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Heegaard Reducing Spheres for the 3-Sphere

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Dedicated to the memory of Marco Reni

Summary. - *There is a natural way of indexing the difference between two collections of reducing spheres for the standard genus g Heegard splitting of S^3. Any two collections are related by a sequence of collections, any two of which differ by low index, hence in a simple way.*

1. Background

In [4] an invariant in $\mathbb{Q}/2\mathbb{Z}$ was defined for a knot K with unknotting tunnel γ. An important ingredient in the proof that the invariant depended only on the pair (K, γ) was a characterization, in some sense due to Goeritz [1], of reducing spheres for the genus two Heegaard splitting $H_1 \cup H_2$ of S^3. It was shown there that any pair of reducing spheres for $H_1 \cup H_2$ can be connected by a sequence of reducing spheres, so that any successive pair in the sequence intersect very simply. Here we will use Powell's [3] extension of Goeritz' result similarly to describe reducing spheres for arbitrary genus Heegaard splittings of S^3.

We begin by extending notation and definitions.

Let $H_1 \cup_F H_2$ be a genus $g > 1$ Heegaard splitting of S^3. It is well-known that $H_1 \cup_F H_2$ is the standard genus g splitting (see [6] or [5]) so in particular $H_1 \cup_F H_2$ is reducible.

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Definition 1.1. *Disjoint reducing spheres* S, S' *for* $H_1 \cup_F H_2 = S^3$ *are* parallel *if the splitting surface* F *intersects the collar between* S *and* S' *in* S^3 *in a spanning annulus.*

A collection $\mathcal{S}$ *of disjoint pairwise non-parallel reducing spheres for* $H_1 \cup_F H_2$ *is* complete *if, for any reducing sphere* S' *disjoint from* $\mathcal{S}$*, some component of* $F - (\mathcal{S} \cup \{S'\})$ *is a planar surface.*

A complete collection $\mathcal{S}$ *of reducing spheres for* $H_1 \cup_F H_2$ *is* minimal *if* $|\mathcal{S}| = g-1$*. It is* maximal *if any reducing sphere for* $H_1 \cup_F H_2$ *that is disjoint from* $\mathcal{S}$ *is parallel to an element of* $\mathcal{S}$*.*

Since $H_1 \cup_F H_2$ is a standard splitting, it is easy to see that $H_1 \cup_F H_2$ does have a minimal complete collection of reducing spheres and that a maximal complete collection has $2g-3$ elements. We wish to understand how two complete collections of reducing spheres are related.

The first observation is that if $\mathcal{S}$ and $\mathcal{S}'$ are two (possibly inteersecting) collections of disjoint reducing spheres, then any simple closed curve in $\mathcal{S} \cap \mathcal{S}'$ that does not intersect F can be removed by an isotopy rel F. Indeed, suppose $c \subset (\mathcal{S} \cap \mathcal{S}')$ is such a curve disjoint from F. Then since each component of $\mathcal{S}$ intersects F in a single closed curve, one disk D bounded by c in $\mathcal{S}$ is disjoint from F. Hence, with no loss of generality, we may assume (by taking an innermost such circle) that D is also disjoint from $\mathcal{S}'$ and that c bounds a disk D' in $\mathcal{S}'$ that is disjoint from F. Then the imbedded sphere $D \cup D'$ lies entirely in H_1 or H_2, so D' can be isotoped to D rel F. The isotopy removes the component c of intersection, and perhaps other curves of intersection as well. So henceforth we will assume that any component of $\mathcal{S} \cap \mathcal{S}'$ intersects F. The curve necessarily intersects F in an even number of points since F is separating.

For $\mathcal{S}$ a collection of disjoint reducing spheres, let $\mathcal{C}_{\mathcal{S}}$ be the collection of separating curves $\mathcal{S} \cap F$ in F. Define the *intersection number* $\mathcal{S} \cdot \mathcal{S}'$ of two collections $\mathcal{S}$ and $\mathcal{S}'$ of disjoint reducing spheres to be the minimum number of points in $\mathcal{C}_{\mathcal{S}} \cap \mathcal{C}_{\mathcal{S}'}$ that can be achieved by isotopy of $\mathcal{C}_{\mathcal{S}}$ and $\mathcal{C}_{\mathcal{S}'}$ in F.

