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Decomposition of some Hypergeometric Polynomials with respect to the Cyclic Group of Order n

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Summary. - *Let* $\{P_m\}_{m\geq 0}$ *be a sequence of polynomials with complex coefficients and let* n *be an arbitrary positive integer. The components with respect to the cyclic group of order* n *of the polynomial* P_m, $m = 0, 1, \ldots$, *are given by*

$$(P_m)_{[n,k]}(z) = \frac{1}{n}\sum_{\ell=0}^{n-1} \omega_n^{-k\ell}\, P_m\left(\omega_n^{\ell} z\right), \quad k = 0, 1, \ldots, n-1,$$

where $\omega_n = exp(\frac{2i\pi}{n})$. *In this paper, we consider two class of hypergeometric polynomials, the Brafman polynomials and the Srivastava-Panda polynomials. For the components of these polynomials, we establish hypergeometric representations, differential equations and generating functions.*

1. Introduction

Let n be an arbitrary positive integer. Denote by $\omega_n = exp(\frac{2i\pi}{n})$ the complex n-root of unity and by $\mathbb{N}_n = \{0, 1, \ldots, n-1\}$ the set of the first n integers. Let $\mathcal{H}(C) = \mathcal{H}$ be the vector space of

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holomorphic functions on an annulus C with center in the origin. Define an operator $\sigma : \mathcal{H} \longrightarrow \mathcal{H}$ by:

$$\sigma f(z) = f(\omega_n z) \qquad \text{for} \qquad f \in \mathcal{H} \quad \text{and} \quad z \in C.$$

The following decomposition in direct sum holds (cf. [21], p.43):

$$\mathcal{H} \;=\; \bigoplus_{k=0}^{n-1} \mathcal{H}_{[n,k]}, \tag{1}$$

where

$$\mathcal{H}_{[n,k]} \;=\; \ker\left(\sigma - \omega_n^k I\right) \;=\; \ker\left(I - \Pi_{[n,k]}\right),$$

with

$$\Pi_{[n,k]} \;=\; \frac{1}{n}\sum_{\ell=0}^{n-1} \omega_n^{-k\ell}\sigma^{\ell}. \tag{2}$$

The subspaces $\mathcal{H}_{[2,0]}$ and $\mathcal{H}_{[2,1]}$ amount, respectively, to subspaces of even functions, and odd functions.
It follows that every function f belonging to $\mathcal{H}$ may be written as

$$f \;=\; \sum_{k=0}^{n-1} f_{[n,k]}, \tag{3}$$

with

$$f_{[n,k]}(z) \;=\; \Pi_{[n,k]}(f)(z) \;=\; \frac{1}{n}\sum_{\ell=0}^{n-1} \omega_n^{-k\ell} f(\omega_n^{\ell} z), \quad k \in \mathbb{N}_n. \tag{4}$$

The identity (3) is called the decomposition of the function f with respect to the cyclic group $\{\omega_n^k,\ k \in \mathbb{N}_n\}$ and the functions $f_{[n,k]}$ defined by (4) will be referred to as the components (with respect to the cyclic group of order n) of the function f.
The function $\tilde{f}_{[n,k]}$ defined by

$$\tilde{f}_{[n,k]}(x) \;=\; \tilde{\Pi}_{[n,k]}(f)(x) \;=\; x^{\frac{-k}{n}}\, f_{[n,k]}\left(x^{\frac{1}{n}}\right) \tag{5}$$

is called the associated with the component $f_{[n,k]}$.

In the special functions theory, the decomposition (3) was used for various reasons by many authors. Some special functions were introduced by this tool. For instance, the components with respect to the cyclic group of order n of the exponential function $z \longrightarrow \exp(z)$ (resp. $z \longrightarrow \exp\left((-1)^{\frac{1}{n}}z\right)$) are the hyperbolic (resp. trigonometric) functions of order n. More generally, the Osler-Srivastava identity (cf. (8), below) provides the components of the generalized hypergeomatric function ${}_pF_q$. The associated with these components are expressed by ${}_{np}F_{nq+n-1}$. It follows then the subclass of generalized hypergeometric funcions $\{{}_0F_r,\ r = 0, 1, \dots\}$ is invariant under the action of the operator $\tilde{\Pi}_{[n,k]}$ given by (5). In an earlier work [2], we investigated the properties of the components of the functions ${}_0F_q$ which correspond to the hyper-Bessel functions. The particular case $q = 0$, correponding to the exponential function, was considered by Erdélyi et al [7], Good [9], Muldoon and Ungar [17], Oniga [18], Ricci [21], Silverman [23], Ungar [35],[36], and Zachary [37].

