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Three-fold Coverings and Hyperelliptic Manifolds: a Three-Dimensional Version of a Result of Accola

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We dedicate this paper to the memory of our friend and collegue Marco Reni. We discussed the paper when the first and the third authors were visiting Trieste in february and may 2000, and it was Marco who had the decisive idea for the proof of our main result, shortly before his tragical death in june 2000.

Summary. - *It has been proved by Accola that any 3-fold unbranched covering of a Riemann surface of genus two is hyperelliptic (a 2-fold branched covering of the 2-sphere) if the covering is non-regular, and 1-hyperelliptic (a 2-fold branched covering of a torus) if it is regular. In the present paper, we show that the corresponding result holds for closed 3-manifolds when replacing the genus by the Heegaard genus.*

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1. Introduction

A closed Riemann surface is said to be *hyperelliptic* if it admits a conformal involution such that the quotient space of the surface by the action of the involution is the 2-sphere S^2. Similarly, a closed orientable 3-manifold is *hyperelliptic* if it admits an involution such that the quotient space of the manifold by the action of the involution is homeomorphic to the 3-sphere S^3. It is well known that both Riemann surfaces of genus two and 3-manifolds of Heegaard genus two are hyperelliptic, and it is interesting to ask which properties of hyperelliptic surfaces carry over to hyperelliptic 3-manifolds (see [8],[10] for some results on hyperelliptic 3-manifolds, and [7] for hyperelliptic Riemann surfaces).

It is known that any 2-fold unbranched covering of a Riemann surface of genus two is hyperelliptic ([5],[6],[1]), and it is proved in [9] that this result remains true in dimension three: any 2-fold unbranched covering of a closed orientable 3-manifold of (Heegaard) genus two is hyperelliptic. In the present paper, we study the case of 3-fold unbranched coverings. In the case of Riemann surfaces, the following result was obtained in [2, Corollary 1].

THEOREM 1.1. (Accola) *Let $S_4 \to S_2$ be an unbranched 3-fold covering of closed Riemann surfaces of genus four and two, respectively.*

i) If the covering is regular (Galois), then S_4 is 1-hyperelliptic (called elliptic hyperelliptic in [2]: a 2-fold covering of a Riemann surface of genus one, see also [7, p.249]).

ii) If the covering is non-regular, then S_4 is hyperelliptic.

In the present paper we will show that the corresponding result holds also in dimension three. Our main result is as follows.

THEOREM 1.2. *Let W_2 be a 3-manifold of Heegaard genus two, and let W be an unbranched 3-fold covering of W_2.*

i) If the covering is regular then W is hyperelliptic or a 2-fold branched covering of a 3-manifold of Heegaard genus one (that is of a lens space or of $S^2 \times S^1$).

ii) If the covering is non-regular then W is hyperelliptic.

We recall that a (branched or unbranched) covering is *regular* if the group of covering transformations acts transitively on each fiber, so the base space is obtained as the quotient by the action of the covering group; equivalently, the covering corresponds to a normal subgroup of the (orbifold) fundamental group of the base space, and the covering group is isomorphic to the factor group.

So there are some common features between hyperelliptic Riemann surfaces and hyperelliptic 3-manifolds when considering the Heegaard genus instead of the genus. However, in general the situation for 3-manifolds is much more complicated. For example, a hyerelliptic Riemann surface has a unique hyperelliptic involution which lies in the center of its automorphism group. On the other hand, hyperelliptic 3-manifolds may have an arbitrarily high number of non-conjugate hyperelliptic involutions (however it has been shown in [10] that, in the case of hyperbolic 3-manifolds, there is a universal, in fact quite small bound on the number of conjugacy classes of hyperelliptic involutions, independent of the 3-manifold).

2. Preliminary results

In this section we collect some results which are needed for the proof of Theorem 1.2.

We first sketch the proof of a Lemma which can be found in standard books on finite transformation groups. To state this Lemma we recall some terminology about abelian groups.

Any finitely generated abelian group G can be expressed as a direct sum $G = \mathbb{Z}^r \oplus T_2 \oplus T_{2'}$ where T_2 is the subgroup of elements of G whose order is a power of two, and $T_{2'}$ the subgroup of elements of G whose orders are odd. We call T_2 the *2-torsion*, $T_{2'}$ the *odd torsion* and r the *rank* of G. We note that T_2 and $T_{2'}$ are characteristic subgroups of G (but not $\mathbb{Z}^r$, in general).

