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Filters and Pathwise Connectifications

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Summary. - *Let p be a free open-filter on a Hausdorff space X. In this paper we investigate when $X \cup \{p\}$ can be densely embedded in a pathwise connected T_2-space. The main part of the paper is devoted to the cases where X is the rational or the real line.*

1. Introduction

In the last years, the subject of connectifications has swiftly developed and expanded, proving to be one of the most attractive and fruitful in modern general topology (as basic references, see [6], [5] and [1]). In a similar vein, it would seem very natural to investigate the related topic of pathwise connectifications; all the same that, while the notions of connected and pathwise connected space are in some sense two rather close ones, some very recent results show how much connectfiability and pathwise connectifiability may be far to each other (cfr., for example, [3, Remark 2.3 and Example 2.4]) Actually, pathwise connectifications still look as an almost untouched

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field, and they certainly offer an ample room for further, extensive researches.

In this paper, we deal with the following general problem. Let X be a T_2-space, and p a free open filter on X (so that p may be considered, in the standard way, as a point added to X, and $X \cup \{p\}$ is a T_2-space): which kind of links are there between the inner properties of p as a filter, and the (T_2)-pathwise connectifiabilty of $X \cup \{p\}$? (We recall that, according to [3] and [4], a T_2-space is called pathwise connectifiable if it can be densely embedded in a pathwise connected T_2-space).

On the one hand, if the space X and the open-filter p have especially good properties, it is possible that $X \cup \{p\}$ is already pathwise connected. Actually, the one-point compactification of a locally compact, σ-compact, pathwise connected T_2-space X is just a space of that kind; and the same holds for the real line R with the added point $+\infty$. On the other hand, it is proved in [3, Example 2.1] that if p is a maximal free open-filter on X, then $X \cup \{p\}$ is not pathwise connectifiable. What about other, less trivial situations?

Clearly an ideal result in this vein would be a necessary and sufficient condition for the space $X \cup \{p\}$ to be pathwise connectifiable. Such a condition should be expressed using both the topological structure of X and the set-theoretical properties of p — which is, after all, a collection of (open) subsets of X. This is noteworthy because, on the contrary, the possibility of envisaging $X \cup \{p\}$ as a dense subset of a pathwise connected T_2-space is a kind of quite outer property.

As we will see later, however, such an effective and enlightening characterization looks really hard to be obtained, for the general case. In this paper, after giving some very basic results valid for every T_2-space X, we investigate in detail the cases where X is either the rational or the real line. Even in these quite special situations (which turn out to be essentially different from each other), we do not get a complete characterization. But several results in both directions show, on the one hand, how promising this line of investigation might be; and, on the other hand, they make the complexity of the problem stand out. Let us also observe that, for the case of the real line, it would be trivial wondering about the connectifiability of $\mathbf{R} \cup \{p\}$, where p is a free open filter on $\mathbf{R}$, because a space of this kind is always connected.

We refer the reader to [2] for notations and terminology not explicitly given.

2. Definitions and general results

Let (X, τ) be a topological space. A collection p of open subsets of X is said to be an *open-filter* if:

a) $p \neq \emptyset$, $p \neq \tau$;

b) $\forall A, B \in p \colon A \cap B \in p$;

c) $\forall A \in p \colon \forall B \in \tau \colon (A \subseteq B \Longrightarrow B \in p)$.

If p is a open-filter on X and $\mathcal{B}$ is a subcollection of p such that every element of p contains some element of $\mathcal{B}$, and for every $A, B \in \mathcal{B}$ there is a $C \in \mathcal{B}$ with $C \subseteq A \cap B$, then we say that $\mathcal{B}$ is a *base* for p. On the other hand, given any collection $\mathcal{B}$ of open subsets of X, such that for every $A, B \in \mathcal{B}$ there exists $C \in \mathcal{B}$ with $C \subseteq A \cap B$, it "generates" a (unique) open-filter p on X, for which $\mathcal{B}$ is a base.

For p an open-filter on X, we may consider the topological space $X \cup \{p\}$, where X — as a subspace of $X \cup \{p\}$ — has the same original topology, while the point p is endowed with the local base: $\{F \cup \{p\} \mid F \in p\}$.

We say that the open-filter p on X is *free* if

$$\forall x \in X \colon \exists V \text{ neighbourhood of } x \colon \exists F \in p \colon V \cap F = \emptyset.$$

Clearly, if X is a T_2-space, then the open-filter p is free if and only if $X \cup \{p\}$ is T_2.

Let X be a topological space, and Y a subspace of X. Then, if p is an open-filter on X, the collection $\{F \cap Y \mid F \in p\}$ is an open-filter on Y. It will be denoted by $p|_Y$. Observe that if p is free, then $p|_Y$ is free, too.

We have already recalled the result of [3], where it is proved that if p is a free maximal open-filter on a T_2 topological space X, then $X \cup \{p\}$ is not pathwise connectifiable. We will prove now other general results, using similar techniques.

Let $\{A_n \mid n \in \mathbf{N}\}$ be a cellular family in Z (i.e., a family of non-empty open subsets of Z such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$),

$\Phi_n \subseteq \mathcal{P}(Z)$ for every $n \in \mathbf{N}$ and let $\mathcal{U}$ be a free ultrafilter on $\mathbf{N}$. For every $F \in \mathcal{U}$ and $\phi \in \Phi = \prod_n \Phi_n$ set $A_F = \bigcup\{A_i \mid i \in F\}$ and $\phi^* = \bigcup\{\phi(i) \mid i \in \mathbf{N}\}$.

