Rend. Istit. Mat. Univ. Trieste
Vol. XXXIII, 1–18 (2001)

Filters, Nets and Cofinal Types

Camillo Costantini and Enrico Priola (*)

Dedicated to the memory of Prof. Davide Carlo Demaria

Summary. - *In this paper we investigate functionals relating filters and nets on a given set X, with special respect to the problem of monotonicity. In particular, we provide three different functionals Ψ_k $(k = 1, 2, 3)$ from the collection of the filters on X to the class of the nets on X, such that if $\mathcal{F} \supseteq \mathcal{G}$ then $\Psi_k(\mathcal{F})$ is a subnet of $\Psi_k(\mathcal{G})$. We also compare them with the standard functional N, which fails to be monotone. To this end, we often use the theory of cofinal types.*

1. Introduction

In this paper we study relationships and mutual reversibility between the notion of filter and that of net, with special regard to their natural (or, possibly, newly defined) order structures.

The main reasons for such an investigation come from general topology, where establishing a close link between the above notions may allow to use them in a parallel way, exploiting the peculiarities of each one (the reader is also referred to [10, Remark 13.5]). Nevertheless, the treatment we are going to provide here is essentially

(*) Authors' addresses: Camillo Costantini, Department of Mathematics, University of Torino, Via Carlo Alberto 10, 10123 Torino, Italy, e-mail: costanti@dm.unito.it
Enrico Priola, Department of Mathematics, University of Torino, Via Carlo Alberto 10, 10123 Torino, Italy, e-mail: priola@dm.unito.it
AMS Subject Classification: 03E05, 06A07, 54A20
Keywords: filter, directed set, net, subnet, convergent function, cofinal type, monotone functional, ponet

set-theoretic, inasmuch as the partial orders (or preorders) we will consider, on the collection (or *class*) of all filters (nets) on a set X, do not need any supplementary structures on X itself.

Let X be a set: a net on X is a triple $(R, D, \leq)$, where $(D, \leq)$ is a (preordered) directed set and R is a function from D to X. We will often write (R, D) or simply R, instead of $(R, D, \leq)$, if the other elements of the triple are clear from the context.

In a directed set $(D, \leq)$ we use the notation $\uparrow a$ to denote, for every $a \in D$, the *terminal set* $\{a' \in D \mid a' \geq a\}$. If (R, D) and (S, E) are two nets in X, we will say (according to [4]) that $R \succeq S$ if there exists $\varphi\colon D \to E$ such that $R = S \circ \varphi$ and

$$\forall b \in E\colon \exists a \in D\colon \varphi(\uparrow a) \subseteq \uparrow b \tag{$*$}$$

This clearly corresponds to the usual notion of subnet or finer net which is used in general topology (see, for example, [7, ch. 2] or [6, §1.6]).

Observe that the class $\Gamma(X)$ of all nets on X is only preordered by the above relation $\preceq$. Two nets R, S are said to be equivalent, and we write $R \sim S$, if $R \preceq S$ and $S \preceq R$.

We will also call $\Phi(X)$ the collection of all filters on X. It turns out that $\Phi(X)$ is a complete semi-lattice with respect to set-theoretic inclusion (and a complete, atomic distributive lattice, if we add to $\Phi(X)$ the "null filter" $\mathcal{P}(X)$; see [10, Remark 13.8]).

A first, well-known connection between filters and nets can be extablished by associating to every net (R, D) the filter $\Im(R)$ generated by the collection $\{R(\uparrow i) \mid i \in D\}$. Such an association looks extremely natural; and we can give here some arguments to think that it is, in some sense, the *only right* one.

First of all, we observe that if $R \succeq S$, then $\Im(R) \supseteq \Im(S)$. On the other hand, in the case where X is a topological space it is easily seen that, following the standard terminology, the limit and cluster points of R and $\Im(R)$ are always the same.

Furthemore, it can be proved that for every net R on any set X and for every filter $\mathcal{F}$ on X with $\mathcal{F} \neq \Im(R)$, there exists a suitable topology τ such that the limit points of $\mathcal{F}$ and $\Im(R)$ fail to coincide. Indeed if for example $F \in \mathcal{F} \backslash \Im(R)$ then, putting $\tau = \{\emptyset, F, X\}$, we have that $\mathcal{F}$ converges to any point of X while $\Im(R)$ (hence R) converges only to the points of $X \backslash F$.

Let us now deal with the converse problem of finding a suitable correspondence from filters to nets, that is to say a functional Θ associating to every filter $\mathcal{F}$ on X a net $\Theta(\mathcal{F})$ on X. Which are reasonable conditions for Θ? A basic requirement is

$$\Im(\Theta(\mathcal{F})) = \mathcal{F}; \tag{1}$$

observe that, by the above remarks, this automatically gives that, for every topology τ on X, $\mathcal{F}$ and $\Theta(\mathcal{F})$ have the same convergence behaviour (i.e., the same limit and cluster points). What seems harder to be obtained is the monotonicity of Θ, that is:

$$\Theta(\mathcal{F}) \succeq \Theta(\mathcal{G}) \quad \text{whenever} \quad \mathcal{F} \supseteq \mathcal{G}. \tag{2}$$

It turns out that the two main ways of defining such a Θ, which can be found in the literature (see §2), satisfy (1) but not (2). The central result of this article provides three different functionals fulfilling both the above conditions. At our knowledge, the only partial results in this directions that can be found in the literature are [3, Proposition 2.5'] and [4, Propositions 3 and 5].

We also establish a link with the theory of the cofinal types of directed sets, to investigate more in detail the whole subject, and especially in which cases the newly defined functionals turn out to act in an essentially different way with respect to the old ones (for general references about cofinal types, see [13, ch. 2], [11] and [12]).

On the other hand, we observe that it would be too strong to require the functional Θ to satisfy the condition

$$\Theta(\Im(R)) \sim R, \tag{3}$$

since it is possible to find nonequivalent nets R, S on X such that $\Im(R) = \Im(S)$ (see next section).

Finally, in the same spirit of [5, §4], we will tackle the problem of whether the restriction to nets indexed by partially ordered sets — that we will call *partially ordered nets*, or simply *ponets* — would be equally satisfactory in order to get suitable connections with filters.

2. Associating a filter to a net.

Let $(D, \leq_1)$, $(E, \leq_2)$ be two (preordered) directed sets: a function $\varphi : D \to E$ satisfying condition $(*)$ of the introduction — that is, such that:

$$\forall b \in E \colon \exists a \in D \colon \forall a' \geq_1 a \colon \varphi\,(a') \geq_2 b$$

is said to be *convergent*; we say that the directed set D is cofinally finer than E — and we write $D \geq E$ — if there exists a convergent function from D to E. We say that D and E are equivalent, or that they have the same *cofinal type*, if $D \geq E$ and $E \geq D$; in this case, we write $D \approx E$.

