

Finite Simple Groups Acting on 3-Manifolds and Homology Spheres

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SUMMARY. - *Any finite group admits actions on closed 3-manifolds, and in particular free actions. For actions with fixed points, assumptions on the type of the fixed point sets of elements drastically reduce the types of the possible groups. Concentrating on the basic case of finite simple groups we show in the present paper that, if some involution of a finite simple group G acting orientation-preservingly on a closed orientable 3-manifold has nonempty connected fixed point set, then G is isomorphic to a projective linear group $\mathrm{PSL}(2, q)$, and thus of a very restricted type. The question was motivated by our work on the possible types of isometry groups of hyperbolic 3-manifolds occurring as cyclic branched coverings of knots in the 3-sphere. We characterize also finite groups which admit actions on \mathbb{Z}_2 -homology spheres, generalizing corresponding results for integer homology spheres.*

1. Introduction

The main result of the present paper is the following

THEOREM 1.1. *Let G be a finite nonabelian simple group of diffeomorphisms of a closed orientable 3-manifold. Then, if G contains an*

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involution with nonempty connected fixed point set, G is isomorphic to a projective linear group $\mathrm{PSL}(2, q)$, for some odd prime power q .

We study here only the most basic case of finite simple groups, using quite involved results from the classification of finite simple groups. Considering suitable composition series, also the case of arbitrary finite groups can be analyzed; as the proofs in the general case become much longer and more technical we do not follow this line here, illustrating the methods only for the most basic case of simple groups. In the situation of Theorem 1.1, it remains open which of the linear fractional groups $\mathrm{PSL}(2, q)$ really occur; for the moment, the only known example seems to be the alternating group $\mathbb{A}_5 \cong \mathrm{PSL}(2, 5)$.

Theorem 1.1 shows that, if a finite simple group of diffeomorphisms of a 3-manifold contains an involution with nonempty connected fixed point set, then this simple group is of a very restricted type. On the other hand, it is easy to see that every finite group admits actions on closed 3-manifolds, for example free actions (and also isometric actions on closed hyperbolic 3-manifolds). It is shown in [5] that every finite group admits free actions also on rational homology 3-spheres. The situation changes drastically for integer homology 3-spheres. If a finite group G acts freely on a \mathbb{Z} -homology (3-) sphere then G has periodic cohomology (of period four) and is again of a restricted type (see [1], [8] or [17]). The same occurs for arbitrary (i.e. not necessarily free) finite group actions on \mathbb{Z} -homology 3-spheres; a characterization of the groups is given in [10] and [19].

We consider now the case of \mathbb{Z}_2 -homology 3-spheres. Theorem 1.1 implies the following

COROLLARY 1.2. *Let G be a finite nonabelian simple group of diffeomorphisms of a \mathbb{Z}_2 -homology 3-sphere. Then some involution in G has nonempty connected fixed point set and G is isomorphic to a projective linear group $\mathrm{PSL}(2, q)$, for an odd prime power q .*

As a consequence of a result in [16], free actions of arbitrary finite groups on \mathbb{Z}_2 -homology n -spheres (i.e. in arbitrary dimensions) can be characterized as follows.

For a group G , we denote by $\mathcal{O}(G)$ the maximal normal subgroup of odd order of G (see [16, p. 293]); note that $\mathcal{O}(G)$ is solvable by the Feit-Thompson theorem.

THEOREM 1.3. *Let G be a finite group of diffeomorphisms of a \mathbb{Z}_2 -homology n -sphere such that every involution in G has empty fixed point set. Then either G is solvable or it is of the following type:*

$G/\mathcal{O}(G)$ contains a normal subgroup G_0 of odd index such that the center $Z(G_0)$ of G_0 has order two and $G_0/Z(G_0)$ is one of the following groups:

$$\mathrm{PSL}(2, q), \quad \mathrm{PGL}(2, q), \quad \text{or the alternating group } \mathbb{A}_7,$$

for an odd prime power q greater than three.

