Rend. Istit. Mat. Univ. Trieste
Suppl. 1 Vol. XXXII, 257–288 (2001)

Finite Quotients of the Picard Group and Related Hyperbolic Tetrahedral and Bianchi Groups

Luisa Paoluzzi and Bruno Zimmermann (*)

Summary. - *There is an extensive literature on the finite index subgroups and the finite quotient groups of the Picard group* $PSL(2,\mathbb{Z}[i])$. *The main result of the present paper is the classification of all linear fractional groups* $PSL(2,p^m)$ *which occur as finite quotients of the Picard group. We classify also the finite quotients of linear fractional type of various related hyperbolic tetrahedral groups which uniformize the cusped orientable hyperbolic 3-orbifolds of minimal volumes. Also these cusped tetrahedral groups are of Bianchi type, that is of the form* $PSL(2,\mathbb{Z}[\omega])$ *or* $PGL(2,\mathbb{Z}[\omega])$, *for suitable* $\omega \in \mathbb{C}$. *It turns out that all finite quotients of linear fractional type of these tetrahedral groups are obtained by reduction of matrix coefficients mod p whereas for the Picard group most quotients do not arise in this way (as in the case of the classical modular group* $PSL(2,\mathbb{Z})$*). From a geometric point of view, we are looking for hyperbolic 3-manifolds which are regular coverings, with covering groups isomorphic to* $PSL(2,q)$ *or* $PGL(2,q)$ *and acting by isometries, of the cusped hyperbolic 3-orbifolds of minimal volumes. So these are the cusped hyperbolic 3-manifolds of minimal volumes admitting actions of linear fractional groups. We also give some application to the construction of closed hyperbolic 3-manifolds with large group actions. We are concentrating in this work on quotients of linear fractional type because all finite quotients of relatively small order of the above groups are of this or closely related types (similar to the case of Hurwitz actions on Riemann surfaces), so the linear fractional groups are the first and most important class of finite simple groups to take into consideration.*

1. Introduction

In this work we shall study the finite quotients of linear fractional type $PSL(2,q)$ and $PGL(2,q)$ of the Picard group and of some related hyperbolic tetrahedral and Bianchi groups uniformizing the cusped orientable hyperbolic 3-orbifolds of minimal volumes. We are interested only in quotients by torsion-free subgroups, so from a geometric point of view we are looking for hyperbolic 3-manifolds which are finite regular coverings of the minimal volume hyperbolic 3-orbifolds. Consequently these are the cusped hyperbolic 3-manifolds of minimal volumes admitting actions of linear fractional groups. We give also some application to the construction of closed hyperbolic 3-manifolds admitting large group actions. We are concentrating in this work on quotients of linear fractional type because all finite quotients of relatively small order of the above groups are of this or closely related types (similar to the case of Hurwitz actions on Riemann surfaces), so the linear fractional groups are the first and most important class of finite simple groups to take into consideration.

By [18] the cusped (non-compact of finite volume, without boundary) orientable hyperbolic 3-orbifold of minimal volume is a *tetrahedral orbifold*: the quotient of hyperbolic 3-space $\mathbf{H}^3$ by a tetrahedral group. Also the next smallest cusped hyperbolic 3-orbifolds are of tetrahedral type or closely related to such, see [2], [20]. Recall that a *tetrahedral group* is defined as the subgroup of index two of the orientation-preserving elements in the Coxeter group generated by the reflections in the faces of a hyperbolic *Coxeter polyhedron*; this is a tetrahedron all of whose dihedral angles are integer submultiples π/n of π. Then to each edge is associated an integer $n \geq 2$, and to

(*) Authors' addresses: Luisa Paoluzzi, Laboratoire de Topologie, Université de Bourgogne, avenue Alain Savary – BP 47870, 21078 Dijon cédex, France e-mail: paoluzzi@u-bourgogne.fr
Bruno Zimmermann, Dipartimento di Scienze Matematiche, Università degli Studi di Trieste, via Valerio 12/B, 34127 Trieste, Italy, e-mail: zimmer@univ.trieste.it
Keywords: Picard group, hyperbolic tetrahedral group, hyperbolic 3-orbifold and 3-manifold.
AMS Subject Classification: 57M60, 57S17, 20F38.

each vertex three integers n, m and k. We distinguish three types of hyperbolic tetrahedra: bounded tetrahedra all of whose vertices lie in $\mathbf{H}^3$ (spherical vertices: $1/n + 1/m + 1/k > 1$), cusped tetrahedra which have at least one vertex on the sphere at infinity of $\mathbf{H}^3$ (ideal or euclidean vertices: $1/n+1/m+1/k = 1$), and finally infinite-volume tetrahedra which have at least one vertex outside infinity (hyperbolic vertices: $1/n + 1/m + 1/k < 1$). Up to isometries, there are exactly 9 bounded hyperbolic Coxeter tetrahedra (the "Lannér-tetrahedra") and 23 cusped ones, see [26] or [19] for a list. Associated to each Coxeter tetrahedron is a graph of groups which is the 1-skeleton of the tetrahedron (including also the ideal and hyperbolic vertices). To each edge with dihedral angle π/n resp. vertex as above is associated its stabilizer in the tetrahedral group; this stabilizer is a cyclic group of order n resp. a spherical, euclidean or hyperbolic triangle group (n, m, k) generated by two elements of orders n and m whose product has order k.

Given a tetrahedral group T associated to a hyperbolic Coxeter polyhedron $\mathcal{T}$, the tetrahedral orbifold $\mathbf{H}^3/T$ is a hyperbolic 3-orbifold obtained by identifying two copies of T along their boundaries. Topologically we obtain the 3-sphere minus a finite number points (the ideal and hyperbolic vertices), and the singular set of the orbifold is the 1-skeleton of the tetrahedron resp. the above graph of groups, again with the ideal and hyperbolic vertices deleted. One may also consider compact versions of these tetrahedral orbifolds by truncating the ideal vertices of the Coxeter tetrahedron by horospheres, and the hyperbolic vertices by hyperbolic planes orthogonal to the three faces meeting at a vertex. Then the boundary of the corresponding truncated tetrahedral orbifold consists of euclidean resp. totally geodesic hyperbolic triangular 2-orbifolds (2-spheres with three branch points).

As noted above, the cusped hyperbolic 3-orbifolds of minimal volumes are among the cusped tetrahedral orbifolds. We remark that the closed hyperbolic 3-orbifolds of minimal volumes are still unknown but that also in the closed case the candidates are the tetrahedral orbifolds associated to the Lannér tetrahedra or are double covered by these.

In this work we shall classify the quotients of type $PSL(2,q)$ and $PGL(2,q)$, with torsion-free kernel, of the smallest cusped tetrahedral groups, and also of some related groups as the Picard group. We shall call *admissible* a group homomorphism from a tetrahedral (or polyhedral) group to a finite group which is injective on the torsion elements of the vertex groups. In particular, if the tetrahedral (resp. polyhedral) group is infinite, an admissible homomorphism has torsion-free kernel. We denote by $PSL(2,q)$ (resp. $PGL(2,q)$) the quotient of the group of 2×2 matrices with entries in the Galois field $\mathbb{F}_q$ ($q = p^m$, p prime) and determinant 1 (resp. non 0), modulo its centre. For the main results concerning these groups see [24], [7] and [11]. From a geometric point of view, we are looking for hyperbolic 3-manifolds which are regular coverings, with covering groups of type $PSL(2,q)$ or $PGL(2,q)$ acting by isometries, of the smallest cusped 3-orbifolds. These manifolds are obtained as the quotients of $\mathbf{H}^3$ by the actions of the torsion-free kernels we find.

The cusped tetrahedral groups we are going to study are interesting also from a number-theoretical viewpoint. It is well known that the group $PSL(2,\mathbb{C}) = PGL(2,\mathbb{C})$ can be identified with the group of orientation preserving isometries of $\mathbf{H}^3$. Let $\mathcal{O}_d$ denote the ring of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ where d is a square-free positive integer. An important class of discrete subgroups of $PSL(2,\mathbb{C})$ consists of the groups of the form $PSL(2,\mathcal{O}_d)$ and $PGL(2,\mathcal{O}_d)$; the first are the so called *Bianchi groups* (see e.g. [9]) which can be considered as generalizations of the classical modular group $PSL(2,\mathbb{Z})$. The universal covering groups of the three smallest cusped hyperbolic 3-orbifolds are the (extended) Bianchi groups $PGL(2,\mathcal{O}_3)$, $PGL(2,\mathcal{O}_1)$ and $PSL(2,\mathcal{O}_3)$ (see [20]). These groups admit natural homomorphisms to $PSL(2,q)$ or $PGL(2,q)$ given by reduction of matrix coefficients mod p. Our first main result is the classification of all admissible quotients of type $PSL(2,q)$ and $PGL(2,q)$ of these groups (it turns out that only values of the form $q = p$ and $q = p^2$ occur), and also the proof that all these quotients are obtained as reductions mod p. This answers also the first half of the question at the end of [3]. The second half of the question regards the other Bianchi groups; here the corresponding statement is not true in general, as we shall see: for example for the Picard group most finite quotients of linear fractional type are not obtained by reduction of coefficients mod p.

One of the main results of the present paper is the classification of all admissible quotients of type $PSL(2,q)$ of another remarkable Bianchi group: the Picard group $PSL(2,\mathcal{O}_1) = PSL(2,\mathbb{Z}[i])$ (which is a polyhedral group associated to a cusped Coxeter pyramid). There exists an extensive literature on the Picard group and its finite index subgroups and finite quotient groups, see the monograph [9] and the references given there. In fact most of the finite admissible quotients of not too large order are (closely related to) linear fractional groups which generically constitute the typical class of finite simple groups. The Picard group plays an exceptional role also from a geometric viewpoint since it is the fundamental group of the limit orbifold of minimal volume ([1]). The situation for the Picard group is more similar to that of the classical modular group (see [16]) rather than to that of the small Bianchi groups of tetrahedral type. In fact we shall see that $PSL(2,q)$ is an admissible quotient of the Picard group for most values of q (in contrast to the tetrahedral groups, and also to the cocompact triangle groups in dimension two). The reason is that the Picard orbifold (the quotient of hyperbolic space by the Picard group) has a non-rigid cusp on which hyperbolic Dehn surgery can be performed. Moreover the tetrahedral groups are unsplittable as a free product with amalgamation or HNN-extension (as are the cocompact triangle groups in dimension two), see [29], whereas the Bianchi groups which are not of tetrahedral type, including the Picard group, are splittable (as is the classical modular group which is a cusped triangle group uniformizing the cusped orientable hyperbolic 2-orbifold of minimal volume), see [9] or [10].