If the family $\{A_{F,\phi} \mid F \in \mathcal{U},\, \phi \in \Phi\}$, where $A_{F,\phi} = (A_F \cap \phi^*) \cup \{p\}$, is a base for the space Z at a point p, then we have the following

LEMMA 2.1. *If Z is dense in a Hausdorff space Y, then there are no sequences in $Y \setminus \{p\}$ converging to p.*

Proof. Let us suppose that there is a sequence $\{q_n\}_{n \in \mathbf{N}}$ in $Y \setminus \{p\}$ converging to p, with $q_n \neq q_m$ whenever $n \neq m$.

Without loss of generality we may assume that for every $n \in \mathbf{N}$ there is a pair of disjoint open subsets C_n and D_n of Y such that: $q_n \in C_n$, $p \in D_n$, $C_{n+1} \cup D_{n+1} \subseteq D_n$, $D_n \cap Z = A_{F_n,\phi_n}$ with $F_{n+1} \subset F_n$.

Since $\mathcal{U}$ is an ultrafilter, we may assume that $F = \bigcup\{F_{2n} \setminus F_{2n+1} \mid n \in \mathbf{N}\} \in \mathcal{U}$.

Let $G = \bigcup\{D_{2n} \cap A_{F_{2n} \setminus F_{2n+1}} \mid n \in \mathbf{N}\} \cup \{p\}$. Now $q_{2n} \notin \mathrm{Cl}_Y(G)$ for every $n \in \mathbf{N}$, in fact $q_{2n} \in C_{2n}$ and

$$\begin{aligned} C_{2n} \cap G \subseteq \left(C_{2n} \cap \Big(\bigcup_{i=1}^{n-1} A_{F_{2i} \setminus F_{2i+1}}\Big)\right) \;&\cup\; \left(C_{2n} \cap \Big(\bigcup_{i \geq n} D_{2i}\Big)\right) \cup \\ &\cup\; (C_{2n} \cap \{p\}) = \emptyset. \end{aligned}$$

Since G is a neighbourhood of p in Z, we reach a contradiction. $\square$

COROLLARY 2.2. *Let X be a T_2 topological space and $\{A_n \mid n \in \mathbf{N}\}$ a discrete family of open subsets of X. Fix a free ultrafilter $\mathcal{U}$ on $\mathbf{N}$ and let p be the free open-filter on X generated by $\{\bigcup_{n \in F} A_n \mid F \in \mathcal{U}\}$. Then $X \cup \{p\}$ is not pathwise connectifiable.*

Proof. Trivial. $\square$

COROLLARY 2.3. *Let X be a T_3-space and $\{x_n \mid n \in \mathbf{N}\}$ a closed and discrete subset of X (with $n \mapsto x_n$ one-to-one). Let $\mathcal{U}$ be a free ultrafilter on $\mathbf{N}$, and p the (free) open-filter on X defined by: $p = \{A \text{ open in } X \mid \{n \in \mathbf{N} \mid x_n \in A\} \in \mathcal{U}\}$. Then $X \cup \{p\}$ is not pathwise connectifiable.*

Proof. Since X is T_3, there is a cellular family $\{A_n \mid n \in \mathbf{N}\}$ in X such that $x_n \in A_n$ for every $n \in \mathbf{N}$. Take as Φ_n the family of all open neighbourhoods of x_n in X, $Z = X \cup \{p\}$ and apply Lemma 2.1. □

We conclude this section by putting a natural general question, which will be solved at the end of the paper. Suppose τ_1, τ_2 are two topologies on a set X, with τ_1 coarser that τ_2: then the pathwise connectedness of (X, τ_2) implies that of (X, τ_1). If we suppose that (X, τ_1) (and hence also (X, τ_2)) is T_2, then does the pathwise connectifiability of (X, τ_2) implies that of (X, τ_1)? In particular, if p_1, p_2 are two free open filters on a T_2-space X, with $p_1 \subseteq p_2$, one could ask whether the pathwise connectifiability of $X \cup \{p_2\}$ implies that of $X \cup \{p_1\}$.

The answer is no, in fact Example 4.10 together with Corollary 2.2 will provide us with a counterexample.

3. The case of the rational line

In this section we tackle the general problem considered in the introduction, in the special case when $X = \mathbf{Q}$. The main result will be that $\mathbf{Q} \cup \{p\}$ is pathwise connectifiable, if p has a countable base when restricted to some open subset of $\mathbf{Q}$.

We need a preliminary result.

Lemma 3.1. *Let A be an open (nonempty) subset of $\mathbf{Q}$, and let p be a free open-filter on $\mathbf{Q}$ such that $p \in \mathrm{Cl}_{X \cup \{p\}} A$ and the filter $p|_A$ has a countable base. Then there exists a subset B of A such that:*

a) B is clopen in $\mathbf{Q}$;

b) $p \in \mathrm{Cl}_{X \cup \{p\}} B$;

c) there exists a countable, strictly decreasing base $\{B_n \mid n \in \mathbf{N}\}$ for $p|_B$, such that every B_n is clopen in B (hence in $\mathbf{Q}$) and $B_1 = B$.

Proof. Since the open-filter p is free, by hypothesis it is possible to get a countable, strictly decreasing base $\{A_n \mid n \in \mathbf{N}\}$ for $p|_A$, such that for every $n \in \mathbf{N}$ the set $A_n \setminus A_{n+1}$ has nonempty interior in $\mathbf{Q}$. Choosing, for every $n \in \mathbf{N}$, a nonempty $M_n \subseteq A_n \setminus A_{n+1}$ which is clopen in $\mathbf{Q}$, we have that $B = \bigcup_{n \in \mathbf{N}} M_n$ is in turn clopen in $\mathbf{Q}$. Indeed, let $a \in \mathbf{Q} \setminus B$: then there exists a neighbourhood V of a in $\mathbf{Q}$

and an $n \in \mathbf{N}$ such that $V \cap A_n = \emptyset$. Therefore, $V \cap (\mathbf{Q} \setminus \bigcup_{h=1}^{n-1} M_h)$ is a neighbourhood of a in $\mathbf{Q}$ which misses B.