In this paper, we deal with two other subclasses of generalized hypergeometric functions invariant under the action of the operator $\tilde{\Pi}_{[n,k]}$. That concerns the Brafman polynomials and the Srivastava-Panda polynomials. For the components of these polynomials, we establish hypergeometric representations, differential equations and generating functions. In the last section, we list some examples of hypergeometric polynomial families belonging to these two subclasses.

2. Brafman polynomials and Srivastava-Panda polynomials

For the sake of brevity, we adopt throughout this paper the following notations:

. (a_p) abbreviates the set of p complex parameters $a_1, a_2, ..., a_p$.

. $(a)_m$ is the Pochhammer symbol given by:

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = \begin{cases} 1 & \text{if } \ m = 0 \\ a(a+1)\cdots(a+m-1) & \text{if } \ m = 1, 2, 3, \dots \end{cases},$$

$a \neq 0, -1, -2, \dots$.

. $\Delta(r, \rho)$, r being a positive integer, denotes the r parameters

$\frac{\rho+j}{r}$, $j=0,1,\dots,r-1$.
The asterisk in $\Delta^*(n,k+1)$, k belongs to $\{0,1,\dots,n-1\}$, represents the fact that the parameter $\frac{n}{n}$ is omitted in $\Delta(n,k+1)$, so that the set $\Delta^*(n,k+1)$ obviously contains only $(n-1)$ parameters.

. $\Delta[n,a_p]$ designates the np parameters:

$$\frac{a_i+j}{n}, \quad i=1,2,\dots,p \quad \text{and} \quad j=0,1,\dots,n-1.$$

Recall that the generalized hypergeometric functions are defined by (see, for instance, [15], p.136 Eq.(1)):

$$ {}_pF_q\begin{pmatrix}(a_p), & \\ & z\\ (b_q), & \end{pmatrix} = \sum_{m=0}^{+\infty}\frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\cdot\frac{z^m}{m!},$$

where p and q are positive integers or 0 (interpreting an empty product as 1) and z is the complex variable. The numerator parameters (a_p) and the denominator parameters (b_q) take on complex values provides that $b_j \neq 0,-1,-2,\dots,\ j=1,\dots,q$.

Next, we deal with two particular generalized hypergeometric polynomials.

Definition 2.1. *The Brafman polynomials are defined by (cf. [5], p.186, Eq.(52)):*

$$\mathcal{B}_m^{\ell}[(a_r);(b_s);x] = {}_{\ell+r}F_s\begin{pmatrix}\Delta(\ell,-m), & (a_r), & \\ & & x\\ & (b_s), & \end{pmatrix}, \qquad (6)$$

where $(\ell,m)\in\mathbb{N}^*\times\mathbb{N}$, and a_i; $i=1,\dots,r$, and b_j, $j=1,\dots,s$, are complex parameters independent of m and x.

Definition 2.2. *The Srivastava-Panda polynomials are defined by the hypergeometric representation (cf. [30], p.471, Eq.(4.2)) :*

$$J_m^{(\lambda)}\big(x;r;(a_p);(b_q)\big)$$
$$= {}_{p+2r}F_q\begin{pmatrix}\Delta(r,-m), & \Delta(r,\lambda+m), & (a_p), & \\ & & & x\\ & & (b_q), & \end{pmatrix}, \qquad (7)$$

where $(r,m)\in\mathbb{N}^*\times\mathbb{N}$ and a_i, $i=1,\dots,p$, and b_j, $j=1,\dots,q$, are complex parameters independent of m and x.

Notice that these polynomials are related to the Brafman polynomials (6) by the confluent relation:

$$\lim_{\lambda\to\infty} J_m^{(\lambda)}\Big(\big(\frac{r}{\lambda}\big)^r x; r; (a_p); (b_q)\Big) \;=\; \mathcal{B}_m^r[(a_p);(b_q);x]$$

since we have

$$\lim_{\lambda\to\infty} \frac{r^{rm}}{\lambda^{rm}} \prod_{j=0}^{r-1} \Big(\frac{m+\lambda+j}{r}\Big)_m = \lim_{\lambda\to\infty} \frac{\Gamma(\lambda+m+rm)}{\lambda^{rm}\Gamma(\lambda+m)} = 1.$$

Some applications of Brafman polynomials and Srivastava-Panda polynomials are mentioned in the Srivastava-Shreshtha's paper[32].