In Section 5 we use the fact that the Picard orbifold has a non-rigid cusp for the construction of closed hyperbolic 3-manifolds with $PSL(2,q)$-actions. By the above minimality property of the Picard orbifold it seems reasonable that for many of the groups $PSL(2,q)$ the minimal volume of a manifold with such an action is realized in this way.

We close with the remark that subgroups of small index in some Bianchi and tetrahedral groups have been classified in [14] and [6], by computational methods using the group-theory packages GAP and Cayley, see also [4] and [9] for the Picard group.

2. The smallest cusped hyperbolic 3-orbifold

Consider the Coxeter tetrahedron $\mathcal{T}_{k,n}$ shown in Figure 1 where an integer n at an edge denotes a dihedral angle π/n.

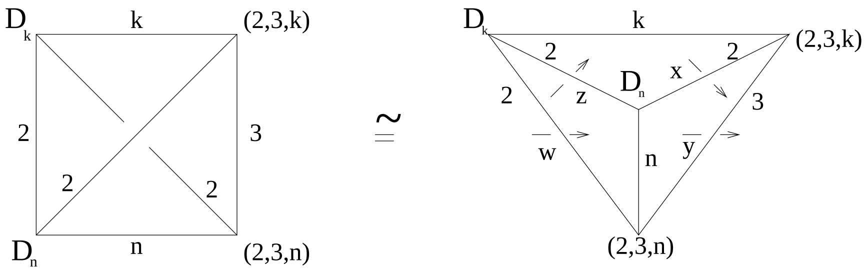


Figure 1.

We shall denote by $T_{k,n}$ the corresponding tetrahedral group. We note that the tetrahedral orbifolds $\mathcal{O}_{k,n} := \mathbf{H}^3/T_{k,n}$ are exactly the tetrahedral orbifolds which admit a Heegaard splitting along a 2-orbifold which is a 2-sphere with four branch points of orders 2, 2, 2 and 3. This is the minimal possibility for Heegaard splittings of 3-orbifolds which is not too special (i.e. along hyperbolic Heegaard 2-orbifolds), see [32]. Among these tetrahedra there are the two bounded hyperbolic tetrahedra $\mathcal{T}_{4,5}$ and $\mathcal{T}_{5,5}$ and the four cusped hyperbolic tetrahedra $\mathcal{T}_{k,6}$, for $k = 3, 4, 5$ and 6; for all larger values of k, n the tetrahedra are unbounded hyperbolic, for smaller values they belong to other geometries. The tetrahedron $\mathcal{T}_{4,5}$ has minimal volume among the 9 Lannér tetrahedra (see the list in [19]), and finite admissible quotients of the tetrahedral groups $T_{4,5}$ and $T_{5,5}$ have been investigated in [12]. Here we are interested in the cusped and infinite volume cases. The tetrahedral orbifold $\mathcal{O}_{3,6}$ is the unique cusped orientable hyperbolic 3-orbifold of minimal volume ([18]).

As indicated in Figure 1, the tetrahedron $\mathcal{T}_{k,n}$ has the triangle group $(2,3,k)$ and the dihedral group $\mathbf{D}_k = (2,2,k)$ of order $2k$ as vertex groups. We shall use the following presentations of these groups:

$$\mathbf{D}_k = \langle w, z \mid w^2, z^2, (wz)^k \rangle; \quad (2,3,k) = \langle x, y \mid x^2, y^3, (xy)^k \rangle.$$

We define the free products with amalgamation

$$\mathbf{G}_k := \mathbf{D}_k *_{\mathbb{Z}_k} (2,3,k),$$

where $\mathbb{Z}_k = \langle xy \rangle = \langle zw \rangle$.

Apart from the dihedral groups $(2,2,n)$, the other finite vertex groups which may occur are the tetrahedral group $\mathbf{A}_4 = (2,3,3)$, the octahedral group $\mathbf{S}_4 = (2,3,4)$ and the dodecahedral group $\mathbf{A}_5 = (2,3,5)$. Also, each element of finite order in the tetrahedral group is conjugate to an element in some vertex group.

A presentation for $T_{k,n}$ can be obtained from the presentation of the associated Coxeter group -which is computed using Poincaré's theorem for fundamental polyhedra- by applying the Reidemeister-Schreier subgroup method. By [28], [5] one obtains exactly the graph amalgamation or polygonal product over the graph of groups associated to the Coxeter polyhedron, that is the iterated free product with amalgamation of the vertex groups amalgamated over the edge groups (which is the quotient of the fundamental group of the graph of groups, by setting the HNN-generators equal to 1). Also, one may delete one of the vertices and all edges emanating from it. As result one obtains the presentation

$$\begin{aligned} T_{k,n} &= \mathbf{D}_k *_{\mathbb{Z}_k} (2,3,k)/\langle (zx)^n \rangle = \mathbf{G}_k/\langle (zx)^n \rangle \\ &= \langle x, y, z \mid x^2, y^3, z^2, (xy)^k, (xyz)^2, (zx)^n \rangle. \end{aligned}$$

The generators x, y, z and w are shown in Figure 1; they are rotations around the corresponding edges. The group $T_{2,n}$ is isomorphic to the extended triangle group $[2,3,n]$ generated by the reflections in the sides of a triangle with angles $\pi/2, \pi/3$ and π/n (see [13]).

We want to investigate admissible homomorphisms from $T_{k,n}$ to $PSL(2,q)$ and to $PGL(2,q)$. We need some preliminary remarks

concerning these groups; proofs can be found in [24] or [7]. As usual, we represent elements by matrices. Then the trace of an element of $PSL(2,q)$ is an element of $\mathbb{F}_q$ which is well-defined up to sign. There exists an element of order k in $PSL(2,q)$ if and only if k divides either $(q-1)/2$ or $(q+1)/2$, or if $k=p$ (where $q=p^m$); in this last case the element is parabolic, that is, has trace ± 2. Two non-parabolic elements are conjugate if and only if they have the same trace (in the whole paper, traces will always be considered up to sign). For all integers $k \geq 2$ there exists a polynomial $P_k \in \mathbb{Z}[t]$ with the following property:
Consider an element of $PSL(2,q)$ of order $k \geq 2$. Then its trace is a root of the polynomial $P_{k,q} \in \mathbb{F}_q[t]$ which is the image of P_k in the canonical unitary ring homomorphism $\mathbb{Z}[t] \longrightarrow \mathbb{F}_q[t]$ (see [27]). Vice versa if there exists a root t_0 of $P_{k,q}$ in $\mathbb{F}_q$ then the image in $PSL(2,q)$ of the matrix

$$\begin{pmatrix} t_0 & 1 \\ -1 & 0 \end{pmatrix}$$

has order a divisor of k.
Note that $\mathbb{F}_{q^2}$ admits an involution (Frobenius automorphism of order two) defined by associating to each element its q-th power. This behaves like conjugation for complex numbers. The elements of the field $\mathbb{F}_q$ are exactly the fixed points of this involution. If q is odd, the set $\iota\mathbb{F}_q$ of elements that are mapped to their negatives consists of 0 and all elements which are square roots of non-square elements of $\mathbb{F}_q$; these are all of the form $\iota\alpha$, $\alpha \in \mathbb{F}_q$, for a fixed $\iota \in \mathbb{F}_{q^2} - \mathbb{F}_q$ such that $\iota^2 \in \mathbb{F}_q$. Note that for all $\alpha \in \mathbb{F}_{q^2}$, $\alpha + \alpha^q \in \mathbb{F}_q$ while $\alpha - \alpha^q \in \iota\mathbb{F}_q$. In the following, let $\mathbb{F}_q^* := \mathbb{F}_q - \{0\}$. We can now define the unitary group

$$U(2,q) := \{\begin{pmatrix} a & b \\ -b^q & a^q \end{pmatrix} \mid a, b \in \mathbb{F}_{q^2},\ a^{q+1} + b^{q+1} \in \mathbb{F}_q^*\}$$

and the special unitary group

$$SU(2,q) := \{\begin{pmatrix} a & b \\ -b^q & a^q \end{pmatrix} \in U(2,q) \mid a^{q+1} + b^{q+1} = 1\}.$$

The quotients of these two groups by their centres are the groups $PU(2,q)$ resp. $PSU(2,q)$ which are isomorphic to the groups $PGL(2,q)$ resp. $PSL(2,q)$ (see [24], [7] for these isomorphisms).

Note that $PSL(2,q)$ and $PSU(2,q)$ are subgroups of $PSL(2,q^2)$. Because every element of $\mathbb{F}_q$ is a square in $\mathbb{F}_{q^2}$ also $PGL(2,q)$ and $PU(2,q)$ can be considered as subgroups of $PSL(2,q^2)$, by normalizing determinants to 1 (note however that after normalization the elements in $PU(2,q) - PSU(2,q)$ are no longer in unitary form). After this normalization it will then make sense to talk of traces, parabolic elements and so on also for elements of $PGL(2,q)$ and $PU(2,q)$; observe that the trace of an element in $PGL(2,q) - PSL(2,q)$ is an element in $\iota\mathbb{F}_q$. The orders of non-parabolic elements in $PGL(2,q)$ divide either $q-1$ or $q+1$. Non-parabolic elements of $PSL(2,q)$ (resp. $PGL(2,q)$) are conjugated to diagonal matrices if their orders divide $(q-1)/2$ (resp. $q-1$); they are conjugated to diagonal matrices in $PSU(2,q) \cong PSL(2,q)$ (resp. $PU(2,q) \cong PGL(2,q)$) if their order divides $(q+1)/2$ (resp. $q+1$).

