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Degrees of Self-Mappings of Seifert Manifolds with Finite Fundamental Groups

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Summary. - *We calculate all degrees of selfmaps $f\colon M \to M$ of a Seifert manifold M with fundamental group of finite order N, provided that f induces an automorphism of $\pi_1(M)$. The answer is given in terms of N, namely, the possible degrees* degf *form the set* $\{k^2 + \ell N \mid \gcd(k,N) = 1,\ k,\ell \in \mathbb{Z}\}$.

1. Introduction

Given a pair of n-manifolds M_1 and M_2 and an integer d, there arises the basic and natural question whether there exists a continuous mapping $f\colon\ M_1 \to M_2$ of degree d and the problem to give an answer

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using topological invariants. In the dimensions $n = 1$ or 2 the answer is well-known while, contrarily, for dimensions $n \geq 4$ one cannot expect a general answer because there are non-decidable properties for their fundamental groups. The remaining case $n = 3$ is very attractive since the manifolds with a geometric structure seem to be "handable" and – by Thurston Geometrization Conjecture – it is expected that all 3-manifolds can be decomposed by tori and spheres into manifolds admitting a geometric structure, see [10].

In a first and already interesting step one can study the following question:

Question (*) *What are the possible degrees of continuous self-mappings of a closed* 3*-manifold* M *inducing automorphisms of* $\pi_1(M)$*?*

Waldhausen investigated Question (*) in the case when M is irreducible (that is, is not a connected sum of two 3-manifolds both of which are different from the 3-sphere) and sufficiently large – this means that M admits an incompressible embedded surface different from the sphere. According to Waldhausen [13], for such a manifold M all self-mappings from (*) are homotopic to homeomorphisms and, thus, have degrees ± 1. For manifolds with a hyperbolic geometric structure the same property follows from Mostow's rigidity theorem [5]. Exceptions can be expected from Seifert manifolds, that is, closed 3-manifolds admitting a fibration by circles possibly with exceptional fibres. (For a definition see [8, 5.2].) In fact, according to [14, Section 4], a manifold M of the form $M_1 \# \ldots \# M_n$ with geometric M_i admits a self-mapping of degree different from ± 1 inducing an automorphism of $\pi_1(M)$ if and only if each M_i is either a Seifert manifold with a finite fundamental group or $M_i = S^2 \times S^1$.

Let us restrict ourselves to the case of Seifert manifolds with finite fundamental groups, that is, to geometric 3-manifolds the universal covering of which is the 3-sphere, and the covering transformations are motions of the standard metric on S^3, see [11] or [8, 6.2].

Question (**) *What are the possible degrees of continuous self-mappings of a Seifert manifold* M *with finite fundamental group inducing automorphisms of* $\pi_1(M)$*?*

Now the situation is very different from the case of sufficiently large manifolds. Firstly, since the universal cover of M is S^3, M is orientable and a given self-mapping can be altered within a ball such that the degree changes $\bmod |\pi_1(M)|$, for the well known argument see [1, Lemma 2.3]. On the other hand, Plotnik [9] presented a selfmap of the Poincaré homology sphere inducing an automorphism on the fundamental group of order 120, but having degree $49 \not\equiv \pm 1 \bmod 120$. Further, Question (**) was solved in some other partial cases, namely, for lens spaces and for generalized prism manifolds, see [3, 12, 2], respectively.

We give a full answer to Question (**) in terms of $N = |\pi_1(M)|$, see Theorem 2.2: the possible degrees form the set $\{k^2 + \ell N \mid \gcd(k, N) = 1,\ k, \ell \in \mathbb{Z}\}$.
In most cases there are mappings with a degree not congruent to ± 1 modulo N. The table below contains a list of the possible degrees of self-mappings inducing automorphisms of the fundamental groups for the Seifert manifolds with non-trivial finite fundamental groups.

2. Results

Definition 2.1. *Let M be an orientable closed manifold. Let $\mathcal{F}(M)$ be the system of continuous mappings $f\colon M \to M$ fixing the base-point and inducing an automorphism $f_\#\colon \pi_1(M) \to \pi_1(M)$ and let $\mathcal{D}(M) = \{\deg f \mid f \in \mathcal{F}(M)\}$. If $\pi_1(M)$ is of finite order N then we consider the classes $\deg f \bmod N$, $f \in \mathcal{F}(M)$. Let $\mathcal{D}(M) \bmod N$ denote the $\bmod N$ classes of $\mathcal{D}(M)$ and $\|\mathcal{D}(M)\|_N$ their number.*

Suppose that $\pi_i(M) = 0$, $1 < i < \dim M$. If $|\pi_1(M)| = \infty$ then $\mathcal{D}(M)$ is a multiplicative group by [6, Theorem I]; hence, $\mathcal{D}(M)$ is a subgroup of $\mathbb{Z}^* = \{1, -1\}$. If $|\pi_1(M)| = N < \infty$ then $\mathcal{D}(M) \bmod N$ is a multiplicative group by [6, Theorem IIa]; hence, $\mathcal{D}(M) \bmod N \subset \mathbb{Z}_N^*$.

