

## Towers of Covers of Hyperbolic 3-Manifolds

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SUMMARY. - *Our main result is the construction of an infinite tower of covers of hyperbolic integral homology spheres.*

### 1. Introduction

The following conjectures are well-known and important in 3-manifold theory.

CONJECTURES 1.1 - 1.3. *Let  $M$  be a closed orientable hyperbolic 3-manifold. Then  $M$  has:*

- 1.1. *Infinite virtual first Betti number*
- 1.2. *A finite covering  $\tilde{M}$  with positive first Betti number*
- 1.3. *A finite covering  $\tilde{M}$  which is Haken*

We consider a "local version" of these conjectures by restricting attention to towers of covers of the given hyperbolic 3-manifold. In this context, we have the following questions:

QUESTIONS 1.4 - 1.6. *Given a closed orientable hyperbolic 3-manifold  $M$  and an infinite tower of (finite-sheeted) covers  $\dots \rightarrow M_n \rightarrow M_{n-1} \rightarrow \dots \rightarrow M_1 \rightarrow M_0 = M$ :*

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- 1.4. *For any integer  $b > 0$  is there an integer  $n = n(b)$  such that the first Betti number  $b_1(M_n) \geq b$ ?*
- 1.5. *Is there an integer  $n$  such that  $b_1(M_n) \geq 1$ ?*
- 1.6. *Is there an integer  $n$  such that  $M_n$  is Haken?*

In this note, we give a negative answer to Questions 1.4 and 1.5 by constructing the first example (to our knowledge) of an infinite tower of covers with all the  $M_n$  hyperbolic integral homology spheres (Proposition 3.1).

Using the same techniques, we construct a closed orientable hyperbolic 3-manifold  $M$  with the property that, for any tower of abelian covers starting with  $M$ ,  $b_1(M_n) = 3$  for all  $n$  (Proposition 3.3).

While we suspect that the answer to Question 1.6 is 'no', we are unable to produce an infinite tower of covers with all the  $M_n$  hyperbolic non-Haken 3-manifolds. However we can construct such towers of arbitrary finite length (Proposition 4.1).

Concerning the relation between Betti number and injectivity radius for covering towers of hyperbolic manifolds, we mention the following question due to D. Cooper:

**QUESTION 1.7.** *Given an infinite tower of covers of hyperbolic rational homology spheres, is the injectivity radius of every  $M_n$  in the tower always uniformly bounded above?*

Note that a positive answer to Question 1.7 implies that Conjectures 1.1 and 1.2 are true. Indeed, one can use the residual finiteness of hyperbolic 3-manifold groups to construct, for any hyperbolic manifold, a tower of covers in which the injectivity radius of the  $M_n$  tends to infinity with  $n$  (see [4, 3.58] for further details).

Although the answer to Question 1.7 is probably 'no', in all of our towers the injectivity radius is indeed uniformly bounded above.

## 2. Totally null-homotopic knots and coverings

In this section we give definitions and the two key propositions used to construct our towers of hyperbolic manifolds (see also [1, §4]).

DEFINITION 2.1. *Let  $M$  be a closed orientable 3-manifold. A knot  $k \subset M$  is totally null-homotopic if it bounds a singular disk  $D \subset M$  whose regular neighborhood  $N(D)$  is null-homotopic in  $M$  (i.e.  $i_*(\pi_1 N(D)) = \{1\}$ , where  $i_* : \pi_1 N(D) \rightarrow \pi_1 M$  is induced by the inclusion of  $N(D)$  in  $M$ ).*

REMARK 2.2. *If  $k \subset M$  is null-homotopic, there is a preferred meridian-longitude coordinate system  $(m, l)$  on  $\partial(M - \text{int}N(k))$  such that  $l$  is null-homologous in  $M - \text{int}N(k)$ . Therefore any simple closed curve on  $\partial(M - \text{int}N(k))$  has a unique slope  $(p, q)$  in this system, where  $p$  and  $q$  are coprime.*

PROPOSITION 2.3. *Every closed orientable 3-manifold contains a totally null-homotopic hyperbolic knot.*

This proposition is proved in [1, Prop. 4.2], using results of R. Myers [6]. It allows us to transform any tower of coverings of 3-manifolds into a tower coverings of hyperbolic 3-manifolds with the same homology groups and covering automorphism groups in the case of regular covers. More precisely, we have the following:

PROPOSITION 2.4. *Given a tower of (finite-sheeted) covers of closed orientable 3-manifolds  $\dots \rightarrow N_n \rightarrow N_{n-1} \rightarrow \dots \rightarrow N_0$ , there is a tower of finite coverings of closed orientable hyperbolic 3-manifolds,  $\dots \rightarrow M_n \rightarrow M_{n-1} \rightarrow \dots \rightarrow M_0$  such that:*

- i) *For each integer  $n$ , there is a homology equivalence  $f_n : M_n \rightarrow N_n$ .*
- ii) *The covering map  $p_n^* : M_n \rightarrow M_{n-1}$  is the pull-back of the covering map  $p_n : N_n \rightarrow N_{n-1}$  by the homology equivalence  $f_n : M_n \rightarrow N_n$ .*

As a corollary of Proposition 2.4 we have:

COROLLARY 2.5. *Any finite orientation preserving group acting freely on a closed orientable 3-manifold acts freely and isometrically on some closed orientable hyperbolic 3-manifold of the same homology type.*

*Proof of Proposition 2.4.* Let

$$\dots \rightarrow N_n \rightarrow N_{n-1} \rightarrow N_1 \rightarrow N_0. \quad (1)$$

be a tower of covers of closed orientable 3-manifolds. By Proposition 2.3, there exists a totally null-homotopic hyperbolic knot  $k$  in  $N_0$ . We choose a simple closed curve  $\alpha$  in  $\partial N(k)$  of slope  $(1, m)$ , such that the surgery manifold  $M_0 = N_0(k, \alpha)$  is hyperbolic. There are infinitely many such curves by Thurston's hyperbolic surgery Theorem [7].

Since  $k$  is null-homotopic in  $N_0$ , it follows (from [1, Prop. 3.2]) that there is a degree one map  $f_0 : M_0 \rightarrow N_0$  which is also an integer homology equivalence due to the choice of  $(1, m)$  surgery slope.

The fact that  $k$  is totally null-homotopic in  $N_0$  implies that, for the covering  $p_1 : N_1 \rightarrow N_0$ , each component of  $p_1^{-1}(k)$  is mapped homeomorphically to  $k$ , hence  $p_1$  induces a covering:

$$p_1^* : M_1 = N_1(p_1^{-1}(k), p_1^{-1}(\alpha)) \rightarrow M_0 = N_0(k, \alpha) \quad (2)$$

which is the pull-back of the covering  $p_1 : N_1 \rightarrow N_0$  by the homology equivalence  $f_0 : M_0 \rightarrow N_0$ .

Now  $f_0$  lifts to a degree-one map  $f_1 : M_1 = N_1(p_1^{-1}(k), p_1^{-1}(\alpha)) \rightarrow N_1$ , which is still an integer homology equivalence because each component of  $p_1^{-1}(k)$  bounds a singular disk in  $N_1$  and all these singular disks are mutually disjoint (see [1, Lemma 4.3]).

By repeating this construction for  $p_2, p_3$  and so on, we get the desired induced tower of coverings of hyperbolic 3-manifolds :

$$\begin{aligned} \dots \rightarrow N_n(q_n^{-1}(k), q_n^{-1}(\alpha)) \rightarrow N_{n-1}(q_{n-1}^{-1}(k), q_{n-1}^{-1}(\alpha)) \rightarrow \\ \dots \rightarrow N_1(p_1^{-1}(k), p_1^{-1}(\alpha)) \rightarrow N_0(k, \alpha) \end{aligned} \quad (3)$$

where  $q_n = p_n \circ p_{n-1} \circ \dots \circ p_1$ . This completes the proof of Proposition 2.4.  $\square$

**REMARK 2.6.** *Let  $k_\alpha \subset M_0 = N_0(k, \alpha)$  be the core of the surgery solid torus. Then  $k_\alpha$  is lifted isometrically to any manifold in the tower of coverings (3), therefore the injectivity radius of the hyperbolic 3-manifolds in the tower is uniformly bounded by the length of  $k_\alpha$ .*

### 3. Towers of hyperbolic integral homology spheres

In this section we prove the following:

**PROPOSITION 3.1.** *There exists an infinite tower of regular covers of hyperbolic integral homology spheres.*

*Proof.* By Proposition 2.4, it suffices to construct an infinite tower of regular covers of non-hyperbolic integral homology 3-spheres.

Let  $p : S^3 \rightarrow \Sigma^3$  denote the 120-sheeted regular covering from  $S^3$  to the Poincare homology sphere  $\Sigma^3$ , and  $N_0 = \Sigma^3 \# \Sigma^3$  be the connected sum of two Poincare homology 3-spheres. Starting with  $N_0$ , we construct an infinite tower of regular covers of (non-prime) homology 3-spheres as follows.