For the proofs of our main results we need the following computational Lemma which we state only for odd prime powers q; the methods apply (and the main theorems will be formulated) also for powers of 2 but we will avoid the proofs in this case.

LEMMA 2.1. *Let $\gamma, \tau \in \mathbb{F}_q \cup \iota\mathbb{F}_q^*$ be the traces of elements of order k resp. n in $PSL(2,q^2)$ where q is odd and $\gamma \neq \pm 2$. Let $C(\gamma,\tau) := \gamma^2\tau^2 - 4\gamma^2 - 4\tau^2 + 12 \in \mathbb{F}_q$.*
Then, except in the case $\gamma = \tau = 0$ and $q \equiv 0 \pmod 3$, there exists an admissible homomorphism Φ from $T_{k,n}$ to $PSL(2,q^2)$ such that $\Phi(xy)$ has trace γ and $\Phi(zx)$ has trace τ. Such a homomorphism has image in $PSL(2,q)$ if and only if $\gamma, \tau \in \mathbb{F}_q$ and $C(\gamma,\tau)$ is a square in $\mathbb{F}_q$. It has image in $PGL(2,q)$ if and only if one of the following conditions holds:

i) $\gamma, \tau \in \iota\mathbb{F}_q^*$ *and* $C(\gamma,\tau)$ *is a square in* $\mathbb{F}_q$*;*

ii) one of γ, τ *is in* $\mathbb{F}_q^*$*, the other in* $\iota\mathbb{F}_q^*$*, and* $C(\gamma,\tau)$ *is a non-square in* $\mathbb{F}_q$*;*

iii) $\gamma, \tau \in \mathbb{F}_q$*, one is* 0*, and* $C(\gamma,\tau)$ *is a non-square in* $\mathbb{F}_q$*;*

iv) one of γ, τ *is* 0*, the other is in* $\iota\mathbb{F}_q^*$*; ¿*

v) at least one of γ, τ *is in* $\iota\mathbb{F}_q^*$ *and* $C(\gamma,\tau) = 0$*.*

In each case, up to conjugation in $PSL(2,q^2)$ resp. $PGL(2,q)$ there are at most two admissible homomorphisms Φ from $T_{k,n}$ to $PSL(2,q^2)$ resp. $PSL(2,q)$ or $PGL(2,q)$ such that $\Phi(xy)$ has trace γ and $\Phi(zx)$ has trace τ. In particular if $C(\gamma,\tau)=0$ there is exactly one homomorphism Φ.

Proof. We present the proof for the case where the image of Φ is in $PSL(2,q)$, the other statements of the Lemma are obtained by similar considerations.
In the following we shall write $X := \Phi(x)$, $Y := \Phi(y)$ and $Z := \Phi(z)$. We want to define an admissible homomorphism $\Phi : T_{k,n} \longrightarrow PSL(2,q)$. First of all we look for an element XY with trace γ in $PSL(2,q)$. Necessarily $\gamma \in \mathbb{F}_q$ and the same holds for τ because ZX must belong to $PSL(2,q)$. Since XY is not parabolic, up to conjugation in $PSL(2,q^2)$, we can assume that it is in diagonal form either working in $PSL(2,q)$ or in $PSU(2,q)$. So

$$XY = \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

where $\gamma = \lambda + \lambda^{-1}$ and $\lambda \in \mathbb{F}_q$ if we are working in $PSL(2,q)$ or $\lambda^{-1} = \lambda^q$ if we are working in $PSU(2,q)$. Now we must be able to find an element X of order 2 or equivalently trace 0 such that the element $Y = XXY$ has order 3 or equivalently has trace $\pm 1 =: \epsilon$ (if $q = 3^m$ elements of order 3 are parabolic but still the condition holds since $\pm 1 \equiv \mp 2 \pmod 3$). An element of order 2 must be of the form

$$X := \pm \begin{pmatrix} \alpha & \beta \\ \delta & -\alpha \end{pmatrix}$$

(with $\alpha, \beta, \delta \in \mathbb{F}_q$ if we are working in $PSL(2,q)$ or $\alpha^q = -\alpha$ and $\delta = -\beta^q$ if we are working in $PSU(2,q)$), thus

$$Y = \pm \begin{pmatrix} \alpha\lambda & \beta\lambda^{-1} \\ \delta\lambda & -\alpha\lambda^{-1} \end{pmatrix}$$

Imposing the trace of Y to be ϵ we obtain $\alpha = \epsilon/(\lambda - \lambda^{-1}) \neq 0$ and $\beta\delta = (3-\gamma^2)/(\gamma^2-4)$. Notice that the condition $\alpha^q = -\alpha$ is satisfied when we work in $PSU(2,q)$ since $\lambda^q = \lambda^{-1}$ in this case. Remark that if $q \equiv 0 \pmod 3$ we have to check that Y is not the trivial element in which case the homomorphism would not be admissible: this is equivalent to require that one of β, δ is not 0.

We wish to stress that if $\gamma^2 \neq 3$, up to conjugation with an element in the centralizer of XY in $PSL(2, q^2)$ there is exactly one element X satisfying this condition. In fact, under this assumption, $\delta \neq 0$ (and $\beta \neq 0$) and it can be rescaled to be 1 if we are working in $PSL(2, q)$ or to be a fixed $(q+1)$-root of $(\gamma^2 - 3)/(\gamma^2 - 4)$ if we are working in $PSU(2, q)$. Elements in the centralizer of XY are of the form

$$\begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}$$

and we must choose those where ρ^2 belongs to $\mathbb{F}_q$ if we are working in $PSL(2, q)$ (and the centralizer is contained in $PGL(2, q)$) or $\rho^{q+1} = 1$ if we are working in $PSU(2, q)$ (and the centralizer is contained in $PSU(2, q)$). If $\gamma = 0$, the centralizer of XY contains also the element

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

If $\gamma^2 = 3$ we have three different choices according as $\beta = \delta = 0$, $\beta = 0, \delta \neq 0$ or $\beta \neq 0, \delta = 0$ -note that the latter two cases cannot happen in $PSU(2, q)$.

Now we must find an element Z of order 2 such that XYZ has order 2 or equivalently trace 0 and ZX has trace τ. We have

$$Z := \pm \begin{pmatrix} \mu & \varphi \\ \psi & -\mu \end{pmatrix}$$

Computing XYZ we see that it has order 2 if and only if $\mu = 0$ since $\lambda - \lambda^{-1} \neq 0$ for the element XY is non trivial. Next we compute

$$ZX = \pm \begin{pmatrix} \varphi\delta & -\varphi\alpha \\ \psi\alpha & \psi\beta \end{pmatrix}$$

We have $\psi = -\varphi^{-1}$. Let us impose the trace to be τ. Note that even if $\tau = \pm 2$, ZX is non-trivial since $\varphi\alpha \neq 0$. If $3 - \gamma^2 \neq 0$, we are led to solve a second degree equation

$$\delta\varphi^2 - \tau\varphi - \beta = 0$$

in the unknown φ (note that since $\beta \neq 0$ we have that $\varphi \neq 0$) whose discriminant is $\Delta = (\tau^2\gamma^2 - 4\tau^2 - 4\gamma^2 + 12)/(\gamma^2 - 4)$. Note that we are assuming q odd so it make sense to consider the discriminant; anyhow all other considerations hold true also when q is even. If we are working in $PSL(2, q)$ we need to find a solution in $\mathbb{F}_q$, so Δ must be a square in $\mathbb{F}_q$. Since $\gamma^2 - 4 = (\lambda - \lambda^{-1})^2$ is a square in $\mathbb{F}_q$ we obtain the given condition with $C(\gamma, \tau) := \Delta(\gamma^2 - 4)$.
If we are working in $PSU(2, q)$, again we have to solve the same equation which always admits a solution in $\mathbb{F}_{q^2}$, but we have to check that $\varphi^{-1} = \varphi^q$. This is easily seen to be verified whenever Δ is 0 or a non-square in $\mathbb{F}_q$ -just compare the two expressions for ψ given by $(\tau - \delta\varphi)/\beta$ and $-\varphi^q$ where $\varphi = (\tau \pm \sqrt{\Delta})/2\delta$. Since in this case $\gamma^2 - 4$ is a non-square in $\mathbb{F}_q$, we get the same condition found for $PSL(2, q)$.
If $3 - \gamma^2 = 0$ we have $C(\gamma, \tau) = \tau^2(\gamma^2 - 4)$. There are different cases to consider. If $\tau = 0$ then to find a solution we must choose X such that $\beta = \delta = 0$. In this case any $\varphi \neq 0$ is an admissible solution. Anyway up to conjugation with an element in the centralizer of XY -note that these elements centralize also X- the choice is unique (and $C(\gamma, \tau) = 0$ in this case). We can assume $\varphi = 1$ and this is acceptable for both $PSL(2, q)$ and $PSU(2, q)$. However if $q \equiv 0 \pmod 3$, the homomorphism is not admissible since the element Y is trivial and we must exclude this case. If $\tau \neq 0$ instead, we cannot choose X in such a way that $\beta = \delta = 0$ and we have exactly one solution for both possible choices of X. Indeed if $\gamma = 0$, the two possible choices for X are conjugated by the extra element in the centralizer of XY. Remark that the elements Z that we find are in $PSL(2, q)$ but not in $PSU(2, q)$. In the latter case we do not have a solution in $PSU(2, q)$ and in fact $C(\gamma, \tau)$ is a non-square in $\mathbb{F}_q$.
It is now obvious that the given condition is necessary and sufficient.

The other parts of Lemma 2.1 are proved by similar methods. □

The proof of Lemma 2.1 gives also the following result which we shall need later.

LEMMA 2.2. *Let γ be the trace of an element of order k in $PSL(2, q^2)$.*

There exists an admissible homomorphism Φ from the triangle group $(2,3,k)$ to $PSL(2,q)$ (resp. to $PGL(2,q)$) such that the trace of $\Phi(xy)$ is γ if and only if $\gamma \in \mathbb{F}_q$ (resp. $\gamma \in \iota\mathbb{F}_q^*$; in this case assume $q \neq 2^m$), except in the case $k=2$ and $q=3^{2m+1}$. In this latter case an admissible homomorphism exists to $PGL(2,q)$. Moreover if $\gamma^2 \neq 3$, up to conjugation in $PGL(2,q)$ there is exactly one such homomorphism.