It is clear that B satisfies b); putting, for every $n \in \mathbf{N}$, $B_n = \bigcup_{h=n}^{\infty} M_h$, we get a countable base for $p|_B$ fulfilling c). $\square$

THEOREM 3.2. *Let p be a free open-filter on $\mathbf{Q}$, and suppose there exists an open subset A of $\mathbf{Q}$ such that $p \in \mathrm{Cl}_{\mathbf{Q}\cup\{p\}}A$ and $p|_A$ has a countable base. Then $\mathbf{Q} \cup \{p\}$ is pathwise connectifiable.*

Proof. By Lemma 3.1, there exists a clopen $B \subseteq \mathbf{Q}$, with $p \in \mathrm{Cl}_{X\cup\{p\}}B$, such that $p|_B$ admits a countable, strictly decreasing base $\{B_n \mid n \in \mathbf{N}\}$ consisting of clopen subsets of $\mathbf{Q}$, with $B_1 = B$. We may also suppose, up to replacing B with B_2, that $Q \setminus B \neq \emptyset$. For every $n \in \mathbf{N}$, put $L_n = B_n \setminus B_{n+1}$.

Since $S = \mathbf{Q} \setminus B$ is countable and p is free, it is easy to find by induction a sequence $(S_n)_{n\in\mathbf{N}}$ of clopen subsets of S, such that:

1) $S_n \cap S_{n'} = \emptyset$ for $n \neq n'$;

2) $\bigcup_{n\in\mathbf{N}} S_n = S$;

3) $p \notin \mathrm{Cl}_{\mathbf{Q}\cup\{p\}}S_n$ for every $n \in \mathbf{N}$.

Let h be a dense topological embedding of $\mathbf{Q}$ in $\mathbf{R}$ such that, for every $n \in \mathbf{N}$, $h|_{L_n}$ is a (dense) topological embedding of L_n in $]n-1, n[$, while $h|_{S_n}$ is a (dense) topological embedding of S_n in $]-n, -n+1[$. Let also $Q = h(\mathbf{Q})$ and p_h be the open-filter on Q defined by $A \in p_h \iff h^{-1}(A) \in p$, so that $Q\cup\{p_h\}$ is homeomorphic to $\mathbf{Q} \cup \{p\}$.

Let $\check{p}$ be the open-filter on $\mathbf{R}$ defined by:

$$\check{p} = \{\Omega \text{ open in } \mathbf{R} \mid \Omega \cap Q \in p_h \text{ and } \exists n \in \mathbf{N}\colon]n, +\infty[\subseteq \Omega\}.$$

Observe that $\check{p}$ is free. Indeed, given any $r \in \mathbf{R}$, there exists a neighbourhood V of r such that the set $T = \{m \in \mathbf{Z} \mid V \cap]m-1, m[\neq \emptyset\}$ contains either one or two consecutive numbers; then put $\mathcal{M} = \{h^{-1}(\,]m-1, m[\,) \mid m \in T\} \subseteq \{L_n \mid n \in \mathbf{N}\} \cup \{S_n \mid n \in \mathbf{N}\}$. Since $p \notin \mathrm{Cl}_{\mathbf{Q}\cup\{p\}}L_n$ and $p \notin \mathrm{Cl}_{\mathbf{Q}\cup\{p\}}S_n$ for every $n \in \mathbf{N}$, there exists an element F of p missing every element of $\mathcal{M}$, so that $h(F)$ misses every element of $\{h(M) \mid M \in \mathcal{M}\}$. Let $I = \mathrm{Cl}_{\mathbf{R}}(\bigcup_{M\in\mathcal{M}} h(M)) = \mathrm{Cl}_{\mathbf{R}}(\bigcup_{m\in T}]m-1, m[\,)$, then I is a closed bounded interval of $\mathbf{R}$ including V, and $\mathbf{R}\setminus I$ belongs to $\check{p}$ because $(\mathbf{R}\setminus I)\cap Q \supseteq h(F) \in p_h$.

Since it is clear that $\mathbf{R}\cup\{\check{p}\}$ is pathwise connected, if we can prove that $\check{p}|_Q = p_h$, we will have that $\mathbf{R}\cup\{\check{p}\}$ is a pathwise connectification of $Q\cup\{p_h\}$, which is homeomorphic to $\mathbf{Q}\cup\{p\}$. Of course, it suffices to show that $p_h \subseteq \check{p}|Q$. Let $\tilde{F} = h(F)$ be any element of p_h, with $F \in p$, and let Ω' be an open subset of $\mathbf{R}$ with $\Omega' \cap Q = \tilde{F}$; since, for some $n \in \mathbf{N}$, $B_n = \bigcup_{n' \geq n} L_{n'} \subseteq F$, we have that $h(\bigcup_{n' \geq n} L_{n'}) = \,]n, +\infty[\cap Q \subseteq h(F) = \tilde{F}$. Let $\Omega = \Omega' \cup]n, +\infty[$, then $\Omega \cap Q = \tilde{F} \cup (\,]n, +\infty[\cap Q) = \tilde{F}$, so that $\Omega \in \check{p}$ and $\tilde{F} \in \check{p}|_Q$. $\square$