It is clear that if (R, D) and (S, E) are two nets such that $R \succeq S$, then in particular $D \geq E$ in the sense of directed sets. We will often use this fact to prove the incomparability of some particular nets.

In the introduction, we have already defined the functional $\Im$ from $\Gamma\,(X)$ to $\Phi\,(X)$, and observed that it is monotone, i.e., $R \succeq S$ implies $\Im(R) \supseteq \Im(S)$. Note that the reverse implication does not hold, in general; in fact, it is possible to find R, S such that $\Im(R) = \Im(S)$, although R and S are incomparable as nets. For example, we may consider as D and E any two incomparable directed sets — say $D = \omega$ and $E = \omega_1$ — and put R and S to be the functions on D and E, respectively, both constantly equal to a fixed $x \in X$.

Thus, a natural problem arises of providing an internal characterization to the class $\Gamma\,(X)$, for two nets R, S to have the same associated filter. Such a problem is connected with the possibility of introducing a different notion of "subnet" which can be found in the literature (see [1] and also [9]) and which could render more manageable some applications of nets to topology. According to such a definition, a net R is said to be a subnet of S simply if $\Im(R) \supseteq \Im(S)$.

In the following, we will provide a solution to the above problem.

Definition 2.1. *Two nets R, S in X are said to be compatible, and we write $R \bowtie S$, if there exists a net T in X such that $T \succeq R$ and $T \succeq S$. Two filters $\mathcal{F}, \mathcal{G}$ are said to be compatible, and we write $\mathcal{F} \star \mathcal{G}$, if $\mathcal{F} \vee \mathcal{G} = \{F \cap G \,|\, F \in \mathcal{F},\, G \in \mathcal{G}\}$ is still a filter on X.*

The next result is just a restatement of [8, Lemma 4.1].

Lemma 2.2. *Let $R, S \in \Gamma\,(X)$: then $R \bowtie S \iff \Im(R) \star \Im(S)$.*

LEMMA 2.3. *Let* $\mathcal{F}, \mathcal{F}'$ *be filters on* X. *If for every* $\mathcal{G} \in \Phi(X)$ *with* $\mathcal{F} \star \mathcal{G}$ *we have that* $\mathcal{F}' \star \mathcal{G}$, *then* $\mathcal{F} \supseteq \mathcal{F}'$.

Proof. By contradiction, suppose $F' \in \mathcal{F}'$ is such that $F' \notin \mathcal{F}$. Let $F = X \backslash F'$ and $\mathcal{G} = \{M \subseteq X \mid F \subseteq M\}$: then $\mathcal{G} \star \mathcal{F}$ (otherwise, there exists $F^* \in \mathcal{F}$ with $F^* \cap F = \emptyset$, whence $F^* \subseteq F'$ and $F' \in \mathcal{F}$), while it is not true that $\mathcal{G} \star \mathcal{F}'$. This contradicts the hypothesis. □

DEFINITION 2.4. *For every filter* $\mathcal{F}$ *on* X, *we denote by* $\mathrm{N}(\mathcal{F})$ *the net* $(R, D, \leq)$, *where* $D = \{(F,x) \mid F \in \mathcal{F},\, x \in F\}$, $(F,x) \leq (F',x') \iff F' \subseteq F$ *and* $R \colon D \to X$ *is defined by:* $R(F,x) = x$.

The functional N will be studied in detail in the next sections. It is easy to check that — as it is well-known — $\mathcal{F} = \Im(\mathrm{N}(\mathcal{F}))$ for every $\mathcal{F} \in \Phi(X)$.

PROPOSITION 2.5. *Let* $R, S \in \Gamma(X)$. *Then* $\Im(R) \supseteq \Im(S)$ *if and only if:*

$$\forall T \in \Gamma(X) : (R \bowtie T \Longrightarrow S \bowtie T)\,.$$

Proof. If $\Im(R) \supseteq \Im(S)$ and $T \in \Gamma(X)$ is such that $R \bowtie T$, then by Lemma 2.2: $\Im(R) \star \Im(T)$, whence $\Im(S) \star \Im(T)$ and $S \bowtie T$.

Conversely, if $R, S \in \Gamma(X)$ are such that

$$\forall T \in \Gamma(X) : (R \bowtie T \Longrightarrow S \bowtie T)\,,$$

then we prove that $\Im(R) \supseteq \Im(S)$. To this end, taking Lemma 2.3 into account, let $\mathcal{G}$ be any filter on X such that $\mathcal{G} \star \Im(R)$: since $\mathcal{G} = \Im(\mathrm{N}(\mathcal{G}))$, we have that $\mathrm{N}(\mathcal{G}) \bowtie R$, whence $\mathrm{N}(\mathcal{G}) \bowtie S$ and $\Im(\mathrm{N}(\mathcal{G}))(= \mathcal{G}) \star \Im(S)$; therefore, $\Im(R) \supseteq \Im(S)$. □

3. Associating a net to a filter.

The functional N we have introduced in the previous section is by far the most common in General Topology and other fields of Mathematics, to associate a net to a filter (cfr. [6, Theorem 1.6.13]). Such a functional is studied, for example, in [2, 3, 4] and [5], where some basic properties are pointed out; its peculiar role is also emphasized by the fact that it is "minimal", in the sense of the following proposition.

PROPOSITION 3.1. *Let $\mathcal{F}$ be a filter on X and $R = (R, D, \leq) \in \Gamma(X)$ such that $\Im(R) \supseteq \mathcal{F}$: then $R \succeq \mathrm{N}(\mathcal{F})$.*

Proof. Let $\mathrm{N}(\mathcal{F}) = (R', D', \leq')$. Define a function $\varphi: D \to D'$ by:

$$\varphi(a) = (R(\uparrow a), R(a))$$

for every $a \in D$. It is clear that $R' \circ \varphi = R$ and that φ is convergent. $\square$

In the literature, there exists another way to associate a net to a filter [5, §4]. Following such a definition, given any $\mathcal{F} \in \Phi(X)$ and fixed a total order $\preceq$ on X, we consider the set $D = \mathcal{F} \times \omega \times X$, endowed with the lexicographic (directed) order $\sqsubseteq$ generated by the partial directed orders $\supseteq$, $\leq$ and $\preceq$ on $\mathcal{F}$, ω and X, respectively. Then we put $\mathrm{B}(\mathcal{F}) = (R, D, \sqsubseteq)$, where the function $R: D \to X$ is defined by $R(F, n, x) = x$ (in the following, as usual, the symbol $\mathrm{B}(\mathcal{F})$ will be often intended to denote only the function R).

