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Continuous Dependence Results for an Inverse Problem in the Theory of Combustion of Materials with Memory

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Summary. - *We prove theorems of continuous dependence on the data for both direct and inverse problems for semilinear integrodifferential equations. Such results are applied to the specific case of the combustion of a material with memory.*

1. Introduction

This paper is a natural continuation of [2] in which we have proved existence and uniqueness results for an inverse problem in the theory of combustion.
In fact, in [2] we have developed the theory for general integrodifferential semilinear systems of type (6)-(9), whose unknowns have values in a Banach space X. The combustion of a material with memory is a particular case of such system. We use as fundamental tool the analytic semigroup theory, and, recalling that the first abstract approach to parabolic linear identification problems using semigroup theory has been done in [4], the results in [2] generalize the ones in [4]. More precisely, in [2] we have proved that, under suitable

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assumptions on the data, the abstract semilinear non local inverse problem (6)–(9) admits a unique solution in the space Hölder continuous functions. Then we have applied the previous theorem to the specific case of combustion (1)–(5) proving existence and uniqueness of a solution; but we have not consider the continuous dependence on the data for the abstract as well as for the concrete case. In this note we are going to complete [2] giving the continuous dependence results for both direct and inverse problems in the abstract case as well as in the concrete case for the combustion system. Moreover, since the equations governing the evolution of the temperature and the density of a material with memory are not very well known, in the form we consider in [2], in section 6 of this note, we give the physical deduction of equations (1)–(2).

The considerations reported in section 6 leads to equations governing the evolution of both the temperature u and the density ρ of a material with a thermal memory, which is represented by the convolution kernel h. So we are now in the position to state our direct and inverse problems. Let Ω be an open bounded set in $\mathbf{R}^n$ ($n \in \mathbf{N}\backslash\{0\}$) with a smooth boundary $\partial\Omega$ and consider the following system:

$$\begin{aligned} D_t u(t,x) &= \operatorname{div}\left(d_1(x)\nabla u(t,x)\right) \\ &+ \int_0^t h(t-s)\operatorname{div}\left(d_1(x)\nabla u(s,x)\right) ds \\ &+ f(u(t,x),\rho(t,x)), \quad (t,x) \in [0,T]\times\Omega, \end{aligned} \tag{1}$$

$$\begin{aligned} D_t \rho(t,x) &= \operatorname{div}\left(d_2(x)\nabla \rho(t,x)\right) \\ &+ g(u(t,x),\rho(t,x)), \qquad (t,x) \in [0,T]\times\Omega, \end{aligned} \tag{2}$$

$$u(0,x) = u_0(x), \qquad \rho(0,x) = \rho_0(x), \qquad x \in \Omega, \tag{3}$$

$$D_\nu u(t,x) = D_\nu \rho(t,x) = 0, \qquad (t,x) \in [0,T]\times\partial\Omega, \tag{4}$$

$$\int_\Omega \varphi(x) u(t,x)\, dx = \ell(t), \qquad t \in [0,T]. \tag{5}$$

Here $f, g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, d_1, d_2, u_0, $\rho_0 : \Omega \to \mathbf{R}$, $\ell : [0,T] \to \mathbf{R}$, $\varphi : \Omega \to \Omega$ are given functions, while D_ν denotes the outward normal derivative on $\partial\Omega$.
The direct problem consists in: *given* $h : [0,T] \to \mathbf{R}$, *find two functions* $u : [0,T] \times \Omega \to \mathbf{R}$, $\rho : [0,T] \times \Omega \to \mathbf{R}$ *satisfying the equations (1)–(4).*
The inverse problem consists in: *find three functions* $u : [0,T] \times \Omega \to \mathbf{R}$, $\rho : [0,T] \times \Omega \to \mathbf{R}$ *and* $h : [0,T] \to \mathbf{R}$ *satisfying the equations (1)–(5).*
Along with the specific identification problem (1)–(5) we will consider the following abstract versions related to a Banach space X:

$$u'(t) = Au(t) + \int_0^t h(t-s)Au(s)\,ds + f(u(t),\rho(t)), \quad t \in [0,T], \tag{6}$$

$$\rho'(t) = B\rho(t) + g(u(t),\rho(t)), \qquad t \in [0,T], \tag{7}$$

$$u(0) = u_0, \qquad \rho(0) = \rho_0, \tag{8}$$

$$\Phi(u(t)) = \ell(t), \qquad t \in [0,T], \tag{9}$$

where $A : \mathcal{D}(A) \subset X \to X$ and $B : \mathcal{D}(B) \subset X \to X$ are two linear closed operators and Φ is a known linear bounded functional on X. We assume that $f : X \times X \to X$ and $g : X \times X \to X$ are known nonlinear operators, u_0 and $\rho_0 \in X$ are given elements as well as the function $\ell : [0,T] \to \mathbf{R}$. The abstract direct problem consists in: *given* $h : [0,T] \to \mathbf{R}$, *find two functions* $u : [0,T] \to X$, $\rho : [0,T] \to X$ *satisfying the equations (6)–(8).*
The abstract inverse problem consists in: *find three functions* $u : [0,T] \to X$, $\rho : [0,T] \to X$ *and* $h : [0,T] \to \mathbf{R}$ *satisfying the equations (6)–(9)*

The plan of the paper is as follows. In section 2 we recall some basic results of the analytic semigroup theory, which is a fundamental tool in our approach and we give the main properties of the operators in (6)-(9) so that we can state our main abstract theorems for the

direct (c.f.(6)-(8)) and for the inverse problem (c.f.(6)-(9)). In section 3 we give an application of the abstract theorems of section 2 in the case $X = L^p(\Omega)$. In section 4 we recall a theorem, proved in [2], that assures the equivalence of problem (6)–(9) with a suitable fixed point system which is the starting point to prove the continuous dependence theorems. Moreover, we recall the main lemmas to estimate operators appearing in the equivalent fixed point system (51)-(53).
In section 5 we state and prove the continuous dependence theorems for both direct and inverse problems in the fixed point form (51)-(53) from which we easily deduce theorems 2.1 and 2.2. Finally in section 6 we deduce the evolution equations governing the combustion of a material with memory.

2. Notation and main abstract results.

Let X be a Banach space with norm $\|\cdot\|$ and let $T > 0$. We denote by $C([0,T];X)$ the usual space of continuous functions with values in X equipped with the sup–norm.
For $\beta \in (0,1)$ we define

$$C^\beta([0,T];X) = \{u \in C([0,T];X) : |u|_{\beta,T,X} = \sup_{0\le s<t\le T} (t-s)^{-\beta}\|u(t)-u(s)\| < \infty\} \tag{10}$$

and, setting $\|u\|_{0,T,X} = \|u\|_{C([0,T];X)}$, we endow it with the norm

$$\|u\|_{\beta,T,X} = \|u\|_{0,T,X} + |u|_{\beta,T,X}. \tag{11}$$

We now recall some results from the analytic semigroup theory. Let $A : \mathcal{D}(A) \subset X \to X$, $B : \mathcal{D}(B) \subset X \to X$ be two linear closed operators (possibly with $\overline{\mathcal{D}}(A) \neq X$, $\overline{\mathcal{D}}(B) \neq X$) satisfying the following assumptions:
H1 there exists $\theta \in (\pi/2, \pi)$ *such that any*
$\lambda \in \mathbf{C}\backslash\{0\}$ *with* $|\arg\lambda| \le \theta$ *and* $\lambda = 0$
belong to the resolvent sets of A *and* B*;*
H2 there exists $M > 0$ *such that*
$\max\left(\|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)}, \|\lambda(\lambda I - B)^{-1}\|_{\mathcal{L}(X)}\right) \le M$
for any $\lambda \in \mathbf{C}\backslash\{0\}$ *with* $|\arg\lambda| \le \theta$.

For any pair of Banach spaces X_1 and X_2, $\mathcal{L}(X_1; X_2)$ denotes the space of all bounded linear operators from X_1 to X_2 equipped with the sup–norm. When $X_1 = X_2 = X$, we set $\mathcal{L}(X) = \mathcal{L}(X; X)$.
According to assumptions *H1, H2*, it is possible to define the two semigroups $\{e^{tA}\}_{t\geq 0}$, $\{e^{tB}\}_{t\geq 0}$ of bounded linear operators in $\mathcal{L}(X)$ so that $t \to e^{tA}$, $t \to e^{tB}$ are analytic functions from $(0,\infty)$ to $\mathcal{L}(X)$ (for more details see [6], [7]). Moreover there exist positive constants $\tilde{c}_k(\theta)$ for $k \in \mathbf{N}$ such that

$$\|t^k A_i^k e^{tA_i}\|_{\mathcal{L}(X)} \leq \tilde{c}_k(\theta) M, \qquad t > 0, \quad (i = 1, 2) \tag{12}$$

where we have set, for the sake of simplicity, $A_1 := A$ and $A_2 := B$. Let us now endow $\mathcal{D}(A_i)$ with the graph–norms and let us define the two families of interpolation spaces $\mathcal{D}_{A_i}(\beta, p)$ ($\beta \in (0,1)$, $p \in (1, +\infty]$, $i = 1, 2$) between $\mathcal{D}(A_i)$ and X by the following equations according as $p \in (1, +\infty)$ or $p = +\infty$:

$$\mathcal{D}_{A_i}(\beta, p) = \Big\{ x \in X : \ |x|_{\mathcal{D}_{A_i}(\beta,p)}$$
$$:= \Big(\int_0^{+\infty} t^{(1-\beta)p-1} \|A_i e^{tA_i} x\|^p \, dt \Big)^{1/p} < \infty \Big\}, \tag{13a}$$

$$|x|_{\mathcal{D}_A(\beta,\infty)} = \sup_{t>0} t^{1-\beta} \|A_i e^{tA_i} x\|. \tag{13b}$$