Now we consider arbitrary finite group actions on \mathbb{Z}_2 -homology 3-spheres. Let M be a \mathbb{Z}_2 -homology 3-sphere and G a finite group of orientation preserving diffeomorphisms of M . As above, we denote by $\mathcal{O}(G)$ the maximal normal subgroup of odd order of G . It is easy to see that also the quotient manifold $M/\mathcal{O}(G)$ is a \mathbb{Z}_2 -homology 3-sphere on which $G/\mathcal{O}(G)$ acts.

Given two groups G_1 and G_2 and a central involution in each of them, we denote by

$$G_1 \times_{\mathbb{Z}_2} G_2$$

the central product of G_1 and G_2 with the two central involutions identified, i.e. G_1 and G_2 commute elementwise and $G_1 \cap G_2 = \mathbb{Z}_2$ is generated by the identified central involutions (so $G_1 \times_{\mathbb{Z}_2} G_2$ is a factor group of the direct product of the two groups, see [15, p. 173]).

THEOREM 1.4. *Let G be a finite group of orientation preserving diffeomorphisms of a \mathbb{Z}_2 -homology 3-sphere M . Then G is of one of the following three types (where q and q' denote odd prime powers).*

1) $G/\mathcal{O}(G)$ is solvable, and hence also G (as a consequence of the Feit-Thompson theorem).

2) $G/\mathcal{O}(G)$ contains a subgroup of index at most two isomorphic to $\mathrm{PSL}(2, q)$ or to $\mathbb{Z}_2 \times \mathrm{PSL}(2, q)$; any involution of $\mathrm{PSL}(2, q)$ has nonempty connected fixed point set (acting on $M/\mathcal{O}(G)$).

3) *i) $G/\mathcal{O}(G)$ has a subgroup of index at most two isomorphic to a central product*

$$\mathrm{SL}(2, q) \times_{\mathbb{Z}_2} G_0$$

where G_0 is solvable with a unique involution; the unique involutions of $\mathrm{SL}(2, q)$ and G_0 act freely.

ii) $G/\mathcal{O}(G)$ has a normal subgroup of index at most four isomorphic to a central product

$$\mathrm{SL}(2, q) \times_{\mathbb{Z}_2} \mathrm{SL}(2, q')$$

where the central involution acts freely.

Theorem 1.4 generalizes corresponding results for integer homology 3-spheres in [10] and [19] to the case of \mathbb{Z}_2 -homology 3-spheres. As the proof is quite long and involved, we will not give the details here and refer to the proofs of the main results in [10] and [19] which have to be modified suitably.

Interesting examples of \mathbb{Z}_2 -homology 3-spheres are obtained as 2-fold (or 2^n -fold) cyclic branched coverings of knots in S^3 (see [7, p. 16]), and by Dehn surgery of type a/b on knots in S^3 , with a odd. In fact, our results were motivated by our work on cyclic branched coverings of hyperbolic knots and links in S^3 . In [18], [11] and [13] (see also the survey article [12] where Theorem 1.1 was announced), the following problem is considered: in how many different ways can the same hyperbolic 3-manifold M occur as a cyclic branched covering of different knots and links in the 3-sphere. The answer depends largely on the structure of the finite isometry group of M , and also on the symmetry group of the knot or link. A solution was obtained for various situations; for example, the situation is well-understood if M is a cyclic branched covering of a non-invertible knot, or if the finite isometry group of M is solvable (moreover, under a mild restriction, the isometry group of M is solvable for even branching orders $n > 4$). The solvable case is generic for isometry groups of hyperbolic 3-manifolds which are cyclic branched coverings of knots: for a given hyperbolic knot and large branching orders this is an easy consequence of a shortest geodesic argument in Thurston's hyperbolic Dehn surgery theorem. For small branching