In the following $\varphi(N)$ is the number of coprime classes $\bmod N$, furthermore

$$\psi(N) = \begin{cases} (\text{number of primes dividing } N) - 1 & \text{if } N \equiv 2 \bmod 4, \\ \text{number of primes dividing } N & \text{if } N \equiv 4 \bmod 8 \\ & \text{or } N \text{ odd}, \\ (\text{number of primes dividing } N) + 1 & \text{if } N \equiv 0 \bmod 8. \end{cases}$$

THEOREM 2.2. *Let $M = M^3$ be a 3-dimensional closed Seifert manifold with fundamental group of finite order N. Then M is orientable and*

$$\mathcal{D}(M) = \{k^2 + \ell N \mid \gcd(k, N) = 1, \ k, \ell \in \mathbb{Z}\}. \quad (1)$$

In particular,

$$\|\mathcal{D}(M)\|_N = \frac{\varphi(N)}{2^{\psi(N)}}.$$

Theorem 2.2 is proved in section 4. For some special Seifert manifolds – lens spaces, generalized prism manifolds, and the Poincaré homology sphere – the result follows from [3, 2, 9] (see Cases 4.1, 4.2, 4.5), respectively.

The universal covering of M is S^3, see [8, 6.2]. Then M is orientable and to every mapping $f \in \mathcal{F}(M)$ of degree d and for every $\ell \in \mathbb{Z}$ there is a mapping $g \in \mathcal{F}(M)$ of degree $d+\ell N$; for the argument see [1, Lemma 2.3]. Therefore it suffices to find the mod N-classes of the possible degrees. The finite fundamental groups of Seifert manifolds different from S^3 are listed in the following table together with presentations according to [8, 6.2].

$\pi_1(M)$	presentation	$\mathcal{D}(M)$
$\mathbb{Z}_n$	$= \langle z \mid z^n = 1 \rangle, \ n \geq 2$	see (1)
D^*_{4n}	$= \langle x, y \mid x^2 = (xy)^2 = y^n \rangle$ here $n = 2^q n' \geq 2, \ n'$ odd	see (1)
T^*_{24}	$= \langle x, y \mid x^2 = (xy)^3 = y^3, \ x^4 = 1 \rangle$	$\{1\} + 24\mathbb{Z}$
O^*_{48}	$= \langle x, y \mid x^2 = (xy)^3 = y^4, \ x^4 = 1 \rangle$	$\{1, \ 25\} + 48\mathbb{Z}$
I^*_{120}	$= \langle x, y \mid x^2 = (xy)^3 = y^5, \ x^4 = 1 \rangle$	$\{1, \ 49\} + 120\mathbb{Z}$
$D'_{2^q(2n+1)}$	$= \langle x, y \mid x^{2^q} = y^{2n+1} = 1, \ yxy = x \rangle$ $q \geq 2, \ n \geq 1$	see (1)
$T'_{8 \cdot 3^q}$	$= \langle x, y, z \mid x^2 = (xy)^2 = y^2, \ z^{3^q} = 1, \ zxz^{-1} = y, \ zyz^{-1} = xy \rangle$	$\{1\} + 24\mathbb{Z}$
$\mathbb{Z}_m \times G$	G any group from above, $m \geq 2$, $\gcd(\lvert G \rvert, m) = 1$	see (1)

Table

Let us point out that $\mathcal{D}(M)$ has the form (1) in Theorem 2.2 for every manifold listed in the table, the explicit result is given for the cases with only few classes. In particular, for the group $T'_{8\cdot 3^q}$ of order $N = 8\cdot 3^q$ there are actually 3^{q-1} classes of degrees $\bmod N$. Namely, $\mathcal{D}(M) \bmod 8\cdot 3^q = \{(1+24\ell) \bmod 8\cdot 3^q \,|\, 0 \le \ell < 3^{q-1}\}$.

For a group G and a system of elements $a_1,\dots,a_k \in G$, let us denote by $\langle a_1,\dots,a_k\rangle_n$ the subgroup of G generated by the elements $a_1,\dots,a_k$; n is the order of this subgroup. The groups D^*_{4n}, $D'_{2^q(2n+1)}$ and $T'_{8\cdot 3^q}$ of the table admit the decompositions

$$D^*_{4n} = \langle y^{2^{q+1}}\rangle_{n'} \rtimes \langle x, y^{n'}\rangle_{2^{q+2}} = \mathbb{Z}_{n'} \rtimes D^*_{4\cdot 2^q},$$

$$D'_{2^q(2n+1)} = \langle y\rangle_{2n+1} \rtimes \langle x\rangle_{2^q} = \mathbb{Z}_{2n+1} \rtimes \mathbb{Z}_{2^q},$$

$$T'_{8\cdot 3^q} = \langle x, y\rangle_8 \rtimes \langle z\rangle_{3^q} = Q_8 \rtimes \mathbb{Z}_{3^q}.$$

The group T^*_{24} is isomorphic to $T'_{8\cdot 3}$. For odd n the groups D^*_{4n} and D'_{4n} are isomorphic. The group Q_8 of the quaternions $\pm 1, \pm i, \pm j, \pm k$ is counted as D^*_8. The groups D^*_{4n} and $T^*_{24}, O^*_{48}, I^*_{120}$ are extensions of $\mathbb{Z}_2$ by the dihedral groups and the classical Platonic groups, respectively.

3. Preliminaries

The universal covering of any Seifert manifold M with finite fundamental group is 3-sphere, see [8, 6.2]. Hence, $\pi_2(M) \cong \pi_2(S^3) = \{1\}$. By obstruction theory arguments, P. Olum showed in [7, (4,6)] the first and in [6, Theorem IIa] the second part of the following proposition.

Proposition 3.1. (P. Olum [7, 6]) *Let M be an orientable 3-manifold with finite fundamental group and trivial $\pi_2(M)$. Every endomorphism $\varphi\colon \pi_1(M) \to \pi_1(M)$ is induced by a (basepoint preserving) continuous map $f\colon M \to M$. Furthermore, if g is also a continuous self-mapping of M such that $f_\# = g_\# = \varphi$ then* $\deg f \equiv \deg g \bmod |\pi_1(M)|$.