3. Components

The fact that the Brafman polynomials and the Srivastava-Panda polynomials are two subclasses of hypergeometric polynomials invariant under the action of the operators $\tilde{\Pi}_{[n,k]}$ are given by the following

THEOREM 3.1. *The associated polynomials with the components with respect to the cyclic group of order n of Brafman polynomials (resp. Srivastava-Panda polynomials) are also Brafman polynomials (resp. Srivastava-Panda polynomials).*

Proof. The Osler-Srivastava identity provides an explicit expression of the components of a generalized hypergeometric function. In fact, we have (cf. [19], p. 890 Eq. (5), [27], p. 194, Eq. (12) or [29], p. 213)

$$\mathcal{F}\left((a_p),(b_q),n,k,z\right) = \Pi_{[n,k]}\left(z \longrightarrow {}_pF_q\begin{pmatrix}(a_p), & \\ & z\\ (b_q), & \end{pmatrix}\right)$$

$$= \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\cdot\frac{z^k}{k!}\cdot$$

$$\cdot {}_{np}F_{nq+n-1}\left(\begin{matrix} \Delta(n,a_1+k),\,\ldots\,,\Delta(n,a_p+k), \\ \Delta^*(n,k+1),\ \Delta(n,b_1+k),\,\ldots\,,\Delta(n,b_q+k),\end{matrix}\ \frac{z^n}{n^{(1-p+q)n}}\right). \tag{8}$$

Now, the application of the operator $\Pi_{[n,k]}$ to the two members, considered as functions of the variable x, of the identity (6), the use of the Osler-Srivastava identity (8), and the following equality which one can easily verify:

$$\bigcup_{j=0}^{\ell-1} \Delta\big(n, \frac{\alpha+j}{\ell}\big) \; = \; \Delta(n\ell,\alpha) \quad \text{with} \quad \alpha \in \mathbb{C} \quad \text{and} \quad (n,\ell) \in {\mathbb{N}^*}^2$$

lead to

$$\begin{aligned}
&\Big(x \to \mathcal{B}_m^{\ell}\left[(a_p);(b_q);x\right]\Big)_{[n,k]} \\
&\quad = \left(x \to {}_{\ell+p}F_p \begin{pmatrix} \Delta(\ell,-m), & (a_p), & \\ & & x \\ & (b_q), & \end{pmatrix} \right)_{[n,k]} \\
&\quad = \frac{\prod_{j=0}^{\ell-1}\left(\frac{-m+j}{\ell}\right)_k \cdot \prod_{i=1}^{p}(a_i)_k}{\prod_{s=1}^{q}(b_s)_k} \cdot \frac{x^k}{k!} \cdot \\
&\qquad \cdot {}_{n\ell+np}F_{nq+n-1} \begin{pmatrix} \Delta(n\ell,-m+k\ell), \Delta[n,a_p], & \\ & \left(\frac{x}{n^{(1-\ell-p+q)}}\right)^n \\ \Delta^*(n,k+1), \Delta[n,b_q], & \end{pmatrix} \\
&\quad = \frac{\prod_{j=0}^{\ell-1}\left(\frac{-m+j}{\ell}\right)_k \cdot \prod_{i=1}^{p}(a_i)_k}{\prod_{s=1}^{q}(b_s)_k} \cdot \frac{x^k}{k!} \cdot \\
&\qquad \cdot \mathcal{B}_{m-k\ell}^{n\ell}\left[\Delta[n,a_p]; \Delta^*(n,k+1), \Delta[n,b_q]; \Big(\frac{x}{n^{(1-\ell-p+q)}}\Big)^n\right]
\end{aligned}$$

if $k\ell \leq m$.

A simular calculus with the identity (7) gives us

$$\begin{aligned}
&\Big(x \to J_m^{(\lambda)}\left(x;r;(a_p);(b_q)\right)\Big)_{[n,k]} \\
&\quad = \frac{\prod_{j=0}^{r-1}\left(\frac{-m+j}{r}\right)_k \left(\frac{\lambda+m+j}{r}\right)_k \cdot \prod_{i=1}^{p}(a_i)_k}{\prod_{s=1}^{q}(b_s)_k} \cdot \frac{x^k}{k!} \cdot \\
&\qquad \cdot J_{m-k\ell}^{(\lambda)}\left(\frac{x^n}{n^{(1-p-2r+q)n}}; nr; \Delta[n,a_p]; \Delta^*(n,k+1), \Delta[n,b_p]\right)
\end{aligned} \tag{9}$$

if $kr \leq m$. □

4. Generating functions

Starting from a generating function of a polynomial family, we derive the corresponding ones for the components. In each case, we compare the obtained result with other already established in the literature. For the sake of brevity, we limit ourselves to the case $k = 0$.

