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Hidden Symmetries of Cyclic Branched Coverings of 2-Bridge Knots

Marco Reni and Andrei Vesnin (*)

Dedicated to the memory of Marco Reni,
a good friend and a nice mathematician

Summary. - *We consider hyperbolic 3-manifolds $M_n(K)$, which are n-fold cyclic branched coverings of 2-bridge knots K. We show that for $n \geq 5$ the orientation-preserving isometry group of $M_n(K)$ either is a lift of a symmetry group of K or has a very special structure.*

1. Introduction

The class of hyperbolic 3-manifolds with cyclic symmetry have been subject of extensive literature. It is known that the isometry group of a hyperbolic 3-manifold with finite volume is a finite group and, conversely, that any finite group can be realized in this way [4]. There is no general method for computing the isometry group of a given hyperbolic manifold. Sometimes it is possibly by algebraic methods (see, for example [5]), by geometrical methods (see, for example [7]), or by computer systems (see, for example [2]). On the other hand

(*) Author's addresses: Marco Reni, Università degli Studi di Trieste, Dipartimento di Scienze Matematiche, 34100 Trieste, Italy
Andrei Vesnin, Sobolev Institute of Mathematics, Novosibirsk 630090, Russia
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very much is known about the symmetry groups of knots and links (see [1], [3]). Particularly, the class of 2-bridge knots is completely studded.

Thus a natural question is if one can compute the isometry group of a branched covering of a hyperbolic knot from the symmetry group of a knot.

The case of 2-fold branched coverings of π-hyperbolic knots admitting strongly invertible involution, was investigated in [8].

In the present paper, we consider the isometry groups of manifolds $M_n(K)$ which are n-fold cyclic coverings of 3-sphere S^3 branched over a 2-bridge knot K. Recall that if a 2-bridge knot K is not torus then it is hyperbolic, i.e. $S^3 \setminus K$ is a hyperbolic manifold [13]. We consider only hyperbolic 2-bridge knots below. So, $M_n(K)$ is hyperbolic for $n \geq 3$ with only exceptional case $n = 3$ if K is the figure-eight knot. Denote by $\mathcal{O}_n(K)$ a 3-orbifold whose underlying space is S^3 and singular set is a 2-bridge knot K with singularity index n.

We would like to relate the orientation-preserving isometry group $G = \mathrm{Iso}^+(M_n(K))$ of the manifold $M_n(K)$ to the isometry group of the orbifold $\mathcal{O}_n(K)$.

We say that $M_n(K)$ *has no hidden symmetries* (with respect to the given branched covering) if every isometry of $M_n(K)$ is the lift of some isometry of $\mathcal{O}_n(K)$.

As a consequence of Thurston's hyperbolic surgery theorem [13], for large values of n any $M_n(K)$ has no hidden symmetries. The following estimation of such n for hyperbolic manifolds which are cyclic branched coverings of links was given in [9].

Theorem 1.1. [9] *(a) Let $M = M_n(L)$ be a (hyperbolic) n-fold cyclic branched covering of a hyperbolic link L. Let $v = \mathrm{vol}(S^3 \setminus L)$ and v_n the volume of the smallest hyperbolic 3-orbifold with torsion of order n. If $n \geq (v/v_n) - 1$, then M has no hidden symmetries.*

(b) For sufficiently large n there exists a hyperbolic knot whose (hyperbolic) n-fold cyclic branched covering have hidden symmetries.

Among famous examples of manifolds with hidden symmetries are Hantzsche–Wendt Euclidean manifold (that is the 3-fold cyclic branched covering of the figure-eight knot) and Seifert–Weber hyperbolic dodecahedral manifold (that is the 5-fold strongly cyclic branched covering of the Whitehead link) (see discussion in [9])

The aim of the present paper is to improve the part (a) of Theorem 1.1 in the case of cyclic branched coverings of 2-bridge knots. More exactly, the following statement holds.