Seifert manifolds with finite fundamental group are well known, see [8, 6.2]:

Proposition 3.2. *If a Seifert manifold has finite cyclic fundamental group then it is a lens space. If two Seifert manifolds M_1, M_2 have isomorphic fundamental groups of finite order which are not cyclic then M_1 and M_2 are homeomorphic. The non-trivial finite fundamental groups of Seifert manifolds are listed in the table.*

In the following we have to prove some properties of finite fundamental groups which are known for suitable subgroups. The required properties can be obtained from [4, VIII-2] where the arguments are based on facts of homological algebra. We present a detailed more geometric elementary proof.

Lemma 3.3. *Let a finite group G contain a proper subgroup H such that*

$$\gcd(|H|, [G : H]) = 1.$$

If H is normal then H is characteristic, that is, every automorphism of G maps H to itself. If H is a Sylow subgroup (that is, of maximal prime power order) then every automorphism of G maps H to a conjugate of it.

Proof. Let $\alpha \in \operatorname{Aut} G$. Assume that H is normal and put $H_0 = \alpha(H)$. Then $H \cap H_0$ is normal in H_0 and, by the first isomorphism theorem, $H_0/(H \cap H_0)$ can be identified with a subgroup of G/H. However the orders of these two factor groups are relatively prime; hence, $H_0/(H \cap H_0)$ is trivial, that is, $H = H_0$.

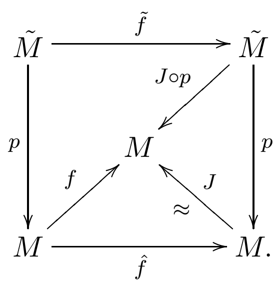
If H is a Sylow subgroup, H_0 is also a Sylow subgroup of the same order and, thus, they are conjugate by the second Sylow theorem. □

Lemma 3.4. *Consider $f \in \mathcal{F}(M)$. Let $G = \pi_1(M)$ contain a subgroup H such that $f_\#(H)$ is conjugate to H. Consider the covering $p\colon \tilde{M} \to M$ corresponding to H, that is, $H = p_\#(\pi_1(\tilde{M}))$. Then there is a map $\tilde{f}\colon \tilde{M} \to \tilde{M}$ and a homeomorphism $J\colon M \to M$ isotopic to id_M such that the following diagram is commutative:*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{M} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle J \circ p} \\ M & \xrightarrow[f]{} & M. \end{array}$$

A consequence is that $\deg \tilde{f} = \deg f$*; hence,* $\deg f \in \mathcal{D}(\tilde{M})$*.*

Proof. By hypothesis, $f_\#(H) = j_g(H)$ for some element $g \in \pi_1(M)$ where j_g is the inner automorphism $j_g(h) = g^{-1}hg$, $h \in \pi_1(M)$. Let $J\colon M \to M$ be a homeomorphism isotopic to the identity with $J_\# = j_g$. (During the isotopy the basepoint is moved along a path belonging to the class g.) Then the composition $\hat{f} = J^{-1} \circ f$ belongs to $\mathcal{F}(M)$ and $\hat{f}_\#(H) = j_g^{-1} \circ f_\#(H) = H$. Since the covering $p\colon \tilde{M} \to M$ corresponds to the subgroup H, that is $p_\#(\pi_1(\tilde{M})) = H$, there is a lifting $\tilde{f}$ of the map $\hat{f}$ to the covering, and we obtain the commutative diagram



□

As above let $j_g\colon \pi_1(M) \to \pi_1(M)$, $g \in \pi_1(M)$ denote the inner automorphism $h \mapsto g^{-1}hg$, $h \in \pi_1(M)$. Proposition 3.1 and the Lemmata 3.3, 3.4 provide the following algorithm for proving Theorem 2.2:

Proposition 3.5. *Let* M *be a Seifert manifold with finite fundamental group of order* $N = N_1 \cdot \ldots \cdot N_m$ *where* $N_1, \ldots, N_m$ *are pairwise coprime. Let* $H_1, \ldots, H_m \subset \pi_1(M)$ *be subgroups of* $\pi_1(M)$ *of orders* $N_1, \ldots, N_m$*, respectively, and* $p_i\colon M_i \to M$ *the corresponding coverings, that is* $p_{i\#}(\pi_1(M_i)) = H_i$*,* $1 \le i \le m$*; we identify* H_i *and* $\pi_1(M_i)$*.*

(a) *If each* H_i *is a normal or a Sylow subgroup then*

$$\mathcal{D}(M) \subset \mathcal{D}(M_1) \cap \ldots \cap \mathcal{D}(M_m) \quad \textit{and}$$

$$\|\mathcal{D}(M)\|_N \le \|\mathcal{D}(M_1)\|_{N_1} \cdot \ldots \cdot \|\mathcal{D}(M_m)\|_{N_m}.$$

(b) *Assume that for each system* $d_1 \in \mathcal{D}(M_1), \dots, d_m \in \mathcal{D}(M_m)$ *there are elements* $g_i \in \pi_1(M)$*, continuous basepoint preserving mappings* $f_i\colon M_i \to M_i$*,* $1 \le i \le m$ *and an automorphism* φ *of* $\pi_1(M)$ *such that for* $1 \le i \le m$*:*

$$j_{g_i}^{-1} \circ \varphi(H_i) = H_i,\ j_{g_i}^{-1} \circ \varphi|_{H_i} = f_{i\#} \quad \textit{and} \quad \deg f_i \equiv d_i \bmod N_i.$$