Moreover, we set

$$\mathcal{D}_{A_i}(1 + \beta, p) = \{ x \in \mathcal{D}(A_i) : A_i x \in \mathcal{D}_{A_i}(\beta, p) \} \tag{14}$$

$\mathcal{D}_{A_i}(n + \beta, p)$ ($n = 0, 1, \ i = 1, 2$) turn out to be Banach spaces when equipped with the norms

$$\|x\|_{\mathcal{D}_{A_i}(n+\beta,p)} = \sum_{j=0}^{n} \|A_i^j x\| + |A_i^n x|_{\mathcal{D}_{A_i}(\beta,p)} \qquad (i = 1, 2). \tag{15}$$

For more details about interpolation spaces see [8].
If the given data and the operators appearing in (17)–(20) do not satisfy suitable regularity conditions and if we do not require suitable relations among the data the direct problem (6)–(8) as well as the

inverse problem (6)–(9) in general do not have solution. So we require that $f, g : X \times X \to X$ be a pair of nonlinear operators with the following properties:

H3 $f, g \in C^2(X \times X; X)$;

H4 $f \in C^2(\mathcal{D}(A) \times \mathcal{D}(B); \mathcal{D}(A)), \quad g \in C^2(\mathcal{D}(A) \times \mathcal{D}(B); \mathcal{D}(B))$.

H5 *f and g are bounded and Lipschitz continuous operators on each closed ball in $X \times X$ along with their derivatives up to the second order.*

More exactly, there exist the following functions

$$\eta, \beta : \overline{\mathbf{R}}^+ \times \overline{\mathbf{R}}^+ \to \overline{\mathbf{R}}^+, \tag{16}$$

which are continuous and nondecreasing in each of their arguments, such that

$$\begin{aligned} \|D_i f\|_{\mathcal{L}(X\times X,X)} + \|D_i g\|_{\mathcal{L}(X\times X,X)} + \|D_j D_i f\|_{\mathcal{L}(X\times X,\mathcal{L}(X\times X,X))} \\ + \|D_j D_i g\|_{\mathcal{L}(X\times X,\mathcal{L}(X\times X,X))} \\ \le \beta(\|u\|, \|\rho\|), \qquad i = u, \rho, \quad j = u, \rho, \end{aligned} \tag{17}$$

$$\begin{aligned} \|D_i f(u_2, \rho_2) - D_i f(u_1, \rho_1)\|_{\mathcal{L}(X\times X,X)} \\ \|D_i g(u_2, \rho_2) - D_i g(u_1, \rho_1)\|_{\mathcal{L}(X\times X,X)} \\ \le \beta\Big(\sum_{\ell=1}^{2} \|u_\ell\|, \sum_{\ell=1}^{2} \|\rho_\ell\|\Big) \\ \times(\|u_2 - u_1\| + \|\rho_2 - \rho_1\|), \qquad i = u, \rho, \end{aligned} \tag{18}$$

$$\begin{aligned} \|D_j D_i f(u_2, \rho_2) - D_j D_i f(u_1, \rho_1)\|_{\mathcal{L}(X\times X,\mathcal{L}(X\times X,X))} \\ + \|D_j D_i g(u_2, \rho_2) - D_j D_i g(u_1, \rho_1)\|_{\mathcal{L}(X\times X,\mathcal{L}(X\times X,X))} \\ \le \beta\Big(\sum_{\ell=1}^{2} \|u_\ell\|, \sum_{\ell=1}^{2} \|\rho_\ell\|\Big) \\ \times(\|u_2 - u_1\| + \|\rho_2 - \rho_1\|), \qquad i = u, \rho, \quad j = u, \rho, \end{aligned} \tag{19}$$

$$\begin{aligned} \|D_i f(u, \rho)\|_{\mathcal{L}(\mathcal{D}(A)\times\mathcal{D}(B),\mathcal{D}(A))} + \|D_i g(u, \rho)\|_{\mathcal{L}(\mathcal{D}(A)\times\mathcal{D}(B),\mathcal{D}(B))} \\ \le \eta(\|u\|_{\mathcal{D}(A)}, \|\rho\|_{\mathcal{D}(B)}), \quad i = u, \rho, \end{aligned} \tag{20}$$

$$\begin{aligned}&\|D_i f(u_2,\rho_2) - D_i f(u_1,\rho_1)\|_{\mathcal{L}(\mathcal{D}(A)\times\mathcal{D}(B),\mathcal{D}(A))}\\ &+\|D_i g(u_2,\rho_2) - D_i g(u_1,\rho_1)\|_{\mathcal{L}(\mathcal{D}(A)\times\mathcal{D}(B),\mathcal{D}(B)))}\\ &\le \eta\Big(\sum_{\ell=1}^{2}\|u_\ell\|,\sum_{\ell=1}^{2}\|\rho_\ell\|\Big)(\|u_2-u_1\|+\|\rho_2-\rho_1\|),\qquad i=u,\rho,\end{aligned} \tag{21}$$

$$\begin{aligned}&\|D_i D_j f(u,\rho)\|_{\mathcal{L}(\mathcal{D}(A)\times\mathcal{D}(B),\mathcal{L}(\mathcal{D}(A)\times\mathcal{D}(B),\mathcal{D}(A)))}\\ &+\|D_i D_j g(u,\rho)\|_{\mathcal{L}(\mathcal{D}(A)\times\mathcal{D}(B),\mathcal{L}(\mathcal{D}(A)\times\mathcal{D}(B),\mathcal{D}(B)))}\\ &\le \eta(\|u\|_{\mathcal{D}(A)},\|\rho\|_{\mathcal{D}(B)}),\qquad i=u,\rho,\quad j=u,\rho,\end{aligned} \tag{22}$$

$$\begin{aligned}&\|D_i D_j f(u_1,\rho_1) - D_i D_j f(u_2,\rho_2)]\|_{\mathcal{L}(\mathcal{D}(A)\times\mathcal{D}(B),\mathcal{L}(\mathcal{D}(A)\times\mathcal{D}(B),\mathcal{D}(A)))}\\ &+\|D_i D_j g(u_1,\rho_1) - D_i D_j g(u_2,\rho_2)]\|_{\mathcal{L}(\mathcal{D}(A)\times\mathcal{D}(B),\mathcal{L}(\mathcal{D}(A)\times\mathcal{D}(B),\mathcal{D}(B)))}\\ &\le \eta(\sum_{\ell=1}^{2}\|u_\ell\|_{\mathcal{D}(A)},\sum_{\ell=1}^{2}\|\rho_\ell\|_{\mathcal{D}(B)})\\ &\times(\|u_2-u_1\|_{\mathcal{D}(A)}+\|\rho_2-\rho_1\|_{\mathcal{D}(B)}),\qquad i=u,\rho,\quad j=u,\rho.\end{aligned} \tag{23}$$

And finally we make the following assumptions related to some pair $\beta\in(0,1)$ and $\varepsilon\in(0,1-\beta)$:

H6 $\mathcal{D}(A)=\mathcal{D}(B)$;

H7 $u_0\in\mathcal{D}_A(1+\beta,p)$, $Au_0\in\mathcal{D}_A(\beta+\varepsilon,p)$, $\rho_0\in\mathcal{D}_B(1+\beta,p)$;

H8 $Au_0+f(u_0,\rho_0)\in\mathcal{D}_A(1+\beta,p)$, $B\rho_0+g(u_0,\rho_0)\in\mathcal{D}_B(1+\beta,p)$;

H9 $\Phi\in\mathcal{L}(X;\mathbf{R})$;

H10 $\Phi[Au_0]\neq 0$;

H11 $\ell\in C^{2+\beta}([0,T];\mathbf{R})$.

Remark 2.1. *Observe that H3 ensures, via the closed graph theorem, that $A^{-1}B$, $B^{-1}A\in\mathcal{L}(X)$. Conditions H7 and H8, appear in hypothesis of lemma 4.3 and point i) of theorem 2.2 in [2], and they are easily satisfied if we require that $u_0\in D_A(2+\beta,p)$, $v_0\in D_B(2+\beta,p)$. Moreover, conditions H9–H11 are indispensable to obtain a suitable fixed point equation for the unknown convolution kernel h.*

We are now in position to state our main results. For the direct problem and for the inverse one, respectively, we have

THEOREM 2.2. *Let H1-H8 and estimates (18)-(23) hold. The map*

$$(u_0, \rho_0, h) \to (u, \rho) \tag{24}$$

is continuous from $\mathcal{D}_A(2+\beta, p) \times \mathcal{D}_B(2+\beta, p) \times C^{\beta}([0,T]; \mathbf{R})$ *to* $C^{2+\beta}([0,T]; X)^2$. *Moreover the following estimate holds*

$$\begin{aligned} &\|u - u'\|_{2+\beta,T,X} + \|\rho - \rho'\|_{2+\beta,T,X} \le a_1(\beta, \varepsilon, \theta, p, M, T, m) \\ &\times \Big(\|h - h'\|_{0,T,\mathbf{R}} + \|u_0 - u_0'\|_{\mathcal{D}_A(2+\beta,p)} + \|\rho_0 - \rho_0'\|_{\mathcal{D}_B(2+\beta,p)} \Big), \end{aligned} \tag{25}$$

where a_1 *is continuous in its arguments for small* T.