Proof. Note that at least one among x, y and xy must have non-parabolic image in $PSL(2,q^2)$. Assume that the image of xy is non-parabolic and repeat the same reasoning seen in proof of Lemma 2.1. Note that all considerations hold true also when q is even. Uniqueness follows from the fact that the entries of all matrices considered are completely determined up to conjugation in $PGL(2,q)$ under our hypothesis. This is due to the fact that the centralizer of the image of the element xy is indeed contained in $PGL(2,q)$. Obviously, the same procedure works also if we start with the element x or y in the case when xy has parabolic image. This finishes the proof of Lemma 2.2. □

The next Theorem gives the classification of all finite admissible quotients of linear fractional type of the three tetrahedral groups associated to the Coxeter tetrahedra $\mathcal{T}_{k,n}$ with exactly one cusp, and in particular of the tetrahedral group $T_{3,6}$ uniformizing the smallest orientable cusped hyperbolic 3-orbifold.

THEOREM 2.3. *The groups $PSL(2,q)$ and $PGL(2,q)$ are admissible quotients of the tetrahedral group $T_{k,6}$ exactly in the following cases:*

$k=3$:

i)	$p \equiv 1 \pmod{12}$:	$PSL(2,p)$
ii)	$p \equiv 7 \pmod{12}$:	$PGL(2,p)$
iii)	$p \equiv 5, 11 \pmod{12}$:	$PSL(2,p^2)$

$k=4$:

i)	$p \equiv 1 \pmod{24}$:	$PSL(2,p)$
ii)	$p \equiv 5,7,11 \pmod{24}$:	$PGL(2,p)$
iii)	$p \equiv 13,17,19,23 \pmod{24}$:	$PSL(2,p^2)$

$k = 5$:

i)	$p \equiv 1, 49 \pmod{60}$:	$PSL(2, p)$
ii)	$p \equiv 19, 31 \pmod{60}$:	$PGL(2, p)$
iii)	for all other values of $p \neq 2, 3$:	$PSL(2, p^2)$

Up to conjugation in $PGL(2, q)$ resp. in $PSL(2, q^2)$ there are at most two admissible surjections of $T_{k,6}$ if $k = 3, 4$ and at most four admissible surjections if $k = 5$.

Proof. $k = 3$:
Since we can find elements of order 3 in $PSL(2, q)$ for all q, up to conjugation, we can assume that the image of the element xy sits in $PSL(2, p)$ ($q = p^m$). For $n = 6$ we have $\tau^2 = 3$ and $C(\gamma, \tau) = -1$ for $\gamma = \pm 1$. We can apply Lemma 2.1 with $q := p$ to see where the images of the admissible homomorphisms lie.
Surjectivity follows from the fact that an element of order 6 cannot belong to $\mathbf{S}_4$ or to $\mathbf{A}_5$ the only other possible subgroups in $PSL(2, p)$ containing the vertex group $\mathbf{A}_4$. Note that up to conjugation there are no other admissible homomorphisms.
$k = 4$:
Note that an element of order 4 exists in $PSL(2, q)$ if and only if $q \equiv \pm 1 \pmod 8$. Thus if $p \equiv \pm 1 \pmod 8$ we can assume that the image of the element xy sits in $PSL(2, p)$ and as before we can apply Lemma 2.1 to the case $\gamma^2 = 2$, $\tau^2 = 3$ and $C(\gamma, \tau) = -2$ with $q := p$. If $p \not\equiv \pm 1 \pmod 8$, we can find an element of order 4 in $PGL(2, p) - PSL(2, p)$. Applying Lemma 2.1 we are again able to say when the image is contained in $PGL(2, p)$ itself. To prove surjectivity one exploits the maximality of the vertex group $\mathbf{S}_4$ in $PSL(2, p)$ if $p \equiv \pm 1 \pmod 8$ or in $PGL(2, p)$ if $p \equiv \pm 3 \pmod 8$.
$k = 5$:
We can find elements of order 5 in $PSL(2, q)$ if and only if $q(q^2 - 1) \equiv \pm 1 \pmod 5$. Moreover elements of odd order cannot lie in $PGL(2, q) - PSL(2, q)$. This means that we can apply Lemma 2.1 to the case $q := p$ if $p \equiv \pm 1 \pmod 5$ or $q := p^2$ if $p \equiv \pm 3 \pmod 5$ with $\gamma^2 = (3 \pm \sqrt{5})/2$, $\tau^2 = 3$ and $C(\gamma, \tau) = -(\pm 1(\pm\sqrt{5})/2)^2$. The only case that must be considered aside is when $q = 5^m$ and $k = 5$,

since in this situation the element of order k is parabolic. This can be done, for instance, exchanging the roles of n and k. Surjectivity follows from the maximality of the vertex group $\mathbf{A}_5$. Note that in this case we can have four possible quotients because we find two different values for $C(\gamma, \tau)$ associated to the two different traces of non conjugated elements of order 5.

This finishes the proof of Theorem 2.3. □

Up to conjugation, the tetrahedral group $T_{3,6}$ uniformizing the smallest cusped 3-orbifold is equal to the extended Bianchi group $PGL(2, \mathcal{O}_3) = PGL(2, \mathbb{Z}[\omega])$, considered as a subgroup of the isometry group $PSL(2, \mathbb{C})$ of hyperbolic 3-space, where ω is a primitive cubic root of unity and thus satisfies $\omega^2 + \omega + 1 = 0$. Suppose $p \neq 3$. Then there are two possibilities: either 3 divides $p - 1$ and a primitive cubic root of unity σ exists in $\mathbb{F}_p$, or 3 divides $p + 1$ and a primitive cubic root of unity σ exists in $\mathbb{F}_{p^2}$ (but not in $\mathbb{F}_p$). In both cases we have two unitary ring homomorphisms $\phi_j : \mathbb{Z}[\omega] \longrightarrow \mathbb{F}_{p^2}$ defined by $\phi_j(\omega) = \sigma^j$, $j = 1, 2$. These induce group homomorphisms $\Phi_j : T_{3,6} = PGL(2, \mathbb{Z}[\omega]) \longrightarrow PGL(2, p^2)$ by applying ϕ_j to the entries of the matrices which represent the elements of $T_{3,6}$. We shall say that the two homomorphisms are obtained by *reduction of coefficients mod p*.

THEOREM 2.4. *Let p be a prime different from* 2, 3. *Then, by reduction of coefficients* mod p, *we obtain two admissible surjections Φ_j, $j = 1, 2$, from $T_{3,6} = PGL(2, \mathbb{Z}[\omega])$ onto one of the following groups:*

i) $p \equiv 1 \pmod{12}$: $PSL(2, p)$
ii) $p \equiv 7 \pmod{12}$: $PGL(2, p)$
iii) $p \equiv 5, 11 \pmod{12}$: $PSL(2, p^2)$

Up to conjugation, all admissible homomorphisms from $T_{3,6}$ to a linear fractional group $PSL(2, q)$ or $PGL(2, q)$ are obtained by reduction of coefficients mod p *and, for each p, there are exactly two such homomorphisms. For each of the above finite groups, the kernels of the corresponding surjections are the universal covering groups of the cusped hyperbolic* 3*-manifolds of minimal volume admitting an action of the group.*

Proof. First we show that, for every p different from 2 and 3, the group homomorphisms Φ_j, $j = 1, 2$, are admissible, that is have torsion-free kernel. An element of finite order in $T_{3,6} = PGL(2, \mathbb{Z}[\omega])$ has order 2, 3 or 6. An element of order 2 in $PGL(2, \mathbb{Z}[\omega])$ has trace 0, so also its image has trace 0. Because $p \neq 2$, it cannot lie in the kernel of Φ_j. An element of order 3 is in $PSL(2, \mathbb{Z}[\omega])$, and the square of its trace is equal to 1. Again, because $p \neq 3$, it does not lie in the kernel of Φ_j which is therefore torsion-free.

We determine the image of Φ_j in $PGL(2, p^2)$. Note that the determinants of elements of $PGL(2, \mathbb{Z}[\omega])$ are $\pm 1, \pm\omega$ and $\pm\omega^2$ (these are the units in $\mathbb{Z}[\omega]$), therefore all elements have representatives with determinant 1 or -1. Now -1 is a square in $\mathbb{F}_p$ if and only if $p \equiv 1 \pmod 4$, otherwise it is a square in $\mathbb{F}_{p^2}$. It follows that the image of Φ_j lies in the groups listed in the three cases of the Theorem, and it remains to prove surjectivity. This follows from the fact (see [7]) that for $q \neq 2^m$ and $q \neq 3^2$ the group $PSL(2, q)$ is generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -\xi & 1 \end{pmatrix}$$

where ξ generates $\mathbb{F}_q$ over $\mathbb{F}_p$. With $\xi = \sigma^j$ these matrices belong to the image of Φ_j, and Φ_j results surjective in each of the three cases.

The two homomorphisms Φ_1 and Φ_2 are not conjugate since the images of the element

$$\begin{pmatrix} 0 & 1 \\ -1 & \omega \end{pmatrix},$$

of determinant 1, have different traces σ and σ^2 and thus are not conjugate. Now, by Theorem 2.3, each admissible homomorphism is obtained by reduction mod p. □

3. Some infinite series of tetrahedral groups $T_{k,n}$

In this Section, for a fixed $2 \leq k \leq 5$ we shall consider the series of tetrahedral groups $T_{k,n}$ for arbitrary n. For $n > 6$ these tetrahedral groups have exactly one hyperbolic vertex group which is the triangle group $(2, 3, n)$. For any single $n > 6$ it is rather difficult to give a complete classification of the finite quotients of $T_{k,n}$ of linear fractional type (see [13] for the "Hurwitz-case" $n = 7$), so we will discuss simultaneously the whole series of groups. As noted above, the groups $T_{2,n}$ are isomorphic to the extended triangle groups $[2, 3, n]$. The groups $T_{3,n}$ have been considered in detail in [6]. Both the groups $T_{2,n}$ and $T_{3,n}$ will serve us in the next Section to determine the finite quotients of the Picard group.