Then there is a mapping $f\colon M \to M$ *with* $f_\# = \varphi$ *and* $\deg f \equiv d_i \bmod N_i$*,* $1 \le i \le m$*. Hence,*

$$\mathcal{D}(M) \supset \mathcal{D}(M_1) \cap \dots \cap \mathcal{D}(M_m) \quad \textit{and}$$

$$\|\mathcal{D}(M)\|_N \ge \|\mathcal{D}(M_1)\|_{N_1} \cdot \ldots \cdot \|\mathcal{D}(M_m)\|_{N_m}.$$

Proof. (a) Consider $f \in \mathcal{F}(M)$. By the condition on the H_i and Lemma 3.3, the subgroup $f_\#(H_i)$ of $\pi_1(M)$ is conjugate to H_i, $1 \le i \le m$. We apply Lemma 3.4 to the subgroups $H_1, \dots, H_m$ and obtain $\deg f \in \mathcal{D}(M_i)$, $1 \le i \le m$. Hence,

$$\mathcal{D}(M) \subset \mathcal{D}(M_1) \cap \dots \cap \mathcal{D}(M_m).$$

Define

$$\Phi\colon \mathcal{D}(M) \bmod N \to (\mathcal{D}(M_1) \bmod N_1) \times \dots \times (\mathcal{D}(M_m) \bmod N_m),$$

$$d \bmod N \mapsto (d \bmod N_1, \dots, d \bmod N_m).$$

Since $N_1, \dots, N_m$ are pairwise coprime the map Φ is injective; hence

$$\|\mathcal{D}(M)\|_N \le \|\mathcal{D}(M_1)\|_{N_1} \cdot \ldots \cdot \|\mathcal{D}(M_m)\|_{N_m}.$$

(b) Let $d \in \mathcal{D}(M_1) \cap \dots \cap \mathcal{D}(M_m)$. By hypothesis, there is an automorphism φ satisfying the conditions of the hypothesis where now all d_i are equal to d. By the first assertion of Proposition 3.1, there is a continuous map $f\colon M \to M$ with $f_\# = \varphi$. Since $\varphi(H_i)$ is conjugate to H_i, $1 \le i \le m$ we obtain, by Lemma 3.4, mappings $\tilde{f}_i\colon M_i \to M_i$ and homeomorphisms $J_i\colon M \to M$ isotopic to id_M, $1 \le i \le m$ with the following commutative diagrams for $1 \le i \le m$:

$$\begin{array}{ccc} M_i & \xrightarrow{\tilde{f}_i} & M_i \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle J_i \circ p_i} \\ M & \xrightarrow[f]{} & M \end{array} \qquad \begin{array}{ccc} H_i & \xrightarrow{j_{g_i}^{-1} \circ \varphi|_{H_i}} & H_i \\ {\scriptstyle p_{i\#}}\downarrow & & \downarrow{\scriptstyle j_{g_i} \circ p_{i\#}} \\ \pi_1(M) & \xrightarrow[f_\# = \varphi]{} & \pi_1(M). \end{array}$$

Here we use $\tilde{f}_{i\#} = j_{g_i}^{-1} \circ \varphi|_{H_i}$ and $J_{i\#} = j_{g_i}$. By Lemma 3.4,

$$\deg f = \deg \tilde{f}_1 = \ldots = \deg \tilde{f}_m$$

and, by the second assertion of Proposition 3.1,

$$\deg \tilde{f}_i \equiv \deg f_i \bmod N_i, \quad 1 \le i \le m.$$

Hence, $\deg f \equiv d_i \bmod N_i$, $1 \le i \le m$. Since $N_1, \ldots, N_m$ are pairwise coprime, we obtain $\deg f \equiv d \bmod N$, that is, $d \in \mathcal{D}(M)$, and it follows that

$$\mathcal{D}(M) \supset \mathcal{D}(M_1) \cap \ldots \cap \mathcal{D}(M_m). \tag{2}$$

Let $d_i \in \mathcal{D}(M_i)$, $1 \le i \le m$. Since $N_1, \ldots, N_m$ are pairwise coprime, it follows from the Chinese remainder theorem that there exists a natural number d with the following two equivalent properties:

$$d \equiv d_i \bmod N_i,\ 1 \le i \le m \quad \text{and}$$

$$\{d + \ell N | \ell \in \mathbb{Z}\} = \{d_1 + \ell N_1 | \ell \in \mathbb{Z}\} \cap \ldots \cap \{d_m + \ell N_m | \ell \in \mathbb{Z}\}.$$

By (2), $d \in \mathcal{D}(M)$ and $\Phi(d \bmod N) = (d_1 \bmod N_1, \ldots, d_m \bmod N_m)$; now it follows that Φ is surjective and

$$\|\mathcal{D}(M)\|_N \ge \|\mathcal{D}(M_1)\|_{N_1} \cdot \ldots \cdot \|\mathcal{D}(M_m)\|_{N_m}.$$

□

Corollary 3.6. *Under the conditions of Theorem 2.2,*

$$\mathcal{D}(M) \subset \{k^2 + \ell N \mid \gcd(k, N) = 1,\ k, \ell \in \mathbb{Z}\} \quad \textit{and}$$

$$\|\mathcal{D}(M)\|_N \le \|\{k^2 + \ell N \mid \gcd(k, N) = 1,\ k, \ell \in \mathbb{Z}\}\|_N = \frac{\varphi(N)}{2^{\psi(N)}}.$$

Proof. Let $N = p_1^{n_1} \cdot \ldots \cdot p_m^{n_m}$ be the prime decomposition of N. For $1 \le i \le m$, let $H_i \subset \pi_1(M)$ be a p_i-Sylow subgroup (that is, of order $p_i^{n_i}$); by the first Sylow theorem such a subgroup exists. Define $N_i = p_i^{n_i}$.