Proof. It is a consequence of theorem 5.1. □

THEOREM 2.3. *Let assumptions H1-H11 and estimates (18)-(23) hold. Let* (u, ρ, h) *and* (u', ρ', h') *be the solutions of problem (6)-(9) related to data* $(u_0, \rho_0, \Phi, \ell)$ *and* $(u_0', \rho_0', \Phi', \ell')$, *respectively. Then the map*

$$(u_0, \rho_0, \Phi, \ell) \to (u, \rho, h) \tag{26}$$

is continuous from $\mathcal{D}_A(2+\beta, p) \times \mathcal{D}_B(2+\beta, p) \times \mathcal{L}(X; \mathbf{R}) \times C^{2+\beta}([0,T])$ *to* $C^{2+\beta}([0,T]; X)^2 \times C^{\beta}([0,T])$. *Moreover the following estimate holds*

$$\begin{aligned} &\|u - u'\|_{\beta,T,X} + \|\rho - \rho'\|_{3+\beta,T,X} + \|h - h'\|_{3+\beta,T,\mathbf{R}} \\ &\le a_2(\beta, \varepsilon, \theta, p, M, T, m) \max\{T^{1-\beta}, T^{\varepsilon}\} \\ &\times \Big(\|\Phi - \Phi'\|_{\mathcal{L}(X,\mathbf{R})} + |[\Phi(Au_0)]^{-1} - [\Phi(Au_0')]^{-1}| \\ &+ \|\ell - \ell'\|_{C^2([0,T])} + \|u_0 - u_0'\|_{\mathcal{D}_A(2+\beta,p)} + \|\rho_0 - \rho_0'\|_{\mathcal{D}_B(2+\beta,p)} \Big) \end{aligned} \tag{27}$$

where a_2 *is continuous in its arguments for small* T.

Proof. It is an immediate consequence of theorem 5.3 and (53). □

3. An application of the abstract results to the combustion system.

We are going to apply the abstract continuous dependence results of the previous section to problem (1)–(5). We give the stability result for the example in [2]. Let Ω be an open bounded set in $\mathbf{R}^n$ with boundary of class C^2 and let $d_1, d_2 \in C^1(\overline{\Omega};\mathbf{R})$, $a_{h,k}, D_{x_h} a_{h,k} \in C(\overline{\Omega};\mathbf{R})$ $(h,k = 1,...,n)$, $\zeta_1 \in C^2(\overline{\Omega};\mathbf{R})$, $\zeta_2 \in C(\overline{\Omega};\mathbf{R})$ be given functions satisfying

$$d_i(x) \geq \delta_i, \qquad \zeta_i(x) < 0, \qquad \forall x \in \overline{\Omega},\ i = 1,2, \tag{28}$$

$$\sum_{h,k=1}^{n} a_{h,k}(x)\xi_h\xi_k \geq \mu|\xi|^2, \qquad \forall x \in \overline{\Omega},\ \forall \xi \in \mathbf{R}^n, \tag{29}$$

for some constants $\mu > 0$, $\delta_i > 0$ $(i = 1,2)$. Then the linear differential operators

$$A_i(x, D_x) = \sum_{h,k=1}^{n} D_{x_h}[d_i(x)a_{h,k}(x)D_{x_k}] + \zeta_i(x)I, \qquad (i = 1,2), \tag{30}$$

are uniformly elliptic in $\overline{\Omega}$. We now choose

$$X = L^p(\Omega) \qquad (p \in (n, +\infty)) \tag{31}$$

as our reference Banach space. Consequently, we can choose $(i = 1,2)$

$$\mathcal{D}(A_{i,p}) = \{u \in W^{2,p}(\Omega) : A_i u \in L^p(\Omega), \quad (D_{\tilde{\nu}} u)|_{\partial\Omega} = 0\}, \tag{32}$$

as domains of $A_{i,p}$ $(i = 1,2)$ defined by (30), where $\widetilde{\nu}$ is the *conormal* outward unit vector related to $\partial\Omega$ ($\widetilde{\nu}$ is proportional to the vector with components $\sum_{h=1}^{n} \nu_h(x)a_{h,k}(x)$ $(k = 1,\dots,n)$, ν denoting the outward *normal* unit vector relative to $\partial\Omega$).
Then we recall the following characterizations concerning the interpolation spaces related to A_i ([8]), where $\beta \neq 1/(2p)$ and $p \in (n, +\infty)$:

$$\mathcal{D}_{A_i}(\beta, p) = W^{2\beta,p}(\overline{\Omega}) \quad if \quad \beta \in (0, 1/(2p)), \tag{33a}$$

$$\mathcal{D}_{A_i}(\beta,p) = W^{2\beta,p}_{\tilde{\nu}}(\overline{\Omega}) \quad if \quad \beta \in (1/(2p),1), \tag{33b}$$

where

$$W^{2\beta,p}_{\tilde{\nu}}(\overline{\Omega}) = \{u \in W^{2\beta,p}(\overline{\Omega}) : \ (D_{\tilde{\nu}}u)|_{\partial\Omega} = 0 \ \}. \tag{34}$$

Finally, we can define the set $\mathcal{G}$ of admissible data consisting of all functions u_0, ρ_0, ϕ, ℓ_1, f and g satisfying the following assumptions, for some $\beta \in (0,1)\backslash\{1/(2p)\}$:

H12 $f, g \in W^{3+[2\beta],\infty}(\mathbf{R}^2)$, $[r]$ *denoting the integer part of* $r \in \mathbf{R}$;

H13 $u_0, \rho_0 \in W^{4+2\beta,p}(\Omega)$;

H14 $D_\nu A^j u_0 = D_\nu B^j \rho_0 = 0$ *on* $\partial\Omega$ $j = 0, 1$;

H15 $D_\nu A[Au_0 + f(u_0,\rho_0)] = D_\nu B[B\rho_0 + g(u_0,\rho_0)] = 0$ *on* $\partial\Omega$ *if* $\beta \in (1/(2p),1)$;

H16 $\phi \in L^1(\Omega)$, $\ell \in C^{2+\beta}([0,T])$;

H17 $\int_\Omega \phi(x) Au_0(x)\,dx \neq 0$. Moreover, we consider the pair of functions related to the Arrehenius kinetics, (cf. (98))

$$f(u,\rho) = -g(u,\rho) = \rho^\eta \exp(\gamma - \gamma/u) \tag{35}$$

where γ is a positive constant and depends on the material under consideration.

THEOREM 3.1. *Let* $\eta > 3 + [2\beta]$ *and let assumptions H12–H17 be satisfied. Then there exists* $T^* \in (0,T]$, *depending on the norms of the data only, such that for every* $\tau \in (0,T^*)$ *problem (6)–(9) admits a unique solution*

$$(u,\rho,h) \in [C^{2+\beta}([0,\tau];L^p(\Omega)) \cap C^{1+\beta}([0,\tau];W^{2,p}(\Omega))]$$
$$\times [C^{2+\beta}([0,\tau];L^p(\Omega)) \cap C^{1+\beta}([0,\tau];W^{2,p}(\Omega))] \times C^\beta([0,\tau];\mathbf{R}).$$

Let (u,ρ,h) *and* (u',ρ',h') *be the unique solutions related to the data* (u_0,ρ_0,ϕ,ℓ) *and* $(u_0',\rho_0',\phi',\ell')$, *respectively. Then there exists a pos-*

itive constant $C(T)$, continuous in T for small $T > 0$ such that

$$\begin{aligned}
&\|u-u'\|_{C^{2+\beta}([0,\tau];W^{2,p}(\Omega))}+\|\rho-\rho'\|_{C^{2+\beta}([0,\tau];W^{2,p}(\Omega))}\\
&+\|h-h'\|_{C^{\beta}([0,\tau];\mathbf{R})}\le C(T)\Big\{\sup_{v\in L^p(\Omega)}\\
&\times\Big(\int_\Omega |v|^p\,dx\Big)^{-1/p}\Big(\int_\Omega |(\phi-\phi')v|^p\,dx\Big)^{1/p}\\
&+\Big|\Big(\int_\Omega \phi(x)A(x)u_0(x)|^p\,dx\Big)^{-1}-\Big(\int_\Omega \phi(x)A(x)u_0'(x)|^p\,dx\Big)^{-1}\Big|\\
&+\|\ell-\ell'\|_{C^2([0,\tau])}+\|u_0-u_0'\|_{W^{4+2\beta,p}(\overline{\Omega})}\\
&+\|\rho_0-\rho_0'\|_{W^{4+2\beta,p}(\overline{\Omega})}\Big\},\quad if\quad \beta\in(0,1/(2p)).
\end{aligned}\tag{36}$$

For the case $\beta \in (1/(2p), 1)$ we replace $W^{4+2\beta,p}(\overline{\Omega})$ by $W_\nu^{4+2\beta,p}(\overline{\Omega})$ in estimate (36).