Another reason for considering these series is the following. Note that, for a fixed k and arbitrary n, the finite admissible quotients of the groups $T_{k,n}$ are exactly the $\mathbf{G}_k$-groups, i.e. the finite admissible quotients of the group $\mathbf{G}_k = \mathbf{D}_k *_{\mathbb{Z}_k} (2, 3, k)$. For $k = 2, 3, 4$ and 5, the $\mathbf{G}_k$-groups are exactly the finite groups occurring as orientation-preserving symmetry groups of maximal possible order $12(g - 1)$ of 3-dimensional handlebodies of genus $g > 1$, and also in maximally symmetric group actions on closed 3-manifolds, see [30], [31]. The $\mathbf{G}_2$-groups are also the finite symmetry groups of maximal possible order $12(g-1)$ of compact bounded surfaces of algebraic genus $g > 1$. The group $\mathbf{G}_2 = \mathbf{D}_2 *_{\mathbb{Z}_2} \mathbf{D}_3$ is isomorphic to the extended modular group $PGL(2, \mathbb{Z})$, and its finite quotients of linear fractional type have been classified in [22]. This is contained as the case $k = 2$ in the following Theorem.

THEOREM 3.1. *Let $2 \leq k \leq 5$ be fixed. Then $PSL(2, q)$ is an admissible quotient of $T_{k,n}$, for some n, exactly in the following cases; equivalently, $PSL(2, q)$ is a $\mathbf{G}_k$-group exactly in the following cases:*

$k = 2$:	$q \neq 2, 7, 3^2, 11, 3^{2m+1}$;
$k = 3$:	$q \neq 2, 7, 3^2, 11, 3^{2m+1}$;
$k = 4$:	$q \equiv \pm 1 \pmod 8$ *but* $q \neq 7, 3^2$;
$k = 5$:	$q(q^2 - 1) \equiv 0 \pmod 5$ *but* $q \neq 3^2$.

Proof. We present the proof only for the case $q = p^n$ is odd. From Lemma 2.1 we know that, if $\gamma \neq \pm 2$, an admissible homomorphism from $T_{k,n}$ to $PSL(2, q)$ exists if and only if we can find traces $\gamma, \tau \in \mathbb{F}_q$ of elements of order k resp. n in $PSL(2, q)$ such that $C(\gamma, \tau)$ is a square in $\mathbb{F}_q$. If we want a solution for some n all we need to check is that, fixed an appropriate γ according to the cases $k = 2, 3, 4$ resp. 5, there exists $\tau \in \mathbb{F}_q$ such that $C(\gamma, \tau)$ is a square in $\mathbb{F}_q$. The trace γ of an element of order $k = 2, 3, 4$ resp. 5 satisfies $\gamma = 0$, $\gamma^2 = 1$,

$\gamma^2 = 2$ resp. $\gamma^2 \pm \gamma - 1 = 0$. In particular, there are always elements of order $k = 2$ and 3 in $PSL(2,q)$, and there are elements of order 4 (resp. 5) if and only if $q \equiv \pm 1 \pmod 8$ (resp. $q(q^2-1) \equiv 0 \pmod 5$). In the following assume that $\gamma \neq \pm 2$ and that, fixed k, $PSL(2,q)$ contains elements of order k. In the different cases $C(\gamma,\tau)$ is given by:

$k = 2$: $-4\tau^2 + 12$, which is a square in $\mathbb{F}_q$ if and only if $3 - \tau^2$ is, since is always a square;
$k = 3$: $8 - 3\tau^2$;
$k = 4$: $-2\tau^2 + 4$, which is a square in $\mathbb{F}_q$ if and only if $2 - \tau^2$ is a square, since in this case $2 = \gamma^2$ is a square;
$k = 5$: $(-5 \pm \sqrt{5}/2)\tau^2 \mp 2\sqrt{5}$;

By Lemma 2.1, there exists an admissible homomorphism $T_{k,n} \longrightarrow PSL(2,q)$ if and only if the conic
$k = 2$: $a^2 + \tau^2 = 3$
$k = 3$: $a^2 + 3\tau^2 = 8$
$k = 4$: $a^2 + \tau^2 = 2$
$k = 5$: $a^2 + ((5 \mp \sqrt{5})/2)\tau^2 = \mp 2\sqrt{5}$
in the affine space $(\mathbb{F}_q)^2$ is non-empty. This question is answered in [7]: the number of points on the conic is $q \pm 1$, in particular the conic is non-empty. Here we assume $q \not\equiv 0 \pmod 3$ if $k = 2$ so that no coefficient is 0.
However we want to know when this solution is surjective. This will always be the case if the elements of order 2 and 3 whose product has order n are a generating pair. All we must do is discard the cases when they generate a proper subgroup, in particular we shall assume $q \neq 3^2$. Using the classification of subgroups in the linear fractional groups, we see that if the elements of order 2 and 3 generate a proper subgroup then we are in one of the following situations:
i) $\tau^2 = 0, 1, 2, 3$ or $\tau^2 \pm \tau - 1 = 0$ where they generate an exceptional subgroup or an affine subgroup;
ii) $\tau \in \mathbb{F}_{q'} \subset \mathbb{F}_q$ where they generate a subgroup contained in $PSL(2,q') \subset PSL(2,q)$;
iii) $\tau^2 \in \mathbb{F}_{\sqrt{q}}$ (this condition makes sense only if q is an even power of p) where the image is a group inside $PGL(2,\sqrt{q})$.
We shall call τ *admissible* if it is not of one of the forms in i), ii) and iii). The idea now is to count all solutions τ and see when their

number is larger than the number of non-admissible solutions.
The number of solutions τ is at least $(q-1)/2$ (because with (a,τ) also $(-a,\tau)$ is on the conic) while the number of τ's satisfying i), ii) or iii) is at most $11+\sqrt{q}+\varepsilon\sqrt{q}$ where ε is 0 if q is an odd power of p and 1 otherwise. Note that if $q=p$ is a prime there are at most 11 non-admissible τ's belonging all to case i). Now as in [22] we see that there is a solution for all $q>25$, $q\neq 49$. Just consider the function $f(q):=q-2(1+\epsilon)\sqrt{q}-23$ (resp. $g(q):=q-23$ when q is prime). We want the function to be positive. Redefining $t:=\sqrt{q}$ in the first case and studying the functions for all positive real numbers, it is not difficult to see that they are non-positive only for the given values. This means that for $q>25$ but $q\neq 7^2$ $PSL(2,q)$ is an admissible quotient of $T_{k,n}$, $k=2,3,4,5$, for some n. In the cases $q\leq 25$, $q=7^2$ as well as $q=3^2$, an admissible τ is looked for directly. Note that if $k=2$ the condition that the elements of order 2 and 3 are a generating pair is not only sufficient but also necessary since $(2,3,n)$ has index 2 in $T_{2,n}$ and the linear fractional groups are simple (apart from $PSL(2,2), PSL(2,3)$).
Let us now study the case $k=2$, $p=3$. The conic becomes: $a^2=-\tau^2$. There is only the point $(0,0)$ on it if -1 is a non-square (i.e. q is an odd power of 3), while there are $2q-1$ points if -1 is a square. If $q\equiv 1 \pmod 4$ we can repeat the same estimate as above and conclude that we always find an admisible τ apart from $q=3^2$. If $q\equiv -1$ instead, there is no admissible homomorphism from $T_{2,2}$ to $PSL(2,3^m)$ (see Lemma 2.1).
Suppose now that $\gamma=\pm 2$. If we require $\tau\neq\pm 2$ then we can exchange the roles of γ and τ. We obtain $C(\gamma,\tau)=-4$ which is a square if and only if -1 is, independently of τ. We conclude that there are always admissible homomorphism for all $\tau\neq\pm 2$. To ensure surjectivity it is enough to see if $q-2>(1+\varepsilon)\sqrt{q}+11$ and this is always the case for $q\neq 3,5$. In these two cases the existence of an admissible homomorphism is sufficient to ensure surjectivity since we are mapping $T_{k,n}$ to one of its vertex groups. □

Remark 3.2. *Note that by reduction of coefficients mod p for the extended modular group $PGL(2,\mathbb{Z})\cong \mathbf{G}_2$ we obtain admissible quotients of type $PGL(2,p)$ if $p\equiv -1 \pmod 4$, i.e. -1 is a non-square in $\mathbb{F}_p$, and of type $PSL(2,p)$ if $p\equiv 1 \pmod 4$. Now the case $k=2$ of*

Theorem 3.1 implies that most admissible quotients of the extended modular group are not obtained by reduction of coefficients mod p.

Corollary 3.3. *$PSL(2,q)$ is a maximal handlebody group of order $12(g-1)$ exactly for all q different from 2, 7, 3^2 and 3^{2m+1}.*

4. The Picard group

In the next Theorem we shall classify the groups $PSL(2,q)$ which are admissible quotients of the Picard group $PSL(2,\mathcal{O}_1) = PSL(2,\mathbb{Z}[i])$. The Picard group is a polygonal product (see Figure 2) isomorphic to the free product with amalgamation

$$\mathbf{G}_2 *_{(\mathbb{Z}_2 * \mathbb{Z}_3)} \mathbf{G}_3 = (\mathbf{D}_2 *_{\mathbb{Z}_2} \mathbf{D}_3) *_{(\mathbb{Z}_2*\mathbb{Z}_3)} (\mathbf{D}_3 *_{\mathbb{Z}_2} \mathbf{A}_4),$$

where the amalgam $(\mathbb{Z}_2 = \langle w \rangle) * (\mathbb{Z}_3 = \langle y \rangle)$, isomorphic to the modular group $PSL(2,\mathbb{Z})$, identifies the generators called w and y in the above presentations of $\mathbf{G}_2$ and $\mathbf{G}_3$ (actually, of $\mathbf{G}_k$), see [28], [9], [10].