REMARK 3.3. *It is easy to check that in the above result the assumption that A is open may be replaced by the weaker condition:* $\mathrm{Cl}_{\mathbf{Q}}(\mathrm{Int}_{\mathbf{Q}}(\mathrm{Cl}_{\mathbf{Q}} A)) = \mathrm{Cl}_{\mathbf{Q}} A$ *— which means that A is a dense subset of some open $B \subseteq \mathbf{Q}$. Indeed, in this case $p|_B$ has a countable base, once $p|_A$ has.*

We are now going to provide two examples showing, on the one hand, that the sufficient condition for pathwise connectifiability of $\mathbf{Q} \cup \{p\}$, given by Theorem 3.2, is not necessary; and, on the other hand, that in such a condition we cannot drop the assumption that A is open (or, at least, dense in an open set). Notice that another example of the first fact — obtained by a quite different construction — will be given at the end of the paper (cfr. Remark 4.11 after Example 4.10).

EXAMPLE 3.4. *There exists a free open-filter p on $\mathbf{Q}$ such that $\mathbf{Q} \cup \{p\}$ is pathwise connectifiable, but for every $A \subseteq \mathbf{Q}$ with $p \in \mathrm{Cl}_{\mathbf{Q}\cup\{p\}} A$, $p|_A$ has no countable base.*

Proof. Let $Y = [0, 1[\, \times [0, 1] \cup \{p\}$ be the space in which $Y \setminus \{p\}$ has the euclidean topology and a base for Y at the point p consists of the sets of the form $G \cup \{p\}$, where G is an open subset of $Y \setminus \{p\}$ containing $]\varepsilon, 1[\, \times \{1\}$ for some $\varepsilon \in [0, 1[$.

Clearly $\mathbf{Q}$ is homeomorphic to the dense subspace

$$X = ([0, 1[\, \cap \mathbf{Q})^2$$

of Y and Y is a pathwise connected Hausdorff space.

It remains to show that for every $A \subseteq X$ such that $p \in \mathrm{Cl}_{X\cup\{p\}} A$, the filter $p|_A$ has no countable base. Let $\{A_n \mid n \in \omega\}$ be a countable subfamily of $p|_A$ and take $q_n = (x_n, y_n) \in A_n$ so that $x_n < x_{n+1}$ for every $n \in \omega$ and $x_n \to 1$. Since $G = Y \setminus \{q_n \mid n \in \omega\}$ is an open set which includes $]0,1[\times\{1\}$ and no A_n, it follows that $\{A_n \mid n \in \omega\}$ is not a base for $p|_A$. $\square$

EXAMPLE 3.5. *There exist an infinite subset A of $\mathbf{Q}$ and a free open-filter p on $\mathbf{Q}$ such that $p \in \mathrm{Cl}_{\mathbf{Q}\cup\{p\}} A$ and $p|_A$ has a countable base, but $\mathbf{Q} \cup \{p\}$ is not pathwise connectifiable.*

Proof. Put $A = \mathbf{N}$. Let p be the free open filter on $\mathbf{Q}$ generated by the family $\mathcal{B}$ of all open subsets of $\mathbf{Q}$ containing a tail of $\mathbf{N}$.

Since the restriction of $\mathcal{B}$ to $\mathbf{N}$ is countable, $p|_{\mathbf{N}}$ has a countable base; also, it is clear that $p \in \mathrm{Cl}_{\mathbf{Q}\cup\{p\}}\mathbf{N}$. Thus, we only have to prove that $\mathbf{Q} \cup \{p\}$ is not pathwise connectifiable.

By contradiction, suppose $\mathbf{Q} \cup \{p\}$ may be envisaged as a dense subspace of a pathwise connected, T_2-space Z. Let us fix an arc ϑ in Z, with $\vartheta(0) = z \neq p$ and $\vartheta(1) = p$. Take a strictly increasing sequence $\{s_n \mid n \in \mathbf{N}\}$ of elements of $[0,1[$, such that $\sup_{n\in\mathbf{N}} s_n = 1$ and $\{\vartheta(s_n) \mid n \in \mathbf{N}\} \cap \mathbf{N} = \emptyset$. Observe that $K_1 = \{\vartheta(s_n) \mid n \in \mathbf{N}\} \cup \{p\}$ and $K_2 = \mathbf{N} \cup \{p\}$ are compact subsets of Z (the second fact depends on the definition of the filter p). Therefore, $(\mathrm{Cl}_Z\mathbf{N}) \cap \{\vartheta(s_n) \mid n \in \mathbf{N}\} = \emptyset$ and $(\mathrm{Cl}_Z\{\vartheta(s_n) \mid n \in \mathbf{N}\}) \cap \mathbf{N} = \emptyset$; thus it is possible to associate to every $n \in \mathbf{N}$ two open subsets U_n, V_n of Z, in such a way that:

1) $\forall n \in \mathbf{N}\colon (n \in U_n \text{ and } \vartheta(s_n) \in V_n)$;

2) $\forall n, n' \in \mathbf{N}\colon U_n \cap V_n = \emptyset$;

3) $\forall n, n' \in \mathbf{N}\colon (n \neq n' \Longrightarrow (U_n \cap U_{n'} = \emptyset \text{ and } V_n \cap V_{n'} = \emptyset))$.