Theorem 1.2. *Let* $n \geq 5$ *and* $M = M_n(K)$ *be a (hyperbolic)* n*-fold cyclic branched covering of a hyperbolic 2-bridge knot* K*. Let* $v = \mathrm{vol}(S^3 \setminus K)$ *and* v_n *the volume of the smallest orientable hyperbolic 3-orbifold with torsion of order* n*. If* $n \geq \sqrt{v/(4v_n)} + 1$*, then* M *has no hidden symmetries.*

Concerning the structure of the isometry group G we can indeed provide much more information than what is stated in Theorem 1.2 above, supporting evidence that hidden symmetries may occur only in exceptional situations. Indeed we do not know any explicit example of hidden symmetries for cyclic branched coverings of 2-bridge knots, and we formulate

Question 1.3. *Does any (hyperbolic)* n*-fold cyclic branched covering of a (hyperbolic) 2-bridge knot has no hidden symmetries?*

The Theorem 1.2 is a corollary of the following result.

Theorem 1.4. *For* $n \geq 5$ *let* $M = M_n(K)$ *be the (hyperbolic)* n*-fold cyclic branched covering of a 2-bridge knot* K *and* G *its orientation-preserving isometry group. Then one of the following three cases occurs:*

(i) M *has no hidden symmetries;*

(ii) G *is large;*

(iii) n *is odd and* G *is special.*

By "G is large", we mean that G contains a normal abelian subgroup of order p^2 which does not descend to the 2-bridge knot K where p is an odd prime number such that n divides $(p-1)$ or $(p+1)$. So the order of G satisfies the inequality $|G| \geq 4np^2$.

By "G is special", we mean that M is an integral homology 3-sphere, G is isomorphic to $\mathrm{PSL}(2,p)$ where p is an odd prime number such that n divides $(p-1)/4$ or $(p+1)/4$. So the order of G satisfies the inequality $|G| \geq p(p-1)(p+1)/2$.

Proof of Theorem 1.2. In both cases when G is large or special in above sense, we have an inequality $|G| \geq 4n(n-1)^2$. Consider the quotient orbifold $M_n(K)/G$. Since G is a group of orientation-preserving isometries, $M_n(K)/G$ is a orientable hyperbolic orbifold with torsion of order n and

$$\mathrm{vol}(M_n(K)/G) = \frac{n \cdot \mathrm{vol}(\mathcal{O}_n(K))}{|G|} < \frac{n \cdot \mathrm{vol}(S^3 \setminus K)}{4n(n-1)^2} = \frac{v}{4(n-1)^2} \leq v_n,$$

where we used that $n \geq \sqrt{v/(4v_n)} + 1$. Thus, we got a contradiction with the assumption the volume v_n is smallest.

This complete the proof of Theorem 1.2. □

We refer [1] and [3] for properties of 2-bridge knots and [13] for the hyperbolic manifolds and orbifolds theory. The technique used for the proof of Theorem 1.4 will be developed in Sections 2–4, and the proof will be given in Section 5.

2. Auxiliary results from finite group theory

The technique of the proof of Theorem 1.4 is based on properties of Sylow subgroups, and we refer to [11] and [12] for facts on finite group. For reader convenience, in this section we list basic properties which will be used below. Recall that a subgroup is called a *p-subgroup* if its order is a power of p.

PROPOSITION 2.1. [11, p. 88, Theorem 1.6] *If H is a proper subgroup of a p-group G, then we have $N_G(H) = \{g \in G \mid H^g = H\} \neq H$. Thus, the normalizer of a proper subgroup H is strictly larger than H.*

A subgroup of a group G is said to be a *Sylow p-subgroup* if it is of order p^n where $|G| = p^n m$ with $(p, m) = 1$.

PROPOSITION 2.2. [11, p. 99, Theorem 2.7] *Let H be a normal subgroup of a group G. If S is a Sylow p-subgroup of H, then G is generated by H and the normalizer $N_G(S)$ of S in G.*

Let p be a prime number. An abelian group E is said to be an *elementary abelian p-group* if every element x of E satisfies $x^p = 1$. The following important property of elementary abelian p-groups will be used below.