Proof. The continuous dependence estimate is a consequence of theorem 2.3 and the existence and uniqueness of a solution of theorem 4.1 in [2]. □

4. An equivalent problem and preliminary lemmas

In this section we reformulate the integrodifferential problem (6)–(9) in terms of an equivalent fixed point system by suitable operators. Moreover we recall the equivalence theorem and all the suitable lemmas to estimate operators defined in the sequel. Such lemmas are proved in [2] and they will let us show our main results in the following sections 4 and 5. Let

$$\begin{aligned}
&(u,\rho,h)\in[C^{2+\beta}([0,T];X)\cap C^{1+\beta}([0,T];\mathcal{D}(A))]\\
&\times[C^{2+\beta}([0,T];X)\cap C^{1+\beta}([0,T];\mathcal{D}(B))]\times C^{\beta}([0,T];\mathbf{R})
\end{aligned}$$

be a solution to problem (6)–(9). With (u,ρ,h) we associate the triplet (v,π,h) defined by

$$u'(t)=v(t)\Longleftrightarrow u(t)=u_0+1*v(t),\tag{37}$$

$$\rho'(t) = \pi(t) \Longleftrightarrow \rho(t) = \rho_0 + 1 * \pi(t). \tag{38}$$

We define the operators

$$L_1[w](t) := A\int_0^t \mathrm{e}^{(t-s)A} w(s)\,ds,\ \ \widetilde{L}_1[w](t) := \int_0^t \mathrm{e}^{(t-s)A} A w(s)\,ds \tag{39}$$

$$L_2[w](t) := B\int_0^t \mathrm{e}^{(t-s)B} w(s)\,ds,\ \ \widetilde{L}_2[w](t) := \int_0^t \mathrm{e}^{(t-s)B} B w(s)\,ds \tag{40}$$

$$L_3[h,x](t) := \int_0^t h(s) A \mathrm{e}^{(t-s)A} x\,ds \tag{41}$$

$$N_1[h,w](t) := \int_0^t h(t-s) w(s)\,ds \tag{42}$$

$$N_2[w,z](t) = D_u f(u_0 + 1 * A^{-1} w(t), \rho_0 + 1 * B^{-1} z(t)) - D_u f(u_0, \rho_0) \tag{43}$$

$$N_3[w,z](t) = D_\rho f(u_0 + 1 * A^{-1} w(t), \rho_0 + 1 * B^{-1} z(t)) - D_\rho f(u_0, \rho_0) \tag{44}$$

$$N_4[w,z](t) = D_u g(u_0 + 1 * A^{-1} w(t), \rho_0 + 1 * B^{-1} z(t)) - D_u g(u_0, \rho_0) \tag{45}$$

$$N_5[w,z](t) = D_u g(u_0 + 1 * A^{-1} w(t), \rho_0 + 1 * B^{-1} z(t)) - D_\rho g(u_0, \rho_0) \tag{46}$$

$$M_1(u_0,\rho_0) = D_u f(u_0,\rho_0) A^{-1},\quad M_2(u_0,\rho_0) = D_\rho f(u_0,\rho_0) B^{-1} \tag{47}$$

$$M_3(u_0,\rho_0) = D_u g(u_0,\rho_0) A^{-1},\quad M_4(u_0,\rho_0) = D_\rho g(u_0,\rho_0) B^{-1} \tag{48}$$

and we introduce the functions

$$w_0(t) := A\mathrm{e}^{tA}[Au_0 + f(u_0, \rho_0)], \qquad z_0(t) := B\mathrm{e}^{tB}[B\rho_0 + g(u_0, \rho_0)], \tag{49}$$

$$\chi := [\Phi(Au_0)]^{-1}. \tag{50}$$

We can now rewrite our identification problem in the following fixed–point form:

$$\begin{aligned} w = w_0 + L_3[h, Au_0] + L_1[N_1(h, w)] + L_1[N_2(w, z)(A^{-1}w)] \\ + L_1[N_3(w, z)B^{-1}z] + \widetilde{L}_1[M_1(u_0, \rho_0)w] \\ + \widetilde{L}_1[M_2(u_0, \rho_0)z)] := w_0 + S_1(w, z, h), \end{aligned} \tag{51}$$

$$\begin{aligned} z = z_0 + L_2[N_4(w, z)(A^{-1}w)] + L_2[N_5(w, z)B^{-1}z] \\ + \widetilde{L}_3[M_3(u_0, \rho_0)w] + \widetilde{L}_3[M_4(u_0, \rho_0)z] := z_0 + S_2(w, z, h), \end{aligned} \tag{52}$$

$$\begin{aligned} h = h_0 - \chi\Phi\Big\{ L_3[h, Au_0] + L_1[N_1(h, w)] \\ + L_1[N_2(w, z)A^{-1}w] + L_1[N_3(w, z)B^{-1}z] + N_1(h, w) \\ + N_2(w, z)A^{-1}w + N_3(w, z)B^{-1}z + M_1(u_0, \rho_0)S_1(w, z, h) \\ + M_2(u_0, \rho_0)S_2(w, z, h)\Big\} := h_0 + S_3(w, z, h), \end{aligned} \tag{53}$$

where

$$\begin{aligned} h_0(t) = \chi\{\ell''(t) - \Phi[w_0(t)]\} - \chi\Phi\big[D_u f(u_0, \rho_0)A^{-1}w_0(t) \\ + D_\rho f(u_0, \rho_0)B^{-1}z_0(t)\big]. \end{aligned} \tag{54}$$

Thanks to definitions (37)-(50) and (54) we can state the following theorem 4.1 and we recall lemmas 4.2–4.9 whose proofs are in [2].

THEOREM 4.1. *Let assumptions H1, H2 and H11 be satisfied. Let* $(u, \rho, h) \in [C^{2+\beta}([0, T]; X) \cap C^{1+\beta}([0, T]; \mathcal{D}(A))] \times [C^{2+\beta}([0, T]; X) \cap C^{1+\beta}([0, T]; \mathcal{D}(B))] \times C^{\beta}([0, T]; \mathbf{R})$ *be a solution to problem (6)–(9). Then the triplet* (w, z, h)*, where*

$$w = Au', \qquad z = B\rho', \tag{55}$$

belongs to $C^\beta([0,T];X) \times C^\beta([0,T];X) \times C^\beta([0,T];\mathbf{R})$ *and solves problem (51), (52), (53). Conversely, if* $(w,z,h) \in C^\beta([0,T];X) \times C^\beta([0,T];X) \times C^\beta([0,T];\mathbf{R})$ *is a solution to problem (51), (52), (53), then the triplet* (u,ρ,h)*, where*

$$u = u_0 + 1 * A^{-1}w, \qquad \rho = \rho_0 + 1 * B^{-1}z, \tag{56}$$

belongs to $[C^{2+\beta}([0,T];X) \cap C^{1+\beta}([0,T];\mathcal{D}(A))] \times [C^{2+\beta}([0,T];X) \cap C^{1+\beta}([0,T];\mathcal{D}(B))] \times C^\beta([0,T];\mathbf{R})$ *and solves problem (6)–(9).*

LEMMA 4.2. *Let* $h \in C^\beta([0,T];\mathbf{R})$*,* $g \in C^\beta([0,T];X)$*. Then* hg *belongs to* $C^\beta([0,T];X)$ *and satisfies the estimate*

$$\|hg\|_{\beta,T,X} \le \|h\|_{\beta,T,\mathbf{R}} \|g\|_{\beta,T,X}. \tag{57}$$

LEMMA 4.3. *Let* L_3 *be the operator defined in (41). If* $h \in C([0,T];\mathbf{R})$ *and* $Au_0 \in \mathcal{D}_A(\beta+\varepsilon,p)$*, then the following estimate holds for any* $\varepsilon \in (0,1-\beta)$ *:*

$$\|L_3[h,Au_0]\|_{\beta,T,X} \le T^\varepsilon c_1(\beta,\varepsilon,\theta,M,T)\|h\|_{0,T,\mathbf{R}} \|Au_0\|_{\mathcal{D}_A(\beta+\varepsilon,p)}. \tag{58}$$

The function c_1 *is continuous and nondecreasing in* T*.*

LEMMA 4.4. *Let* L_1 *and* N_1 *be the operators defined in (40) and (42), respectively. Then operator* L_1N_1 *maps* $C^\beta([0,T];\mathbf{R}) \times C([0,T];X)$ *into* $C^\beta([0,T];X)$ *and satisfies the following estimate for any* $(h,w) \in C^\beta([0,T];\mathbf{R}) \times C([0,T];X)$ *:*

$$\|L_1[N_1(h,w)]\|_{\beta,T,X} \le c_2(\beta,\varepsilon,\theta,p,M,T)T^\beta \|h\|_{\beta,T,\mathbf{R}} \|w\|_{0,T,X}. \tag{59}$$