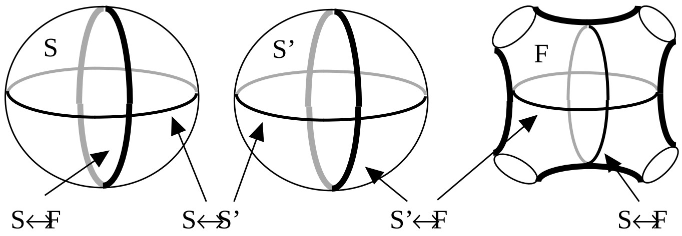

Figure 1.

2. When $\mathcal{S} \cdot \mathcal{S}' \leq 4$

In this section we begin by identifying how minimal or maximal complete collections of reducing spheres are related, if their intersection number is simply 2. Suppose S and S' are reducing spheres such that $S \cdot S' = 2$. Since each component of $S \cap S'$ intersects F in an even number of points, this means that $S \cap S'$ is a single circle, intersecting F in two points and intersecting each H_i in a single arc. We say S and S' are *orthogonal*. See Figure 1.

Definition 2.1. *Suppose $\mathcal{S}$ is a collection of reducing spheres and S' is another reducing sphere so that $\mathcal{S} \cdot S' = 2$. Let $S \in \mathcal{S}$ be the sphere that S' intersects. Then the collection $\mathcal{S}'$ of disjoint reducing spheres obtained from $\mathcal{S}$ by replacing S with S' is obtained by* orthogonal replacement.

Suppose $\mathcal{S}$ and $\mathcal{S}'$ are complete collections of disjoint reducing spheres and $\mathcal{S} \cap \mathcal{S}' = 2$. Let $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ be the spheres in each collection that intersect, in a curve $\gamma = S \cap S'$. Arbitrarily orient S and S' so that each has a $+$ and a $-$ side in F. The curve $c' = S' \cap F$ intersects each side in F of the curve $c = S \cap F$ in an arc. Let $b(\pm, \pm)$ be the 4-bigons in F formed by attaching an arc of $c - c'$ to an arc of $c' - c$, with, say both bigons $b(+, \pm)$ on the $+$ side of S and both bigons $b(\pm, +)$ being on the $+$ side of S'.

Lemma 2.2. *Each of the 4 bigons $b(\pm, \pm)$ bounds a disk in both H_1 and H_2.*

Proof. $F \cup S'$ divides S into 4 bigons, each with one side on an arc of $c' - c$ and the other side on an arc of $\gamma - c$. $F \cup S$ similarly divides up S'. Attaching a bigon from $S-(\gamma\cup c)$ to a bigon from $S-(\gamma\cup c)$ along a segment of $\gamma\subset H_i$ that they share gives a meridian disk for H_i bounded by one of the bigons $b(\pm,\pm)\subset F$. With this operation we can create a meridian in either handlebody bounding any of the meridians. □

Denote by $S(\pm,\pm)$ the 4 reducing spheres obtained from the bigons $b(\pm,\pm)$ by attaching a meridian in each handlebody to each.

PROPOSITION 2.3. *If $\mathcal{S}'$ and $\mathcal{S}$ are maximal complete collections with $\mathcal{S}'\cdot\mathcal{S}=2$ then $\mathcal{S}'$ is obtained from $\mathcal{S}$ by orthogonal replacement.*

If $\mathcal{S}'$ and $\mathcal{S}$ are minimal complete collections with $\mathcal{S}'\cdot\mathcal{S}\le 2$, then there is a minimal complete collection $\mathcal{S}_1$ that is disjoint from both $\mathcal{S}$ and $\mathcal{S}'$.

Proof. If $\mathcal{S}$ and $\mathcal{S}'$ are maximal complete collections then all four spheres $S(\pm,\pm')$ must be parallel to spheres in $\mathcal{S}$ and $\mathcal{S}'$. Since all the other spheres in $\mathcal{S}$ and $\mathcal{S}'$ are disjoint, in fact the only difference (up to parallelism) between the collections is that S is replaced by S', as required.