LEMMA 2.1. *Let N be a compact 3-manifold and u an involution acting on N. Denote by N_u the underlying topological space of the quotient N/u and by $H_1(N)^u$ the subgroup of the elements of the first homology group $H_1(N)$ of N which are fixed by u for the induced action on $H_1(N)$. Then the rank and the odd torsion of the first homology group $H_1(N_u)$ of N_u are equal, respectively, to the rank and the odd torsion of $H_1(N)^u$.*

Proof. The Lemma is standard in finite transformation group theory (see, for example, [3, p. 119-120]). It follows from a transfer argument [3, p. 119, 2.2] that there exist two homomorphisms $\pi_* : H_1(N)^u \to H_1(N_u)$ and $\mu_* : H_1(N_u) \to H_1(N)^u$ such that the two maps

$$\pi_*\mu_* : H_1(N_u) \to H_1(N_u)$$

$$\mu_*\pi_* : H_1(N)^u \to H_1(N)^u$$

are 'multiplication times 2'. The Lemma follows from the remark that 'multiplication times 2' is an isomorphism when restricted to the odd torsion of $H_1(N_u)$ and $H_1(N)^u$. A similar kind of argument proves that the ranks of $H_1(N_u)$ and $H_1(N)^u$ are equal.

This finishes the proof. □

By an n-fold branched covering of a link L in the 3-sphere S^3, we mean an n-fold branched covering of S^3 branched along the link L. We use the notation $M \to S^3(L)$.

By applying the Lemma to the case that N_u is the 3-sphere S^3, we get the following:

PROPOSITION 2.2. *Let N be the 2-fold branched covering of a link L in S^3 and u the covering involution of the covering $N \to S^3(L)$. Then the involution u lifts to any regular unbranched cyclic covering M of odd order of N. Moreover the set of all lifts of u to M generates a dihedral group of order $2n$.*

Proof. It follows from the Lemma and the fact that the underlying topological space of the quotient N/u is S^3, that the subgroup $H_1(N)^u$ of elements of $H_1(N)$ which are fixed by the induced action of u has trivial odd torsion and rank zero. Any regular odd order cyclic covering of N corresponds to an epimorphism ψ of $H_1(N)$ onto a cyclic group of odd order (so the existence of such a covering implies that either the rank r or the odd torsion $T_{2'}$ of $H_1(N)$ is not trivial). We will show that the kernel of ψ is invariant under the action of u.

The kernel of ψ contains the 2-torsion T_2 (which is invariant under the action of u). Suppose that the element $t \in \operatorname{kernel}\psi$ has infinite order. If u maps t to t' then it maps t' to t (because u has order two), and hence tt' is fixed by u. As u has no fixed points of infinite or odd order, it follows easily that tt' is in the 2-torsion T_2 of $H_1(N)$, and consequently also t' is in the kernel of ψ. Finally, since u fixes no non-trivial element of the odd torsion $T_{2'}$ (which is also invariant under the action of u), it is easy to see that u sends each element of $T_{2'}$ to its inverse, and in particular leaves invariant every subgroup of $T_{2'}$. It follows that the kernel of ψ is invariant under the action of u, and consequently u lifts to M.

Let now $\mathbb{Z}_n$ be the covering group of a regular odd order cyclic covering M of N. The covering group $\mathbb{Z}_n$ is isomorphic to the quotient of $H_1(N)$ by the kernel of ψ, and it follows from the above that a lift of u to M, acting on $\mathbb{Z}_n$ by conjugation, sends each element of $\mathbb{Z}_n$ to its inverse. It follows that the lifts of u to M generate a dihedral group $\mathbb{D}_n$ of order $2n$.

This finishes the proof of Proposition 2.2. $\square$

We shall need a result which estimates the Heegaard genus of a 3-manifold occuring as a branched covering of a link L in S^3 in terms of n and the bridge number of the link L.

Let M be a closed orientable 3-manifold. Recall that a pair (H_g, H'_g) of handlebodies of genus g is called a *Heegaard splitting of genus* g of M if $M = H_g \cup H'_g$ and $H_g \cap H'_g = \partial H_g = \partial H'_g$ is a closed orientable surface of genus g. The minimal genus among the genera of all Heegaard splittings of M is called the *Heegaard genus* of M. The 3-sphere S^3 is the only 3-manifold of Heegaard genus zero. The Heegaard genus of M is equal to one if and only if M is a lens space $L(p,q)$ or $S^2 \times S^1$; it is natural to consider these manifolds of Heegaard genus one as 3-dimensional analogous of the Riemann surface of genus one, i.e. the torus T^2.