Observe that the net $\mathrm{B}(\mathcal{F})$ is always defined on a partially ordered set (i.e., it is a *ponet*). Since $(\mathrm{B}(\mathcal{F}))(\uparrow(F, n, x)) = F$ for every $(F, n, x) \in \mathcal{F} \times \omega \times X$, we have — like the functional N — that $\Im(\mathrm{B}(\mathcal{F})) = \mathcal{F}$. Such an equality emphasizes the fact that, in a convergence framework, the use of either filters, nets or ponets are completely equivalent. Nevertheless, from the point of view of the relative order structures, the class of nets is essentially "larger" than that of ponets. Indeed, while for every (preordered) directed set there exists a partially ordered directed set having the same cofinal type [13, Lemmas 4.1 and 4.2], the same does not hold for nets and ponets, as the following characterization shows (observe that the only sets for which each net on it is equivalent to a suitable ponet are the singletons and the empty set).

We recall that in a preordered set $(M, \leq)$, the definition of *maximum* is the same as in an ordered set, i.e. m is a maximum of $(M, \leq)$ if $x \leq m$ for every $x \in M$. Of course, a preordered set may have several different maxima, because the anti-symmetric property does not hold, in general.

THEOREM 3.2. *A net $(R, D, \leq)$ is equivalent to a ponet if and only if there exists no pair a, b of elements of D, such that $R(a) \neq R(b)$ and a, b are both maxima in $(D, \leq)$.*

Proof. Suppose first that there are a, b as before and let φ be a convergent function from D to D', where $(R', D', \leq')$ is a ponet. Then we know that there exists $\tilde{a} \in D$ such that $\forall c \geq \tilde{a}: \varphi(c) \geq' \varphi(a)$, and in particular $\varphi(b) \geq' \varphi(a)$ (because b is a maximum); in a symmetric way, we obtain that $\varphi(a) \geq' \varphi(b)$, whence $\varphi(a) = \varphi(b)$. Therefore it is impossible that $R' \circ \varphi = R$.

Suppose now that there exist no a, b as before: call M the set of all maxima of D.
1^0 case: $M = \emptyset$. Fix a total order $\sqsubseteq$ on D, and let $\leq^*$ be obtained from $\leq$ and $\sqsubseteq$ by:

$$x \leq^* y \Longleftrightarrow (\,(x \leq y \ \text{ and } \ y \not\leq x) \quad \text{or} \quad (x \leq y,\ y \leq x \ \text{ and } \ x \sqsubseteq y)\,);$$

it is a routine verification that $(D, \leq^*)$ is a directed poset. The identity maps $i': (D, \leq) \to (D, \leq^*)$ and $i'': (D, \leq^*) \to (D, \leq)$ are both convergent, so that $(R, D, \leq)$ and $(R, D, \leq^*)$ are equivalent.
2^0 case: $M \neq \emptyset$. By hypothesis, there exists $\bar{x} \in X$ such that $R(t) = \bar{x}$ for every $t \in M$. Fix any partial order $\leq'$ on $D \backslash M$ and extend it to $D' = (D \backslash M) \cup \{d'\}$, where $d' \notin D \backslash M$, in such a way that $t \leq' d'$ for every $t \in D'$. Fix an element $\bar{m}$ of M and define $R': D' \to X$, $\varphi_1: D \to D'$ and $\varphi_2: D' \to D$ by:

$$R'(t) = \begin{cases} R(t) & \text{if } t \in D \backslash M, \\ \bar{x} & \text{if } t = d', \end{cases}$$

$$\varphi_1(t) = \begin{cases} t & \text{if } t \in D \backslash M, \\ d' & \text{if } t \in M, \end{cases} \qquad \text{and} \qquad \varphi_2(t) = \begin{cases} t & \text{if } t \in D \backslash M, \\ \bar{m} & \text{if } t = d'. \end{cases}$$

Thus it is clear that the ponet $(R', D', \leq')$ is equivalent to $(R, D, \leq)$. $\square$

Since, by Proposition 3.1, $\mathrm{B}(\mathcal{F}) \succeq \mathrm{N}(\mathcal{F})$ for every $\mathcal{F} \in \Phi(X)$, the question arises of characterizing the cases where the reverse inequality holds — so that we have equivalence.

Henceforth, we will denote the product of two directed sets D_1, D_2, endowed with the lexicographic order, by $D_1 \cdot D_2$.

LEMMA 3.3. *Let D, E be two directed sets: then, if D has no last element, $D \cdot E \approx D$, and if D has last element, then $D \cdot E \approx E$.*

Proof. Easy. $\square$

PROPOSITION 3.4. *Let $\mathcal{F}$ be a filter on X: then* $\mathrm{N}(\mathcal{F}) \sim \mathrm{B}(\mathcal{F})$ *if and only if* $(\mathcal{F}, \supseteq)$ *has no last element.*

Proof. Let $\mathrm{N}(\mathcal{F}) = (R, D, \leq)$ and $\mathrm{B}(\mathcal{F}) = (R', D', \sqsubseteq)$. If $\mathcal{F}$ has a last element F, then by the above lemma it is easily seen that $(D', \sqsubseteq)$ has no last element, while $(D, \leq)$ has a maximum (take any element of the kind (F, a), with $a \in F$). Thus $(D, \leq) \not\geq (D', \sqsubseteq)$, and hence $\mathrm{N}(\mathcal{F}) \not\succeq \mathrm{B}(\mathcal{F})$.

Suppose now that $\mathcal{F}$ has no last element, and consider the function φ from D to D' defined by $\varphi(U, x) = (U, 1, x)$ for every $U \in \mathcal{F}$ and $x \in U$. Then φ is convergent and $R = R' \circ \varphi$. $\square$

In the following two results we use the notion of cofinal type to establish exactly for which pairs of filters the functionals N and B are actually monotone, and for which they are not.

THEOREM 3.5. *Let $\mathcal{F}, \mathcal{G} \in \Phi(X)$: then* $\mathrm{N}(\mathcal{F}) \succeq \mathrm{N}(\mathcal{G})$ *if and only if* $\mathcal{F} \supseteq \mathcal{G}$ *and* $(\mathcal{F}, \supseteq) \geq (\mathcal{G}, \supseteq)$.

Proof. Let $\mathrm{N}(\mathcal{F}) = (R, D, \leq)$ and $\mathrm{N}(\mathcal{G}) = (R', D', \leq')$. If there exists a convergent $\varphi: D \to D'$ such that $R' \circ \varphi = R$, then in particular $(D, \leq) \geq (D', \leq')$, whence $(\mathcal{F}, \supseteq) \geq (\mathcal{G}, \supseteq)$ — because it is easily seen that $(\mathcal{F}, \supseteq) \approx (D, \leq)$ and $(\mathcal{G}, \supseteq) \approx (D', \leq')$. Moreover, from $\mathrm{N}(\mathcal{F}) \succeq \mathrm{N}(\mathcal{G})$ we get $\mathcal{F} = \Im(\mathrm{N}(\mathcal{F})) \supseteq \Im(\mathrm{N}(\mathcal{G})) = \mathcal{G}$.