The first stage in the tower is given by

$$p_1 : N_1 = S^3 \# \{120 \text{ copies of } \Sigma^3\} \rightarrow N_0 = \Sigma^3 \# \Sigma^3 \quad (4)$$

where the restriction of  $p_1$  to  $S^3$  is the covering  $p$  which acts transitively on the 120 two-spheres of connected sums (see also [5]).

Noting that  $N_1$  is homeomorphic to the connect sum of 120 copies of  $\Sigma^3$ , we choose one of the  $\Sigma^3$  in  $N_1$  and construct the covering

$$\begin{aligned} p_2 : N_2 &= S^3 \# \{120 \times 119 \text{ copies of } \Sigma^3\} \rightarrow N_1 \\ &= \Sigma^3 \# \{119 \text{ copies of } \Sigma^3\} \end{aligned} \quad (5)$$

where the restriction of  $p_2$  to  $S^3$  is the covering  $p$  which exchanges 120 sets of 119 two-spheres of connected sums.

This process can be infinitely repeated: setting  $r_0 = 2$ , and defining inductively  $r_n = 120(r_{n-1} - 1)$ , the cover  $p_n$  has the form

$$\begin{aligned} p_n : M_n &= S^3 \# \{r_n \text{ copies of } \Sigma^3\} \rightarrow M_{n-1} \\ &= \Sigma^3 \# \{(r_{n-1} - 1) \text{ copies of } \Sigma^3\} \end{aligned} \quad (6)$$

Hence we get an infinite tower of regular covers between non-prime homology spheres and applying Proposition 2.4 to this tower yields the desired infinite tower of coverings of hyperbolic homology spheres.  $\square$

REMARK 3.2. *Each homology sphere in the first tower of covers has a fundamental group that is a non-trivial free product. It follows that each hyperbolic homology sphere in the tower of covers obtained by Proposition 2.4 has a fundamental group which is a non-trivial amalgamated product and hence is Haken. Moreover by Remark 2.6 there is a uniform upper bound for the injectivity radius of all the hyperbolic homology 3-spheres in this tower of covers.*

The arguments used in the proof of Proposition 2.4 also allow us to show:

PROPOSITION 3.3. *There exists a closed orientable hyperbolic 3-manifold with first Betti number  $b_1(M) = 3$  such that in any tower of abelian covers starting at  $M$ , the Betti numbers remain constant and equal to 3.*

*Proof.* Consider the 3-torus  $T^3$ . Since any finite cover of  $T^3$  is abelian and homeomorphic to  $T^3$ , any tower of abelian covers starting at  $N_0 = T^3$  consists of manifolds homeomorphic to  $T^3$ .

Now, as in the proof of Proposition 2.4, we choose a totally null-homotopic hyperbolic knot  $k$  in  $T^3$  and a simple closed curve  $\alpha$  in  $\partial N(k)$  of slope  $(1, m)$ , such that the surgery manifold  $M = T^3(k, \alpha)$  is hyperbolic. Since  $k$  is totally null-homotopic in  $T^3$ , there exists a degree one map  $f : M \rightarrow T^3$  which is an integral homology equivalence. In particular  $H_1(M; \mathbb{Z}) = \mathbb{Z}^3$  and the core  $k_\alpha$  of the surgery solid torus is null-homologous in  $M$ . It follows that any epimorphism from  $\pi_1(M)$  to a finite abelian group factors through the epimorphism  $f_* : \pi_1(M) \rightarrow \pi_1(T^3)$ . Therefore any abelian covering of  $M$  is the pull-back by  $f$  of an abelian covering of  $T^3$ .

By repeating this construction we show, as in the proof of Proposition 2.4, that any tower of abelian covers starting at  $M_0 = M$  is the pull-back by  $f$  of a tower of abelian covers starting at  $N_0 = T^3$ . In particular all the hyperbolic 3-manifolds in these towers of covers are homology equivalent to the 3-torus  $T^3$ , and hence have first Betti number equal to 3.  $\square$

#### 4. Towers of hyperbolic non-Haken 3-manifolds

Recall that a closed orientable 3-manifold is Haken if it is irreducible and it contains an incompressible orientable embedded surface.

As remarked above, the hyperbolic integral homology 3-spheres in the infinite tower of covers provided in Proposition 3.1 are all Haken manifolds. We are currently unable to construct an infinite tower of covers of non-Haken hyperbolic 3-manifolds. However we can construct such towers of covers of arbitrary finite length.