Let $U = \bigcup_{n\in\mathbf{N}} U_n$: since $W = Y \cap (U \cup \{p\})$ is an open neighbourhood of p in Y, there exists A open in Z such that $A \cap Y = W$. Then the density of Y in Z implies that $A \subseteq \mathrm{Cl}_Z W$; since, for every $n \in \mathbf{N}$, $V_n \cap \mathrm{Cl}_Z U = \emptyset$, we have that $\{\vartheta(s_n) \mid n \in \mathbf{N}\} \cap A = \emptyset$. Clearly, this contradicts the continuity of ϑ at 1. $\square$

4. The case of the real line

If p is a free open-filter on the real line $\mathbf{R}$, the pathwise connectifiability of $\mathbf{R} \cup \{p\}$ implies that of $\mathbf{Q} \cup \{p|_{\mathbf{Q}}\}$. The converse is not true, as it easily follows from the main result of this section. This corresponds, in some sense, to our intuitive feeling that the real line is much more "rigid" than the rational line.

Actually, we will characterize here the free open-filters p, having a *countable base*, for which $\mathbf{R} \cup \{p\}$ is pathwise connectifiable; and we will give an example to show that the assumption on the countable base cannot be dropped.

LEMMA 4.1. *Let Y be a T_2-space containing the real line $\mathbf{R}$ as a subspace, and let $y \in (\mathrm{Cl}_Y \mathbf{R}) \setminus \mathbf{R}$. Then for every open neighbourhood V of y in Y we have that $V \cap \mathbf{R}$ is unbounded.*

Proof. If, by contradiction, $V \cap \mathbf{R} \subseteq [a, b]$ with $a, b \in \mathbf{R}$, $a < b$, then $U = Y \setminus [a, b]$ is an open neighbourhood of y in Y, and the same holds for $U \cap V$. But this contradicts $y \in \mathrm{Cl}_Y \mathbf{R}$, because $(U \cap V) \cap \mathbf{R} = \emptyset$. $\square$

LEMMA 4.2. *Let X be a topological space, $n \in \mathbf{N}$ and $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n$ open subsets of X. If $A_0 \setminus A_n$ has nonempty interior, then there exists $j \in \{0, \ldots n-1\}$ such that $A_j \setminus A_{j+1}$ has nonempty interior.*

Moreover, if (X, d) is a metric space and $A_0 \setminus A_n$ has unbounded interior, then there exists $j \in \{0, \ldots n-1\}$ such that $A_j \setminus A_{j+1}$ has unbounded interior.

Proof. Suppose first that there exists an open nonempty V such that $V \subseteq A_0 \setminus A_n$, and let $j = \max\{i \in \{0, \ldots, n\} \mid V \cap A_i \neq \emptyset\}$: this definition is correct because $V \cap A_0 = V \neq \emptyset$. Moreover, $j < n$, because $V \subseteq A_0 \setminus A_n$. Then $j + 1 \in \{0, \ldots, n\}$ and $V \cap A_{j+1} = \emptyset$; putting $W = V \cap A_j$, we have a nonempty open set which is contained in $A_j \setminus A_{j+1}$.

Suppose now that (X, d) is a metric space, and that the open nonempty V above is also unbounded in (X, d). Fix an $\bar{x} \in X$, and for every $\varepsilon > 0$ let $\overline{S}_d(\bar{x}, \varepsilon) = \{y \in X \mid d(\bar{x}, y) \leq \varepsilon\}$; also, for every $m \in \mathbf{N}$ and $i \in \{0, \ldots, n\}$, put $A_i^m = A_i \setminus \overline{S}_d(\bar{x}, m)$. It follows from the above result that for every $m \in \mathbf{N}$ there is an $i_m \in \{0, \ldots, n\}$ such that $A_{i_m}^m \setminus A_{i_m+1}^m = A_{i_m}^m \setminus A_{i_m+1}$ has nonempty interior. Let $j \in \{0, \ldots, n\}$ such that $i_m = j$ for infinitely many $m \in \mathbf{N}$: then $A_j \setminus A_{j+1}$ has unbounded interior. $\square$

DEFINITION 4.3. *We will call p_+ (respectively, p_-) the free open-filter on* $\mathbf{R}$ *having as a (countable) base the collection* $\{\,]n, +\infty[\, |\, n \in \mathbf{N}\}$ *(respectively,* $\{\,]-\infty, -n[\, |\, n \in \mathbf{N}\}$*).*

PROPOSITION 4.4. *Let p be a free open-filter on* $\mathbf{R}$*. Then* $X = \mathbf{R} \cup \{p\}$ *is pathwise connected if and only if either* $p \subseteq p_+$ *or* $p \subseteq p_-$*.*

Proof. If $p \subseteq p_+$, let $Y =]-\infty, +\infty]$ with the standard topology. Since Y is pathwise connected, X also is, as a continuous image of Y. If $p \subseteq p_-$, the argument is symmetric.

Suppose now $\mathbf{R} \cup \{p\}$ pathwise connected and let φ be a one-to-one path in $\mathbf{R} \cup \{p\}$ with $\varphi(1) = p$. Then $\varphi|_{[0,1[}$ is either strictly increasing or strictly decreasing. In the first case, given any $G \in p$, we have that $G \cup \{p\}$ is a neighbourhood of p in $\mathbf{R} \cup \{p\}$, thus there exists $t \in]0, 1[$ such that $\varphi(\,]t, 1]) \subseteq G \cup \{p\}$, whence $\varphi(\,]t, 1[\,) \subseteq G$. Clearly, $\varphi(\,]t, 1[\,) =]f(t), l[$ for some $l \in]f(t), +\infty]$; but it is impossible that $l \in \mathbf{R}$, because this would contradict the free character of p. Therefore, $\varphi(\,]t, 1[\,) =]\varphi(t), +\infty[\subseteq G$ and $G \in p_+$.