Suppose now that $\mathcal{F} \supseteq \mathcal{G}$ and $(\mathcal{F}, \supseteq) \geq (\mathcal{G}, \supseteq)$. Let $\eta: \mathcal{F} \to \mathcal{G}$ be convergent, and define $\varphi: D \to D'$ by $\varphi(F, x) = (\eta(F) \cup F, x)$. Then $R' \circ \varphi = R$ and φ is convergent — given any $(G, y) \in D'$, if $F \in \mathcal{F}$ is such that $\eta(\uparrow F) \subseteq \uparrow G$, then choosing any $x \in G \cap F$ we have that $\varphi(\uparrow(G \cap F, x)) \subseteq \uparrow(G, y)$. $\square$

THEOREM 3.6. *For $\mathcal{F}, \mathcal{G} \in \Phi(X)$, we have that* $\mathrm{B}(\mathcal{F}) \succeq \mathrm{B}(\mathcal{G})$ *if and only if* $\mathcal{F} \supseteq \mathcal{G}$ *and* $(\mathcal{F}, \supseteq) \cdot \omega \geq (\mathcal{G}, \supseteq) \cdot \omega$.

Proof. Put, like above, $\mathrm{B}(\mathcal{F}) = (R, D, \leq)$ and $\mathrm{B}(\mathcal{G}) = (R', D', \leq')$. If $\mathrm{B}(\mathcal{F}) \succeq \mathrm{B}(\mathcal{G})$, then $\mathcal{F} = \Im(\mathrm{B}(\mathcal{F})) \supseteq \Im(\mathrm{B}(\mathcal{G})) = \mathcal{G}$ and $(\mathcal{F}, \supseteq) \cdot \omega \approx (D, \leq) \geq (D', \leq') \approx (\mathcal{G}, \supseteq) \cdot \omega$.

Suppose now that $\mathcal{F} \supseteq \mathcal{G}$ and that there exists a convergent function $\eta = (\eta_1, \eta_2)$ from $(\mathcal{F}, \supseteq) \cdot \omega$ to $(\mathcal{G}, \supseteq) \cdot \omega$, where obviously

$\eta_1\colon (\mathcal{F}, \supseteq) \cdot \omega \to \mathcal{G}$ and $\eta_2\colon (\mathcal{F}, \supseteq) \cdot \omega \to \omega$. Define $\varphi\colon \mathrm{B}(\mathcal{F}) \to \mathrm{B}(\mathcal{G})$ by $\varphi(F, n, x) = (\eta_1(F, n, x) \cup F,\ \eta_2(F, n, x),\ x)$: it is easily shown as in the previous proof that $R' \circ \varphi = R$ and that φ is convergent. □

Remark 3.7. *If $\mathcal{F}, \mathcal{G}$ are filters on X with $\mathcal{F} \supseteq \mathcal{G}$, it is clear from the above propositions that $\mathrm{N}(\mathcal{F}) \succeq \mathrm{N}(\mathcal{G}) \Longrightarrow \mathrm{B}(\mathcal{F}) \succeq \mathrm{B}(\mathcal{G})$. This means, in some sense, that the functional* B *preserves monotonicity in more cases than the functional* N *— e.g., when $\mathcal{F}$ has a last element and $(\mathcal{G}, \supseteq) \approx \omega$. To find an example of this situation, let $X = \omega$, $\mathcal{F} = \{F \subseteq \omega \mid 0 \in F\}$ and*

$$\mathcal{G} = \{G \subseteq \omega \mid 0 \in G \quad \text{and} \quad \exists n \in \omega\colon \forall m \geq n\colon m \in G\}.$$

Corollary 3.8. *The following are equivalent:*

1) the functional N is monotone on $\Phi(X)$;

2) the functional B is monotone on $\Phi(X)$;

3) the set X if finite.

Proof. We only prove (2)$\Longrightarrow$(3). If X is infinite, fix $\bar{x} \in X$ and a non-principal ultrafilter $\mathcal{U}$ on X; let $\mathcal{F} = \{F \subseteq X \mid \bar{x} \in F\}$ and $\mathcal{G} = \{U \cup \{\bar{x}\} \mid U \in \mathcal{U}\}$. Put $\mathrm{B}(\mathcal{F}) = (R, D, \leq)$ and $\mathrm{B}(\mathcal{G}) = (R', D', \leq')$, then $\mathcal{F} \supseteq \mathcal{G}$, $D \approx \omega$ and $D' \approx (\mathcal{U}, \supseteq)$; it is easy to prove by an inductive construction that ω cannot be cofinally finer than $(\mathcal{U}, \supseteq)$. □

4. Central results.

In this section we are going to introduce three different functionals Ψ_1, Ψ_2, Ψ_3 from $\Phi(X)$ to $\Gamma(X)$, fulfilling conditions (1) and (2) of the introduction; then we compare them with the standard functional N. It turns out that, in some sense, they stray less and less from N, so that $\Psi_1(\mathcal{F}) \succeq \Psi_2(\mathcal{F}) \succeq \Psi_3(\mathcal{F}) \succeq \mathrm{N}(\mathcal{F})$ for every $\mathcal{F} \in \Phi(X)$; and we give necessary and sufficient conditions on $\mathcal{F}$ for $\Psi_i(\mathcal{F}) \sim \mathrm{N}(\mathcal{F})$, for $i = 1, 2, 3$.

Definition 4.1. *Let $\mathcal{F}$ be any filter on X, and let $(\mathcal{F}_i)_{i \in I}$ be a (one-to-one) indexing of $\Phi(X)$, where $\hat{\imath}$ is the element of I such that $\mathcal{F}_{\hat{\imath}} =$*

$\mathcal{F}$; consider the cartesian product $\Pi_{i\in I}\mathcal{F}_i$ and let D_1 be the set of all pairs (f,x), where $f\in\Pi_{i\in I}\mathcal{F}_i$ and $x\in f(\hat{\imath})$. We declare $\Psi_1(\mathcal{F})$ to be the triple $(R_1,D_1,\leq_1)$, where $R_1(f,x)=x$ for every $(f,x)\in D_1$ and $(f,x)\leq_1(g,y)$ iff $f\leq g$ with respect to the componentwise product order generated by the directed sets $(\mathcal{F}_i,\supseteq)$.