PROPOSITION 2.3. [11, p. 160, (5.23)] *Let E be a finite elementary abelian p-group which is not cyclic, and let $\mathcal{M}$ be the set of all maximal subgroups of E: $\mathcal{M} = \{E_0 \mid |E : E_0| = p\}$. If E acts on a finite additive group A and if p is prime to the order $|A|$ of A, then we have $A = \langle C_A(E_0)\rangle$, i.e. A is generated by the centralizers $C_A(E_0)$ of the subgroups E_0, where E_0 ranges over $\mathcal{M}$.*

Consider the case when a Sylow 2-subgroup is dihedral. Simple groups, i.e. without proper normal subgroups, having dihedral Sylow 2-subgroups are classified.

PROPOSITION 2.4. [12, p. 505, Theorem 8.6] *Let G be a simple group with a dihedral Sylow 2-subgroup. Then, either G is isomorphic to* $\mathrm{PSL}(2, p)$ *for an odd prime power $p > 3$ or $G \cong \mathbb{A}_7$.*

The following result is usually referred as the Z^*-theorem.

PROPOSITION 2.5. [12, p. 315, Theorem 2.14] *Let $t \in G$ be an involution and S_2 a Sylow 2-subgroup of G containing t. Denote by $O(G)$ the maximal normal subgroup of G of odd order ("the core of G"). Then the canonical image of t is central in the factor group $G/O(G)$ if and only if t does not conjugate in G to any element of $S_2 \setminus \{t\}$ ("t does not fuse to any different element of S_2").*

3. Auxiliary results about cyclic branched covering of 2-bridge knots

We will use many times the following two properties of the symmetries of a 2-bridge knot.

(P1) The symmetry group of a 2-bridge knot is dihedral of order four or eight; in particular a 2-bridge knot has no periods of order greater than four and no periods of odd order.

(P2) A 2-bridge knot has a cyclic period of order two and one lift of this cyclic period to the n-fold cyclic branched covering M, which will be denoted by r in the proof, is a hyperelliptic involution, that is the quotient $M/\langle r\rangle$ is the 3-sphere S^3.

The following result was obtained in [10].

PROPOSITION 3.1. [10, Lemma 1] *Let $M = M_n(K)$ be hyperbolic and let p an odd prime dividing n. Then a Sylow p-subgroup of the orientation-preserving isometry group of M is either cyclic or a direct product of two cyclic groups. Moreover if K has no periods of order p, the Sylow p-subgroup is cyclic.*

Applying Proposition 3.1 to the case of 2-bridge knots, which do not have odd order periods (see property (P1)), we get the following fact.

COROLLARY 3.2. *Let $M = M_n(K)$ be hyperbolic, where K is a 2-bridge knot. For any odd prime number p dividing n the covering group contains a Sylow p-subgroup S_p of the orientation-preserving isometry group of M. In particular S_p is cyclic and has connected fixed point set.*

Another useful property is the following.

PROPOSITION 3.3. *Let $M = M_n(K)$ be hyperbolic, G be its orientation-preserving isometry group, and $H \subset G$ be the covering group of K.*

(i) If $g \in G$ normalizes a nontrivial subgroup of H, then g normalizes H.

(ii) Let $P \subset G$ be a cyclic subgroup of odd order with connected fixed point set. If an element of H, which is not an involution, normalizes P, then H and P commute (so P descends to K).

Proof. (i) The fixed point set of any nontrivial subgroup H_1 of the covering group H (which is cyclic) is the (connected) preimage $\widetilde{K}$ of K in M and H_1 acts as a group of local rotations around $\widetilde{K}$. If $g \in G$ normalizes H_1 then g maps $\widetilde{K}$ (that is the fixed point set of H_1) to itself. So gHg^{-1} is a cyclic group of rotations with fixed point set $g(\widetilde{K}) = \widetilde{K}$. It follows $gHg^{-1} = H$, that is g normalizes H.