Suppose then that $\mathcal{S}$ and $\mathcal{S}'$ are a minimal complete collection. If $\mathcal{S}'\cdot\mathcal{S}=0$ there is nothing to prove. If $\mathcal{S}'\cdot\mathcal{S}=2$ let S and S' be the elements of each so that $S\cap S'$ is a single circle. Each component of $F-\mathcal{S}$ is a punctured torus. One such component C contains both bigons $c(+,\pm')$ and exactly one of these circles, say $c(+,+')$ cuts off a planar surface from C not containing $c=S\cap F$. Then the collection $\mathcal{S}_1$ of reducing spheres obtained by replacing S by $S(+,-')$ is still a minimal complete collection and it is disjoint from $\mathcal{S}$ and $\mathcal{S}'$. □

A similar characterization is possible also when $\mathcal{S}\cdot\mathcal{S}'=4$, but the argument is more complicated. We begin with the easiest case:

PROPOSITION 2.4. *Suppose $\mathcal{S}'$ and $\mathcal{S}$ are complete collections of reducing spheres with $\mathcal{S}\cdot\mathcal{S}'=4$. Suppose further that there are spheres $S_1,S_2\in\mathcal{S}$ and $S_1',S_2'\in\mathcal{S}'$ so that $S_i\cdot S_j'=2\delta_{ij}$.*

If $\mathcal{S}'$ and $\mathcal{S}$ are maximal complete collections, then there is a maximal complete collection $\mathcal{S}''$ which can be obtained from both $\mathcal{S}$ and $\mathcal{S}'$ by orthogonal replacement.

If $\mathcal{S}'$ and $\mathcal{S}$ are minimal complete collections, then there are disjoint minimal complete collections $\mathcal{S}_0$ and $\mathcal{S}'_0$ such that $\mathcal{S}$ is disjoint from $\mathcal{S}_0$ and $\mathcal{S}'_0$ is disjoint from $\mathcal{S}'$.

Proof. Suppose first that $\mathcal{S}$ and $\mathcal{S}'$ are maximal. Then the 4 reducing spheres $S(\pm,\pm)$ obtained, as above, from the pair of spheres S_1 and S'_1 must be in both $\mathcal{S}$ and $\mathcal{S}'$. Let $\mathcal{S}''$ be obtained from $\mathcal{S}$ by replacing S_1 by S'_1. The result now follows from Proposition 2.3.

Next suppose that $\mathcal{S}$ and $\mathcal{S}'$ are minimal and let C be a component of $F - \mathcal{S}$ incident to S_1 that contains both the bigons $b_1(+,\pm)$ obtained from the spheres S_1 and S'_1, as described above. Just as in the proof of Proposition 2.3, one of the bigons, say $b_1(+,+)$ cuts off from C a planar component disjoint from $c_1 = S_1 \cap F$. Then let $\mathcal{S}_0$ be obtained from $\mathcal{S}$ by replacing S_1 by the sphere $S_1(+,-)$ obtained by attaching together meridian disks in H_1 and H_2 that are bounded by $b_1(+,-)$. Then $\mathcal{S}_0$ is a minimal complete collection of reducing spheres that is disjoint from $\mathcal{S}$ and, moreover, $\mathcal{S}_0 \cdot \mathcal{S}' = 2$. The result now follows from Proposition 2.3. □

Not covered by Proposition 2.4 is the possibility that $\mathcal{S} \cdot \mathcal{S}' = 4$ because there is a sphere in one collection, $S \in \mathcal{S}$, say, such that $S \cdot \mathcal{S}' = 4$. In that case, there are still three subcases to consider: S intersects two different spheres in $\mathcal{S}'$, each in a single circle, S intersects a single sphere in $\mathcal{S}'$ in two different circles, and S intersects a single sphere in $\mathcal{S}'$ in a single circle.

Let's begin by considering $F \cap \mathcal{S}$ and $F \cap \mathcal{S}'$. Let c be the circle $S \cap F$, separating F into two surfaces F_+ and F_-. Since $S \cdot \mathcal{S}' = 4$, $F_+ \cap \mathcal{S}'$ contains two arc components α_1, α_2 with ends on c and, since each circle component of $F \cap \mathcal{S}'$ is separating, it follows that there are disjoint subarcs β_1 and β_2 of c so that the α_i and β_i have common end points, $i = 1, 2$. Let $b_i \subset F$ be the bigon $\alpha_i \cup \beta_i$ and $r \subset F$ be the 4-gon obtained by replacing the two arcs β_i in c by the arcs α_i. See Figure 2.