orders, the isometry group may be nonsolvable; an example is the Seifert-Weber hyperbolic dodecahedral space which is a 5-fold cyclic branched covering of the Whitehead link and has the alternating group $\mathbb{A}_5 \cong \text{PSL}(2, 5)$ as a group of isometries. A nonhyperbolic example is the spherical Poincaré homology 3-sphere which occurs as a cyclic branched covering of torus knots and again has the group $\mathbb{A}_5 \cong \text{PSL}(2, 5)$ as a group of isometries. All such examples belong to the phenomenon of "hidden symmetries" considered in [14]. However, such nonsolvable situations seem to be quite rare and are not yet well understood, in particular there remains the interesting question which nonsolvable or simple groups really occur as finite isometry or diffeomorphism groups of 3-manifolds which are cyclic branched coverings. At present, the only known example of such a simple group seems to be $\mathbb{A}_5 \cong \text{PSL}(2, 5)$, and in fact this might be the only one.

2. Preliminaries

We need to describe the local action of a finite group of diffeomorphisms in the neighbourhood of an invariant simple closed curve in a 3-manifold. More precisely, let G be a finite group of orientation preserving diffeomorphisms of a closed orientable 3-manifold M , and suppose that G contains an element h with nonempty connected fixed point set K . Any element g of G which commutes with h maps the fixed point set K of h to itself. Therefore g induces a reflection (strong inversion) or a rotation on K : we shall call such elements *h-reflections* resp. *h-rotations* (an element which acts trivially on K is also an *h-rotation*).

So the centralizer $C_G h$ of h in G is the set of *h-rotations* and *h-reflections*, and the subgroup of *h-rotations* is normal in $C_G h$ of index one or two. We have

PROPOSITION 2.1. *Let G be a finite group of orientation preserving diffeomorphisms of a closed orientable 3-manifold; suppose that G contains an element h with nonempty connected fixed point set K . Then the normalizer $N_G h$ of h in G is isomorphic to a subgroup of a semidirect product $\mathbb{Z}_2 \ltimes (\mathbb{Z}_a \times \mathbb{Z}_b)$, for some nonnegative integers*

a and b , where \mathbb{Z}_2 operates on the normal subgroup $\mathbb{Z}_a \times \mathbb{Z}_b$ (the subgroup of h -rotations) by sending each element to its inverse.

Proof. By a result of Newman [9] (see also [6]) a periodic transformation of a manifold which is the identity on an open subset is the identity. Thus the action of an element of the normalizer $N_G h$ (which maps K to itself) is determined by its action in a neighbourhood of K where it acts as a standard rotation or reflection, and the group of rotations is an abelian group of rank one or two. \square

The proof of Theorem 1.1 will be based on a difficult result which was an important step in the 70's towards the classification of finite simple groups: the classification of simple groups with sectional 2-rank at most four (that is every 2-subgroup is generated by at most four elements). This is the culmination of the efforts of many mathematicians and was completed by Gorenstein and Harada, see [16, p. 513, Theorem 8.12]). It seems likely that for the proof of Theorem 1.1 one could avoid this deep result and use more elementary arguments; but at present this would result in a very long and technical proof.

The first step in the proof of Theorem 1.1 will be a Lemma which enables us to use the Gorenstein-Harada classification.

LEMMA 2.2. *Let G be a finite group of orientation preserving diffeomorphisms of a closed orientable 3-manifold. If G contains an involution with nonempty connected fixed point set, then it has sectional 2-rank at most four.*

Proof. The proof of Lemma 2.2 depends on two algebraic results which are commonly used in finite group theory to determine the structure of Sylow 2-subgroups from local conditions. The first result is due to MacWilliams, the second one to Harada. Recall that an elementary abelian group is a direct product of cyclic groups of the same prime order.