The function c_2 *is continuous and nondecreasing in* T*.*

LEMMA 4.5. *Let* N_j $(j=2,3,4,5)$ *be the operators defined in (43)–(46). Under assumptions (18)–(23)* N_j $(j=2,3,4,5)$ *maps* $C([0,T];X)^2$ *into* $C^\beta([0,T];\mathcal{L}(X))$ *and satisfies the following estimates for any* $w,z \in C([0,T];X)$ *:*

$$\|N_j(w,z)\|_{\beta,T,\mathcal{L}(X)} \le T^{1-\beta} c_3\big(\|w\|_{0,T,X}, \|z\|_{0,T,X}, T\big). \tag{60}$$

The function c_3 *is continuous and nondecreasing in each of the arguments pointed out.*

LEMMA 4.6. *Let L_1 and L_2 be the operators defined in (39) and (40), respectively. Then operators L_iN_j ($i = 1,2$, $j = 2,3,4,5$) map $C([0,T];X)^2$ into $C^\beta([0,T];\mathcal{L}(X))$ and satisfy the following estimates for any pair $w, z \in C^\beta([0,T];X)$:*

$$\|L_i[N_j(w,z)]\|_{\beta,T,\mathcal{L}(X)} \le T^{1-\beta} \times c_4\big(\|w\|_{0,T,X}, \|z\|_{0,T,X}, T\big), \quad i = 1,2,\ j = 2,3,4,5. \tag{61}$$

The function c_4 is continuous and nondecreasing in each of the arguments pointed out.

LEMMA 4.7. *Let M_j ($j = 1,2,3,4$) and $\widetilde{L}_i$ ($i = 1,3$) be the operators defined in (47), (48) and (39), (40), respectively. Then operators $\widetilde{L}_iM_j$ ($i = 1$, $i = 3$, $j = 2,3,4,5$) map $C^\beta([0,T];X)^2$ into $C^\beta([0,T];\mathcal{L}(X))$ and satisfy the following estimates for any pair $w, z \in C^\beta([0,T];X)$:*

$$\|\widetilde{L}_i[M_{i+2j}(u_0,\rho_0)w]\|_{\beta,T,X} \le c_5(\|u_0\|_{\mathcal{D}(A)}, \|\rho_0\|_{\mathcal{D}(B)}, T)T^{1-\beta}\|w\|_{\beta,T,X}, \quad i = 1,2,\ j = 0,1. \tag{62}$$

The function c_5 is continuous and nondecreasing in each of the arguments pointed out.

LEMMA 4.8. *Let N_j ($j = 2,...,5$) be the operators defined in (43)–(46). Under assumptions (18)–(23) operators N_j ($j = 2,3,4,5$) map $C^\beta([0,T];X)^2$ into $C^\beta([0,T];\mathcal{L}(X))$ and satisfy the following estimates for any quadruplet $w_1, w_2, z_1, z_2 \in C^\beta([0,T];X)$:*

$$\|L_i^k[N_j(w_2,z_2) - N_j(w_1,z_1)]\|_{0,T,\mathcal{L}(X)} \le T^{1+\beta}c_6\Big(\|u_0\|, \|\rho_0\|, \sum_{\ell=1}^{2}\|w_\ell\|_{0,T,X}, \sum_{\ell=1}^{2}\|z_\ell\|_{0,T,X,T}\Big) \times\big(\|w_2 - w_1\|_{0,T,X} + \|z_2 - z_1\|_{0,T,X}\big), \qquad i,k = 0,1,\ j = 2,3,4,5, \tag{63}$$

$$
\begin{aligned}
&|L_i^k[N_j(w_2,z_2)-N_j(w_1,z_1)]|_{\beta,T,\mathcal{L}(X)}\\
&\le T^{1-\beta}c_7\Big(\|u_0\|,\|\rho_0\|,\sum_{\ell=1}^{2}\|w_\ell\|_{0,T,X},\sum_{\ell=1}^{2}\|z_\ell\|_{0,T,X},T\Big)\\
&\times\big(\|w_2-w_1\|_{0,T,X}+\|z_2-z_1\|_{0,T,X}\big),\qquad i,k=0,1,\ \ j=2,3,4,5,
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
&\|L_i^k[N_j(w_2,z_2)-N_j(w_1,z_1)]\|_{\beta,T,\mathcal{L}(X)}\\
&\le T^{1-\beta}c_8\Big(\|u_0\|,\|\rho_0\|,\sum_{\ell=1}^{2}\|w_\ell\|_{0,T,X},\sum_{\ell=1}^{2}\|z_\ell\|_{0,T,X},T\Big)\\
&\times\big(\|w_2-w_1\|_{0,T,X}+\|z_2-z_1\|_{0,T,X}\big),\qquad i,k=0,1,\ \ j=2,3,4,5,
\end{aligned}
\tag{65}
$$

where c_6, c_7, c_8 are continuous and nondecreasing functions in each of the arguments pointed out.

LEMMA 4.9. *Let M_j $(j=1,2,3,4)$ and $\widetilde{L}_i$ $(i=1,3)$ be the operators defined in (47), (48) and (39), (40), respectively. Then operators $\widetilde{L}_iM_j$ $(i=1,\ i=3,\ j=2,3,4,5)$ map $C^\beta([0,T];X)^2$ into $C^\beta([0,T];\mathcal{L}(X))$ and satisfy the following estimates for any pair $w,z\in C^\beta([0,T];X)$: for $i=1,2,\ j=0,1$*

$$
\begin{aligned}
&\|\widetilde{L}_i[(M_{i+2j}(u_0,\rho_0)-M_{i+2j}(u_0',\rho_0'))w]\|_{\beta,T,X}\\
&\le c_9(\|u_0\|_{\mathcal{D}(A)},\|\rho_0\|_{\mathcal{D}(B)},\|u_0'\|_{\mathcal{D}(A)},\|\rho_0'\|_{\mathcal{D}(B)},T)\\
&\times T^{1-\beta}\Big(\|u_0-u_0'\|_{\mathcal{D}(A)}+\|\rho_0-\rho_0'\|_{\mathcal{D}(B)}\Big)\|w\|_{\beta,T,X}.
\end{aligned}
\tag{66}
$$

The function c_9 is continuous and nondecreasing in each of the arguments pointed out.

LEMMA 4.10. *The convolution operator*

$$
N_1(w,h):=\int_0^t h(t-s)w(s)\,ds \tag{67}
$$

maps $C([0,T];X)\times C^\beta([0,T];\mathbf{R})$ into $C^\beta([0,T];\mathbf{R})$ and satisfies the following estimate

$$
\|N_1(w,h)\|_{\beta,T,X}\le(2T+T^{1-\beta})\|w\|_{\beta,T,X}\|h\|_{0,T,\mathbf{R}}. \tag{68}
$$

Proof. It is a particular case of proposition 3.1 in [4]. □

5. Proofs of the main abstract results

We are now in position to state and prove the main results related to the direct abstract problem in fixed point formulation (51)–(53), i.e. we suppose that h is a given data and we study the continuity of the map $(u_0, \rho_0, h) \to (u, \rho)$ which is a consequence of the following continuous dependence result.

THEOREM 5.1. *Let H1-H8 and estimates (18)-(23) hold. Let (w, z) and (w', z') be the unique solutions to problem (51) and (52) related to data (w_0, z_0, h) and (w_0', z_0', h'), respectively. Then the map $(u_0, \rho_0, h) \to (w, z)$ is continuous from $\mathcal{D}_A(2+\beta, p) \times \mathcal{D}_B(2+\beta, p) \times C^\beta([0,T]; X)$ to $C^\beta([0,T]; X)^2$. Moreover the following estimate holds*

$$\|w - w'\|_{\beta,T,X} + \|z - z'\|_{\beta,T,X} \leq c_{13}(\beta, \varepsilon, \theta, p, M, T, m)$$
$$\times \Big(\|h - h'\|_{0,T,\mathbf{R}} + \|u_0 - u_0'\|_{\mathcal{D}_A(2+\beta,p)} + \|\rho_0 - \rho_0'\|_{\mathcal{D}_B(2+\beta,p)} \Big). \quad (69)$$

REMARK 5.2. *We point out that, theorem 5.1 can be obviously generalized studying the continuity of the map $(u_0, \rho_0, h, f, g) \to (w, z)$. Such generalization is reasonable because, in the combustion case the operators f and g are given by the Arrehenius kinetic (c.f.(98)) that contains the two parameters η and γ which are experimentally determined with some error. However in the sequel we prove theorem 5.1, only, since the generalization is obvious, but needs too many calculations.*