Proposition 2.5. *Suppose $\mathcal{S}$ and $\mathcal{S}'$ are complete collections of reducing spheres so that, for some sphere $S \in \mathcal{S}$, $S \cdot \mathcal{S}' = \mathcal{S} \cdot \mathcal{S}' = 4$. Then the bigons b_i just described both bound disks in the same handlebody (H_1 or H_2). In fact, if α_1 and α_2 lie in different components*

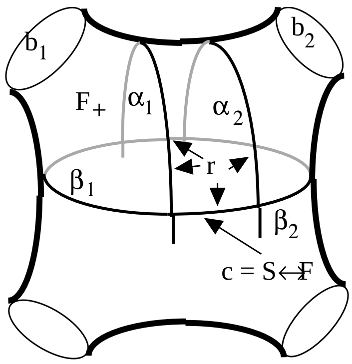

Figure 2.

of $\mathcal{S}' \cap F$ then the b_i bound disks in both H_1 and H_2. In any case, the 4-gon r bounds disks in both H_1 and H_2.

Proof. Suppose first that α_1 and α_2 lie in two different components c_1 and c_2 respectively of $\mathcal{S}' \cap F$, corresponding to the intersections of spheres $S'_i \in \mathcal{S}'$ with F. Then $S \cap S'_i$ is a circle in S'_i that intersects c_i in exactly two points. In particular, in each of H_1 and H_2 there is a bigon disk in S'_i for which one side is a_i and the other is an arc of $S'_i \cap S$ with the same end points. Now consider the picture in S. The two spheres S'_i each intersect $c \subset S$ in two points, so each arc of $(S \cap \S'_i) - F$ is parallel, via bigons in S, to either β_1 or β_2. Putting together a pair of bigons, one in S and one in S', sharing a common arc (in our choice of H_1 or H_2) we get a disk whose boundary is the bigon $b_i \subset F$. This proves the proposition in this case: the first and second statements are immediate; the last follows by observing that, following the second statement, r can be made parallel to c in either H_1 or H_2.

So now assume that α_1 and α_2 lie in the same component c' of $\mathcal{S}' \cap F$, corresponding to the intersections of a sphere $S' \in \mathcal{S}'$ with F. In this situation, the roles of S and S' are now symmetric, moreover the arcs $S' \cap F_-$ form bigons in F with the sides of r not incident to the bigons b_i. $S \cap S'$ may consist of two circles or one.

If $S \cap S'$ is two circles, then for one pair of arcs of $S - S'$ (without loss of generality, the pair $\beta_i, i = 1, 2$) and either choice of H_1 or H_2, say H_1, there is a bigon of S in H_1 with one side on β_i and other side an arc of $(S \cap S') - F$. In S' that arc of $(S \cap S') - F$ is the side

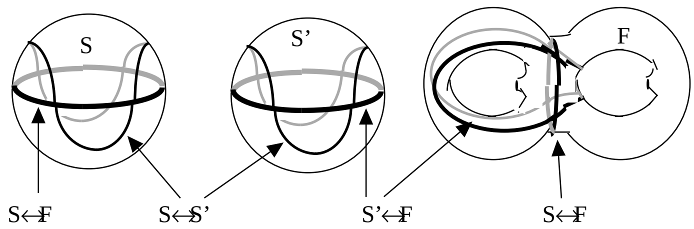

Figure 3.

of a bigon whose other side lies in c', hence is α_i. Putting the bigons together, as above, produces a meridian of H_1 whose boundary is b_i. The choice of H_1 was arbitrary so, just as in the previous case, the b_i bound meridians of both H_i and the proposition is established.

Finally, if $S \cap S'$ is a single circle γ, then note that each arc of $\gamma - F$ is the side of a single bigon in S (resp. S') whose other side is an arc of $c - \gamma$ (resp. $c' - \gamma$). The bigon lies in H_1 or H_2 (we have no choice) and which handlebody it lies on alternates as we progress around arcs of $\gamma - F$. Gluing together the corresponding bigons, one in S and one in S', we establish that the bigons b_1 and b_2 bound disks in H_1, say, while the bigons in F_- described above each bound disks in H_2. The former fact tells us that r is parallel in H_1 to c and hence bounds a disk in H_1. Symmetrically, the second fact tells us that r is parallel in H_2 to c' and so bounds a disk in H_2. $\square$