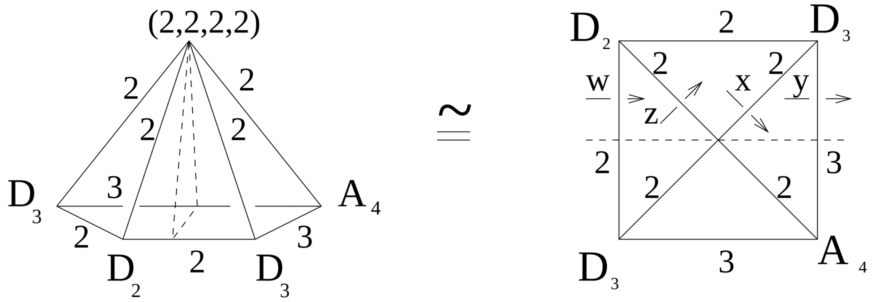


Figure 2.

Given a homomorphism of the Picard group $\mathbf{G}_2 *_{(\mathbb{Z}_2*\mathbb{Z}_3)} \mathbf{G}_3$ onto a finite group, the image of the element wy^{-1} has a certain order n, and then the homomorphism factors through the free product with amalgamation

$$T_{2,n} *_{(2,3,n)} T_{3,n}$$

over the triangle group $(2,3,n) = \langle w, y \mid w^2, y^3, (wy^{-1})^n \rangle$ (note that, in the above generators of both $\mathbf{G}_2$ and $\mathbf{G}_3$, one has $wy^{-1} = zx$, and that $T_{k,n} = \mathbf{G}_k / \langle\langle (zx)^n \rangle\rangle$. Thus we are in a situation where we can apply Lemma 2.1 and Theorems 2.3 and 3.1 to the factors $T_{2,n}$ and $T_{3,n}$ of the product. The result is as follows.

THEOREM 4.1. *The group $PSL(2,q)$ is an admissible quotient of the Picard group exactly for the values of q different from $2, 7, 3^2, 11$ and 3^{2m+1}.*

Proof. We present the proof for the case where q is not a power of 2 or 3.
Suppose there exists an admissible homomorphism from the Picard group to $PSL(2,q)$. As noted above, the homomorphism factors through $T_{2,n} *_{(2,3,n)} T_{3,n}$ and induces admissible homomorphisms from both $T_{2,n}$ and $T_{3,n}$ to $PSL(2,q)$. By Lemma 2.1 resp. the proof of Theorem 3.1, we find a solution of the following system of equations

$$\begin{cases} a^2 = 3 - \tau^2 \\ b^2 = 8 - 3\tau^2 \end{cases}$$

where τ is the trace of the element of order n in $PSL(2,q)$ which is the image of zx.
Conversely, suppose we have a solution of this system. By Lemma 2.1, there exist admissible homomorphisms from both $T_{2,n}$ and $T_{3,n}$ to $PSL(2,q)$ such that the image of the element zx has trace τ. By Lemma 2.2 the restrictions of these homomorphisms to the common triangle subgroup $(2,3,n)$ generated by z and x are the same up to conjugation whenever $\tau^2 \neq 3$. Then the two homomorphisms combine giving an admissible homomorphism from $T_{2,n} *_{(2,3,n)} T_{3,n}$ to $PSL(2,q)$, and thus also from the Picard group $\mathbf{G}_2 *_{(\mathbb{Z}_2 * \mathbb{Z}_3)} \mathbf{G}_3$. So we want to find a solution of the above system which implies also surjectivity (as in the proof of Theorem 3.1).

The solutions of the system are the points of an affine variety in $(\mathbb{F}_q)^3$. It is easy to see that this variety is a smooth curve of degree 4 (the number of points of intersection of the curve with a generical plane in the algebraic closure of the field) and genus 1. The genus can be computed considering the projection of the curve on the plane $a = 0$ and using Riemann-Hurwitz formula. The number of points of a projective variety over a Galois field can be estimated (see [15]). For curves this estimate is the Hasse-Weil bound (see [23, page 170]) and becomes

$$|N-q| \leq 2\sqrt{q}+1$$

where N is the number of points on the projective closure of our curve. The number of solutions to our system is then grater than or equal to $q-2\sqrt{q}-5$ (here we are excluding the points at infinity which are four at the most) and since the curve has degree 4, the number of possible τ's is not less than $(q-2\sqrt{q}-5)/4$: fix τ there are at the most four possible (a,b) such that (τ,a,b) is on the curve. We want this number to be larger than $(1+\varepsilon)\sqrt{q}+11$ (equivalently $q-(6+4\varepsilon)\sqrt{q}-49>0$) or 11 (equivalently $q-2\sqrt{q}-49>0$) if $q=p$ is a prime just like in Theorem 3.1. Studying the functions $f(q):=q-6\sqrt{q}-49$ (for q an odd, non-trivial power of a prime), $f'(q):=f(q)-4\sqrt{q}$ (for q an even power of a prime) and $g(q):=q-2\sqrt{q}-49$ (q prime) defined over the positive real numbers (one can replace $t:=\sqrt{q}$ for simplicity), we see that they are positive for all $q\geq 67$ but $q\neq 121,169$ and in these cases an admissible τ does exist.

We want now to see what happens when the functions are non-positive. First we remark some facts. First of all, by reduction of coefficients mod p we find admissible quotients of the Picard group of type $PSL(2,p)$ if $p\equiv 1 \pmod 4$ and of type $PSL(2,p^2)$ if $p\equiv -1 \pmod 4$ (compare Lemma 2.2).

Secondarily it is easy to see that our system of equations is equivalent to

$$\begin{cases} \tau^2 = 3-a^2 \\ 3a^2-b^2=1 \end{cases}$$

and we are able to solve the second equation and give an explicit expression for a and b (see [7] for details). Substituting a in the first equation we obtain $\tau^2=(34-t^2-t^{-2})/12$ where $t\in\mathbb{F}_q^*$ if 3 is a square in $\mathbb{F}_q$ or $t\in\mathbb{F}_{q^2}^*$ is a $(q+1)$-root of unity otherwise. Note that in both cases we can put $t=1$ and we have an admissible solution τ (i.e. $\tau^2\neq 0,1,2,3$ and $\tau^2\pm\tau-1\neq 0$) for all prime numbers $p\neq 5$, $p\equiv \pm 1,\pm 5 \pmod{24}$.

At this point we only need to check if $PSL(2,q)$ is an admissible quotient of the Picard group for $q=7,11,25,31,59,169$ and this is done by direct computation. We do not find an admissible τ only when $q=7,11$.

This finishes the proof of Theorem 4.1. □

The proof of Theorem 4.1 works for different amalgams of the groups $\mathbf{G}_k$ generalizing the Picard group (some of the corresponding orbifolds and their volumes occur in [6, p.169-170]). For example the following holds.

THEOREM 4.2. *The group* $PSL(2,q)$ *is an admissible quotient of the extended Bianchi group* $PGL(2,\mathcal{O}_2) = PGL(2,\mathbb{Z}[i\sqrt{2}]) \cong \mathbf{G}_2 *_{(\mathbb{Z}_2 * \mathbb{Z}_3)} \mathbf{G}_4$ *exactly for all* $q \equiv \pm 1 \pmod 8$ *different from* $7, 3^2$.

The Picard group has various torsion-free subgroups of small index uniformizing the complements of hyperbolic links in the 3-sphere, for example the Whitehead link and the Borromean rings (see [4]). Recall that the group of a link is defined as the fundamental group of its complement. The group of the Whitehead link is a subgroup of index 12 in the Picard group. By restricting the surjections from Theorem 4.1 to this subgroup we get

COROLLARY 4.3. *The group* $PSL(2,q)$ *is a quotient of the group of the Whitehead link for all values* $q > 11$ *different from* 3^{2m+1}.

Proof. Surjectivity of the restrictions follows from the fact that, for $q > 11$, a proper subgroup of $PSL(2,q)$ has index at least $q+1$ (see [7]). □

A similar result holds for the group of the Borromean rings which has index 24 in the Picard group. However here one can say much more. The group of the Borromean rings has the free group of rank 2 as a quotient (see the proof of Theorem 5.1), and hence also every 2-generator group is a quotient, in particular every finite simple group. Note that the group of the 2-bridge Whitehead link is 2-generated and does not have the free group of rank 2 as a quotient. The group of the figure-8-knot is a subgroup of index 12 in the tetrahedral Bianchi group $PSL(2,\mathcal{O}_3)$ which we shall consider in the last Section. As for the group $PGL(2,\mathcal{O}_3)$ considered in Section 2, almost all finite admissible quotients of linear fractional type of this tetrahedral group are obtained by reduction mod p, so one obtains quite a restricted set of quotients in this way. It would be interesting to know which finite simple groups are quotients of the group of the figure-8-knot (see also [25]).

5. On the construction of closed hyperbolic 3-manifolds with large group actions

By [1] the Picard orbifold $\mathbf{H}^3/PSL(2,\mathcal{O}_1)$ is the smallest hyperbolic 3-orbifold whose volume is a limit of other volumes or, equivalently, the smallest hyperbolic 3-orbifold with a non-rigid cusp on which Dehn surgery can be performed (see also [8] for the notion of Dehn surgery on orbifolds). Thus hyperbolic Dehn surgery on the Picard orbifold can be used to construct small hyperbolic 3-manifolds admitting $PSL(2,q)$-actions, i.e. the quotient of the volume of the manifold by the order $|PSL(2,q)|$ of the group is small.

The closed hyperbolic 3-orbifolds of minimal volumes are not known. The probable candidates in the orientable case are the tetrahedral orbifolds associated to some of the nine Lannér tetrahedra resp. quotients of these by involutions. Finite quotients of type $PSL(2,q)$ and $PGL(2,q)$ of the corresponding tetrahedral groups have been studied in [12] and [21]. As in the case of cusped tetrahedral groups only a restricted set of values of q occurs. So, as in Section 3 it seems reasonable to consider infinite series of small volume hyperbolic 3-orbifolds simultaneously. As the smallest limit volume is that of the Picard orbifold we shall consider closed hyperbolic 3-orbifolds obtained by generalized hyperbolic Dehn surgery on the cusp of the Picard orbifold. The volumes of these closed orbifolds are smaller than the volume of the Picard orbifold which they have as a limit value. Also, the closed hyperbolic 3-orbifolds of smallest known volumes are obtained in this way. We denote by $v = 0,30532\ldots$ the volume of the Picard orbifold. We shall also consider 3-orbifolds obtained by surgery on the Borromean rings.