(ii) The fixed point set of any element $h \in H$ (as well as the fixed point set of H) is the preimage $\widetilde{K}$ of K. If h normalizes P, it fixes setwise the connected fixed point set, say F, of P. As h is not an involution, it acts as a rotation on F and commutes with P which locally acts as a group of rotations on F. Also, since P commutes with h, it fixes setwise its fixed point set $\widetilde{K}$. Moreover P acts as a group of rotations around $\widetilde{K}$ because it is of odd order. Therefore P commutes with H which locally acts as a group of rotations on $\widetilde{K}$.

The proof of Proposition 3.3 is completed. □

4. Two basic lemmas

Now we will prove Lemma 4.1 and 4.2, which contain the most technical part of our arguments.

LEMMA 4.1. *For $n \geq 3$ let $M = M_n(K)$, where K is a 2-bridge knot, and G be the orientation-preserving isometry group of M. Suppose that:*

(i) G contains an involution u such that the centralizer $C_G(u)$ of u coincides with the normalizer $N_G(H)$ of the covering group H;

(ii) G has a nontrivial normal subgroup of odd order.

Then either M has no hidden symmetries or G is large.

Proof. The proof follows from statements (a) – (d) below.

(a) *The group G contains a normal elementary p-subgroup (that is a direct product of cyclic groups) for some odd prime number p.*

It is a routine matter in finite group theory to show that a finite group which contains a normal subgroup of odd order contains also a normal elementary p-subgroup for some odd prime number p. Indeed a nontrivial normal subgroup of odd order is solvable by the Feit-Thompson Theorem (see, for example, [12, p. 356, Theorem 3.1]) and it contains a characteristic nontrivial abelian subgroup (see, for example, [11, p. 121, Corollary 2]), so it also contains a characteristic nontrivial elementary p-subgroup for some odd prime number p.

For a fixed odd prime number p, denote by P a normal elementary p-subgroup of G. The number p may divide the order n of the covering group H of the 2-bridge knot or not: the first case will imply that M has no hidden symmetries and the second case that G is large. We start with the first case which is easier.

(b) *If p divides n, M has no hidden symmetries.*

If p divides n a Sylow p-subgroup of G is cyclic by Corollary 3.2 and has connected fixed point set; in particular P is cyclic and has connected fixed point set. The group H normalizes P, because P is normal in G by construction. By Proposition 3.3(ii), P commutes with H and descends to the 2-bridge knot. By (P1) a 2-bridge knot has no odd order periods. Since P has odd order, this implies that P descends to the trivial group, so P is a subgroup of H. We have thus found a subgroup of H which is normal in G; by Proposition 3.3(i) M has no hidden symmetries. Thus, the statement (b) is proven.

Suppose now that p does not divide the order n of H. By (P1) 2-bridge knot has no odd order periods, so p does not divide the order of $N_G(H)$ and the intersection $P \cap N_G(H)$ is trivial. To complete the proof we shall show that P has rank two, so it has order p^2, and that n divides $(p+1)$ or $(p-1)$. It will give that G is large.

Let u be the involution in the hypothesis such that $C_G(u) = N_G(H)$. In particular u is a central involution of $N_G(H)$. We consider the dihedral subgroup, say D, of order four of $N_G(H)$ generated by u and a lift t of a strong inversion of K (remark that u is not a lift of a strong inversion of K, because lifts of strong inversions do not commute with H, so they are not central in $N_G(H)$). The group D acts by conjugation on P, which is normal in G by the construction. Applying Proposition 2.3 to the cyclic group P (setting $A = P$ and $p = 2$) we will get that P is generated by three centralizers $C_P(u)$, $C_P(t)$, and $C_P(tu)$.

(c) *The centralizer $C_P(u)$ is trivial; the centralizers $C_P(t)$ and $C_P(tu)$ are trivial or cyclic.*

The centralizer $C_P(u)$ is trivial because u is such that $C_G(u)$ coincides with $N_G(H)$, but the intersection $N_G(H) \cap P$ is trivial.