Recall that an *m-bridge presentation* of a link L in S^3 is a decomposition of the pair (S^3, L) into a union $(B_1, \alpha_1) \cup (B_2, \alpha_2)$ where B_i is a 3-ball and α_i is a set of m arcs which is trivial in B_i, for $i = 1, 2$. We say that L is an m-bridge link if m is the minimal number for which L admits an m-bridge presentation.

The following is a special case of a more general result relating Heegaard splittings and bridge numbers for arbitrary branched coverings of links, see e.g. [4, page 169, Proposition 11.3].

Proposition 2.3. *A 3-manifold which is a 3-fold branched covering of an m-bridge link L in S^3, with branching index two at each component of L, has a Heegaard splitting of genus $m-2$.*

Finally we need also the following (see [11, Sublemma 15.4] or [4, page 135, E 9.5])

Proposition 2.4. *The first homology $H_1(M, \mathbb{Z}_2)$ of the 2-fold branched covering M of an r-component link in S^3 is isomorphic to $(\mathbb{Z}_2)^{r-1}$.*

3. Proof of Theorem 1

i) The regular case

We recall that W_2, being a genus two 3-manifold, admits a hyperelliptic involution, say τ; the underlying topological space of the quotient W_2/τ is S^3, and the singular set (branch set) is a 3-bridge link L (see [13]). Let W be a 3-fold regular unbranched covering of W_2.

By Proposition 1, τ lifts to an involution t of W, and the group generated by t and the covering group $\mathbb{Z}_3$ of the covering $W \to W_2$ is isomorphic to the dihedral group $\mathbb{D}_3$ of order six (we remark that the transfer argument used for the proof of the Lemma and Proposition 2.2 can be avoided; we will give an alternative proof at the end of case i).

The quotient W/t is a non-regular 3-fold covering of $S^3(L)$ (that is of S^3 branched along the link L). To complete the proof it is enough to show that the branching index of this non-regular covering is two at each point of L. Case i) of Theorem 1.2 follows then from Proposition 2.3 and the fact that L is a 3-bridge link.

To compute branching indices we study the fixed points of the involutions of $\mathbb{D}_3$ in W.

First of all the fixed point set of τ in W_2 is the preimage $\tilde{L}$ of L. Each component of $\tilde{L}$ lifts to three distinct components in W which are permuted by the action of the covering group $\mathbb{Z}_3$: in fact, denoting by g a generator of $\mathbb{Z}_3$, the three lifts of τ to W are the three conjugate involutions t, gtg^{-1} and g^2tg^{-2}. Being conjugate, the three involutions have homeomorphic fixed point sets which are permuted by the action of $\mathbb{Z}_3$. The fixed point set of any of the three involutions projects onto $\tilde{L}$ in W_2 and onto L in S^3.

When factoring W by t, the fixed point set of t projects to the branch set of the covering $W \to W/t$, and the fixed point set of both gtg^{-1} and g^2tg^{-2} projects to the branching set of the non-regular covering $W/t \to S^3(L)$, with branching index two at each point. This finishes the proof in case i).

Finally we indicate an alternative proof that τ can be lifted to W.

As above, let τ be the hyperelliptic involution of W_2, and let τ^* be the automorphism of $\pi_1(W_2)$ induces by τ.

As the 3-fold covering $W \to W_2$ is regular, the fundamental group $\pi_1(W)$ is the kernel of some epimorphism $\varphi : \pi_1(W_2) \to \mathbb{Z}_3$. We will show that $\pi_1(W) = \operatorname{Ker}\varphi$ is invariant under the action of τ^*.