One case that arises above is special. We will say that $\mathcal{S}$ and $\mathcal{S}'$ have a *classic* 4-*gon intersection* if $\mathcal{S} \cdot \mathcal{S}' = 4$, and there are spheres $S \in \mathcal{S}, S' \in \mathcal{S}'$ so that $S \cdot S' = 4$, each of the 4 bigons in F (two in F_+ and two in F_-) bounds a disk in exactly one of H_1 or H_2, and each 4-gon formed (one in F_+ and one in F_-) bounds a disk in F. If, apart from S and S', the collections $\mathcal{S}$ and $\mathcal{S}'$ coincide, we say that $\mathcal{S}'$ is obtained from $\mathcal{S}$ by *classic* 4-*gon replacement*. See Figure 3.

Proposition 2.6. *Suppose $\mathcal{S}$ and $\mathcal{S}'$ are complete collections of reducing spheres so that, for some sphere $S \in \mathcal{S}$, $S \cdot \mathcal{S}' = \mathcal{S} \cdot \mathcal{S}' = 4$.*

- *If $\mathcal{S}$ and $\mathcal{S}'$ are maximal complete collections then $\mathcal{S}$ is obtained from $\mathcal{S}'$ by classic* 4-*gon replacement.*
- *If $\mathcal{S}'$ and $\mathcal{S}$ are minimal complete collections then there is a minimal complete collection $\mathcal{S}_0$ disjoint from $\mathcal{S}$ and a minimal*

complete collection $\mathcal{S}_0'$ disjoint from $\mathcal{S}'$ so that either $\mathcal{S}_0 \cdot \mathcal{S}' = 2$ or $\mathcal{S}_0$ and $\mathcal{S}_0'$ have a classic 4-gon intersection.

Proof. We continue to use b_i and r to denote the bigons and 4-gon lying in F_+ as described before Proposition 2.5

Suppose $\mathcal{S}$ and $\mathcal{S}'$ are maximal complete collections, and suppose first that there are two spheres in $\mathcal{S}'$ that intersect S. If the 4-gons formed by intersecting both F_+ and F_- with these two spheres both bound disks in F, then the spheres would be parallel, so we assume the 4-gon r in F_+ does not bound a disk in F. Replace the side α_1 in b_1 with the other three sides of r. The resulting circle γ bounds disks in both handlebodies (because both b_1 and r do), is disjoint from $\mathcal{S}$ and intersects $\mathcal{S}'$ in precisely two points, so there is no circle in $\mathcal{S} \cap F$ it can be parallel to. In particular, the sphere obtained by gluing together the disks in each handlebody that γ bounds cannot be parallel to any component of $\mathcal{S}$, contradicting the maximality of $\mathcal{S}$.

Next suppose $\mathcal{S}'$ and $\mathcal{S}$ are maximal complete collections and only one sphere S' of $\mathcal{S}'$ intersects S. Let $c' = S' \cap F$. The first claim is that the 4-gons formed in each of $F_\pm$ by c' bound disks in F. For suppose the 4-gon r in F_+ does not. Then by maximality of $\mathcal{S}$, r is parallel to a component of $\mathcal{S} \cap F$. Moreover maximality of $\mathcal{S}$ would also require that the component F_0 of $F_+ - \mathcal{S}$ incident to S is either a thrice punctured sphere or a once-punctured torus. The latter is ruled out because both S and r are boundary components of F_0 and and the former is ruled out because b_1 and b_2 are both essential in F_0. We conclude that both 4-gons bound disks in F, so $\mathcal{S}$ and $\mathcal{S}'$ have a classic 4-gon intersection.

Now to show that the collections $\mathcal{S} - S$ and $\mathcal{S}' - S'$ are the same, observe that if the component F_0 of $F_+ - \mathcal{S}$ incident to S is planar then both bigons b_1 and b_2 bound disks in both handlebodies, and so these curves correspond to intersections of spheres in both $\mathcal{S}$ and $\mathcal{S}'$, since the bigons are disjoint from both families of spheres. On the other hand, if F_0 is not planar than it must be a once-punctured torus, so there are no other components of $\mathcal{S}$ or $\mathcal{S}'$ incident to it. Hence components of $\mathcal{S} - S$ and $\mathcal{S}' - S'$ that intersect F_+ must coincide. Similarly for F_-. This shows that $\mathcal{S}'$ is obtained from $\mathcal{S}$ by a classic 4-gon replacement.