An element of the centralizer $C_P(t)$ commutes with t, so it fixes setwise the fixed point set, say $F(t)$, of t, which is connected because t is a lift of a strong inversion. A finite group of isometries fixing setwise a simple closed curve, so the group $C_P(t)$ itself, is a semidirect product $\mathbb{Z}_2(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m})$ where $\mathbb{Z}_2$ (a reflection) acts on the normal subgroup $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$ (rotations) by sending each element to its inverse. Remark that $C_P(t)$ is a subgroup of P, so it has odd order. Therefore, $C_P(t)$ is a group of rotations of $F(t)$, so it has rank at most two. Moreover, the case that $C_P(t)$ has rank two occurs only if $C_P(t)$

contains a cyclic subgroup P' of order p fixing pointwise $F(t)$ and acting as a group of rotations around it. But in this case $F(t)$ is fixed also by the cyclic group of order $2p \geq 6$ generated by t and P'. This is impossible because $F(t)$ intersects the preimage $\widetilde{K}$ of K in M in two points and each of the two intersection points would be fixed both by H (which has order $n \geq 3$) and by a cyclic group of order $2p \geq 6$, which is excluded, because the isotropy group of a point in a 3-manifold is a spherical group. We conclude that $C_P(t)$ is trivial or cyclic. An analogous argument holds for $C_P(tu)$ which is also trivial or cyclic.

Thus, the statement (c) is proven.

It follows from (c) and the fact that P is generated by the three centralizers $C_P(u)$, $C_P(t)$ and $C_P(tu)$, that P is either cyclic or the direct product of two cyclic groups. In any case, at least one between the two involutions t and (tr), say t, induces a nontrivial automorphism of P. On the other hand any element h of H acts also by conjugation on P and the induced action is not trivial, because if h commutes with P, by Proposition 3.3(i), P is a subgroup of $N_G(H)$, a contradiction. So each element of the dihedral subgroup of $N_G(H)$ generated by t and H acts nontrivially on P and we get a dihedral subgroup D' of the automorphism group of P with cyclic normal subgroup H.

(d) *If p does not divide n, G is large, i.e. contains a normal abelian subgroup P of order p^2 which does not descend to K and n divides $(p-1)$ or $(p+1)$.*

The case that P is cyclic is impossible, because the automorphism group of a cyclic group is abelian and we know that there exists a dihedral subgroup D' of automorphisms of P. If P has rank two, its automorphism group is the general linear group $\mathrm{GL}(2,p)$ over the finite field with p elements. The group $\mathrm{GL}(2,p)$ contains the normal subgroup $\mathrm{SL}(2,p)$ with index $(p-1)$ and the factor group $\mathrm{GL}(2,p)/\mathrm{SL}(2,p)$ is isomorphic to the multiplicative group of the finite field with p elements. Since this last group does not contain dihedral subgroups, the only possible case is that the intersection $D' \cap \mathrm{SL}(2,p)$ contains a subgroup isomorphic to $H \cong \mathbb{Z}_n$. But $\mathrm{SL}(2,p)$ has, up to conjugation, three maximal cyclic subgroups of order, respectively, $(p-1)$, $(p+1)$ and $(2p)$. Thus n, which is relatively prime with p by hypothesis, divides $(p-1)$ or $(p+1)$.

This finishes the proof of Lemma 4.1. □

Lemma 4.2. *Let M be the (hyperbolic) n-fold cyclic branched covering of a knot, G its orientation-preserving isometry group. Let $u \in G$ be an involution and S_2 a Sylow 2-subgroup of G containing u. Suppose that u is not conjugate by elements of G to a distinct involution in the centralizer $C_{S_2}(u)$ of u in S_2 of G. Then either u is central in G or G has a nontrivial normal subgroup of odd order.*

Proof. If g is an element of S_2 which normalizes $C_{S_2}(u)$, then it acts on $C_{S_2}(u)$ by conjugation, mapping u to an element gug^{-1} of $C_{S_2}(u)$. In our hypothesis $gug^{-1} = u$ because u is not conjugate by elements of G to a distinct involution in the centralizer $C_{S_2}(u)$. So $g \in C_{S_2}(u)$. We have thus shown that the normalizer in S_2 of $C_{S_2}(u)$ is contained in $C_{S_2}(u)$. Whence, by Proposition 2.1, $S_2 = C_{S_2}(u)$.