The singular set of the orbifold $S^3(L) = W_2/\tau$ is a 3-bridge link L. Therefore, the orbifold fundamental group (see [12]) $\pi_1^{\mathrm{orb}}(S^3(L)) = \langle e_1, e_2, e_3\rangle$ is generated by three involutions e_1, e_2, and e_3, corresponding to meridian loops of L. The fundamental group $\pi_1(W_2)$ is a subgroup of index two in the group $\langle e_1, e_2, e_3\rangle$ consisting of all words of even length in these generators; in particular, $\pi_1(W_2)$ is generated by the elements $A = e_1e_2$ and $B = e_1e_3$. Note that $e_2e_3 = e_2e_1 \cdot e_1e_3 = A^{-1}B$. Up to an inner automorphism, the action of τ^* on $\pi_1(W)$ is given by the following rule:

$$\tau^* : A \to e_1^{-1}Ae_1 = A^{-1}, \qquad B \to e_1^{-1}Be_1 = B^{-1}.$$

We will show that τ^* induces an automorphism of the group $\mathbb{Z}_3 = \langle c \,|\, c^3 = 1\rangle$, with respect to the epimorphism $\varphi : \pi_1(W_2) \to \mathbb{Z}_3$. In fact, up to a choice of notations, there are three possibilities for the action of the epimorphism φ:

$$\begin{array}{lll} (i) & \varphi(A) = c, & \varphi(B) = c, \\ (ii) & \varphi(A) = c, & \varphi(B) = c^{-1}, \\ (iii) & \varphi(A) = c, & \varphi(B) = 1. \end{array}$$

Then it is clear that τ^* induces the automorphism $c \to c^{-1}$ of $\mathbb{Z}_3$, and consequently $\operatorname{Ker}\varphi$ is invariant under the action of the automorphism τ^*. Thus, the involution τ lifts to an involution t of W.

The group of covering transformations G of the regular 6-fold covering $W \to W_2/\tau = S^3(L)$ has order 6 and hence is cyclic or dihedral. As the orbifold fundamental group $\pi_1^{\mathrm{orb}}(S^3(L))$ is generated by three involutions and $\pi_1^{\mathrm{orb}}(S^3(L))/\pi_1(W) \cong G$, also G is generated by three involutions, and therefore G is a dihedral group of order six.

ii) The non-regular case

Since W_2 admits a non-regular 3-fold unbranched covering, its fundamental group $\pi_1(W_2)$ contains a non-normal subgroup H of index three which is the fundamental group of the 3-fold covering W. The action of the elements of $\pi_1(W_2)$ on the three cosets of H in $\pi_1(W_2)$ determines an epimorphism $\phi : \pi_1(W_2) \to \mathbb{S}_3$ of $\pi_1(W_2)$ onto the permutation group $\mathbb{S}_3$ of three letters. The kernel of ϕ is a normal subgroup of $\pi_1(W_2)$ and it corresponds to a regular $\mathbb{S}_3$-covering of W_2 which we denote by M. See Figure 1 for the diagram of coverings constructed in the following, where "3" corresponds to a regular 3-fold covering and "(3)" to a non-regular one.

$$
\begin{tikzcd}
 & M \arrow[dl, "2"'] \arrow[dd, "/\mathbb{S}_3"] \arrow[dr, "3"] & \\
W \arrow[d, "2"'] \arrow[dr, "(3)"] & & N \arrow[d, "2"] \arrow[dl, "2"'] \\
W/\bar{u} \arrow[dr, "(3)", "\text{along } L_1"'] & W_2 \arrow[d, "2"] & N/u \cong S^3 \arrow[dl, "2"', "\text{along } L_0"] \\
 & S^3(L_0 \cup L_1) &
\end{tikzcd}
$$

Figure 1.

The covering group of the covering $M \to W_2$ is isomorphic to $\mathbb{S}_3$ and contains three conjugate involutions; factoring M by the action of one of these involutions we get (a manifold homeomorphic to) W. Factoring M by the action of the unique subgroup $\mathbb{Z}_3$ of $\mathbb{S}_3$, we get a manifold, say N, which is a regular unbranched 2-fold covering of W_2.

We have recalled above in case i) that W_2, being a genus two 3-manifold, admits an involution τ such that the underlying topological space of the quotient W_2/τ is S^3 and its singular set a 3-bridge link L. Since W_2 admits the regular 2-fold unbranched covering N, its homology $H_1(W_2, \mathbb{Z}_2)$ is not trivial. By Proposition 3, the link L has at least two components. As L is a 3-bridge link, it has two or three components, and in particular L is the disjoint union $L = L_0 \cup L_1$ where L_0 is the unknot (a 1-bridge link) and L_1 is a 2-bridge link with one or two components.