4.1 Brafman polynomials

In this section, we consider two generating functions for the Brafman polynomials.

Case 1 : The generating function of the family $\left\{\frac{1}{m!}\mathcal{B}^1_m[(a_p);(b_q);x]\right\}_{m\in\mathbb{N}}$ is given by (cf. Erdlyi [7], p. 267, Eq. (25) or Brafman [4], p. 947, Eq. (27)):

$$\sum_{m=0}^{\infty} {}_{p+1}F_q\left(\begin{matrix}-m, & (a_p), & \\ & & x\\ & (b_q), & \end{matrix}\right)\cdot\frac{t^m}{m!} = e^t\ {}_pF_q\left(\begin{matrix}(a_p), & \\ & -xt\\ (b_q), & \end{matrix}\right). \tag{10}$$

If we apply $\Pi_{[n,0]}$ to the two members of (10) considered as functions of the variable x, we obtain:

$$\sum_{m=0}^{\infty} {}_{np+n}F_{nq+n-1}\left(\begin{matrix}\Delta(n,-m),\Delta[n,a_p], & \\ & x\\ \Delta^*(n,1),\Delta[n,b_q], & \end{matrix}\right)\cdot\frac{t^m}{m!}$$
$$= e^t\ {}_{np}F_{nq+n-1}\left(\begin{matrix}\Delta[n,a_p], & \\ & x\left(\frac{-t}{n}\right)^n\\ \Delta^*(n,1),\Delta[n,b_q], & \end{matrix}\right), \tag{11}$$

which one can also deduce from Srivastava identity (cf. [25], p. 203, Eq. (8) or [26], p. 68, Eq. (3.9)):

$$\sum_{m=0}^{\infty} {}_{n+r}F_s\left(\begin{matrix}\Delta(n,-m), & (\alpha_r), & \\ & & x\\ & (\beta_s), & \end{matrix}\right)\cdot\frac{t^m}{m!} = e^t\ {}_rF_s\left(\begin{matrix}(\alpha_r), & \\ & x\left(\frac{-t}{n}\right)^n\\ (\beta_s), & \end{matrix}\right),$$

where (α_r) designates the r parameters $\alpha_1,\alpha_2,\dots,\alpha_r$.

Case 2 : Let us recall the following identity (cf. Erdélyi [7], p. 267, Eq. (22) or Chaundy [6], p. 62, Eq. (25)):

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!}\ {}_{p+1}F_q \begin{pmatrix} -m, & (a_p), & \\ & & x \\ & (b_q), & \end{pmatrix} \cdot t^m$$
$$= \ (1-t)^{-\lambda}\ {}_{p+1}F_q \begin{pmatrix} \lambda, & (a_p), & \\ & & \frac{xt}{t-1} \\ & (b_q), & \end{pmatrix},\ |t|<1. \qquad (12)$$

If we apply the operator $\Pi_{[n,0]}$ to the two members of this identity considered as functions of the variable x, we obtain:

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!}\ {}_{(p+1)n}F_{qn+n-1} \begin{pmatrix} \Delta(n,-m), & \Delta[n,a_p], & \\ & & \left(\frac{x}{n^{q-p}}\right)^n \\ \Delta^*(n,1), & \Delta[n,b_q], & \end{pmatrix} \cdot t^m$$
$$= (1-t)^{-\lambda}\ {}_{(p+1)n}F_{qn+n-1} \begin{pmatrix} \Delta(n,\lambda), & \Delta[n,a_p], & \\ & & \left(\frac{x}{n^{q-p}}\right)^n \cdot \left(\frac{t}{t-1}\right)^n \\ \Delta^*(n,1), & \Delta[n,b_q], & \end{pmatrix},$$
$$|t|<1, \quad (13)$$

which one can also deduce from Brafman identity (cf.[5], p.186, Eq.(55)):

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} \mathcal{B}_m^n[(\alpha_r)\,;\,(\beta_s)\,;\,x] \cdot t^m$$
$$= (1-t)^{-\lambda}\ {}_{n+r}F_s \begin{pmatrix} \Delta(n,\lambda), & (\alpha_r), & \\ & & x\left(\frac{t}{t-1}\right)^n \\ & (\beta_s), & \end{pmatrix}. \quad (14)$$

4.1. Srivastava-Panda polynomials

Recall the generating function (cf. [6], p. 62, Eq. (26)):