The hypothesis that u is not conjugate to any other involution of $C_{S_2}(u)$, implies now that u is not conjugate to a distinct involution in a Sylow 2-subgroup of G. The result follows from Z^*-Theorem (see Proposition 2.5).

This finishes the proof of Lemma 4.2. □

5. Proof of Theorem 1.4

We will consider two separate situations: if n is even then one of the cases (i) or (ii) of Theorem 1.4 can occur, and if n is odd then one of the cases (i), (ii), or (iii) can occur.

Statement 5.1. *If n is even, either M has no hidden symmetries or G is large.*

Since n is even, the cyclic covering group $H \cong \mathbb{Z}_n$ of K contains an involution h with fixed point set the preimage $\widetilde{K}$ of K in M (which is also the fixed point set of H). Denote by $C_{S_2}(h)$ the centralize of h in a Sylow 2-subgroup S_2 of G.

Lemma 5.2. *The involution h is not conjugate to another involution in $C_{S_2}(h)$.*

Proof. If h is conjugate to another involution $ghg^{-1} \in C_{S_2}(h)$, then $g(\widetilde{K})$ is the fixed point set for ghg^{-1}, and therefor for gHg^{-1}. Since $ghg^{-1} \in C_{S_2}(h)$, involutions h and ghg^{-1} commute, so h fixes setwise the fixed point set $g(\widetilde{K})$ of ghg^{-1}. The element h may act as a rotation or a reflection on $g(\widetilde{K})$. We show that both cases are impossible.

If h acts as a rotation on $g(\widetilde{K})$, it commutes with gHg^{-1} which is a group of local rotations around $g(\widetilde{K})$. By Proposition 3.3(i) gHg^{-1} normalizes H and descends to K. Thus, gHg^{-1} would descend to a cyclic group of periods of order $n \geq 6$ (we have assumed n even and $n \geq 5$). This is impossible because K is a 2-bridge knot and, in virtue of property (P1), has no periods of order greater than four.

If h acts as a reflection on $g(\widetilde{K})$, the knots $\widetilde{K}$ and $g(\widetilde{K})$ mutually intersect. Any intersection point is fixed by the action of H and gHg^{-1}, which are cyclic groups of order $n \geq 6$ with trivial intersection. This is impossible because the orientation-preserving isotropy group at a point of a 3-manifold is a spherical group. The Lemma 5.2 is proven. □

By Proposition 3.3(i), the centralizer $C_G(h)$ and the normalizer $N_G(H)$ coincide. Applying Lemma 4.2 for the involution h, we get that either h is central in G or G has a nontrivial normal subgroup of odd order. In the first case each element of G leaves $\widetilde{K}$ setwise fixed and descends to K. In the second case the result follows from Lemma 4.1. Thus, Statement 5.1 is proven.

STATEMENT 5.3. *If n is odd, M has no hidden symmetries or G is large or G is special.*

Let r be the hyperelliptic involution of M described in (P2). The fixed point set F of r may be connected or not.

STATEMENT 5.3 (I). *If F is not connected, either M has no hidden symmetries or G is large.*

By construction r is a lift of a cyclic period of K with connected fixed point set. So the number of components of the fixed point set F of r divides n. In particular, F has an odd number of components which are permuted transitively by the action of H.

LEMMA 5.4. *The centralizer $C_G(r)$ of r and the normalizer $N_G(H)$ of H coincide.*

Proof. The normalizer $N_G(H)$ is the set of lifts of the symmetries of K; in particular $N_G(H)$ is generated by the lifts of the strong inversions of K (see (P1)). A lift of a strong inversion commutes with r; it follows that $N_G(H)$ is a subgroup of $C_G(r)$. We have only to prove the opposite inclusion $C_G(r) \subset N_G(H)$, that is that any element $g \in G$ which commutes with r, normalizes H.