Consider now the subset J of I defined by: $J=\{i\in I\mid \mathcal{F}_i\subseteq\mathcal{F}\}$. Let D_2 be the set of all pairs (f,x), where $f\in\Pi_{j\in J}\mathcal{F}_j$ and $x\in f(\hat{\imath})$, and let

$$D_3=\{(f,x)\in D_2\mid \forall j\in J\colon f(j)\supseteq f(\hat{\imath})\}.$$

Also, let R_k for $k=2,3$ be the function from D_k to X defined by $R_k(f,x)=x$ for every $(f,x)\in D_k$; finally, let $(f,x)\leq_k(g,y)$ for $(f,x),(g,y)\in D_k$ and $k=2,3$ if $f\leq g$ with respect to the componentwise order on D_k generated by the directed sets $(\mathcal{F}_j,\supseteq)$, as j varies in J. For $k=1,2,3$, we put $\Psi_k(\mathcal{F})=(R_k,D_k,\leq_k)$.

THEOREM 4.2. *The functionals Ψ_k, for $k=1,2,3$, fulfil conditions (1) and (2) of the introduction. Moreover, $\Psi_1(\mathcal{F})\succeq\Psi_2(\mathcal{F})\succeq\Psi_3(\mathcal{F})$ for every $\mathcal{F}\in\Phi(X)$.*

Proof. Condition (1) is easily checked; let us turn to condition (2).

Let $\mathcal{F},\mathcal{G}\in\Phi(X)$ with $\mathcal{F}\supseteq\mathcal{G}$, and let $\Psi_k(\mathcal{F})=(R_k,D_k,\leq_k)$ and $\Psi_k(\mathcal{G})=(R'_k,D'_k,\leq'_k)$. First consider the case $k=1$. Let $\hat{\imath},i^*\in I$ be such that $\mathcal{F}_{\hat{\imath}}=\mathcal{F}$ and $\mathcal{F}_{i^*}=\mathcal{G}$; define $\varphi\colon D_1\to D'_1$ by $\varphi(f,x)=(\hat{f},x)$, where

$$\hat{f}(i)=\begin{cases} f(i) & \text{if } i\neq i^*;\\ f(i^*)\cup f(\hat{\imath}) & \text{if } i=i^*.\end{cases}\tag{4}$$

It is clear that $R'_1\circ\varphi=R_1$; we claim that the function φ is also convergent. Indeed, let $(g,y)\in D'_1$, and consider the element (h,z) of D_1, where h is defined by:

$$h(i)=\begin{cases} g(i) & \text{if } i\neq\hat{\imath},\\ g(i^*)\cap g(\hat{\imath}) & \text{if } i=\hat{\imath},\end{cases}\tag{5}$$

and z is any element of $g(i^*)\cap g(\hat{\imath})$. If $(f,x)\in D_1$ is such that $(f,x)\geq_1(h,z)$, and $\varphi(f,x)=(\hat{f},x)$, then for $i\neq\hat{\imath},i^*$ we have that $\hat{f}(i)=f(i)\subseteq h(i)=g(i)$, for $i=\hat{\imath}$ we have that $\hat{f}(\hat{\imath})=$

$f(\hat{\imath}) \subseteq h(\hat{\imath}) = g(i^*) \cap g(\hat{\imath}) \subseteq g(\hat{\imath})$, and for $i = i^*$ we have that $\hat{f}(i^*) = f(i^*) \cup f(\hat{\imath}) \subseteq h(i^*) \cup h(\hat{\imath}) = g(i^*) \cup (g(i^*) \cap g(\hat{\imath})) = g(i^*)$.

For $k = 2, 3$ the proof is similar. Let $\{\mathcal{F}_j \mid j \in J\} = \{\mathcal{F}' \in \Phi(X) \mid \mathcal{F}' \subseteq \mathcal{F}\}$ and $\{\mathcal{F}_i \mid i \in I\} = \{\mathcal{F}' \in \Phi(X) \mid \mathcal{F}' \subseteq \mathcal{G}\}$, with $I \subseteq J$. Let $\hat{\imath}, i^* \in J$ be, like above, such that $\mathcal{F}_{\hat{\imath}} = \mathcal{F}$ and $\mathcal{F}_{i^*} = \mathcal{G}$, and put $\varphi(f, x) = (\hat{f}, x)$, where $\hat{f}$ is defined as in (4) for every $i \in I$: then $R'_k \circ \varphi = R_k$ and φ is convergent. Indeed, given $(g, y) \in D'_k$, take $h \in \Pi_{j \in J} \mathcal{F}_j$ defined by:

$$h(j) = \begin{cases} g(j) & \text{if } j \in I; \\ g(i^*) & \text{if } j = \hat{\imath}; \\ X & \text{if } j \in J \setminus (I \cup \{\hat{\imath}\}). \end{cases}$$

Then, fixed any $z \in g(i^*)$, the element (h, z) of D_k is such that $\varphi(\uparrow(h, z)) \subseteq \uparrow(g, y)$.

To prove the last part of the theorem, let $\mathcal{F}$ be any element of $\Phi(X)$ and $\Psi_k(\mathcal{F}) = (R_k, D_k, \leq_k)$ for $k = 1, 2, 3$. If we put $\Phi(X) = \{\mathcal{F}_i \mid i \in I\}$, $I' = \{i \in I \mid \mathcal{F}_i \subseteq \mathcal{F}\}$ and $\hat{\imath} \in I'$ such that $\mathcal{F}_{\hat{\imath}} = \mathcal{F}$, the function $\vartheta' \colon D_1 \to D_2$ defined by $\vartheta(f, x) = (f|_{I'}, x)$ is convergent. In the same way, defining $\vartheta'' \colon D_2 \to D_3$ by $\vartheta''(f, x) = (g, x)$, where $g(i) = f(i) \cup f(\hat{\imath})$ for every $i \in I'$, we have that ϑ'' is convergent. □

We turn now to compare the functionals Ψ_k with the functional N, defined in section 1. Observe that, by Proposition 3.1, we have $\Psi_k(\mathcal{F}) \succeq \mathrm{N}(\mathcal{F})$ for every $\mathcal{F} \in \Phi(X)$ and $k \in \{1, 2, 3\}$; thus, in this case, the relation $\Psi_k(\mathcal{F}) \sim \mathrm{N}(\mathcal{F})$ is equivalent to $\Psi_k(\mathcal{F}) \preceq \mathrm{N}(\mathcal{F})$.

We first need some preliminary results.

LEMMA 4.3. *Let I be a set of indices and, for every $i \in I$, let D_i be a directed set without any maximum. Then, if we put $D = \Pi_{i \in I} D_i$, endowed with the componentwise order, we have that the cofinality of D is strictly greater than $|I|$.*

As a consequence, for every directed set E with $|E| \leq |I|$ we have that E is not cofinally finer than D.