By Proposition 3.5 (a),

$$\mathcal{D}(M) \subset \mathcal{D}(M_1) \cap \ldots \cap \mathcal{D}(M_m) \quad \text{and}$$

$$\|\mathcal{D}(M)\|_N \leq \|\mathcal{D}(M_1)\|_{N_1} \cdot \ldots \cdot \|\mathcal{D}(M_m)\|_{N_m}$$

where M_i is the covering of M corresponding to the Sylow subgroup H_i. Clearly, each M_i is a Seifert manifold. From the list of finite fundamental groups of Seifert 3-manifolds, see table above, it follows that H_i, $1 \leq i \leq m$ is isomorphic either to a cyclic group $\mathbb{Z}_{N_i}$ or to a generalized dihedral group $D^*_{2^{n_i}}$. For the Seifert manifolds M_i with these fundamental groups the assertion of Theorem 2.2 is already proved in [3] and [12] or [2], respectively, that is,

$$\mathcal{D}(M_i) = \{k^2 + \ell N_i \mid \gcd(k, N_i) = 1,\ k, \ell \in \mathbb{Z}\}.$$

Since $N_1, \ldots, N_m$ are pairwise coprime and $N = N_1 \cdot \ldots \cdot N_m$,

$$\mathcal{D}(M) \subset \bigcap_{i=1}^{m} \mathcal{D}(M_i) = \{k^2 + \ell N \mid \gcd(k, N) = 1,\ k, \ell \in \mathbb{Z}\},$$

and the natural map

$$\left\{k^2 + \ell N \mid \gcd(k, N) = 1,\ k, \ell \in \mathbb{Z}\right\} \bmod N \to \mathcal{D}_1 \times \ldots \times \mathcal{D}_m$$

is bijective, where $\mathcal{D}_i = \mathcal{D}(M_i) \bmod N_i$. Now we use:

$$\|\mathcal{D}_i\|_{N_i} = \|\mathcal{D}_i\|_{p_i^{n_i}} = \frac{\varphi(N_i)}{2^{\psi(N_i)}} = \begin{cases} \varphi(N_i) & ,\quad N_i = 2, \\ \frac{\varphi(N_i)}{2} & ,\quad N_i = 4 \text{ or } N_i \text{ odd}, \\ \frac{\varphi(N_i)}{4} & ,\quad N_i \equiv 0 \bmod 8. \end{cases}$$

This can be proved by looking – for a prime p, $n > 0$, $p \not| k_i$ – at

$$k_1^2 - k_2^2 = (k_1 - k_2)(k_1 + k_2) \equiv 0 \bmod p^n$$

what implies (at least) one of the following congruences:

$$2(k_1 - k_2) \equiv 0 \bmod p^n, \quad 2(k_1 + k_2) \equiv 0 \bmod p^n.$$

Hence, the number of congruence classes $\bmod N$ in

$$\mathcal{D}(M_1) \cap \ldots \cap \mathcal{D}(M_m) = \{k^2 + \ell N \mid \gcd(k, N) = 1,\ k, \ell \in \mathbb{Z}\}$$

is equal to

$$\|\mathcal{D}(M_1)\|_{N_1} \cdot \ldots \cdot \|\mathcal{D}(M_m)\|_{N_m} = \frac{\varphi(N_1)}{2^{\psi(N_1)}} \cdot \ldots \cdot \frac{\varphi(N_m)}{2^{\psi(N_m)}} = \frac{\varphi(N)}{2^{\psi(N)}}.$$

In particular,

$$\|\mathcal{D}(M)\|_N \le \|\{k^2 + \ell N \mid \gcd(k,N) = 1,\ k,\ell \in \mathbb{Z}\}\|_N = \frac{\varphi(N)}{2^{\psi(N)}}.$$

□

Actually, the case "H_i normal" is not needed for the proof of Theorem 2.2, but could be used for the proof of Corollary 3.6 for the cases 4.6, 4.7.

4. Proof of Theorem 2.2

Because of Corollary 3.6 we have the inclusion $\mathcal{D}(M) \subset \{k^2 + \ell N \mid \gcd(k,N) = 1,\ k,\ell \in \mathbb{Z}\}$. Next we will prove "$\supset$" for the different cases quoted in the table.

4.1. Case: $\pi_1(M) = \mathbb{Z}_n$

Let $\pi_1(M) = \mathbb{Z}_n = \langle z \mid z^n = 1\rangle$, $n \ge 2$. Since $\pi_1(M)$ is cyclic, M is a lens space, see Proposition 3.2. Let $f \in \mathcal{F}(M)$. Then $f_\#(z) = z^k$ for some integer k with $\gcd(k,n) = 1$, and conversely, to each such k the endomorphism $\pi_1(M) \to \pi_1(M),\ z \mapsto z^k$ is induced by some $f_k \in \mathcal{F}(M)$, see Proposition 3.1. By [3, Théorème 4.1],

$$\deg f_k \equiv k^2 \bmod n \Longrightarrow \mathcal{D}(M) = \{k^2 + \ell n \mid \gcd(k,n) = 1,\ k,\ell \in \mathbb{Z}\}. \quad (3)$$