If $\varphi|_{[0,1[}$ is strictly decreasing, we prove symmetrically that $p \subseteq p_-$. □

PROPOSITION 4.5. *Let p be a free open-filter on* $\mathbf{R}$*, contained neither in p_+ nor in p_-, and for which there exists $A \in p$ such that for every $B \in p$,* $\mathrm{Int}(A \setminus B)$ *is bounded. Then* $\mathbf{R} \cup \{p\}$ *is not pathwise connectifiable.*

Proof. Suppose we may envisage $\mathbf{R} \cup \{p\}$ as a subspace of a pathwise connected T_2-space Y, and fix a path φ in Y with $\varphi(0) = 0$ and $\varphi(1) = p$. We want to prove that, given any $A \in p$, there exists $B \in p$ such that $\mathrm{Int}_{\mathbf{R}}(A \setminus B)$ is unbounded. Let V be an open subset of Y with $V \cap \mathbf{R} = A$, and let $t \in [0, 1[$ be such that $\varphi([t, 1]) \subseteq V$: then by Proposition 4.4 there exists $\bar{y} \in \varphi([t, 1[\,) \setminus \mathbf{R}$. Let U, W be open neighbourhoods in Y of $p, \bar{y}$, respectively, such that $U \cap W = \emptyset$ and $U, W \subseteq V$. Then by Lemma 3.1 the set $C = W \cap \mathbf{R}$ is unbounded; putting $B = U \cap \mathbf{R}$, we have $B \in p$ and $C \subseteq \mathrm{Int}_{\mathbf{R}}(A \setminus B)$. □

LEMMA 4.6. *Let A be an open subset of* $\mathbf{R}$ *with* $\sup A = +\infty$*. Then there exists a T_2-space Y containing* $[-\infty, +\infty[$ *as a subspace, such that* $Y \setminus [-\infty, +\infty[$ *is homeomorphic to* $[0, 1]$*,* $A \cup (Y \setminus [-\infty, +\infty[)$ *is open in Y and* $\mathbf{R}$ *is open and dense in Y.*

Proof. Up to strictly increasing auto-homeomorphisms of $[-\infty, +\infty[$, we may suppose $A \supseteq \bigcup_{n \in \mathbf{N}} [2n, 2n+1]$. Let $Y = (I \times \{0\}) \cup [-\infty, +\infty[$, where $I = [0,1]$, and σ be the topology of the disjoint sum on Y. We put

$$\tau = \{B \in \sigma \mid \forall (t,0) \in B \colon \exists k \in \mathbf{N} \colon \{t + 2i \mid i \in \mathbf{N}, i \geq k\} \subseteq B\}.$$

It is easily seen that τ is a topology on Y, and is T_2. As for the last fact, observe in particular that if t', t'' are distinct elements of I, then for $\delta < |t' - t''|/2$ we have that

$$((\,]t' - \delta, t' + \delta[\, \cap I) \times \{0\}) \cup \bigcup_{i \in \mathbf{N}}]t' + 2i - \delta, t' + 2i + \delta[$$

and

$$((\,]t'' - \delta, t'' + \delta[\, \cap I) \times \{0\}) \cup \bigcup_{i \in \mathbf{N}}]t'' + 2i - \delta, t'' + 2i + \delta[$$

are disjoint (open) τ-neighbourhoods of $(t', 0)$ and $(t'', 0)$, respectively. Moreover, by the definition of τ it is apparent that

$$Y \setminus [-\infty, +\infty[= I \times \{0\}$$

is homeomorphic to I, that τ induces on $[-\infty, +\infty[$ the usual topology of the extended real line, that $\mathbf{R}$ is open and dense in (Y, τ), and that $A \cup (Y \setminus ([-\infty, +\infty[)$ is τ-open. ☐

Remark 4.7. *A symmetric statement to that of the above lemma holds if A is an open subset of* $\mathbf{R}$ *with* $\inf A = -\infty$.

Theorem 4.8. *Let p be a free open-filter on* $\mathbf{R}$, *such that* $\mathbf{R} \cup \{p\}$ *is not pathwise connected. Then, if* $\mathbf{R} \cup \{p\}$ *is pathwise connectifiable, the following hold:*

1) either $]-\infty, 0[\, \in p$ *or* $]0, +\infty[\, \in p$;

2) $\forall A \in p \colon \exists B \in p \colon \mathrm{Int}(A \setminus B)$ is unbounded.

On the other hand, if 1), 2) hold and p has a countable base, then $\mathbf{R} \cup \{p\}$ *is pathwise connectifiable.*

Proof. Suppose first $\mathbf{R} \cup \{p\}$ pathwise connectifiable; then 2) holds by Proposition 4.5. To prove 1), let Y be a pathwise connected T_2-space containing $\mathbf{R} \cup \{p\}$ as a dense subspace, and fix a one-to-one path φ on Y with $\varphi(0) = 0$ and $\varphi(1) = p$. Then there is a $\hat{t} \in [0,1[$ such that $\varphi(\hat{t}) = \hat{y} \notin \mathbf{R}$ and $\varphi(t) \in \mathbf{R}$ for every $t < \hat{t}$. To prove the existence of such a $\hat{t}$, observe that $\varphi([0,1])$ is not entirely contained in $\mathbf{R} \cup \{p\}$, and $\hat{t} = \inf\{t \in [0,1] \mid \varphi(t) \notin \mathbf{R} \cup \{p\}\}$ must in fact be a minimum; otherwise, fixing an open neighbourhood V of $\varphi(\hat{t})$ in Y such that $V \cap \mathbf{R}$ is bounded, we would have that V is a neighbourhood also for some $\varphi(t) \in Y \setminus (\mathbf{R} \cup \{p\})$ (with $t \in]\hat{t}, 1]$), contradicting Lemma 3.1.