Now suppose that $\mathcal{S}$ and $\mathcal{S}'$ are minimal complete collections. Consider first the case in which one of the bigons, say b_1, is separating in F. (This is automatically the case if b_1 bounds disks in both handlebodies). Since r and c are separating, so then is b_2. At most one of the three components of $F_+ - (\mathcal{S} \cup \alpha_1 \cup \alpha_2)$ containing (push-offs of) one of b_1, b_2 or r can have genus, since $\mathcal{S}$ is complete, so with no loss of generality, the component F_0 incident to b_1 is planar. This implies that b_1 bounds disks in both handlebodies (since every other boundary component of F_0 does). Let c' be the component of $\mathcal{S}' \cap F$ that contains α_1 and c'' be the circle obtained from c' by replacing α_1 by β_1. Let $\mathcal{S}'_0$ be the collection obtained by replacing the sphere that contains c' with the sphere obtained by capping c'' off in both handlebodies. Then $\mathcal{S}'_0$ is complete and minimal (since F_0 is planar), is disjoint from $\mathcal{S}'$ and intersects $\mathcal{S}$ in two fewer points. Here just set $\mathcal{S}_0 = \mathcal{S}$.

Now suppose each of the bigons b_i is non-separating, so in particular each bounds a disk in only one handlebody. We know then that S intersects only one sphere $S' \in \mathcal{S}'$ and that $S \cap S'$ is a single circle γ. Since r bounds disks in both handlebodies, from Proposition 2.5 above, it follows that r is separating in F. The component F_0 of $F - (\mathcal{S} \cup \alpha_1 \cup \alpha_2)$ containing (a push-off of) r can't have genus, since if it did then the component of $F - \mathcal{S}$ containing it would have genus greater than two, since the b_i, contained in it, are non-separating. Let c'' be the circle obtained from $c' = S' \cap F$ by replacing α_1 by the other three sides of r and let $\mathcal{S}'_0$ be the collection obtained from $\mathcal{S}$ by replacing the sphere that contains c' with the sphere obtained by capping c'' off in both handlebodies. Then $\mathcal{S}'_0$ is complete and minimal (since F_0 is planar), is disjoint from $\mathcal{S}'$ and intersects $\mathcal{S}$ in the same four points, now $c'' \cap c$. Indeed, the only relevant change from $\mathcal{S}'$ is that the rectangle r'' formed in F_+ by the arcs $c'' \cap F_+$ now bounds a disk in F. Now repeat the argument on the rectangle formed in F_-, altering an arc component of $c - c'$ instead of $c' - c$, to get the required complete collection $\mathcal{S}_0$. $\square$

Combining the results of Propositions 2.4 and 2.6 above, we summarize:

Theorem 2.7. *If $\mathcal{S}'$ and $\mathcal{S}$ are maximal complete collections with*

$\mathcal{S}' \cdot \mathcal{S} \leq 4$ then either they are isotopic, or one can be obtained from the other by classic 4-gon replacement or there is another maximal complete collection $\mathcal{S}''$ which can be gotten from either $\mathcal{S}$ or $\mathcal{S}'$ by orthogonal replacement.

If $\mathcal{S}'$ and $\mathcal{S}$ are minimal complete collections with $\mathcal{S}' \cdot \mathcal{S} \leq 4$, then there is a sequence of minimal complete collections

$$\mathcal{S} = \mathcal{S}_0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_3 = \mathcal{S}'$$

so that for each $i = 1, \ldots, 3$, $\mathcal{S}_i$ is either disjoint from $\mathcal{S}_{i-1}$ or is obtained from $\mathcal{S}_{i-1}$ by classic 4-gon replacement.

Proof. Suppose $\mathcal{S}$ and $\mathcal{S}'$ are maximal. Then $\mathcal{S} \cdot \mathcal{S}' = 0$ only if $\mathcal{S}$ and $\mathcal{S}'$ are parallel (i. e. isotopic). If $\mathcal{S} \cdot \mathcal{S}' = 2$ the result follows from Proposition 2.3. If $\mathcal{S} \cdot \mathcal{S}' = 4$ and some sphere in one of the collections (say $S \in \mathcal{S}$) has intersection number 4 with the other, then the result follows from Proposition 2.6. If there is no such sphere in either collection then apply Proposition 2.4.