Proof. Use a standard diagonal argument. □

Lemma 4.4. *Let $\mathcal{F}$ be a filter on a set M and $\kappa = |\mathcal{F}|$: then*

$$|\{\mathcal{G} \in \Phi(M) \mid \mathcal{G} \subseteq \mathcal{F}\}| \geq \kappa.$$

Moreover, if $(\mathcal{F}, \supseteq)$ has no maximum, then

$$|\{\mathcal{G} \in \Phi(M) \mid \mathcal{G} \subseteq \mathcal{F},\ (\mathcal{G}, \supseteq) \text{ has no maximum}\}| \geq \kappa.$$

Proof. To prove the first claim, let us associate to every $F \in \mathcal{F}$ the filter $\mathcal{G}(F) = \{F' \subseteq M \mid F \subseteq F'\}$: then $F \mapsto \mathcal{G}(F)$ is one-to-one.

Suppose now that $(\mathcal{F}, \supseteq)$ has no maximum. Let us associate to every $F \in \mathcal{F}$ the filter $\mathcal{H}(F) = \{(M \backslash F) \cup (F' \cap F) \mid F' \in \mathcal{F}\} = \{F'' \in \mathcal{F} \mid F'' \supseteq M \backslash F\}$: then $\mathcal{H}(F) \subseteq \mathcal{F}$, and $(\mathcal{H}(F), \supseteq)$ has no maximum (if G were the maximum of $(\mathcal{H}(F), \supseteq)$, then $G \cap F$ would be the maximum of $(\mathcal{F}, \supseteq)$). We prove that $F \mapsto \mathcal{H}(F)$ is one-to-one. Suppose $F_1, F_2 \in \mathcal{F}$ and $x \in F_1 \backslash F_2$: then $(M \backslash F_1) \cup (F_2 \cap F_1) \in \mathcal{H}(F_1)$ and $x \notin (M \backslash F_1) \cup (F_2 \cap F_1)$. On the contrary, since $x \in M \backslash F_2$, every element of $\mathcal{H}(F_2)$ contains x. Thus $\mathcal{H}(F_1) \neq \mathcal{H}(F_2)$. $\square$

In the following three theorems, we put as usual:

$$\mathrm{N}(\mathcal{F}) = (R, D, \leq) \quad \text{and} \quad \Psi_k(\mathcal{F}) = (R_k, D_k, \leq_k) \quad \text{for} \quad k = 1, 2, 3.$$

We first tackle the case $k = 2$.

Theorem 4.5. *Let $\mathcal{F}$ be any filter on X: then $\Psi_2(\mathcal{F}) \sim \mathrm{N}(\mathcal{F})$ if and only if $\mathcal{F}$ is finite.*

Proof. Let $\{\mathcal{F}_i \mid i \in I\} = \{\mathcal{G} \in \Phi(X) \mid \mathcal{G} \subseteq \mathcal{F}\}$ ($i \mapsto \mathcal{F}_i$ one-to-one) and $\hat{\imath} \in I$ such that $\mathcal{F}_{\hat{\imath}} = \mathcal{F}$. Suppose first that $\mathcal{F}$ is finite: then, for every $i \in I$, there exists a maximum F_i of $(\mathcal{F}_i, \supseteq)$; let $\hat{f} \in \Pi_{i \in I} \mathcal{F}_i$ be defined by $\hat{f}(i) = F_i$ for every $i \in I$. We define $\eta: D \to D_2$ in the following way: for every $(F, x) \in D$ we put $\eta(F, x) = (f, x)$, where $f \in \Pi_{i \in I} \mathcal{F}_i$ is defined by:

$$f(i) = \begin{cases} F_i & \text{for } i \neq \hat{\imath}; \\ F & \text{for } i = \hat{\imath}. \end{cases}$$

Then $R = R_2 \circ \eta$ and η is convergent — because $\eta(F_{\hat{\imath}}, x) = (\hat{f}, x)$ for every $x \in X$.

Suppose now that $\mathcal{F}$ is infinite and let $|\mathcal{F}| = \kappa$: consider first the case where $(\mathcal{F}, \supseteq)$ has not a maximum. By Lemma 4.4, there exists a subset J of I such that $|J| = \kappa$ and for every $j \in J$, $(\mathcal{F}_j, \supseteq)$ has not a maximum. Let $\vartheta\colon D_2 \to \Pi_{j\in J}\mathcal{F}_j$ be defined by $\vartheta(f, x) = f|_J$: then ϑ is convergent with respect to the componentwise order on $\Pi_{j\in J}\mathcal{F}_j$; if, by contradiction, there existed a convergent $\eta\colon D \to D_2$, then $\vartheta \circ \eta$ would be convergent from D to $\Pi_{j\in J}\mathcal{F}_j$. This contradicts Lemma 4.3, as D has the same cofinal type of $(\mathcal{F}, \supseteq)$.

If, on the contrary, $(\mathcal{F}, \supseteq)$ has a maximum $\hat{F}$, then $X\backslash\hat{F}$ is infinite (otherwise, $\mathcal{F}$ would be finite). Fix a filter $\mathcal{H}$ on $X\backslash\hat{F}$ such that $(\mathcal{H}, \supseteq)$ has no maximum and let $\mathcal{G}$ be the filter on X generated by $\left\{\hat{F} \cup H \,|\, H \in \mathcal{H}\right\}$. Then $(\mathcal{G}, \supseteq)$ has no maximum and is contained in $\mathcal{F}$; hence, D_2 has no maximum, too. Thus, $D \not\succeq D_2$. □

Theorem 4.6. *If $\mathcal{F}$ is any filter on X, then $\Psi_1(\mathcal{F}) \sim \mathrm{N}(\mathcal{F})$ if and only if the set X is finite.*

Proof. Let $\{\mathcal{F}_i \,|\, i \in I\} = \Phi(X)$: if X is finite, then for every $i \in I$ there exists a maximum F_i of $(\mathcal{F}_i, \supseteq)$, and we can show as in the previous proof that $\mathrm{N}(\mathcal{F}) \succeq \Psi_1(\mathcal{F})$.

Suppose now that X is infinite: then it is possible to find $2^{|X|}$ (in fact, $2^{2^{|X|}}$) non-principal ultrafilters on X, and since $|\mathcal{F}| \leq 2^{|X|}$, it easily follows — as in the last part of the previous proof — that $D \not\succeq D_1$. □

Theorem 4.7. *Let $\mathcal{C}$ be the filter of all cofinite sets on X. Then $\Psi_3(\mathcal{F}) \sim \mathrm{N}(\mathcal{F})$ if and only if $\mathcal{F} \subseteq \mathcal{C}$.*

Proof. Let, as in Theorem 4.5, $\{\mathcal{F}_i \,|\, i \in I\} = \{\mathcal{G} \in \Phi(X) \,|\, \mathcal{G} \subseteq \mathcal{F}\}$ and $\hat{\imath} \in I$ such that $\mathcal{F}_{\hat{\imath}} = \mathcal{F}$.