4.2. Case: $\pi_1(M) = D^*_{4n}$, in particular $\pi_1(M) = Q_8$

Let $\pi_1(M) = D^*_{4n} = \langle x, y \mid x^2 = (xy)^2 = y^n\rangle$, $n \ge 2$. This case includes the quaternionic group $Q_8 = D^*_8$. By [2, Lemma 4.4], the possible degrees can be realized by mappings $f_k\colon M \to M$ inducing the automorphism $x \mapsto x$, $y \mapsto y^k$ for all k with $\gcd(k,2n) = 1$. From [2, Lemma 4.6] it follows that $\deg f_k \equiv k^2 \bmod 4n \Longrightarrow$

$$\Longrightarrow \mathcal{D}(M) = \{k^2 + 4n\ell \mid \gcd(k,4n) = 1,\ k,\ell \in \mathbb{Z}\}. \quad (4)$$

4.3. Case: $\pi_1(M) = T'_{8\cdot 3^q}$, in particular $\pi_1(M) = T^*_{24}$

Let $\pi_1(M) = T'_{8\cdot 3^q} = \langle x, y, z \,|\, x^2 = (xy)^2 = y^2,\ z^{3^q} = 1,\ zxz^{-1} = y,\ zyz^{-1} = xy\rangle \cong Q_8 \rtimes \mathbb{Z}_{3^q}$, $q \geq 1$. This case includes $\pi_1(M) = T^*_{24}$. Consider the subgroups $H_1 = \langle x, y\rangle \cong Q_8$, $H_2 = \langle z\rangle \cong \mathbb{Z}_{3^q}$ and the covering spaces M_1, M_2 of M corresponding to them. We have already calculated $\mathcal{D}(M_1)$ and $\mathcal{D}(M_2)$, see above. From (3) and (4) we obtain

$$\mathcal{D}(M_1) = \{1 + 8\ell \mid \ell \in \mathbb{Z}\},$$

$$\mathcal{D}(M_2) = \{k^2 + 3^q\ell \mid \gcd(k,3) = 1,\ k, \ell \in \mathbb{Z}\}, \text{ thus}$$

$$\mathcal{D}(M_1) \cap \mathcal{D}(M_2) = \{k^2 + 8\cdot 3^q\ell \mid \gcd(k, 8\cdot 3^q) = 1,\ k, \ell \in \mathbb{Z}\}.$$

Now let us calculate $\mathcal{D}(M_2)$ more explicitly. For $q = 1$ there is only the congruence class $\bmod 3^q$ containing 1. Furthermore, one easily checks that, for any integer ℓ, the number $1 + 3\ell$ fulfills $1 + 3\ell \equiv k^2 \bmod 3^q$ for an appropriate k, $\gcd(k, 3) = 1$. Hence, $\mathcal{D}(M_2) = \{1 + 3\ell \,|\, \ell \in \mathbb{Z}\}$ and

$$\begin{aligned} \mathcal{D}(M_1) \cap \mathcal{D}(M_2) &= \{1 + 8\ell \mid \ell \in \mathbb{Z}\} \cap \{1 + 3\ell \,|\, \ell \in \mathbb{Z}\} \\ &= \{1 + 24\ell \,|\, \ell \in \mathbb{Z}\}. \end{aligned} \tag{5}$$

Now we apply Proposition 3.5 (b). For any integer ℓ take an integer k such that $k^2 \equiv 1 + 24\ell \bmod 8\cdot 3^q$. By

$$\varphi(x) = x, \quad \varphi(y) = y, \quad \varphi(z) = z^k,$$

an automorphism φ of $T'_{8\cdot 3^q}$ is defined. Since $\varphi|_{Q_8}$ is the identity, we can take $f_1 = \mathrm{id}_{M_1}$ and obtain $\deg f_1 = 1 \equiv k^2 \bmod 8$. Since $\varphi(\mathbb{Z}_{3^q}) = \mathbb{Z}_{3^q}$ and $\varphi(z) = z^k$, we obtain, as in Case 4.1, that $\deg f_2 \equiv k^2 \bmod 3^q$ for a map $f_2\colon M_2 \to M_2$ with $f_{2\#} = \varphi|_{\mathbb{Z}_{3^q}}$. Hence, the condition (b) of Proposition 3.5 is fulfilled.

By Proposition 3.5 (b) and (5), there is a map $f\colon M \to M$ with $f_\# = \varphi$ and

$$\deg f \equiv k^2 \bmod 8\cdot 3^q \quad \Longrightarrow \quad \mathcal{D}(M) \supset \{1 + 24\ell \,|\, \ell \in \mathbb{Z}\}.$$

Because of Corollary 3.6 these two sets coincide and moreover

$$\mathcal{D}(M) = \{1 + 24\ell \,|\, \ell \in \mathbb{Z}\} = \{k^2 + 8\cdot 3^q\ell \,|\, \gcd(k, 8\cdot 3^q) = 1,\ k, \ell \in \mathbb{Z}\}.$$

4.4. Case: $\pi_1(M) = O^*_{48}$

Let $\pi_1(M) = O^*_{48} = \langle x, y \mid x^2 = (xy)^3 = y^4,\ x^4 = 1\rangle$. Consider two Sylow subgroups $H_1 = \langle (xy)^2\rangle \cong \mathbb{Z}_3$ and $H_2 = \langle xy^2x^{-1}, y\rangle \cong D^*_{16}$. Let us check the second isomorphism. Consider the epimorphism

$$D^*_{16} = \langle \xi, \eta \mid \xi^2 = (\xi\eta)^2 = \eta^4\rangle \to H_2, \quad \xi \mapsto xy^2x^{-1},\ \eta \mapsto y.$$

Cancelling the center of both groups (namely η^4 and y^4, resp.) we obtain an epimorphism from the dihedral group D_8 to a subgroup of the octahedral group O_{24} which is a Sylow subgroup of order 8 and corresponds to the symmetries of the octahedron which preserve a pair of antipodal vertices. A consequence is that the above epimorphism $D^*_{16} \to H_2$ is an isomorphism.