PROPOSITION 2.3. ([16, p. 514, Theorem 8.15]) *A 2-group with no elementary abelian normal subgroup of rank three has sectional 2-rank at most four.*

PROPOSITION 2.4. ([16, p.514, (8.14)]) *A 2-group S which contains an elementary abelian subgroup A of rank three such that the centralizer $C_S A$ of A in S is contained in A has sectional 2-rank at most four.*

Proposition 2.3 reduces the proof of Lemma 2.2 to the case that a Sylow 2-subgroup S of G contains an elementary abelian normal subgroup A' of rank three. We can assume that S contains an involution h with nonempty connected fixed point set. Now h acts by conjugation on $A' \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and it is easy to see that the subgroup of elements fixed by h is either equal to A' or a subgroup of the form $\mathbb{Z}_2 \times \mathbb{Z}_2$. In any case it follows that the centralizer $C_S h$ of h in S contains an elementary abelian subgroup A of rank three such that $h \in A$. Then $C_S A$ is contained in $C_S h$. By Proposition 2.1, $C_S h$ is a subgroup of a semidirect product $\mathbb{Z}_2 \ltimes (\mathbb{Z}_a \times \mathbb{Z}_b)$. This implies that A , which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, has to contain some h -reflection t , and also a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ of h -rotations. It follows easily that $C_S A$ is equal to A (because in $C_S h$, the h -reflection t commutes only with the elements in A), and an application of Proposition 2.4 finishes the proof of Lemma 2.2. \square

For the proof of Theorem 1.1 we need also the following two results (which were the first steps towards the Gorenstein-Harada Theorem):

PROPOSITION 2.5. ([16, p.505, Theorem 8.6]) *A simple group with a dihedral Sylow 2-subgroup is isomorphic to the alternating group \mathbb{A}_7 or to $\text{PSL}(2, q)$, for an odd prime power $q \geq 5$.*

PROPOSITION 2.6. ([16, p. 582, Theorem 11.1]) *A simple group with an abelian Sylow 2-subgroup of rank two is isomorphic to $\text{PSL}(2, q)$, for $q \equiv 3$ or $5 \pmod{8}$.*

3. Proof of Theorem 1.1

By hypothesis G contains an involution h with nonempty connected fixed point set. By Lemma 2.2, G has sectional 2-rank at most four and hence is one of the groups in the Gorenstein-Harada list of finite simple groups of such type ([16, p. 513, Theorem 8.12]). Now the

centralizer $C_G h$ of the involution h is of the special type given in Proposition 2.1. We will check which of the simple groups in the Gorenstein-Harada list contain an involution with such a centralizer which we will call *admissible* in the following.

We shall first show that no group in the list with more than one conjugacy class of involutions contains an involution with an admissible centralizer. The Gorenstein-Harada Theorem gives the following list of simple groups with sectional 2-rank at most four and more than one conjugacy class of involutions (the algebraic properties of the simple groups can be found in [4]):

M_{12} ; $\text{PSp}(4, q)$, for q odd; J_2 ; \mathbb{A}_n , for $8 \leq n \leq 11$; $\text{PSL}(3, q)$ and $\text{PSU}(3, q)$, for q odd, $q \geq 5$; $\text{PSL}(4, q)$, $\text{PSU}(4, q)$, $\text{PSL}(5, q)$ and $\text{PSU}(5, q)$, for q odd.

We can rule out directly the following groups because the centralizer of any involution in these groups is not solvable:

J_2 ; \mathbb{A}_n , for $10 \leq n \leq 11$; $\text{PSL}(3, q)$ and $\text{PSU}(3, q)$, for q odd, $q \geq 5$; $\text{PSL}(4, q)$, $\text{PSU}(4, q)$, $\text{PSL}(5, q)$ and $\text{PSU}(5, q)$, for q odd.

The remaining groups contain involutions with solvable centralizers.

