=1}^{\infty}$ such that $y\in lim_n x_n$, where the limit is computed with respect to the seminorm $(q^*_{q,p})^s$. Then we can extend the asymmetric seminorm q to $\overline{M}_{q,p}$ in the following way. Note that for each $n\in N$

$$(q^*_{q,p})^s(x_n-y)\geq q^*_{q,p}(x_n-y)\geq q^*_{q,p}(x_n)-q^*_{q,p}(y)$$

and

$$(q^*_{q,p})^s(x_n-y)\geq q^*_{q,p}(y-x_n)\geq q^*_{q,p}(y)-q^*_{q,p}(x_n).$$

Then it is clear that $lim_n q^*_{q,p}(x_n)=q^*_{q,p}(y)$. Taking into account that $q^*_{q,p}|_M=q$, we obtain that the following (topological) extension of q is well defined.

Definition 3.3. *Let* (M,q) *be an asymmetric seminormed ac-space and let* (q,p) *be a compatible couple. Then we define the (topological) extension* $\overline{q}$ *for each* $y\in\overline{M}_{q,p}$ *by mean of the formula*

$$\overline{q}(y):=lim_n q(x_n),$$

where $(x_n)_{n=1}^{\infty}\subset M$ *satisfies that* $y\in lim_n x_n$.

Lemma 3.4. *Let* (M,q) *be an asymmetric seminormed ac-space and let* (q,p) *be a compatible couple. Then* $(\overline{M}_{q,p},\overline{q})$ *is an asymmetric seminormed ac-space.*

Proof. Consider two elements $\overline{x},\overline{y}\in\overline{M}_{q,p}$. Then there are sequences $(x_n)_{n=1}^{\infty}\subset M$ and $(y_n)_{n=1}^{\infty}\subset M$ such that $lim_n(q^*_{q,p})^s(x_n-\overline{x})=0$, $lim_n q(x_n)=\overline{q}(\overline{x})$, $lim_n(q^*_{q,p})^s(y_n-\overline{y})=0$ and $lim_n q(y_n)=\overline{q}(\overline{y})$. Then

$$lim_n q^*_{q,p}(x_n+y_n-x-y)\leq lim_n q^*_{q,p}(x_n-x)+lim_n q^*_{q,p}(y_n-y)=0.$$

This means that $x+y\in\overline{M}_{q,p}$, since $x_n+y_n\in M$ for every $n\in N$. It is also possible to prove that $lim_n q(x_n+y_n)=q^*_{q,p}(x+y)$ in the same way. Finally,

$$\overline{q}(\overline{x}+\overline{y}) \leq lim_n q(x_n) + lim_n q(y_n) = \overline{q}(\overline{x}) + \overline{q}(\overline{y}).$$

The proof for the products $\lambda\overline{x}$, where $\lambda \in R^+$ and $\overline{x} \in \overline{M}$, is similar.

□

Consider a compatible couple (q,p). Then we can define the corresponding closed ac-space $\overline{M}_{q,p}$ endowed with the asymmetric seminorm $\overline{q}$. Since $(q^*_{q,p})^s$ is a seminorm, the ac-space $\overline{(-M)}_{q,p}$ is also closed and $\overline{(-M)}_{q,p} = -\overline{M}_{q,p}$. Thus, we can also consider the closed ac-space $-\overline{M}_{q,p}$ endowed with the asymmetric seminorm $\overline{p}$. Clearly, $X = span\{M\} = span\{\overline{M}_{q,p}\}$. Moreover, the definition of the extension $q^*_{q,p}$ implies $q^*_{q,p} \geq q^*_{\overline{q},\overline{p}}$. This argument shows that the separation properties that are satisfied by $q^*_{\overline{q},\overline{p}}$ are also fulfilled by $q^*_{q,p}$. Therefore, we can suppose that q and p are seminorms defined on the closed ac-spaces M and $-M$ of $(X,(q^*_{q,p})^s)$ in the following theorem. In the general case, the condition that will be required in order to assure that the separation axiom holds for extensions will be obtained as a direct consequence.

Theorem 3.5. *Let (q,p) be a compatible couple of asymmetric norms on the closed ac-spaces M and $-M$ respectively. Then the following are equivalent.*

1) $\psi(x) := max\{q(x), p(-x)\} = 0$ implies $x = 0$ for every $x \in M$.

2) The extension $q^*_{q,p}$ defined by (q,p) is an asymmetric norm.