Consider the quotient $M/\langle r\rangle$. The group $C_G(r)$ descends to a group $\bar{C}$ of orientation-preserving isometries in $M/\langle r\rangle$. Denote by $\bar{H} \subset \bar{C}$ the projection of H. To prove that any element of $C_G(r)$ normalizes H, it is equivalent to prove that any element of $\bar{C}$ normalizes $\bar{H}$, that is that $\bar{H}$ is normal in $\bar{C}$.

We know that the quotient $M/\langle r\rangle$ is homeomorphic to S^3 because r is hyperelliptic. So, by Thurston's Orbifold Geometrization Theorem for finite group actions on S^3, the groups $\bar{C}$ and $\bar{H}$ are subgroups of SO(4).

It follows from the classification of finite subgroups of SO(4) [14] that if a finite subgroup A of SO(4) contains a cyclic subgroup B of order $n \geq 7$, then B is normal in A. So, if $n \geq 7$ the subgroup $\bar{H}$ is a cyclic normal subgroup of $\bar{C}$ and the proof is complete.

Since we consider n odd, the only left case is that $H \cong \mathbb{Z}_5$. A cyclic subgroup $B \cong \mathbb{Z}_5$ in a finite subgroup A of SO(4) is also always normal with the exception of some finite subgroups A containing the alternating group $\mathbb{A}_5$ on five letters. However in our case $\bar{C}$ does not contain $\mathbb{A}_5$, because if $n = 5$ the fixed point set of r in M has five distinct components, so the singular set of the quotient $M/\langle r\rangle$ is also a link with five components. The group $\mathbb{A}_5$ can not be the group of symmetries of such a link. For example $\mathbb{A}_5$ contains ten distinct subgroups of order three. Each of these cyclic subgroups induces a permutation of order three of the five components of the link, fixing setwise at least one component. So there exists at least one component of the link with two distinct cyclic subgroups of order three fixing it setwise. The two cyclic subgroups of order three acts as rotations on the component generating a group $\mathbb{Z}_3 \times \mathbb{Z}_3$. But $\mathbb{A}_5$ does not contain a subgroup of this type. This finishes the proof of Lemma 5.4. □

To conclude the proof of Statement 5.3 (I) it is enough to show that r does not fuse to any other involution of its centralizer and the result follows from Lemma 5.4, and by applying Lemma 4.2 and Lemma 4.1 to the involution r. Indeed note that any involution of $C_G(r)$ normalizes H by Lemma 5.4 and descends to K. By construction n is odd, so an involution of $C_G(r)$ is either r or a lift of a strong inversion of K. A lift of a strong inversion has connected fixed point set, so r can not be conjugate to another involution because by the assumption in Statement 5.3 (I), its fixed point set F is not connected.

STATEMENT 5.3 (II). *If F is connected, M has no hidden symmetries or G is large or G is special.*

In this case H acts as a group of rotations on F and it is easy to show that the two groups $C_G(r)$ and $N_G(H)$ coincide. Indeed an element of G which commutes with r fixes setwise its fixed point set F and so it normalizes H which acts as a group of rotations on F. Thus $C_G(r) = N_G(H)$. In particular, since H has odd order, a Sylow 2-subgroup of $C_G(r)$ is a dihedral group of order four or eight, depending on the symmetry group of K (see (P1)).

We first prove the following property.

LEMMA 5.5. *A Sylow 2-subgroup of G is dihedral.*

Proof. Let S_2 be a Sylow 2-subgroup of G containing r and consider the subgroup $C_{S_2}(r) \subset S_2$. If $C_{S_2}(r) = S_2$, the group S_2 is dihedral and the proof is complete.

If $C_{S_2}(r)$ is a proper subgroup of S_2, by Proposition 2.1 there exists an element g of S_2 which normalizes $C_{S_2}(r)$ and conjugates r to a distinct involution $t = grg^{-1}$ of $C_{S_2}(r)$. Since r is central in $C_{S_2}(r)$, the involution t is also central in $C_{S_2}(r)$: the only possible case is that $C_{S_2}(r)$, which is a dihedral group, has order four. In this case the group $C_{S_2}(r)$ contains three involutions: r, t and the product (tr). The element g, which has order a power of two, acts on $C_{S_2}(r)$ exchanging r and t and commuting with (tr). This implies that (tr) is the unique involution of S_2 which commutes both with g and r. Since every 2-group has a nontrivial center, the only nontrivial element of the center of S_2 is (tr). We have thus proved that $S_2 = C_{S_2}(tr)$.