Next we apply Proposition 3.5 (b). The subgroups H_1 and H_2 have coprime orders and their product is the order of the group O^*_{48}.

From the previous cases it follows that

$$\begin{aligned} \mathcal{D}(M_1) \cap \mathcal{D}(M_2) &= \{k^2 + 48\ell \mid \gcd(k, 48) = 1,\ k, \ell \in \mathbb{Z}\} \\ &= \{1 + 48\ell,\ 25 + 48m \mid \ell, m \in \mathbb{Z}\}. \end{aligned} \tag{6}$$

That there is a map $M \to M$ of degree 1 is evident. Hence, in order to apply Proposition 3.5 (b), we have only to show that there is a map $M \to M$ of degree 25 mod 48. The group $\mathrm{Out}(O^*_{48}) \cong \mathbb{Z}_2$ is generated by the class of the automorphism φ with

$$\varphi(x) = x^{-1} = x^2x, \quad \varphi(y) = y^{-3} = x^2y,$$

see [4, VIII-2(v)]. Both subgroups H_1 and H_2 are invariant under φ and $\varphi|_{H_1} = \mathrm{id}_{H_1}$, $\varphi|_{H_2}(xy^2x^{-1}) = xy^2x^{-1}$, $\varphi|_{H_2}(y) = y^5$. Now we can take $f_1 = \mathrm{id}_{M_1}$; hence, $\deg f_1 = 1$. By Case 4.2, $\deg f_2 \equiv 5^2 \bmod 16$.

By Proposition 3.5 (b) and (6), there is a map $f\colon M \to M$ with $f_\# = \varphi$ and

$$\deg f \equiv 1 \bmod 3,\ \deg f \equiv 5^2 \bmod 16 \quad \Longleftrightarrow \quad \deg f \equiv 25 \bmod 48;$$

hence,

$$\begin{aligned} \mathcal{D}(M) &\supset \{1 + 48\ell,\ 25 + 48m \mid \ell, m \in \mathbb{Z}\} \\ &= \{k^2 + 48\ell \mid \gcd(k, 48) = 1,\ k, \ell \in \mathbb{Z}\}. \end{aligned}$$

Again, due to Corollary 3.6, we have equality.

Let us remark that $\deg f \equiv 25 \bmod 48$ *for every self-map* f *of* M *inducing a non-inner automorphism of* $\pi_1(M)$.

4.5. Case: $\pi_1(M) = I^*_{120}$

Let $\pi_1(M) = I^*_{120} = \langle x, y \mid x^2 = (xy)^3 = y^5,\ x^4 = 1\rangle$. The Seifert manifold M with this fundamental group is called the *Poincaré homology sphere*. The order of I^*_{120} has the prime decomposition $|I^*_{120}| = 2^3 \cdot 3 \cdot 5$. There is only one square mod 8, coprime with 8, and the same is true for 3, but two squares mod 5, prime to 5. Therefore there are two squares mod 120, prime to 120, namely the classes of 1 and 49. Now we apply Proposition 3.5 (b).

We use the notation of Proposition 3.5. The degree $d = 1$ is obtained by the identity. Next we show that $d = 49$ is also the degree of some mapping $M \to M$ inducing an automorphism of $\pi_1(M)$.

The automorphism group of I^*_{120} contains 120 elements [4, VIII-2(vi)], among them 60 inner automorphisms. The group $\mathrm{Out}(I^*_{120})$ is generated by the class of the automorphism $\varphi\colon I^*_{120} \to I^*_{120}$ defined by

$$\varphi(x) = xyx^{-1}y^{-1}x^{-1},\ \varphi(y) = y^7 = x^2y^2.$$

In fact, the above equalities define a homomorphism as follows from the formula

$$\varphi(x)\varphi(y) = y^{-2}x^{-1}y^{-1} \cdot (xy) \cdot yxy^2$$

which is obtained as follows: put $v = y^{-2}x^{-1}y^{-1} \cdot (xy) \cdot yxy^2$,

$$\begin{aligned}\varphi(x)\varphi(y) \cdot v^{-1} &= xyx^{-1}y^{-1}x^{-1}x^2y^2 \ \cdot\ y^{-2}x^{-1}y^{-1}y^{-1}x^{-1}yxy^2 \\ &= xyx^{-1}y^{-3}x^{-1}yxy^2 = xyxy \cdot y^{-5} \cdot yxyxy\ y \\ &= x^2y^{-1}x^{-1} \cdot y^{-5} \cdot y^5x^{-1}\ y = 1.\end{aligned}$$

The image of φ contains y, $y^5 = x^2$ and, since $xyx^{-1}y^{-1}x^{-1} = xy \cdot yxyx^{-2}$, also $xy^2x = xy^7x^{-1}$ as well as $xyx = y^{-1}x^{-1}y^{-1}y^5$; hence, it contains also x, and this proves that φ is surjective. Thus φ is an automorphism.