For a finite group G, we define a *hyperbolic G-manifold* as an orientable complete hyperbolic 3-manifold on which G acts effectively by orientation-preserving isometries. The next result should be compared with [25, Theorem 5] where surgery on the complement of the figure-8 knot is considered.

THEOREM 5.1. *a) For q different from $2, 7, 9$ and 3^{2m+1}, the minimal volume of a closed hyperbolic $PSL(2,q)$-manifold is smaller than*

$v|PSL(2,q)|$. *Moreover, for each fixed q this is the smallest value which is a limit of volumes of hyperbolic $PSL(2,q)$-manifolds.*
b) For any finite r-generator group G, there exist hyperbolic G-manifolds of volume smaller than and arbitrarily close to $24v(r-1)|G|$. Given any real constant c, there exist finite groups G such that the volume of any hyperbolic G-manifold is larger than $c|G|$.

Proof. We start with the proof of part b) of the Theorem.
The group of the Borromean rings is a subgroup of index 24 in the Picard group (see [4]), and so their complement has volume $24v$. Computing the Wirtinger presentation from the standard projection of the link one obtains a group presentation with six generators and six defining relations (one of which may be deleted). Three of the relations can be used to eliminate three of the generators. Setting one of the remaining three generators equal to 1 one obtains a free group of rank 2 which is therefore a quotient of the group of the Borromean rings. Hence every 2-generator group is a quotient of the group of the Borromean rings.

Consider a surjection ϕ of the group of the Borromean rings onto a 2-generator group G. We perform generalized hyperbolic surgeries of the following types on the three components of the Borromean rings. If μ and λ denote a standard meridian-longitude pair for a component of the link, and if ϕ maps $\mu^p\lambda^q$, with $(p,q)=1$, to an element of order n in G then we may perform (np,nq)-surgery on that component, i.e. $\mu^{np}\lambda^{nq}$ becomes trivial after the surgery. The result is a 3-orbifold where the central curve of the added solid torus has branching order n. By Thurston's hyperbolic surgery theorem, excluding finitely many surgeries for each component, the resulting closed 3-orbifolds are hyperbolic, and their volumes are smaller than the volume $24v$ of the complement of the Borromean rings which they have as a limit value. By construction, the surjection ϕ induces admissible surjections of the fundamental groups of these closed 3-orbifolds onto the group G. The G-manifolds of the Theorem are now the regular coverings of the orbifolds corresponding to the kernels of these surjections.

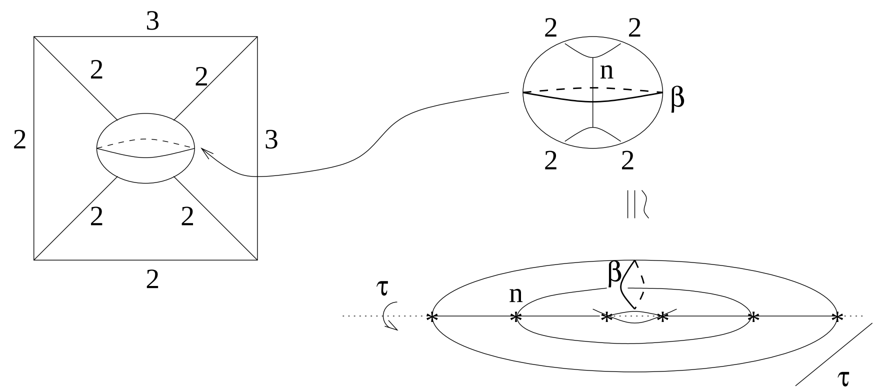
For arbitrary r, we note that the free group of rank r is a subgroup of index $r-1$ in the free group of rank 2. Thus the free group of rank r is a quotient of the fundamental group of an r-fold covering of the complement of the Borromean rings. This covering is a hyperbolic 3-manifold with a finite number of cusps, and the proof is now similar to the case $r = 2$. corresponding to the kernels of these surjections.

For any hyperbolic G-manifold M, the quotient M/G is a hyperbolic 3-orbifold O of volume $\mathrm{vol}(M)/|G|$, and M is the covering of $\mathcal{O}$ corresponding to the kernel of an admissible surjection of $\pi_1(\mathcal{O})$ onto G. By [8, Prop.5.5], all hyperbolic 3-orbifolds whose volumes are smaller than a constant c are obtained by Dehn surgery on one of a finite set of hyperbolic 3-orbifolds. If r denotes the maximal rank of the fundamental groups of these finitely many 3-orbifolds, then also the fundamental group of any 3-orbifold obtained by surgery on one of these has rank less or equal to r. Hence, if the finite group G has rank larger than r, any hyperbolic G-manifold has volume at least $c|G|$.
This finishes the proof of part b) of the Theorem.

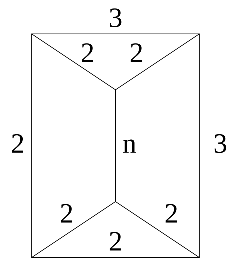
The proof of part a) is similar using surgery on the cusp of the Picard orbifold. Such a surgery is indicated in Figure 3a) where the 3-ball orbifold is glued along its boundary to the boundary-horosphere of the compactified Picard orbifold. Denote by α the curve on the horosphere to which the meridional curve β on the boundary of the 3-ball orbifold is glued. The result is a closed 3-orbifold $\mathcal{O}(\alpha, n)$. Excluding finitely many isotopy classes of curves α, these 3-orbifold are hyperbolic, and their volumes are smaller than the volume of the Picard orbifold which they have as a limit (see [8]).

By Theorem 4.1, for the above values of q there exists an admissible surjection ϕ of the Picard group onto $PSL(2,q)$. If ϕ maps the curve α to an element of order n then it induces an admissible surjection of the fundamental group of the hyperbolic 3-orbifold $\mathcal{O}(\alpha, n)$ onto $PSL(2,q)$. Part a) of the Theorem is proved now by considering the closed hyperbolic $PSL(2,q)$-manifolds which are the coverings of the orbifolds $\mathcal{O}(\alpha, n)$ corresponding to the kernels of these induced surjections.

This finishes the proof of the Theorem. □



a)



b)

Figure 3.

The simplest of the orbifolds $\mathcal{O}(\alpha, n)$ in the proof of Theorem 5.1 are the polyhedral orbifolds shown in Figure 3b) which are hyperbolic for $n > 6$. In this case the curve α of the surgery is represented by the element zx in the Picard group, and ϕ induces an admissible surjection of the corresponding hyperbolic polyhedral group onto $PSL(2, q)$ if and only if $\phi(zx)$ has order n. The volumes of these polyhedra are equal to those of the truncated tetrahedra $\mathcal{T}_{3,n}$ and can be found in [6].

6. Some other cusped tetrahedral groups

Our methods apply to other tetrahedral groups. We state some results without proofs. Consider the tetrahedron represented in Figure 4. We shall denote the tetrahedral group associated to it by $T(k_1, k_2, k_3, n)$. For $k_1 = k_3 = 4$, $k_2 = 2$, $n = 3$ and for $k_1 = k_2 = k_3 = 3$, $n = 3, 4, 5$ the tetrahedron is hyperbolic with one cusp. In particular $\mathcal{O}(4, 2, 4, 3)$ and $\mathcal{O}(3, 3, 3, 3)$ have second and third smallest volumes among hyperbolic cusped orbifolds.

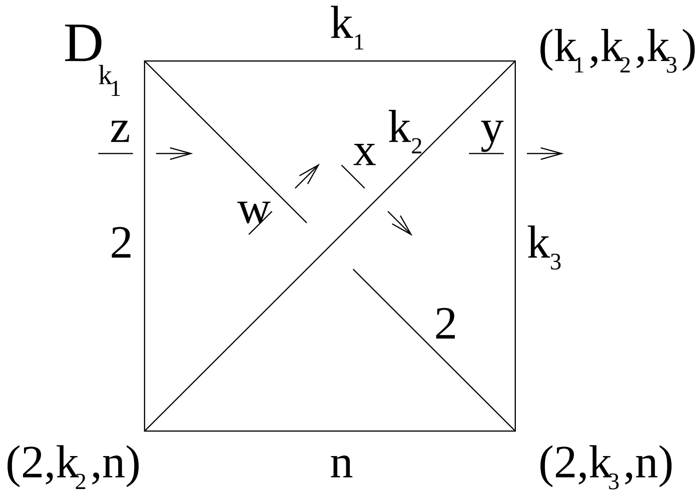


Figure 4.

We have the following generalization of a part of Lemma 2.1.