**PROPOSITION 4.1.** *For any  $n$ , there exists a tower of length  $n$  of covers of hyperbolic non-Haken 3-manifolds.*

*Proof.* Let  $E_0$  be the complement of the figure eight knot. It is well known that  $E_0$  admits a complete hyperbolic structure and that it fibers over the circle with fiber a once-punctured torus and pseudo-Anosov monodromy  $\phi$ . Let  $(m_0, l_0)$  be the preferred meridian-longitude system on  $\partial E_0$ , where  $l_0$  is the boundary of a punctured torus fiber.

Let  $p_n : E_n \rightarrow E_0$  be the cyclic cover of degree  $2^n$ . Then  $E_n$  is a once-punctured torus bundle with monodromy  $\phi^{2^n}$ . It follows that  $p_n^{-1}(l_0)$  has  $2^n$  components and  $p_n^{-1}(m_0)$  is connected. Let  $l_n$  be a component of  $p_n^{-1}(l_0)$  and  $m_n = p_n^{-1}(m_0)$ . Since  $l_n$  bounds a once-punctured torus fiber of the fibration of  $E_n$ , it is null homologous in  $E_n$  and  $(m_n, l_n)$  provides a coordinate system on  $\partial E_n$ . Now each closed simple curve on  $\partial E_n$  has a unique slope given by a pair of coprime integers  $(p, q)$ . We denote by  $E_n(p, q)$  the closed 3-manifold obtained from  $E_n$  by Dehn filling  $\partial E_n$  along the slope  $(p, q)$ .

It is a direct geometric observation that for a closed curve  $\alpha \subset \partial E_0$  of slope  $(2^n r, 1)$ ,  $r \geq 1$ , the preimage  $p_n^{-1}(\alpha)$  has  $2^n$  components of slope  $(r, 1)$ . Hence, for  $k \geq n$  and  $r = 2^{k-n}$ , one sees that the preimage  $p_n^{-1}(\alpha)$  of a closed curve  $\alpha \subset \partial E_0$  of slope  $(2^k, 1)$  has  $2^n$  components of slope  $(2^{k-n}, 1)$ . It follows that the sequence of coverings

$$E_n \rightarrow E_{n-1} \rightarrow \dots \rightarrow E_2 \rightarrow E_1 \tag{7}$$

extends to a sequence of coverings between closed 3-manifolds

$$E_n(2^{k-n}, 1) \rightarrow E_{n-1}(2^{k-n+1}, 1) \rightarrow \dots \rightarrow E_1(2^{k-1}, 1) \rightarrow E_0(2^k, 1). \quad (8)$$

We will make use of three facts:

**Fact 1** The exceptional (i.e. non-hyperbolic) Dehn fillings on the figure eight knot exterior  $E_0$  are known (see [7, Ch.4]). In particular for  $k \geq 3$  the Dehn filled 3-manifold  $E(2^k, 1)$  is hyperbolic, so all the 3-manifolds in the tower (8) are hyperbolic in this case.

**Fact 2** Since each  $E_n$  is a once-punctured torus bundle, there are no closed orientable incompressible embedded surfaces in  $E_n$  and only finitely many boundary-slopes on  $\partial E_n$  (i.e. slopes which are realized by a boundary component of an incompressible orientable surface in  $E_n$ ) (see [3] or [2]).

**Fact 3** If  $(p, q)$  is not a boundary-slope on  $\partial E_n$ , then  $E_n(p, q)$  is irreducible and non-Haken.

Now let

$$\mathcal{S}_n = \{(p, q) \mid (p, q) \text{ is a boundary-slope on } \partial E_i, i = 0, \dots, n\} \quad (9)$$

By Fact 2 above,  $\mathcal{S}_n$  is a finite set for any given  $n$ . In particular there is an integer  $P_n$  such that  $p < P_n$  if  $(p, q) \in \mathcal{S}_n$ .

Given any integer  $n > 0$ , we choose  $k \geq 3$  large enough so that  $2^{k-n} > P_n$ . Then all the 3-manifolds in the tower of coverings (8) of length  $n$  will be non-Haken hyperbolic 3-manifolds by Facts 1 and 3 above. This proves Proposition 4.1.  $\square$

*REMARK 4.2. By considering the cores of the Dehn fillings in the tower of covers (8), one can show that the injectivity radius of the hyperbolic 3-manifolds is uniformly bounded above.*

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