Now suppose $\mathcal{S}$ and $\mathcal{S}'$ are minimal. If $\mathcal{S} \cdot \mathcal{S}' = 0$ there is nothing to prove (just take e. g. $\mathcal{S}_i = \mathcal{S}', i \geq 1$.) If $\mathcal{S} \cdot \mathcal{S}' = 2$ apply Proposition 2.3. If $\mathcal{S} \cdot \mathcal{S}' = 4$ and some sphere in one of the collections (say $S \in \mathcal{S}$) has intersection number 4 with the other, then the result follows from Propositions 2.6 and, if a classic 4-gon intersection isn't encountered, 2.3. If there is no such sphere in either collection then just apply Proposition 2.4. □

3. Sphere collections with more intersections

It's natural to ask if there is a natural extension of Theorem 2.7 when the complete collections of reducing spheres have higher intersection number. In this section we show that there is, but the argument requires the use of a much more sophisticated tool, namely the analysis by Powell [3] (extending Goeritz [1]) of isotopies of S^3 that return the standard genus g surface back to itself. The extension is particularly natural if we do not insist that the complete collections be maximal.

THEOREM 3.1. *If $\mathcal{S}$ and $\mathcal{S}'$ are two complete collections of reducing spheres for a genus g Heegaard splitting F of S^3 then there is a* *sequence of complete collections of reducing spheres*

$$\mathcal{S} = \mathcal{S}_0 \rightarrow \mathcal{S}_1 \rightarrow \ldots \rightarrow \mathcal{S}_m = \mathcal{S}'$$

so that for each $i = 1, \ldots, m$, $\mathcal{S}_i$ *is either disjoint from* $\mathcal{S}_{i-1}$ *or is obtained from* $\mathcal{S}_{i-1}$ *by classic* 4*-gon replacement. Moreover, if* $\mathcal{S}$ *and* $\mathcal{S}'$ *are minimal, so is every collection in the sequence.*

Proof. Any complete collection can be altered to one that is minimal by simply deleting some spheres, so we may as well assume that $\mathcal{S}$ and $\mathcal{S}'$ are minimal.

We say that a minimal complete collection is *standard* if all but one component of $F - \mathcal{S}$ is a punctured torus and the remaining component (needed only if $g \geq 3$) is a $g-1$-punctured torus. Obviously any minimal complete collection is disjoint from a standard one, so we may as well assume that both $\mathcal{S}$ and $\mathcal{S}'$ are standard minimal complete collections.

With this explicit description there is an obvious (but obviously not unique) orientation-preserving homeomorphism $h : S^3 \rightarrow S^3$ with the property that $h(F) = F$ and $h(\mathcal{S}) = \mathcal{S}'$. By the Alexander trick, h is isotopic to the identity.

In [3] Powell shows that any isotopy of S^3 that ends in a homeomorphism carrying F to F is a product of particularly simple such isotopies. Powell's isotopies can be described easily with respect to a particular standard minimal collection of reducing spheres $\mathcal{S}_0$. Indeed, choose $\mathcal{S}_0$ to be the standard minimal collection whose only visible member in [3, Figure 4] is there denoted by $\overline{S}_1$. Of the five generating isotopies shown in [3, Figure 4], the first four visibly move $\mathcal{S}_0$ to a disjoint complete collection. The last, denoted D_θ, visibly does a classic 4-gon replacement on $\overline{S}_1 \in \mathcal{S}_0$. In particular, the theorem is true for the specific example $\mathcal{S}' = h_p(\mathcal{S})$ when h_p is any of the Powell homeomorphisms (acting on $\mathcal{S}$ as identified with $\mathcal{S}_0$ in [3, Figure 4]).