Suppose first that $\mathcal{F} \subseteq \mathcal{C}$: then for every $F \in \mathcal{F}$ and $i \in I$ there exists $H(F, i)$ which is the maximum — with respect to $\supseteq$ — of $\{H' \in \mathcal{F}_i \,|\, H' \supseteq F\}$. Define $\eta\colon D \to D_3$ by $\eta(F, x) = (f, x)$, where $f(i) = H(F, i)$ for every $i \in I$. Clearly, $R = R_3 \circ \eta$, and η is convergent because for every $(g, y) \in D_3$ the element $(g(\hat{\imath}), y)$ of D is such that: $\forall (F, x) \geq (g(\hat{\imath}), y) \colon \eta(F, x) \geq_3 (g, y)$.

Suppose now that there exists $M \in \mathcal{F}$ such that $X\backslash M$ is infinite. Let

$$\mathcal{H} = \{F \cap M \,|\, F \in \mathcal{F}\}$$

be the trace of $\mathcal{F}$ on M, and put $\kappa = |\mathcal{H}|$: then $\mathcal{H}$ is a filter on M and by Lemma 4.4 there exist κ distinct filters on M — say $\{\mathcal{H}_\alpha \mid \alpha \in \kappa\}$, with $\alpha \mapsto \mathcal{H}_\alpha$ one-to-one — each of which is contained in $\mathcal{H}$. Let us fix a filter $\mathcal{G}$ on $X \backslash M$ such that $(\mathcal{G}, \supseteq)$ has no maximum, and for every $\alpha \in \kappa$ put: $\mathcal{E}_\alpha = \{H \cup G \mid H \in \mathcal{H}_\alpha,\ G \in \mathcal{G}\}$. Then $\mathcal{E}_\alpha$ is a filter on X and is contained in $\mathcal{F}$. Therefore, for every $\alpha \in \kappa$ there exists $i(\alpha) \in I$ such that $\mathcal{E}_\alpha = \mathcal{F}_{i(\alpha)}$; clearly, if $\alpha \neq \alpha'$, then $\mathcal{E}_\alpha \neq \mathcal{E}_{\alpha'}$ and hence $i(\alpha) \neq i(\alpha')$. Observe also that $i(\alpha) \neq \hat{\imath}$ for every $\alpha \in \kappa$.

Let us associate to every $(f, x) \in D_3$ a function $\vartheta(f, x)$ from κ to $\mathcal{G}$, defined by:

$$(\vartheta(f,x))(\alpha) = f(i(\alpha)) \cap (X \backslash M) \qquad \text{for every } \alpha \in \kappa.$$

We prove that ϑ is convergent from D_3 to the directed set ${}^\kappa\mathcal{G}$, endowed with the componentwise order generated by $(\mathcal{G}, \supseteq)$. Let $g \in {}^\kappa\mathcal{G}$, and consider the element $(\hat{f}, \hat{x})$ of D_3, where $\hat{x}$ is any element of M and $\hat{f}$ is defined by:

$$\hat{f}(i) = \begin{cases} M \cup g(\alpha) & \text{if } i = i(\alpha) \text{ for a (unique) } \alpha \in \kappa; \\ M & \text{if } i = \hat{\imath}; \\ X & \text{otherwise.} \end{cases}$$

Then, for every $(f, x) \geq (\hat{f}, \hat{x})$, we have in particular that

$$f(i(\alpha)) \subseteq \hat{f}(i(\alpha)) = M \cup g(\alpha)$$

for every $\alpha \in \kappa$, and hence $(\vartheta(f,x))(\alpha) = f(i(\alpha)) \cap (X \backslash M) \subseteq (M \cup g(\alpha)) \cap (X \backslash M) = g(\alpha)$.

If, by contradiction, we had a convergent function $\eta: D \to D_3$, then $\vartheta \circ \eta$ would be convergent from D to ${}^\kappa\mathcal{G}$; this contradicts Lemma 4.3 because D and $(\mathcal{F}, \supseteq)$ have the same cofinal type, and $\mathcal{H}$ is a terminal set in $(\mathcal{F}, \supseteq)$. $\square$

In the last part of this section, we come back to the notion of ponet. Let we call $\Gamma'(X)$ the class of all ponets on a set X. By Theorem 3.2 of the previous section, we see that the notion of ponet cannot replace in every respect that of net. Nevertheless, we can wonder whether it is possible to find a functional $\Theta: \Phi(X) \to \Gamma'(X)$, fulfilling conditions (1) and (2) of the introduction. To give such a question a positive answer, we are going to introduce a suitable functional Ξ from $\Gamma(X)$ to $\Gamma'(X)$; it establishes a rather strict link between the two classes in question, in the sense of the following result.

THEOREM 4.8. *There exists a functional* Ξ *from* $\Gamma(X)$ *to* $\Gamma'(X)$ *such that:*

I) $\forall R_1, R_2 \in \Gamma(X)\colon (R_1 \succeq R_2 \implies \Xi(R_1) \succeq \Xi(R_2))$ *(monotonicity);*

II) $\forall R \in \Gamma(X)\colon \Im(\Xi(R)) = \Im(R)$ *(invariance with respect to the functional* $\Im$*).*

Proof. To define the functional Ξ, suppose to have any net $R = (R, D, \leq) \in \Gamma(X)$. Let, for $a, b \in D$, $a \equiv b \iff (a \leq b \text{ and } b \leq a)$; also, fix any total order $\sqsubseteq$ on D, and for $a, b \in D$ put: $a \leq^* b \iff ((a \leq b \text{ and } b \nleq a) \text{ or } (a \equiv b \text{ and } a \sqsubseteq b))$. Finally, consider $D' = D \times \omega$, and for $(a, n), (b, m) \in D'$ put:

$$(a, n) \leq' (b, m) \iff ((n < m \quad \text{and} \quad a \leq b) \quad \text{or} \quad (n = m \quad \text{and} \quad a \leq^* b)).$$

By checking in the various possible cases the transitive and antisymmetric properties for $\leq'$, and its directed character, we easily see that $(D', \leq')$ is a directed partially ordered set. Define $R'\colon D' \to X$ by $R'(a, n) = R(a)$, and put $\Xi(R, D, \leq) = (R', D', \leq')$. Since for every $(a, n) \in D'$ we have that $R'(\uparrow(a, n)) = R(\uparrow a)$, it follows that $\Im(R) = \Im(R')$. Thus, what only we have to show is monotonicity.

Let $R_1 = (R_1, D_1, \leq_1)$ and $R_2 = (R_2, D_2, \leq_2)$ be nets in X such that there exists a convergent $\varphi\colon D_1 \to D_2$ with $R_2 \circ \varphi = R_1$. Put $\Xi(R_k, D_k, \leq_k) = (R'_k, D'_k, \leq'_k)$ for $k = 1, 2$, and define $\varphi'\colon D'_1 \to D'_2$ by $\varphi'(a, n) = (\varphi(a), n)$: thus it is clear from the properties of φ that $R'_2 \circ \varphi' = R'_1$.