Lemma 6.1. *For $i = 1, 2, 3$, let γ_i resp. $\tau \in \mathbb{F}_q \cup \iota\mathbb{F}_q^*$ be traces of elements of order k_i resp. n in $PSL(2, q^2)$, where q is odd and $\gamma_1 \neq \pm 2$. Let $C(\gamma_1, \gamma_2, \gamma_3, \tau) := \gamma_1^2\tau^2 + 4\gamma_1\gamma_2\gamma_3 - 4\gamma_1^2 - 4\gamma_2^2 - 4\gamma_3^2 - 4\tau^2 + 16$. If $C(\gamma_1, \gamma_2, \gamma_3, \tau)$ is a square in $\mathbb{F}_{q^2}$ then there exists an admissible homomorphism $\Phi : T(k_1, k_2, k_3, n) \longrightarrow PSL(2, q^2)$ such that $\Phi(xy)$ has trace γ_1, $\Phi(x)$ has trace γ_2, $\Phi(y)$ has trace γ_3 and $\Phi(zx)$ has trace τ except in the two cases $\gamma_2 = \gamma_3 = 0$, $\tau = \pm 2$ and γ_2 or $\gamma_3 = \pm 2$, $\gamma_1\gamma_2\gamma_3 - \gamma_2^2 - \gamma_3^2 = 0$, $\tau = 0$.*
The image of Φ lies inside $PSL(2, q)$ if and only if γ_i and τ are in $\mathbb{F}_q$ and $C(\gamma_1, \gamma_2, \gamma_3, \tau)$ is a square in $\mathbb{F}_q$ ($i = 1, 2, 3$).
If $\gamma_i \in \mathbb{F}_q$, $i = 1, 2, 3$, while either $\tau \in \iota\mathbb{F}_q$ and $C(\gamma_1, \gamma_2, \gamma_3, \tau)$ is

a non-square in $\mathbb{F}_q$ *or* τ *is in* $\mathbb{F}_q^*$ *and* $C(\gamma_1, \gamma_2, \gamma_3, \tau) = 0$, *then the image of* Φ *lies in* $PGL(2, q)$ *(but not in* $PSL(2, q)$*).*

Note that we do not give a complete classification of admissible homomorphisms with image contained in $PGL(2, q)$ as in Lemma 2.1, since there are too many different cases to consider according to where the elements x, y and z lie. In Lemma 6.1 we only consider the case when both x and y belong to $PSL(2, q)$ while z belongs to $PGL(2, q) - PSL(2, q)$. Note that for $\gamma_2 = 0$ and $\gamma_3 = \pm 1$ we obtain a part of Lemma 2.1.

Lemma 6.1 has the following applications. In the first we consider the group $T(4, 2, 4, 3)$ which is the extended Picard group $PGL(2, \mathbb{Z}[i])$ (see [5]).

THEOREM 6.2. *The admissible quotients of type* $PSL(2, q)$ *and* $PGL(2, q)$ *of the group* $T(4, 2, 4, 3)$ *are exactly the following:*

$$\begin{array}{lrl} i) & p \equiv 1 \pmod 8: & PSL(2,p) \\ ii) & p \equiv 5 \pmod 8: & PGL(2,p) \\ iii) & p \equiv 3,7 \pmod 8: & PSL(2,p^2) \end{array}$$

All admissible quotients of type $PSL(2, q)$ *and* $PGL(2, q)$ *of* $T(4, 2, 4, 3)$ *are obtained by reduction of coefficients mod* p.

The group $T(3, 3, 3, 3)$ is isomorphic to $PSL(2, \mathbb{Z}[\omega])$ (see [5]). We have the following

THEOREM 6.3. *The groups* $PSL(2, q)$ *and* $PGL(2, q)$ *are admissible quotients of the group* $T(3, 3, 3, n)$ *exactly in the following cases:*
$n = 3$*:*

$$\begin{array}{lrl} i) & p \equiv 1 \pmod 6: & PSL(2,p) \\ ii) & p \equiv -1 \pmod 6: & PSL(2,p^2) \\ iii) & & PSL(2,2^2) \cong PSL(2,5) \cong \mathbf{A}_5 \end{array}$$

$n = 4$*:*

$$\begin{array}{lrl} i) & p \equiv 1,7 \pmod{24}: & PSL(2,p) \\ ii) & p \equiv 3,5,11 \pmod{24}: & PGL(2,p) \\ iii) & p \equiv 13,17,19,23 \pmod{24}: & PSL(2,p^2) \end{array}$$

$n = 5$:

$$\begin{array}{lll} i) & p \equiv \pm 1 \pmod{10}: & \begin{cases} PSL(2,p) & \textit{if } p \equiv 1 \pmod 3 \\ PSL(2,p^2) & \textit{if } p \equiv -1 \pmod 3 \end{cases} \\ ii) & p \equiv \pm 3 \pmod{10}: & PSL(2,p^2) \\ iii) & p = 2: & PSL(2,2^2) \\ iv) & p = 5: & PSL(2,5), PSL(2,5^2) \end{array}$$

In the case $n = 3$, all admissible quotients of $T(3,3,3,3)$ are obtained by reduction of coefficients mod p, with the exception of the group $\mathbf{A}_5$ of the case iii).

Acknowledgements We thank Davide Franco for suggesting the main references concerning algebraic varieties over finite fields and for valuable discussions.

References

[1] C. C. Adams, *Limit volumes of hyperbolic three-orbifolds*, J. Differential Geom. **34** (1991), 115–142.

[2] C. C. Adams, *Noncompact hyperbolic 3-orbifolds of small volume*, Topology '90 (B. Apanasov, W. D. Neumann, A. W. Reid, and L. Siebenmann, eds.), Ohio State University Mathematical Research Institute Publications, vol. 1, De Gruyter, Berlin, 1992, pp. 1–15.

[3] R. C. Alperin, *Normal subgroups of $PSL_2(Z[\sqrt{-3}])$*, Proc. Amer. Math. Soc. **124** (1996), 2935–2941.

[4] A. M. Brunner, M. L. Frame, Y. W. Lee, and N. J. Wielenberg, *Classifying torsion-free subgroups of the Picard group*, Trans. Amer. Math. Soc. **282** (1984), 205–235.

[5] A. M. Brunner, Y. W. Lee, and N. J. Wielenberg, *Polyhedral groups and graph amalgamation products*, Topology Appl. **20** (1985), 289–304.

[6] M. D. E. Conder and G. J. Martin, *Cusps, triangle groups and hyperbolic 3-folds*, J. Austral. Math. Soc. **55** (1993), 149–182.

[7] L. E. Dickson, *Linear groups with an exposition of the Galois field theory*, Dover Publications, New York, 1958.

[8] W. D. Dunbar and G. R. Meyerhoff, *Volumes of hyperbolic 3-orbifolds*, Indiana Univ. Math. J. **43** (1994), 611–637.

[9] B. Fine, *Algebraic theory of the Bianchi groups*, Monographs and textbooks in pure and applied mathematics, vol. 129, Dekker, New York, 1989.

[10] B. Fine and C. Frohman, *Some amalgam structures for Bianchi groups*, Proc. Amer. Math. Soc. **102** (1988), 221–229.

[11] H. Glover and D. Sjerve, *The genus of $PSL_2(q)$*, J. Reine Angew. Math. **380** (1987), 59–86.

[12] M. Gradolato and B. Zimmermann, *Finite quotients of hyperbolic tetrahedral groups*, Rend. Ist. Mat. Univ. Trieste **25** (1993), 211–222.

[13] M. Gradolato and B. Zimmermann, *Extending finite group actions on surfaces to hyperbolic 3-manifolds*, Math. Proc. Cambridge Phil. Soc. **117** (1995), 137–151.

[14] F. Grunewald and J. Schwermer, *Subgroups of Bianchi groups and arithmetic quotients of hyperbolic 3-space*, Trans. Amer. Math. Soc. **335** (1993), 47–78.

[15] S. Lang and A. Weil, *Number of points of varieties in finite fields*, Amer. J. Math. **76** (1954), 819–827.

[16] A. M. Macbeath, *Generators of the linear fractional groups*, Number theory, Proc. Symp. Pure Math., vol. 12, 1969, pp. 14–32.

[17] W. Magnus, *Non-euclidean tesselations and their groups*, Academic Press, New York, 1974.

[18] R. Meyerhoff, *The cusped hyperbolic 3-orbifold of minimum volume*, Bull. Amer. Math. Soc. **13** (1985), 154–156.

[19] R. Meyerhoff, *A lower bound for the volume of hyperbolic 3-orbifolds*, Duke Math. J. **57** (1988), 185–203.

[20] W. D. Neumann and A. W. Reid, *Notes on Adams' small volume orbifolds*, Topology '90 (B. Apanasov, W. D. Neumann, A. W. Reid, and L. Siebenmann, eds.), Ohio State University Mathematical Research Institute Publications, vol. 1, De Gruyter, Berlin, 1992, pp. 311–314.

[21] L. Paoluzzi, *Finite quotients of type $PSL(2,q)$ of hyperbolic and Coxeter groups*, Comm. Algebra **26** (1998), 759–778.

[22] D. Singerman, *$PSL(2,q)$ as an image of the extended modular group with applications to group actions on surfaces*, Proc. Edinburgh Math. Soc. **30** (1987), 143–151.

[23] H. Stichtenoth, *Algebraic function fields and codes*, Universitext, Spinger–Verlag, Berlin, 1993.

[24] M. Suzuki, *Group theory I*, Grundlehren der Mathematischen Wissenschaften, vol. 247, Spinger–Verlag, Berlin, 1982.

[25] C. B. Thomas, *Finite groups of hyperbolic isometries*, Geometry of low-dimensional manifolds: II. Symplectic manifolds and Jones-Witten theory, Proc. Symp. Durham (1989) (Cambridge) (S. K. Donaldson and C. B. Thomas, eds.), London Math. Soc. Lecture Notes, vol. 151, Cambridge Univ. Press, 1990, pp. 163–176.

[26] W. P. Thurston, *Geometry and topology of 3-manifolds (Lecture*

notes), Princeton Univ. Press, Princeton, 1979, new version 1990: Three-dimensional geometry and topology.

[27] S. N. Voon, *The genus of $SL_2(F_q)$*, Michigan Math. J. **40** (1993), 527–544.

[28] N. J. Wielenberg, *Polyhedral orbifold groups*, Low-dimensional topology and Kleinian groups, Symp. Warwick and Durham (1984) (Cambridge) (D. B. A. Epstein, ed.), London Math. Soc. Lecture Notes, vol. 112, Cambridge Univ. Press, 1986, pp. 313–321.

[29] B. Zimmermann, *Generators and relations for discontinuous groups*, Generators and relations in groups and geometries, (Lucca, 1990) (Dordrecht) (A. Barlotti and al., eds.), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 333, Kluwer Acad. Publ., 1991, pp. 407–436.

[30] B. Zimmermann, *Finite group actions on handlebodies and equivariant Heegaard genus for 3-manifolds*, Topology Appl. **43** (1992), 263–274.

[31] B. Zimmermann, *Hurwitz groups and finite group actions on hyperbolic 3-manifolds*, J. London Math. Soc. **52** (1995), 199–208.

[32] B. Zimmermann, *Genus actions of finite groups on 3-manifolds*, Michigan Math. J. **43** (1996), 593–610.

Received November 24, 2000.