Now, if $\varphi|_{[0,\hat{t}[}$ is strictly increasing, then $lim_{t \to \hat{t}} \varphi(t) = +\infty$ (by the free character of p). Thus $g = \varphi|_{[0,\hat{t}]} \colon [0,\hat{t}] \to [0,+\infty[\cup \{\hat{y}\}$ is a homeomorphism; in particular, each neighbourhood of $\hat{y}$ contains some set of the kind $]a,+\infty[$ with $a \in \mathbf{R}$. Since Y is T_2, p must have a neighbourhood disjoint from some $]a,+\infty[$; hence $]-\infty,0[\in p$.

If $\varphi|_{[0,\hat{t}[}$ is strictly decreasing, then we get symmetrically $]0,+\infty[\in p$.

Suppose now that 1), 2) hold and p has a countable base $\{B_n \mid n \in \mathbf{N}\}$; we may restrict ourselves to the case where $]0,+\infty[\in p$ (the case $]-\infty,0[\in p$ is symmetric, and uses Remark 4.7). Let $\{B_n\}_{n \in \mathbf{N}}$ be a decreasing countable base of p, such that $B_1 \subseteq]0,+\infty[$ and, for every $n \in \mathbf{N}$, $B_n \setminus B_{n+1} \supseteq A_n$ with A_n open unbounded in $\mathbf{R}$.

For every $n \in \mathbf{N}$, let $Y_n = [-\infty,+\infty[\cup I_n$, with $I_n = I \times \{n\}$, and let σ_n be the topology on Y_n for which the conclusions of Lemma 4.2 are fulfilled for $A = A_n$. Let

$$Z = [-\infty,+\infty[\cup \left(\bigcup_{n \in \mathbf{N}} I_n \right) \cup \{p\} = \left(\bigcup_{n \in \mathbf{N}} Y_n \right) \cup \{p\},$$

and τ be the topology on Z defined by:

i) $\tau|_{Z \setminus \{p\}} = \hat{\sigma}$, where $\hat{\sigma}$ is the topology generated by the base $\mathcal{S} = \bigcup_{\backslash \in \mathbf{N}} \sigma_\backslash$ (to prove that $\mathcal{S}$ is closed with respect to finite intersections, observe in particular that if $A_{n'}$ is $\sigma_{n'}$-open and $A_{n''}$ is $\sigma_{n''}$-open, with $n' \neq n''$, then $A_{n'} \cap A_{n''}$ is open in $[-\infty,+\infty[$ with respect to the topology of the extended real line, and hence is σ_n-open for every $n \in \mathbf{N}$);

ii) the collection of the open neighbourhoods for p in (Z,τ) is:

$$\left\{A\cup\{p\}\;\middle|\;A\in\hat{\sigma},\ A\cap\mathbf{R}\in p,\ \exists k\in\mathbf{N}\colon \bigcup_{n\geq k} I_n\subseteq A\right\}.$$

Observe that (Z,τ) is T_2. Indeed, given $x,y\in Z$, if $x,y\in Y_n$ for some $n\in\mathbf{N}$, we use the fact that (Y_n,σ_n) is T_2; if $x\in I_{n'}$ and $y\in I_{n''}$ with $n'\neq n''$, then $A_{n'}\cup I_{n'}$ and $A_{n''}\cup I_{n''}$ are disjoint neighbourhoods of x and y, respectively; if $x=p$ and $y\in I_n$ for some $n\in\mathbf{N}$, then $B_n\cup(\bigcup_{n'\geq n} I_{n'})\cup\{p\}$ and $A_n\cup I_n$ do the job; finally, if $x=p$ and $y\in[-\infty,+\infty[$, then since the open-filter p is free and $]0,+\infty[\in p$, choosing V open neighbourhood of y in $[-\infty,+\infty[$ and $\bar{n}\in\mathbf{N}$ with $V\cap B_{\bar{n}}=\emptyset$ we have that $B_{\bar{n}}\cup(\bigcup_{n\geq\bar{n}} I_n)$ and V are disjoints neighbourhoods of x and y in Z.

Let $Y=Z/\sim$, where the equivalence classes of $\sim$ are the set $\{-\infty,(0,1)\}$, the sets $\{(1,k),(0,k+1)\}$ with $k\in\mathbf{N}$, and the singletons of all the other points; let also $j\colon Z\to Y$ be the quotient mapping. Observe that Y inheredits from Z the T_2 character, because every equivalence class of $\sim$ has a finite number of elements (actually, one or two), and each point z of Z has a fundamental system of neighbourhoods of the kind $\{z\}\cup W$, where W is an open saturated set. Indeed, for $z=p$ we may consider the neighbourhoods of the form $\{p\}\cup B_n\cup(\,]0,1]\times\{n\})\cup(\bigcup_{n'>n} I_{n'})$, with $n\in\mathbf{N}$, while for $z\in I_n$ we may consider only neighbourhoods contained in $A_n\cup I_n$ and not containing both the points $(0,n)$ and $(1,n)$.

Clearly, $Y\setminus\{j(p)\}$ is pathwise connected; also, if η is a strictly increasing homeomorphism from $[0,1]$ onto $[1,+\infty]$ and $h\colon[1,+\infty]\to j((\bigcup_{n\in\mathbf{N}} I_n)\cup\{p\})$ is defined by: $h(+\infty)=j(p)$ and $h(x)=j(x-[x],[x])$ if $x\neq+\infty$.