Proof. (of theorem 5.1) We remark that in [2] we have proved existence and uniqueness for the inverse problem only, since the existence and uniqueness theorem for the direct problem follows easily by standard arguments and the lemmas of section 4. We can now define the closed ball (for suitable $m > 0$):

$$B_m(\beta) = \{(u_0, \rho_0, h, w, z) \in B(\beta) : \|(u_0, \rho_0, h, w, z)\|_{B(\beta)} \leq 5m\} \quad (70)$$

where

$$B(\beta) := \mathcal{D}_A(2+\beta, p) \times \mathcal{D}_B(2+\beta, p)$$
$$\times C^\beta([0,T]; \mathbf{R}) \times C^\beta([0,T]; X) \times C^\beta([0,T]; X)$$

and

$$\|(u_0,\rho_0,h,w,z)\|_{B(\beta)} := \|u_0\|_{\mathcal{D}_A(2+\beta,p)} + \|\rho_0\|_{\mathcal{D}_B(2+\beta,p)} + \|h\|_{\beta,T,\mathbf{R}} + \|w\|_{\beta,T,X} + \|z\|_{\beta,T,X}.$$

Keeping in mind the existence and uniqueness results, we let (w,z), (w',z') be the solution of (51) and (52) related to the date (u_0,ρ_0,h), (u_0',ρ_0',h'), respectively. For the differeces $w-w'$ and $z-z'$ from (51) and (52) we derive the following equations:

$$\begin{aligned} w-w' &= w_0 - w_0' + S_1(w,z,h) - S_1(w',z',h') \\ &= w_0 - w_0' + L_3[h, Au_0] - L_3[h', Au_0'] + L_1[N_1(h,w)] - L_1[N_1(h',w')] \\ &\quad + L_1[N_2(w,z)(A^{-1}w)] - L_1[N_2(w',z')(A^{-1}w')] \\ &\quad + L_1[N_3(w,z)B^{-1}z] - L_1[N_3(w',z')B^{-1}z'] \\ &\quad + \widetilde{L}_1[M_1(u_0,\rho_0)w] - \widetilde{L}_1[M_1(u_0',\rho_0')w'] \\ &\quad + \widetilde{L}_1[M_2(u_0,\rho_0)z)] - \widetilde{L}_1[M_2(u_0',\rho_0')z')], \end{aligned} \tag{71}$$

$$\begin{aligned} z-z' &= z_0 - z_0' + S_2(w,z,h) - S_2(w',z',h') \\ &= z_0 - z_0' + L_2[N_4(w,z)(A^{-1}w)] - L_2[N_4(w',z')(A^{-1}w')] \\ &\quad + L_2[N_5(w,z)B^{-1}z] - L_2[N_5(w',z')B^{-1}z'] + \widetilde{L}_3[M_3(u_0,\rho_0)w] \\ &\quad - \widetilde{L}_3[M_3(u_0',\rho_0')w'] + \widetilde{L}_3[M_4(u_0,\rho_0)z] - \widetilde{L}_3[M_4(u_0',\rho_0')z']. \end{aligned} \tag{72}$$

From (71) and (72), after some calculation, we easily deduce the estimates

$$\begin{aligned} \|w-w'\|_{\beta,T,X} &\le \|w_0 - w_0'\|_{\beta,T,X} + \|L_3[h-h', Au_0]\|_{\beta,T,X} \\ &\quad + \|L_3[h', A(u_0-u_0')]\|_{\beta,T,X} + \|L_1[N_1(h-h',w)]\|_{\beta,T,X} \\ &\quad + \|L_1[N_1(h',w-w')]\|_{\beta,T,X} \\ &\quad + \|L_1[(N_2(w,z)-N_2(w',z'))(A^{-1}w)]\|_{\beta,T,X} \\ &\quad + \|L_1N_2(w',z')(A^{-1}(w-w'))]\|_{\beta,T,X} \end{aligned}$$

$$
\begin{aligned}
&+\|L_1[(N_3(w,z)-N_2(w',z'))(B^{-1}z)]\|_{\beta,T,X}\\
&+\|L_1N_3(w',z')(B^{-1}(z-z')]\|_{\beta,T,X}\\
&+\|\widetilde{L}_1[(M_1(u_0,\rho_0)-M_1(u_0',\rho_0'))w]\|_{\beta,T,X}\\
&+\|\widetilde{L}_1[M_1(u_0',\rho_0')(w-w')]\|_{\beta,T,X}\\
&+\|\widetilde{L}_1[(M_2(u_0,\rho_0)-M_2(u_0',\rho_0'))z]\|_{\beta,T,X}\\
&+\|\widetilde{L}_1[M_2(u_0',\rho_0')(z-z')]\|_{\beta,T,X} \qquad (73)
\end{aligned}
$$

$$
\begin{aligned}
&\|z-z'\|_{\beta,T,X}\le\|z_0-z_0'\|_{\beta,T,X}\\
&+\|L_2[(N_4(w,z)-N_4(w',z'))(A^{-1}w)]\|_{\beta,T,X}\\
&+\|L_2[N_4(w',z')(A^{-1}(w-w'))]\|_{\beta,T,X}\\
&+\|L_2[(N_5(w,z)-N_5(w',z'))B^{-1}z]\|_{\beta,T,X}\\
&+\|L_2[N_5(w',z')B^{-1}(z-z')]\|_{\beta,T,X}\\
&+\|\widetilde{L}_3[(M_3(u_0,\rho_0)-M_3(u_0',\rho_0'))w]\|_{\beta,T,X}\\
&+\|\widetilde{L}_3[M_3(u_0',\rho_0')(w-w')]\|_{\beta,T,X}\\
&+\|\widetilde{L}_3[(M_4(u_0,\rho_0)-M_4(u_0',\rho_0'))z]\|_{\beta,T,X}\\
&+\|\widetilde{L}_3[M_4(u_0',\rho_0')(z-z')]\|_{\beta,T,X}. \qquad (74)
\end{aligned}
$$

Thanks to lemmas 4.2- 4.10 we obtain the two following estimates

$$
\begin{aligned}
&\|w-w'\|_{\beta,T,X}\le\|w_0-w_0'\|_{\beta,T,X}\\
&+T^{\varepsilon}c_1(\beta,\varepsilon,\theta,M,T)\Big(\|h-h'\|_{0,T,\mathbf{R}}\|Au_0\|_{\mathcal{D}_A(\beta+\varepsilon,p)}\\
&+\|h'\|_{0,T,\mathbf{R}}\|A(u_0-u_0')\|_{\mathcal{D}_A(\beta+\varepsilon,p)}\Big)\\
&+T^{\beta}c_2(\beta,\varepsilon,\theta,p,M,T)\Big(\|h-h'\|_{\beta,T,\mathbf{R}}\|w\|_{0,T,X}\\
&+\|h'\|_{\beta,T,\mathbf{R}}\|w-w'\|_{0,T,X}\Big)\\
&+T^{1-\beta}\max\{mc_8(m,T),c_4(m,T),\\
&mc_9(m,T),c_5(m,T),\|A^{-1}\|_{L(X)},\|B^{-1}\|_{L(X)}\}\\
&\times\Big(\|w-w'\|_{\beta,T,X}+\|z-z'\|_{\beta,T,X}+\|u_0-u_0'\|_{\mathcal{D}_A(2+\beta,p)}\\
&+\|\rho_0-\rho_0'\|_{\mathcal{D}_B(2+\beta,p)}\Big), \qquad (75)
\end{aligned}
$$

$$\begin{aligned}\|z-z'\|_{\beta,T,X} &\le \|z_0-z_0'\|_{\beta,T,X}\\ &+T^{1-\beta}\max\{mc_8(m,T),c_4(m,T),mc_9(m,T),\\ &c_5(m,T),\|A^{-1}\|_{L(X)},\|B^{-1}\|_{L(X)}\}\\ &\times\Big(\|w-w'\|_{\beta,T,X}+\|z-z'\|_{\beta,T,X}\\ &+\|u_0-u_0'\|_{\mathcal{D}_A(2+\beta,p)}+\|\rho_0-\rho_0'\|_{\mathcal{D}_B(2+\beta,p)}\Big).\end{aligned}\tag{76}$$

Inequalities (75) and (76) can be rewritten as

$$\begin{aligned}\|w-w'\|_{\beta,T,X} &\le \|w_0-w_0'\|_{\beta,T,X}\\ &+c_{10}(\beta,\varepsilon,\theta,p,M,T,m)\max\{T^\beta,T^{1-\beta},T^\varepsilon\}\\ &\times\Big(\|z-z'\|_{\beta,T,X}+\|\rho-\rho'\|_{\beta,T,X}+\|h-h'\|_{0,T,\mathbf{R}}+\|u_0-u_0'\|_{\mathcal{D}_A(2+\beta,p)}\\ &+\|\rho_0-\rho_0'\|_{\mathcal{D}_B(2+\beta,p)}+\|A(u_0-u_0')\|_{\mathcal{D}_A(\beta+\varepsilon,p)}\Big),\end{aligned}\tag{77}$$

$$\begin{aligned}\|z-z'\|_{\beta,T,X} &\le \|z_0-z_0'\|_{\beta,T,X}+T^\beta c_{11}(\beta,\varepsilon,\theta,p,M,T,m)\\ &\times\Big(\|z-z'\|_{\beta,T,X}+\|\rho-\rho'\|_{\beta,T,X}\\ &+\|u_0-u_0'\|_{\mathcal{D}_A(2+\beta,p)}+\|\rho_0-\rho_0'\|_{\mathcal{D}_B(2+\beta,p)}\Big).\end{aligned}\tag{78}$$