Proof. Let us show that 1) implies 2). Suppose that for an element $x \in X$ we have $q^*_{q,p}(x) = 0$ and $q^*_{q,p}(-x) = 0$. Then, as a consequence of the definition of the extension $q^*_{q,p}$, there are sequences $(x_n)_{n=1}^{\infty} \subset M$ and $(y_n)_{n=1}^{\infty} \subset -M$ such that

$$(x-x_n)_{n=1}^{\infty} \subset -M, \quad lim_n q(x_n) = 0, \quad lim_n p(x-x_n) = 0,$$

and

$$(-x-y_n)_{n=1}^{\infty} \subset M, \quad lim_n q(-x-y_n) = 0, \quad lim_n p(y_n) = 0.$$

Let us define the sequence $(z_n)_{n=1}^{\infty} \subset M$, $z_n := x_n - y_n$. Since for every $n \in N$, $-x + z_n = -x - (y_n - x_n) \in M$, we have that

$$q(-x + z_n) \leq q(x_n) + q(-x - y_n)$$

and

$$p(x - z_n)) \leq p(x - x_n) + p(y_n),$$

we deduce that $lim_n q(-x+z_n) = 0$ and $lim_n p(x-z_n) = 0$. Moreover, since $q^*_{q,p}|_M = q$ and $q^*_{q,p}|_{-M} = p$, we get $q(-x + z_n) = q^*_{q,p}(-x + z_n)$ and $p(x - z_n) = q^*_{q,p}(x - z_n)$. Then $\psi(-x + z_n) = (q^*_{q,p})^s(-x + z_n)$ and $lim_n (q^*_{q,p})^s(z_n - x) = 0$. Therefore $x \in M$ since M is closed, and $\psi(x) = (q^*_{q,p})^s(x) = 0$. Then an application of 1) gives 2). For the converse we just need to note that $\psi = (q^*_{q,p})^s|_M$.

□

Corollary 3.6. *Let (q, p) be a compatible couple of asymmetric norms on the ac-spaces M and $-M$. Then the following conditions are equivalent, and imply that $q^*_{q,p}$ is an asymmetric norm:*

1) For every $x \in \overline{M}_{q,p}$, $(q^*_{\overline{q},\overline{p}})^s(x) = 0$ implies $x = 0$.

2) $q^*_{\overline{q},\overline{p}}$ is an asymmetric norm.

4. Applications. Extensions of increasing asymmetric seminorms

To finish this paper we apply the results of the second and the third sections to a particular case. We define a class of asymmetric seminorms that satisfy an increasing condition. Our definition is motivated by the fact that many asymmetric norms that has been used on applied contexts belongs to this class.

Definition 4.1. *Let q be an asymmetric seminorm defined on an ac-space M. We say that q is an increasing asymmetric seminorm if for every pair $x, y \in M$, $q(x) \leq q(x + y)$.*

Note that this property implies a strong restriction on the value of $q(x)$ for the elements $x \in M$ that satisfy that x and $-x$ belong to M, since $q(x) \leq q(x+(-x)) = q(0) = 0$. In particular if M is a linear space, $q = 0$. However, we can find a lot of examples of subsets of Banach lattices that satisfy this property. In particular, the restriction of the norm to an ac-space contained on the positive cone of a Köthe function space satisfies this condition (see [6] for the definition of the Köthe function space). Moreover, the dual complexity space introduced in [7] (see also [10], [5]) satisfies this property too.

Corollary 4.2. *Let q be an increasing asymmetric seminorm on an ac-space M. Then the extension $q^*_{q,p}$ exists for each asymmetric seminorm p defined on $-M$.*

Proof. Since q is increasing, it is obvious that $\Phi_q(x) = sup\{q(y) - q(y+x) : y \in M\} \leq 0$ for every $x \in M$. Then it is clear that each asymmetric seminorm p on $-M$ satisfies $p(-x) \geq \Phi_q(x)$. An application of Theorem 2.4 gives the result. □

Corollary 4.2 is true even in the trivial case $p = 0$. Moreover, consider a normed lattice $(E, \| \|, <)$. Then the canonical asymmetric norm on E is defined as $q_0(y) := \|y \vee 0\|$ for every $y \in E$ (see [2] and [1]). If we define M as the positive cone of E and $q(x) := \|x\|$ for every $x \in M$, it can be easily proved that $q_0(y) = q^*_{q,p}(y)$ for every $y \in E = span\{M\}$, where $p = 0$.

Corollary 4.3. *Let q be an increasing asymmetric seminorm on an ac-space M that satisfies that for every $x \in M$, $q(x) = 0$ implies $x = 0$. Let p be an asymmetric seminorm on $-M$. Then the extension $q^*_{q,p}$ exists and defines an asymmetric norm if M is closed.*

The proof is a direct consequence of Corollary 4.2 and Theorem 3.5. We can use the last result in order to extend the asymmetric norm q_0 defined on the normed lattice E. For instance, Corollary 4.3 can be applied to each ac-space M contained in the positive cone of a Köthe function space $(E, \|.\|, <)$. The properties of this class of normed lattices imply that the asymmetric seminorm q defined as the restriction of $\|.\|$ to M is increasing (see [6], p. 28). (Since the elements of M are positive functions, we have that $|f| \leq |f+g|$ for every $f, g \in M$, and then $\|f\| \leq \|f+g\|$). Moreover, $x = 0$ if and only if $q(x) = 0$ for every $x \in M$. If p is an asymmetric seminorm defined on $-M$ such that M is an ac-space, the extension $q^*_{q,p}$ is an asymmetric norm. Of course, this is also true if $p = 0$. In this case, the asymmetric norm $q^*_{q,0}$ is the natural extension of q_0.

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Received March 30, 2001.