Then $h\circ\eta$ is a path in Y connecting $j(0,1)$ with $j(p)$. Therefore Y is pathwise connected.

Finally, it is easily seen that the subspace $j(\mathbf{R}\cup\{p\})$ of Y is dense in Y and is homeomorphic to $\mathbf{R}\cup\{p\}$; therefore, Y is a pathwise connectification of $\mathbf{R}\cup\{p\}$. $\square$

Corollary 4.9. *Let p be a free open-filter on $\mathbf{R}$ having a countable base $\{B_n\mid n\in\mathbf{N}\}$, such that $\mathbf{R}\cup\{p\}$ is not pathwise connected. Then $\mathbf{R}\cup\{p\}$ is pathwise connectifiable if and only if the following hold:*

1) either $]-\infty,0[\in p$ *or* $]0,+\infty[\in p$;

2) the set $\{n\in\mathbf{N}\,|\,\mathrm{Int}_{\mathbf{R}}(B_n\setminus B_{n+1})$ is unbounded in $\mathbf{R}\}$ *is infinite.*

Proof. On the one hand, it is clear that 1), 2) above imply conditions 1), 2) of Theorem 4.8. On the other hand, suppose condition 2) above fails: then there exists $\bar{n}\in\mathbf{N}$ such that $\mathrm{Int}_{\mathbf{R}}(B_n\setminus B_{n+1})$ is bounded for $n\geq\bar{n}$. We claim that for every $A\in p$, the set $\mathrm{Int}_{\mathbf{R}}(B_{\bar{n}}\setminus A)$ is bounded. Suppose not, and let $\tilde{A}\in p$ be such that $\mathrm{Int}_{\mathbf{R}}(B_{\bar{n}}\setminus\tilde{A})$ is unbounded: taking $n^\sharp>\bar{n}$ with $B_{n^\sharp}\subseteq\tilde{A}$, we have that $\mathrm{Int}_{\mathbf{R}}(B_{\bar{n}}\setminus B_{n^\sharp})$ is unbounded. Hence, by Lemma 4.1, $\mathrm{Int}_{\mathbf{R}}(B_n\setminus B_{n+1})$ is unbounded for some n with $\bar{n}\leq n<n^\sharp$: a contradiction. □

Observe that in Theorem 4.8 we cannot drop the assumption that the open-filter p has a countable base, to show that conditions 1),2) imply the pathwise connectifiability of $\mathbf{R}\cup\{p\}$. Indeed, every maximal free open-filter on $\mathbf{R}$ satisfies conditions 1),2) above, but $\mathbf{R}\cup\{p\}$ is not pathwise connectifiable by Example 2.1 in [3].

On the other hand, we will show now that having a countable base is not a necessary condition for an open-filter p on $\mathbf{R}$, to give rise to a pathwise connectifiable $\mathbf{R}\cup\{p\}$.

EXAMPLE 4.10. *Fix a free ultrafilter $\mathcal{U}$ on $\mathbf{N}$, and let p be the free open-filter on $\mathbf{R}$ generated by the base:*

$$\mathcal{B}=\left\{\bigcup_{n\in F}]n-\varepsilon,n+\varepsilon[\;\middle|\;F\in\mathcal{U},\,\varepsilon>0\right\}.$$

Then:

1) for every $A\subseteq\mathbf{R}$ with $p\in\mathrm{Cl}_{\mathbf{R}\cup\{p\}}A$, the filter $p|_A$ has no countable base;

2) $\mathbf{R}\cup\{p\}$ is pathwise connectifiable.

Proof. Let $\mathbf{N}\cup\{\mathcal{U}\}$ be endowed with the topology in which the points of $\mathbf{N}$ are isolated and a base for $\mathbf{N}\cup\{\mathcal{U}\}$ at the point $\mathcal{U}$ consists of all sets of the form $G\cup\{\mathcal{U}\}$ where $G\in\mathcal{U}$, and let $X=(\mathbf{N}\cup\{\mathcal{U}\})\times[-1,1]$.

Let also $Y=X/\sim$, where the equivalence classes of $\sim$ are

$$\{(0,-1),(\mathcal{U},1)\}$$

and the sets $\{(n,1),(n+1,-1)\}$ with $n \in \mathbf{N}$.

Clearly $\mathbf{R} \cup \{p\}$ is homeomorphic to the dense subspace $((\mathbf{N} \times [-1,1]) \cup \{(\mathcal{U},0)\})/\sim$ of the pathwise connected T_2-space Y.

Moreover it is straightforward to see that for every $A \subseteq \mathbf{R}$, with $p \in \mathrm{Cl}_{\mathbf{R}\cup\{p\}}A$, the filter $p|_A$ has no countable base. $\square$

Remark 4.11. *The restriction of the above filter p to $\mathbf{Q}$ gives another filter satisfying the statement of Example 3.4.*

Remark 4.12. *Let $\mathcal{U}$ be a free ultrafilter on $\mathbf{N}$, and for every $n \in \mathbf{N}$ let $A_n =]n-(1/4), n+(1/4)[$. Then, by Corollary 2.2, if p' is the free open-filter on $\mathbf{R}$ generated by the base $\mathcal{B} = \{\bigcup_{n\in F} A_n \mid F \in \mathcal{U}\}$, the space $\mathbf{R}\cup\{p'\}$ is not pathwise connectifiable. On the other hand, such a filter p' is coarser than the filter p defined in the above example, and $\mathbf{R} \cup \{p\}$ is pathwise connectifiable. Thus, we have a negative answer to the question put at the end of §1.*

Observe also that the restrictions of p and p' to $\mathbf{Q}$ give an analogous example for the rational line.

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