The group M_{12} possesses two conjugacy classes of involutions. The centralizer of an involution which is not central in any Sylow 2-subgroup is not solvable. The centralizer of an involution which is central in some Sylow 2-subgroup has order 192 and contains a subgroup isomorphic to the permutation group on four letters \mathcal{S}_4 which is not admissible.

An analogous argument holds for the remaining groups $\text{PSp}(4, q)$, for q odd, \mathbb{A}_8 and \mathbb{A}_9 which have isomorphic Sylow 2-subgroup and two conjugacy classes of involutions. The centralizer of any involution contains an elementary abelian 2-subgroup of rank four and is not admissible.

So G has only one conjugacy class of involutions, and consequently all involutions in G have connected fixed point set. We can assume that h is a central involution in a Sylow 2-subgroup S of G , so $S = C_S h$ consists of h -rotations and h -reflections. By Proposi-

tion 2.1, the subgroup A of h -rotations of S is abelian of rank at most two. We shall prove that S is dihedral or abelian of rank two.

If A is cyclic then S is dihedral because the Sylow 2-subgroup of a simple group cannot be cyclic.

Suppose that A has rank two. We will show that $S = A$ in this case, that is that S does not contain h -reflections. Suppose, on the contrary, that S contains an h -reflection; then S contains an elementary abelian 2-group B of rank three. The group B contains seven involutions with nonempty connected fixed point set, so there are seven distinct rotation axes which are all invariant under B . We want to count the total number of intersection points between these axes. Note that exactly three axes (of order two) meet in each intersection point. The rotations of each axis form a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ of B , so there are exactly four reflections and four intersection points on each axis. It follows that the total number of intersection points between the seven axes is 28 divided by 3 which is a contradiction.

Thus we have proved that a Sylow 2-subgroup of G is dihedral or abelian of rank two, and we can apply the classification of simple groups with such Sylow 2-subgroups given in Propositions 2.5 and 2.6. To finish the proof of Theorem 1.1, by Propositions 2.5 and 2.6 we have only to prove that G is not isomorphic to \mathbb{A}_7 . The centralizer C of an involution h in \mathbb{A}_7 has order 24; moreover the centralizer in C of any element of order three is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_6$, and no involution contained in $\mathbb{Z}_2 \times \mathbb{Z}_6$ is central in C . By Proposition 2.1, the centralizer C is not admissible, and hence \mathbb{A}_7 does not occur.

This finishes the proof of Theorem 1.1.

4. Proofs of Corollary 1.2 and Theorem 1.3

For the *Proof of Corollary 1.2*, let G be a finite group acting on a \mathbb{Z}_2 -homology sphere M . Assuming that all involutions in G are without fixed points we will show that G is not simple.

As all involutions are free, also a Sylow 2-subgroup S_2 of G acts freely on M . By [2, p. 148, Theorem 8.1], the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ does not act freely on a \mathbb{Z}_2 -homology sphere (of any dimension), therefore S_2

has no subgroups $\mathbb{Z}_2 \times \mathbb{Z}_2$. As the center of the finite 2-group S_2 is nontrivial and contains an involution (unless S_2 is trivial), it follows that S_2 has a unique involution. By a theorem of Burnside ([3, p. 99, Theorem 4.3]), S_2 is either cyclic or a generalized quaternion group. By [16, p. 144, Corollary 2] and [16, p. 306, Example 3], the group G cannot be nonabelian simple.

Thus, for a nonabelian simple group G as in Corollary 1.2, some involution has nonempty fixed point set which is connected by [2, p. 144, Theorem 7.9]. Corollary 1.2 follows now from Theorem 1.1.

The *Proof of Theorem 1.3* is similar. Again the Sylow 2-subgroup S_2 of G is cyclic or a generalized quaternion group. If S_2 is cyclic, by [16, p. 144, Corollary 1] and the Feit-Thompson theorem, G is solvable. If S_2 is quaternionic, by [16, p. 506, Theorem 8.7] the group G is either solvable or of the type described in the Proposition.

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