The involution (tr) is a lift of a strong inversion of K and has connected fixed point set. The group S_2, which commutes with (tr) fixes setwise its fixed point set and so it is a subgroup of a semidirect product $\mathbb{Z}_2(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m})$ where $(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m})$ is the group of (tr)-rotations. The fact that r is a (tr)-reflection and that $C_{S_2}(r)$ is a dihedral group of order four, implies that $m = 0$. This shows that also in this case S_2 is a dihedral group. Thus, Lemma 5.5 is proven. □

To determine the structure of G it is now convenient to consider its simple composition factors and its Fitting subgroup $\mathcal{F}$, that is the maximal nilpotent subgroup of G.

If $\mathcal{F}$ contains elements of odd order, then we can apply Lemma 4.1 to the involution r and we conclude that M has no hidden symmetries or G is large.

If $\mathcal{F}$ is a 2-group, being normal in G, it is contained in any Sylow 2-subgroup of G, in particular in S_2. The group $\mathcal{F}$ can not be dihedral because it follows from the discussion above that any dihedral subgroup of S_2 contains a r-reflection, say (tr); but H does not commute with (tr) so it does not normalize a dihedral 2-group containing (tr). This is a contradiction because by construction any element of G normalizes $\mathcal{F}$. So, if $\mathcal{F}$ is a 2-group, it is cyclic and its unique involution is central in G. This central involution must be r, which is the unique involution of G which commutes with H. Hence $G = C_G(r)$ and M has no hidden symmetries.

The only left case is that $\mathcal{F}$ is trivial. Since a Sylow 2-subgroup of G is dihedral, a minimal normal subgroup E of G is simple and isomorphic to $\mathrm{PSL}(2,p)$ for some p odd (see Proposition 2.4) or to the alternating group $\mathbb{A}_7$ on seven letters.

The case $\mathbb{A}_7$ is ruled out in the following way. We have already remarked above that any dihedral subgroup of S_2 contains a r-reflection, say (tr). Therefore a Sylow 2-subgroup of E contains an element conjugate to (tr). More than that, all involutions of E lie in the same conjugacy class, so they are all conjugate to (tr). This rules out the case $\mathbb{A}_7$ because the centralizer in G of (tr), which has connected fixed point set, is a subgroup of a semidirect product $\mathbb{Z}_2(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m})$ where $\mathbb{Z}_2$ acts on $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$ by sending each element to its inverse, but the centralizer of an involution in $\mathbb{A}_7$ is not of this type.

By Proposition 2.2 the group G is generated by $E \cong \mathrm{PSL}(2,p)$ and the normalizer in G of a Sylow 2-subgroup of E. The normalizer of a dihedral subgroup of S_2 contains only elements of order a power of two, with the exceptional case that it has order four and there exists an element of order three permuting its three involutions; but in this last case such order three elements are already contained in E. This implies that E contains a cyclic subgroup isomorphic to $H \cong \mathbb{Z}_n$.

The group $\mathrm{PSL}(2,p)$ contains, up to conjugation, three maximal cyclic subgroups of order, respectively, p, $(p-1)/2$ and $(p+1)/2$.

The fact that n divides $(p-1)/2$ or $(p+1)/2$ follows because n does not divide p (the normalizer of a Sylow p-subgroup of $\mathrm{PSL}(2,p)$ can not have the structure of $N_G(H)$).

The fact that M is an integral homology 3-sphere follows from the result from [6], because the quotient $M/\langle u\rangle$ is S^3 for any involution u of $\mathrm{PSL}(2,p)$ (all the involutions of G are conjugate to r).

This finishes the proof of Theorem 1.4.

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