Now the homeomorphism h is the composition of homeomorphisms $h = h_1 \circ h_2 \circ \ldots \circ h_m$ where each h_i is one of Powell's homeomorphisms. Construct a sequence of collections of reducing spheres by setting $\mathcal{S}_i = h_1 \circ h_2 \circ \ldots \circ h_i(\mathcal{S}_0)$. Then notice that the pair $\mathcal{S}_{i-1}, \mathcal{S}_i$ is the image under the homeomorphism $h_1 \circ h_2 \circ \ldots \circ h_{i-1}$

of the pair $\mathcal{S}_0, h_i(\mathcal{S}_0)$, for which we know the theorem is true. The theorem follows. □

Of course this theorem implies that part of Theorem 2.7 dealing with minimal complete collections, and its proof is completely independent. So in some sense, it makes much of Theorem 2.7 obsolete. Of course the proof of Theorem 2.7 is much more elementary. Indeed, it is not unreasonable to hope that a more elementary and direct proof of Theorem 3.1 could be found which proceeds by reducing the maximal intersection number between successive complete minimal collections in the sequence. Such a proof, in turn, would in turn likely lead to an elementary proof of Powell's result.

It is more awkward to find a natural extension of Theorem 2.7 to maximal collections. Indeed, consider the (non-maximal) complete collection $\mathcal{S}_0^+$ obtained by adding just one additional sphere (denoted $\overline{S}_2$ in [3, Figure 4]) to the standard minimal collection $\mathcal{S}_0$, so that the components of $F - \mathcal{S}_0^+$ consist of g once-punctured tori and a g-punctured sphere. Then $\mathcal{S}_0^+$ and $D_\theta(\mathcal{S}_0^+)$ have intersection number 16. In fact, for $1 \leq i, j \leq 2$, the spheres $\overline{S}_i$ and $D_\theta(\overline{S}_j)$ have a classic 4-gon intersection.

So we say that two maximal complete collections $\mathcal{S}$ and $\mathcal{S}'$ differ by an *extended* 4*-gon replacement* if there are spheres $S_1, S_2 \in \mathcal{S}$ and spheres $S'_1, S'_2 \in \mathcal{S}'$ so that the two collections coincide except on these spheres and, for $1 \leq i, j \leq 2$, each pair S_i, S'_j have a classic 4-gon intersection.

The definition is contrived so that, for maximal collections, Theorem 2.7 extends to:

Proposition 3.2. *If $\mathcal{S}$ and $\mathcal{S}'$ are any two maximal collections of reducing spheres for a genus n Heegaard splitting F of S^3 then there is a sequence of maximal collections of reducing spheres*

$$\mathcal{S} = \mathcal{S}_0 \rightarrow \mathcal{S}_1 \rightarrow \ldots \rightarrow \mathcal{S}_m = \mathcal{S}'$$

so that $\mathcal{S}_i$ is obtained from $\mathcal{S}_{i-1}$ either by orthogonal replacement or by classic or extended 4*-gon replacement $i = 1, \ldots, m$.*

Proof. Motivated by the preceding example, call a complete collection (typically neither maximal nor minimal) *standard* if the complementary components of F consist of g once-punctured tori and a g-punctured sphere. Any maximal complete collection contains a unique standard collection, and any two standard collections are related by a homeomorphism of S^3 that takes F to itself. Moreover, if we consider the effect of Powell's homeomorphisms on his standard collection, we observe that the first three preserve the collection, the fourth alters it by a single classic 4-gon replacement and the last, as noted above, alters it by an extended 4-gon replacement.

The proof given in Theorem 3.1 is now almost immediately applicable. We are given maximal complete collections $\mathcal{S}$ and $\mathcal{S}'$ and, to begin with, we can picture $\mathcal{S}$ as shown in [3, Figure 4]). (There only two spheres are visible, denoted $\overline{S}_1$ and $\overline{S}_2$.) Let $\mathcal{S}_s$ and $\mathcal{S}'_s$ be the standard complete collections contained in $\mathcal{S}$ and $\mathcal{S}'$ respectively, and $h : S^3 \rightarrow S^3$ be a homeomorphism with $h(F) = F$ and $h(\mathcal{S}_s) = \mathcal{S}'_s$. If we knew that $h(\mathcal{S}) = \mathcal{S}'$ the proof would now exactly replicate the argument in Theorem 3.1.

But, of course, it may not be that $h(\mathcal{S}_s) = \mathcal{S}'_s$ since there are many possible ways of extending a standard collection to a complete collection. But the possible extensions of a given standard collection to a maximal collection correspond precisely to what are often called "markings" of an n-punctured sphere. It is a non-obvious but classical result of Hatcher and Thurston [2, Appendix] that any two markings can be made equivalent by moves in which a single circle is exchanged for one that intersects it in exactly two points. In our context, such moves correspond precisely to orthogonal replacement. □

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