Take Sylow subgroups H_1 of order 8, H_2 of order 3 (they exist by the first Sylow theorem) and $H_3 = \langle y^2\rangle$ of order 5. Their orders are pairwise coprime and their product is $8 \cdot 3 \cdot 5 = 120$, the order of the whole group $\pi_1(M)$.

By Lemma 3.3 (the second Sylow theorem), $\varphi(H_i)$ is conjugate to H_i, $i = 1, 2, 3$. Let f_i be the corresponding self-mapping of the covering manifold M_i (it exists due to Proposition 3.1), $i = 1, 2$. Since 1 is the only coprime square mod 8 and mod 3, it follows from Corollary 3.6 that $\deg f_1 \equiv 1 \bmod 8$, $\deg f_2 \equiv 1 \bmod 3$. Next, the cyclic subgroup H_3 is invariant under φ and $\varphi|_{H_3}(y^2) = y^4$. By Case 4.1, the automorphism $\varphi|_{H_3}$ is realized by a self-mapping $f_3 \colon M_3 \to M_3$ of degree $\deg f_3 \equiv 2^2 \bmod 5$. By Proposition 3.5 (b), there is a mapping $f \colon M \to M$ with $f_\# = \varphi$ and

$$\deg f \equiv 1 \bmod 8,\ \deg f \equiv 1 \bmod 3,\ \deg f \equiv 2^2 \bmod 5 \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad \deg f \equiv 49 \bmod 120;$$

hence,

$$\mathcal{D}(M) \supset \{1 + 120\ell,\ 49 + 120m \mid \ell, m \in \mathbb{Z}\} = \\ = \{k^2 + 120\ell \mid \gcd(k, 120) = 1,\ k, \ell \in \mathbb{Z}\}.$$

Again, by Corollary 3.6, these sets coincide.

Let us formulate the consequence that φ is not an inner automorphism. Furthermore we also obtained the result of Plotnik [9] that there is an automorphism of the fundamental group I^*_{120} of the Poincaré homology sphere which is induced by a self-mapping of degree 49; in fact, this is true for every automorphism which is not an inner automorphism.

4.6. Case: $\pi_1(M) = D'_{2^q(2n+1)}$

Let $\pi_1(M) = D'_{2^q(2n+1)} = \langle x, y \mid x^{2^q} = y^{2n+1} = 1,\ yxy = x\rangle \cong \mathbb{Z}_{2n+1} \rtimes \mathbb{Z}_{2^q}$ where $n \geq 1$, $q \geq 2$. Consider the subgroups $H_1 = \langle x\rangle \cong \mathbb{Z}_{2n+1}$, $H_2 = \langle y\rangle \cong \mathbb{Z}_{2^q}$ and the covering spaces M_1, M_2 of M corresponding to these subgroups. These subgroups have coprime orders, and the product of these orders coincides with the order of the whole group. As above, we apply now Proposition 3.5 (b).

Both subgroups H_1 and H_2 are cyclic. By the arguments in Case 4.3, we obtain

$$\mathcal{D}(M_1)\cap\mathcal{D}(M_2) = \\ = \{k^2+2^q(2n+1)\ell \mid \gcd(k,2^q(2n+1))=1,\ k,\ell\in\mathbb{Z}\}. \quad (7)$$

Take any integer k with $\gcd(k,2^q(2n+1))=1$. One easily checks that there is an automorphism φ of $D'_{2^q(2n+1)}$ with

$$\varphi(x)=x^k,\quad \varphi(y)=y^k.$$

As in Case 4.1, there is a mapping $f_i\colon M_i\to M_i$ on the lens space M_i with $f_{i\#}=\varphi|_{H_i}$ and $\deg f_i\equiv k^2 \bmod N_i$, $i=1,2$ where $N_i=|H_i|$, that is, $N_1=2n+1$, $N_2=2^q$.

By Proposition 3.5 (b), there is a map $f\colon M\to M$ with

$$f_\#=\varphi \quad\text{and}\quad \deg f\equiv k^2 \bmod 2^q(2n+1)$$

and we obtain from (7)

$$\mathcal{D}(M)\supset\mathcal{D}(M_1)\cap\mathcal{D}(M_2) = \\ = \{k^2+2^q(2n+1)\ell \mid \gcd(k,2^q(2n+1))=1,\ k,\ell\in\mathbb{Z}\}.$$

By Corollary 3.6, the sets coincide.

4.7. Case: $\pi_1(M)=\mathbb{Z}_m\times G$

Let $\pi_1(M)=\mathbb{Z}_m\times G$ where G is any group from above with $\gcd(|G|,m)=1$. The subgroups $\mathbb{Z}_m$ and G satisfy all conditions of Proposition 3.5 (b) and, thus,

$$\mathcal{D}(M)\supset\mathcal{D}(M_1)\cap\mathcal{D}(M_2).$$

These two sets coincide by Corollary 3.6. From the above calculations of the sets $\mathcal{D}(M_1)$ and $\mathcal{D}(M_2)$ we obtain

$$\mathcal{D}(M)=\mathcal{D}(M_1)\cap\mathcal{D}(M_2)=\{k^2+|G|m\ell \mid \gcd(k,|G|m)=1,\ k,\ell\in\mathbb{Z}\}.$$

4.8. End of the proof

Using Propositions 3.1, 3.2, we obtain the assertion (1) of Theorem 2.2 from the cases 4.1 – 4.7. Hence, by Corollary 3.6, $\|\mathcal{D}(M)\|_N=\frac{\varphi(N)}{2^{\psi(N)}}$. This finishes the proof of Theorem 2.2.

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