By Proposition 2.4, the homology $H_1(W_2, \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In the first case, there is exactly one 2-fold unbranched covering of W_2 which consequently is N, and the involution τ lifts to an involution of N. In the second case, there are exactly three 2-fold unbranched coverings of W_2 which have been described in [9]. It follows from [9, Proof of the Theorem, case ii] that the involution τ lifts to each of these three coverings of W_2, and hence in particular to N.

Thus in any case the involution τ lifts to N and the manifold N is a regular $\mathbb{Z}_2 \times \mathbb{Z}_2$-covering of L: the covering group of the covering $N \to S^3(L)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and contains three distinct involutions. It is proved in [9, Proof of the Theorem] that exactly one of these three involutions, say v, acts freely on N; this is the covering involution of the unbranched covering $N \to W_2$. The fixed point set of a second involution, say u, is the preimage $\tilde{L}_1$ of L_1 in N, and the fixed point set of the product uv is the preimage $\tilde{L}_0$ of L_0 in N. The underlying topological space of the quotient N/u is the 3-sphere S^3 because N/u is the 2-fold branched covering of the trivial knot L_0 in S^3.

The manifold M is a regular unbranched 3-fold cyclic covering of N. By construction, the free involution v of N lifts to M and the group generated by all lifts of v to M is isomorphic to $\mathbb{S}_3 \cong \mathbb{D}_3$. As the underlying topological space of N/u is S^3, by Proposition 2.2 also the involution u lifts to M and the group generated by all lifts of u to M is also isomorphic to $\mathbb{D}_3$. This determines the structure of the group E generated by all lifts of u and v to M. The group E has order twelve since it contains a normal subgroup $\mathbb{Z}_3$ of index four. A Sylow 2-subgroup S_2 of E has order four and contains two involutions conjugate, respectively, to a lift U of u and to a lift V of v, so the Sylow 2-subgroup of E is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and its action on $\mathbb{Z}_3$ is determined by the dihedral actions of U and V on $\mathbb{Z}_3$. It follows that $E \cong \mathcal{S}_3 \oplus \mathbb{Z}_2$, and we can assume that UV has order two.

By construction, the quotient M/V is the 3-fold non-regular unbranched covering W of W_2. The involution U descends to an involution $\bar{u}$ of W; so $W/\bar{u}$ is a 3-fold non-regular covering of L. To complete the proof we will show that the branching index of the non-regular covering $W/\bar{u} \to S^3(L)$ is one on the component L_0 of L and two on (each component of) L_1, so $W/\bar{u}$ is a 3-fold branched covering of the 2-bridge link L_1. Now Proposition 2.3 implies that $W/\bar{u}$ has Heegaard genus zero and hence is the 3-sphere, so W is hyperelliptic.

Like in the regular case in order to compute branching indices we will study the fixed point sets of the involutions.

We have seen above that the preimage $\tilde{L}$ of L in N splits as $\tilde{L}_0 \cup \tilde{L}_1$ where $\tilde{L}_1$ is the fixed point set of u and $\tilde{L}_0$ the fixed point set of uv. When lifting to M the lifts of u and uv behave in different ways.

Denoting by g a generator of the covering group $\mathbb{Z}_3$ of the covering $M \to N$, the lifts of u to M are three distinct involutions U, gUg^{-1} and g^2Ug^{-2} which are conjugated by the action of $\mathbb{Z}_3$. So U, gUg^{-1} and g^2Ug^{-2} have homeomorphic fixed point sets which are permuted by the action of $\mathbb{Z}_3$. Each of these fixed point sets projects onto $\tilde{L}_1$ in N and onto L_1 in S^3.

On the other hand there is exactly one lift UV of uv which is an involution (and has non-empty fixed poiont set) because the lifts of uv to M generate a cyclic group $\mathbb{Z}_6$. So the fixed point set of UV is the full preimage of $\tilde{L}_0$ in M.

When factoring M by the group S_2 generated by U and V, the union of the fixed point set of UV and the fixed point set of U project to the branching set of the covering $M \to W/\bar{u}$. The fixed point sets of gUg^{-1} and g^2Ug^{-2} (which are conjugate to U or V) project to the branching set of the non-regular covering $W/\bar{u} \to S^3(L)$ which is therefore L_1 with branching index two.

As noted above, an application of Proposition 2.3 finishes now the proof of the Theorem.

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