Now, acting on (77), taking all term $\|w-w'\|_{\beta,T,X}$ to the first hand side, we get

$$\begin{aligned}\|w-w'\|_{\beta,T,X} &\le \Big(1-c_{10}(\beta,\varepsilon,\theta,p,M,T,m)\\ &\times\max\{T^\beta,T^{1-\beta},T^\varepsilon\}\Big)^{-1}\Big\{\|w_0-w_0'\|_{\beta,T,X}\\ &+c_{10}(\beta,\varepsilon,\theta,p,M,T,m)\max\{T^\beta,T^{1-\beta},T^\varepsilon\}\\ &\times\Big(\|z-z'\|_{\beta,T,X}+\|h-h'\|_{0,T,\mathbf{R}}+\|u_0-u_0'\|_{\mathcal{D}_A(2+\beta,p)}\\ &+\|\rho_0-\rho_0'\|_{\mathcal{D}_B(2+\beta,p)}+\|A(u_0-u_0')\|_{\mathcal{D}_A(\beta+\varepsilon,p)}\Big)\Big\},\end{aligned}\tag{79}$$

and acting on (78), taking all term $\|z-z'\|_{\beta,T,X}$ to the first hand

side, we get

$$\|z - z'\|_{\beta,T,X} \le \Big(1 - c_{11}(\beta,\varepsilon,\theta,p,M,T,m)T^{\beta}\Big)^{-1} \times\Big\{\|z_0 - z_0'\|_{\beta,T,X} + c_{11}(\beta,\varepsilon,\theta,p,M,T,m)T^{\varepsilon} \times\Big(\|w - w'\|_{\beta,T,X} + \|u_0 - u_0'\|_{\mathcal{D}_A(2+\beta,p)} + \|\rho_0 - \rho_0'\|_{\mathcal{D}_B(2+\beta,p)}\Big)\Big\}. \tag{80}$$

Finally, solving the system of inequalities (79) in (80) for $\|w - w'\|_{\beta,T,X}$ and $\|z - z'\|_{\beta,T,X}$ we get

$$\|w - w'\|_{\beta,T,X} \le c_{12}(\beta,\varepsilon,\theta,p,M,T,m) \times\Big(\|h - h'\|_{0,T,\mathbf{R}} + \|u_0 - u_0'\|_{\mathcal{D}_A(2+\beta,p)} + \|\rho_0 - \rho_0'\|_{\mathcal{D}_B(2+\beta,p)} + \|A(u_0 - u_0')\|_{\mathcal{D}_A(\beta+\varepsilon,p)} + \|z_0 - z_0'\|_{\beta,T,X}\Big), \tag{81}$$

$$\|z - z'\|_{\beta,T,X} \le c_{13}(\beta,\varepsilon,\theta,p,M,T,m)\Big(\|u_0 - u_0'\|_{\mathcal{D}_A(2+\beta,p)} + \|\rho_0 - \rho_0'\|_{\mathcal{D}_B(2+\beta,p)} + \|z_0 - z_0'\|_{\beta,T,X}\Big), \tag{82}$$

where c_{12} and c_{13} are continuous functions of their arguments for small $T > 0$. To get 38 we observe that w_0 and z_0 are defined in (49) and that H8 holds. □

From the above theorem follows easily theorem 2.2.

Now we study the continuous dependence of the solution for inverse problem (53).

THEOREM 5.3. *Let H1-H11 and estimates (18)-(23) hold. Let h and h' be the solutions to problem (53) related to the data (w_0, z_0, Φ, ℓ) and $(w_0', z_0', \Phi', \ell')$, respectively. Then the map $(u_0, \rho_0, \Phi, \ell) \to h$ is continuous from $\mathcal{D}_A(2+\beta,p) \times \mathcal{D}_B(2+\beta,p) \times \mathcal{L}(X;\mathbf{R}) \times C^{2+\beta}([0,T])$*

to $C^\beta([0,T])$. *Moreover the following estimate holds*

$$\|h-h'\|_{\beta,T,\mathbf{R}} \le c_{15}(\beta,\varepsilon,\theta,p,M,T,m)\max\{T^\beta,T^{1-\beta},T^\varepsilon\}$$
$$\times\Big(\|\Phi-\Phi'\|_{L(X;\mathbf{R})}+\Big|[\Phi(Au_0)]^{-1}-[\Phi(Au_0')]^{-1}\Big|$$
$$+\|\ell-\ell'\|_{C^2([0,T])}+\|u_0-u_0'\|_{\mathcal{D}_A(2+\beta,p)}+\|\rho_0-\rho_0'\|_{\mathcal{D}_B(2+\beta,p)}\Big), \tag{83}$$

where c_{15} *is positive and continuous for small* T.

Proof. Thanks to the existence and uniqueness theorems in [2] and theorem 4.1, we define the closed ball (for suitable $r>0$)

$$Z_r(\beta)=\{(u_0,\rho_0,\Phi,\ell,h)\in Z(\beta):\quad \|(u_0,\rho_0,,\Phi,\ell,h)\|_{Z(\beta)}\le 5r\}$$

where

$$Z(\beta):=\mathcal{D}_A(2+\beta,p)\times\mathcal{D}_B(2+\beta,p)$$
$$\times\mathcal{L}(X;\mathbf{R})\times C^{2+\beta}([0,T])\times C^\beta([0,T])$$

with the norm

$$\|(u_0,\rho_0,\Phi,\ell,h)\|_{Z(\beta)}:=\|u_0\|_{\mathcal{D}_A(2+\beta,p)}+\|\rho_0\|_{\mathcal{D}_B(2+\beta,p)}$$
$$+\|\Phi\|_{\mathcal{L}(X;\mathbf{R})}+\|\ell\|_{C^{2+\beta}([0,T];\mathbf{R})}+\|h\|_{C^\beta([0,T];\mathbf{R})} \tag{84}$$

Let h, h' be the solution of (51) (52) and (53) related to (u_0,ρ_0,Φ,ℓ), $(u_0',\rho_0',\Phi',\ell')$, respectively. For the difference $h-h'$ from (53) we obtain the following equation:

$$h-h'=h_0-h_0'+S_3(w,z,h)-S_3(w',z',h')=h_0-h_0'$$
$$-\chi\Phi\Big\{L_3[h,Au_0]+L_1[N_1(h,w)]+L_1[N_2(w,z)A^{-1}w]$$
$$+L_1[N_3(w,z)B^{-1}z]+N_1(h,w)+N_2(w,z)A^{-1}w+N_3(w,z)B^{-1}z$$
$$+M_1(u_0,\rho_0)S_1(w,z,h)+M_2(u_0,\rho_0)S_2(w,z,h)\Big\}$$
$$-\chi'\Phi'\Big\{L_3[h',Au_0']+L_1[N_1(h',w')]+L_1[N_2(w',z')A^{-1}w']$$
$$+L_1[N_3(w',z')B^{-1}z']+N_1(h',w')$$
$$+N_2(w',z')A^{-1}w'+N_3(w',z')B^{-1}z'$$
$$+M_1(u_0',\rho_0')S_1(w',z',h')+M_2(u_0',\rho_0')S_2(w',z',h')\Big\} \tag{85}$$

From (85), after some calculations, we easily deduce the estimates

$$\begin{aligned}
\|h-h'\|_{\beta,T,\mathbf{R}} &\le \|h_0-h_0'\|_{\beta,T,\mathbf{R}} + \|\chi\Phi-\chi'\Phi'\|_{\mathcal{L}(X,\mathbf{R})} \\
&\times\Big\{ \|L_3[h,Au_0]\|_{\beta,T,X} + \|L_1[N_1(h,w)]\|_{\beta,T,X} \\
&+\|L_1[N_2(w,z)A^{-1}w]\|_{\beta,T,X} + \|L_1[N_3(w,z)B^{-1}z]\| \\
&+\|N_1(h,w)\|_{\beta,T,X} + \|N_2(w,z)A^{-1}w\|_{\beta,T,X} \\
&+\|N_3(w,z)B^{-1}z\|_{\beta,T,X} + \|M_1(u_0,\rho_0)S_1(w,z,h)\|_{\beta,T,X} \\
&+\|M_2(u_0,\rho_0)S_2(w,z,h)\|_{\beta,T,X}\Big\} \\
&+\|\chi'\Phi'\|_{\mathcal{L}(X,\mathbf{R})}\Big\{ \|L_3[h,Au_0]-L_3[h',Au_0']\|\|_{\beta,T,X} \\
&+\|L_1[N_1(h,w)]-L_1[N_1(h',w')]\|_{\beta,T,X} \\
&+\|L_1[N_2(w,z)A^{-1}w]-L_1[N_2(w',z')A^{-1}w']\|_{\beta,T,X} \\
&+\|L_1[N_3(w,z)B^{-1}z]-L_1[N_3(w',z')B^{-1}z']\|_{\beta,T,X} \\
&+\|N_1(h-h',w)\|_{\beta,T,X} + \|N_1(h',w-w')\|_{\beta,T,X} \\
&+\|N_2(w-w',z)A^{-1}w\|_{\beta,T,X} + \|N_2(w,z-z')A^{-1}w\|_{\beta,T,X} \\
&+\|N_2(w',z')A^{-1}(w-w')\|_{\beta,T,X} + \|N_3(w-w',z)B^{-1}z\|_{\beta,T,X} \\
&+\|N_3(w',z-z')B^{-1}z\|_{\beta,T,X} + \|N_3(w',z')B^{-1}(z-z')\|_{\beta,T,X} \\
&+\|[M_1(u_0,\rho_0)-M_1(u_0',\rho_0')]S_1(w,z,h)\|_{\beta,T,X} \\
&+\|M_1(u_0',\rho_0')[S_1(w,z,h)-S_1(w',z',h')]\|_{\beta,T,X} \\
&+\|[M_2(u_0,\rho_0)-M_2(u_0',\rho_0')]S_2(w,z,h)\|_{\beta,T,X} \\
&+\|M_2(u_0',\rho_0')[S_2(w',z',h')-S_2(w,z,h)]\|_{\beta,T,X}\Big\} \qquad (86)
\end{aligned}$$