To show the convergent character of φ', let (a_2, m) be any element of D'_2, and consider the element $(a_1, m+1) \in D'_1$, where $a_1 \in D_1$ is such that $\varphi(\uparrow a_1) \subseteq \uparrow a_2$ in $(D_2, \leq_2)$. Then, if $(b_1, n) \geq'_1 (a_1, m+1)$, we have two possible cases:

1) $n > m + 1$ and $b_1 \geq_1 a_1$, whence $\varphi(b_1) \geq_2 a_2$ and $\varphi'(b_1, n) = (\varphi(b_1), n) \geq'_2 (a_2, m)$;

2) $n = m+1$ and $b_1 \geq_1^* a_1$, whence $b_1 \geq_1 a_1$ and $\varphi'(b_1, m+1) \geq_2' (a_2, m)$. $\square$

COROLLARY 4.9. *For $k = 1, 2, 3$, the functional*

$$\Xi \circ \Psi_k \colon \Phi(X) \to \Gamma'(X)$$

satisfies conditions (1) and (2) of the introduction.

REMARK 4.10. *The definition of the functional Ξ of Theorem 4.8 does not involve directly the set X (it is, in some sense, "intrinsic"). In other words, it is possible to define a functional Λ which associates to every directed set $(D, \leq)$ a directed poset $\Lambda(D, \leq) = (D \times \omega, \leq')$, so that for every set X and every $(R, D) \in \Gamma(X)$ we have that $\Xi(R, D) = (R \circ \pi_1, \Lambda(D))$ (where π_1 is the universal projection on the first component of a product).*

We conclude the paper with a result which shows that the above functional Ξ cannot be "improved".

We know, by Theorem 3.2, that it is impossible to find a functional $\Xi' \colon \Gamma(X) \to \Gamma'(X)$ satisfying the strong condition:
III) $\forall R \in \Gamma(X) \colon \Xi'(R) \sim R$
(which would imply both condition (I) and (II) of Theorem 4.8).

On the other hand, if we restrict to the subclass $\tilde{\Gamma}(X)$ of $\Gamma(X)$, containing all nets on X defined on a directed set which has no maximum, then there exists a functional $\tilde{\Xi}' \colon \tilde{\Gamma}(X) \to \Gamma'(X)$ satisfying condition (III) for all $R \in \tilde{\Gamma}(X)$ (in fact, the 1^o case of the proof of Theorem 3.2 provides us an actual method to construct such a $\tilde{\Xi}'$). Observe that the class $\tilde{\Gamma}(X)$ corresponds, for example, to the nets which are actually used in General Topology.

In the light of such results, it is worth wondering whether it is possible to get a functional $\Xi' \colon \Gamma(X) \to \Gamma'(X)$ satisfying conditions (I) and (II) of Theorem 4.8 on the whole of $\Gamma(X)$, and condition (III) on $\tilde{\Gamma}(X)$. In Theorem 4.12 below, we give such a question a negative answer.

The following fact may be considered as folklore in the theory of cofinal types.

LEMMA 4.11. *Let D' be any directed set for which there exists a convergent map $f \colon \omega \to D'$. Then either D' has a maximum or $D' \approx \omega$.*

THEOREM 4.12. *If X has at least two elements, there exists no Ξ' from $\Gamma(X)$ to $\Gamma'(X)$ such that:*

I) $\forall R_1, R_2 \in \Gamma(X) \colon (R_1 \succeq R_2 \Longrightarrow \Xi'(R_1) \succeq \Xi'(R_2))$;

II) $\forall R \in \Gamma(X) \colon \Im(\Xi'(R)) = \Im(R)$;

III) $\forall R \in \tilde{\Gamma}(X) \colon \Xi'(R) \sim R$.

Proof. Let x, y be two distinct elements of X. If, by contradiction, there exists a Ξ' fulfilling (I), (II), (III), we consider a net (R, D) on X defined on a two-element directed set $D = \{a, b\}$, endowed with the discrete order, such that $R(a) = x$ and $R(b) = y$. Put $\Xi'(R, D) = (R', D')$: then, by (II), D' has no maximum (otherwise, $\Im(R)$ would contain a singleton, which is impossible).

Fix a directed set E without maximum, such that $E \not\geq D'$ (such an E exists because if $E = \omega$ does not work, then by Lemma 4.11 we have $D' \approx \omega$, and hence we can put $E = \omega_1$). Define a $T \colon E \to X$, having constant value x; then $\varphi \colon E \to D$, defined by $\varphi(e) = a$ for every $e \in E$, is convergent and such that $R \circ \varphi = T$. Therefore $(E, T) \succeq (R, D)$, which implies by (I) that $\Xi'(E, T) \succeq \Xi'(R, D) = (R', D')$. But this is impossible because $\Xi'(E, T) \sim (E, T)$ by (III), and hence we would have $E \geq D'$. $\square$

REFERENCES

[1] J.F AARNES AND P.R. ANDENAES, *On nets and filters*, Math. Scand. **31** (1972), 285–292.

[2] R.G. BARTLE, *Nets and filters in topology*, Amer. Math. Monthly **62** (1955), 551–557.

[3] R.G. BARTLE, *A correction for "Nets and filters in topology"*, Amer. Math. Monthly **70** (1963), 52–53.

[4] R.G. BARTLE, *Relations between nets and indexed filter bases*, Colloq. Math. **10** (1963), 211–215.

[5] G. BRUNS AND J. SCHMIDT, *Zur Äquivalenz von Moore-Smith-Folgen und Filtern*, Math. Nachr. **13** (1955), 169–186.

[6] R. ENGELKING, *General Topology*, Heldermann, Berlin, 1989.

[7] J.L. KELLEY, *General Topology*, Van Nostrand Reinhold, New York, 1955.

[8] M. KENNEDY, *The connection between nets and filters*, Irish Math. Soc. Newslett. **11** (1984), 24–28.

[9] M.F. Smiley, *Filters and equivalent nets*, Amer. Math. Monthly **64** (1957), 336–338.

[10] W.J. Thron, *Topological structures*, Rinehart and Winston, Holt, 1966.

[11] S. Todorcevic, *Directed sets and cofinal types*, Trans. Amer. Math. Soc. **290** (1985), 711–723.

[12] S. Todorcevic, *A classification of transitive relations on ω_1*, Proc. London Math. Soc. **73** (1996), 501–533.

[13] J.W. Tukey, *Convergence and uniformity in topology*, Ann. of Math. Studies, vol. 2, Princeton Univ. Press, Princeton, N.J., 1940.

Received September 13, 1999.