Thanks to lemmas 4.2–4.10 and to estimate (70) we get

$$\begin{aligned}
\|h-h'\|_{\beta,T,\mathbf{R}} &\le \|h_0-h_0'\|_{\beta,T,\mathbf{R}} + c_{14}(\beta,\varepsilon,\theta,p,M,T) \\
&\times(|\chi|\|\Phi-\Phi'\|_{\mathcal{L}}(X,\mathbf{R}) + \|\Phi'\|_{\mathcal{L}(X,\mathbf{R})}|\chi-\chi'|) + \|\chi'\Phi'\|_{\mathcal{L}(X,\mathbf{R})} \\
&\times\Big\{ T^{\varepsilon}c_1(\beta,\varepsilon,\theta,M,T)\Big(\|h-h'\|_{0,T,\mathbf{R}}\|Au_0\|_{\mathcal{D}_A(\beta+\varepsilon,p)}
\end{aligned}$$

$$
\begin{aligned}
&+\|h'\|_{0,T,\mathbf{R}}\|A(u_0-u_0')\|_{\mathcal{D}_A(\beta+\varepsilon,p)}\Big)\\
&+T^{\beta}c_2(\beta,\varepsilon,\theta,p,M,T)\Big(\|h-h'\|_{\beta,T,\mathbf{R}}\|w\|_{0,T,X}\\
&+\|h'\|_{\beta,T,\mathbf{R}}\|w-w'\|_{0,T,X}\Big)\\
&+T^{1-\beta}\max\{rc_8(r,T),c_4(r,T),rc_9(r,T),c_5(r,T),\\
&\times\|A^{-1}\|_{L(X)},\|B^{-1}\|_{L(X)}\}\\
&\times\Big(\|w-w'\|_{\beta,T,X}+\|z-z'\|_{\beta,T,X}\\
&\times\|u_0-u_0'\|_{\mathcal{D}_A(2+\beta,p)}+\|\rho_0-\rho_0'\|_{\mathcal{D}_B(2+\beta,p)}\Big)\\
&+(2T+T^{1-\beta})\Big(\|h-h'\|_{C^{(}[0,T];\mathbf{R})}\|w\|_{C^{\beta}([0,T];X)}\\
&+\|h'\|_{C^{(}[0,T];\mathbf{R})}\|w'\|_{C^{\beta}([0,T];X)}\Big)\\
&+T^{1-\beta}2r\max\{c_8(r,T),c_3(r,T)\}\big(\|w-w'\|_{0,T,X}+\|z-z'\|_{0,T,X}\big)\\
&+T^{1-\beta}c_8(r,T)c_{13}(\beta,\varepsilon,\theta,p,M,T,r)\Big(\|h-h'\|_{0,T,\mathbf{R}}\\
&+\|u_0-u_0'\|_{\mathcal{D}_A(2+\beta,p)}+\|\rho_0-\rho_0'\|_{\mathcal{D}_B(2+\beta,p)}\Big)\\
&+T^{1-\beta}2rc_8(r,T)\Big(\|u_0-u_0'\|_{\mathcal{D}_A(2+\beta,p)}+\|\rho_0-\rho_0'\|_{\mathcal{D}_B(2+\beta,p)}\Big). \quad (87)
\end{aligned}
$$

From estimates (82) and (87) we deduce, after some calculation, estimate (83). □

6. The evolution equations for combustion of a material with memory

One of the main tools for the formulation of governing equations for physical problems are the conservation principles. When the problem considered involves a reaction process coupled with diffusion, the conservation principle provides us a set of partial differential equations to be obeyed by the unknown quantities of the system. In our case such quantities are the mass concentration and the temperature in heat conduction. Let us first deduce the equation for the density $\rho(t,x)$ at time t and position x in a material Ω in $\mathbf{R}^n$.

The principle of conservation states that *for any subdomain R of Ω with a smooth boundary S the rate of change of mass density is*

equal to the rate of flux across S plus the rate of generation within R. This statement is a balance relation in which the flux, denoted by the vector J, is the density flow per unit surface area per unit time. Let ν be the outward normal vector on S and q be the rate of generation per unit volume per unit time in R. Assume that ρ, J and q are continuous in x, and that ρ has a continuous derivative in t. Then the conservation principle can be written as

$$D_t \int_R \alpha\, \rho(t,x)\, dx = -\int_S J(t,x)\cdot \nu\, ds + \int_R q(t,x)\, dx \tag{88}$$

where α is a constant. Thanks to the divergence theorem and since R is a domain independent of time we also have

$$\int_R [\alpha\, D_t\rho(t,x) + \operatorname{div} J(t,x) - q(t,x)]\, dx = 0 \tag{89}$$

For the arbitrariness of the subdomain R it follows that

$$\alpha\, D_t\rho(t,x) + \operatorname{div} J(t,x) - q(t,x) = 0 \quad in \ \ \Omega \tag{90}$$

We now have to relate the diffusion flux J to the density function ρ. In the case of a chemical reaction process the Fick's law states that *in absence of convection, the flux is pointwise proportional to the negative gradient of the density i.e.*

$$J = -d^*\nabla\rho \tag{91}$$

where d^* is a strictly positive function in Ω. Replacing (91) in (90) we obtain the evolution equation for the density

$$D_t\rho(t,x) = \operatorname{div}[d\nabla\rho(t,x)] + Q(t,x) \tag{92}$$

where $d = d^*/\alpha$, $Q = q/\alpha$. The function D is called the diffusion coefficient in chemical diffusion processes. We point out that in the case of combustion of several materials, with densities ρ_1,...,ρ_r we have to consider r evolution equations of type (92) for every density ρ_j, $(j = 1, ..., r)$.

Let us now turn our attention to the evolution of the temperature. Also in this case we use the same conservation principle to get the equation of continuity

$$\alpha_1 D_t u(t,x) + \operatorname{div} J_1(t,x) - q_1(t,x) = 0 \quad in \ \ \Omega \tag{93}$$

We point out that the quantities α_1, J_1 and q_1 are defined as above, but they are related to the heat flux. Fick's law (91) in the case of heat conduction is called Fourier's law and is given by

$$J_1(t,x) = -d_1^* \nabla u(t,x) \tag{94}$$

To take the thermal memory effects into account, one of the best modification of Fourier's law, supported by experiments, leads to replace (91) by

$$J_1 = -d_1^* \nabla u(t,x) - k \int_0^t h(t-s) \nabla u(s,x)\, ds \tag{95}$$

where k is a positive function and h is the convolution kernel, which accounts for the thermal memory. To obtain the equation for the evolution governing the temperature we replace (95) into the continuity equation (90) and get

$$D_t u(t,x) = \operatorname{div} [d_1 \nabla u(t,x) + \int_0^t h(t-s) d_1 \nabla u(s,x)\, ds] + Q_1(t,x) \tag{96}$$

where we have set for simplicity $d_1 := d_1^*/\alpha_1$, $k\alpha_1 = d_1$ for and $Q_1 = q_1/\alpha$. We remark that we have set $d_1^* = k$ just for the sake of simplicity, to obtain operator $d_1 \nabla u(t,x)$ also in the integral of (96) so that in the abstract version of the problem (c.f. (6)) we can use the same abstract operator A. What follows still hold in the case we replace operator A in the integral by qA where q is a proportional coefficient. In the concrete case q generally depends on the space variables only.

In the case of nonisothermal chemical reaction process involving a single species of combustible with density ρ the functions Q and Q_1 depend, in turn on the temperature and on the density of the material through a given function f. More precisely

$$Q(t,x) = -a_1 f(u(t,x), \rho(t,x)), \qquad Q_1(t,x) = a_2 f(u(t,x), \rho(t,x)) \tag{97}$$

where a_1 is the Thile number and a_2/a_1 is the Prater temperature. If the reaction is irreversible function f, according to the Arrehenius

kinetic, is given by

$$f(u,\rho) = \rho^{\eta} \exp(\gamma - \gamma/u) \tag{98}$$

where γ is a positive constant related to the activation energy of the reaction, and it is called Arrhenius number, η is called the order of reaction and it is a real number, but for most of the reactions is positive. For more details see for example [1], [3], [5].

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