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} \, {}_{r+2}F_s \begin{pmatrix} -m, & \lambda+m, & (a_r), & \\ & & & x \\ & & (b_s), & \end{pmatrix} \cdot t^m$$
$$= (1-t)^{-\lambda} \, {}_{r+2}F_s \begin{pmatrix} \Delta(2,\lambda), & (a_r), & \\ & & \frac{-4xt}{(1-t)^2} \\ & (b_s), & \end{pmatrix}, \; |t|<1. \quad (15)$$

The application of $\Pi_{[n,0]}$ to the two members of this identity considered as functions of the variable x leads to:

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} \, {}_{n(r+2)}F_{ns+n-1} \begin{pmatrix} \Delta(n,-m), & \Delta(n,\lambda+m), & \Delta[n,a_r], & \\ & & & x \\ & \Delta^*(n,1), & \Delta[n,b_s], & \end{pmatrix} \cdot t^m$$
$$= (1-t)^{-\lambda} \, {}_{n(r+2)}F_{ns+n-1} \begin{pmatrix} \Delta(2n,\lambda), & \Delta[n,a_r], & \\ & & x\left(\frac{-4t}{(1-t)^2}\right)^n \\ \Delta^*(n,1), & \Delta[n,b_s], & \end{pmatrix},$$
$$|t|<1, \quad (16)$$

which one can also deduce from Srivastava-Buschman identity (cf. [28], p. 364, Eq. (19)):

$$\sum_{m=0}^{\infty} \binom{\alpha}{m} {}_{r+p}F_{s+p} \begin{pmatrix} (a_r), \Delta(p-q,-\alpha+m), \Delta(q,-m), & \\ & \frac{(-q)^q (p-q)^{p-q} x}{(p)^p} \\ (b_s), \Delta(p,-\alpha), & \end{pmatrix} \cdot t^m$$
$$= (1+t)^{\alpha} \, {}_rF_s \begin{pmatrix} (a_r), & \\ & \frac{x(-t)^q}{(1+t)^p} \\ (b_s), & \end{pmatrix} \quad \text{if } p \geq q. \quad (17)$$

5. Differential equations

Recall that the components $\mathcal{F}\left((a_p),(b_q),n,k,z\right)$, $n \in \mathbb{N}^*$ and $k \in \mathbb{N}_n$, satisfy the differential equation (cf. [3], Proposition 4.1):

$$\left(\prod_{i=1}^{p}(\vartheta+a_i)_n - \left(\prod_{j=1}^{q}(\vartheta+b_j)_n\right)D^n\right)y=0, \tag{18}$$

where

$$(\vartheta+a)_n = \prod_{r=0}^{n-1}(\vartheta+a+r). \tag{19}$$

In particular:

1/ The components of the Brafman polynomials $\mathcal{B}_m^{\ell}[(a_r);(b_s);x]$ satisfy the differential equation:

$$\left(\prod_{i=1}^{r}(\vartheta+a_i)_n \prod_{h=0}^{\ell n-1}\left(\vartheta+\frac{h-m}{\ell}\right) - \left(\prod_{j=1}^{s}(\vartheta+b_j)_n\right)D^n\right)y=0. \tag{20}$$

2/ The components of the Srivastava-Panda polynomials $J_m^{(\lambda)}\big(x;r;(a_p);(b_q)\big)$ satisfy the differential equation:

$$\left(\prod_{i=1}^{p}(\vartheta+a_i)_n \prod_{h=0}^{rn-1}\left(\vartheta+\frac{h-m}{r}\right)\left(\vartheta+\frac{h+\lambda+m}{r}\right) - \left(\prod_{j=1}^{q}(\vartheta+b_j)_n\right)D^n\right)y=0. \tag{21}$$

6. Some identities

6.1. Neumann expansions

Recall firstly the expansion (cf. [24], p. 425, Eq. (1.3)):

$$x^\lambda\ {}_pF_q\left(\begin{matrix}(\alpha_p), & \\ & -x^2u \\ (\beta_q), & \end{matrix}\right) = \sum_{m=0}^{\infty}\frac{(\lambda+2m)\Gamma(\lambda+m)}{m!}\,J_{\lambda+2m}(2x) \cdot {}_{p+2}F_q\left(\begin{matrix}-m, & \lambda+m & ,(\alpha_p), & \\ & & & u \\ & & ,(\beta_q), & \end{matrix}\right). \quad (22)$$

Apply $\Pi_{[n,0]}$ to the two members of this expansion considered as functions of the variable u to obtain:

$$x^\lambda\ {}_{np}F_{nq+n-1}\left(\begin{matrix} & \Delta[n,\alpha_p], & \\ & & \left(\frac{-x^2u}{n^{1-p+q}}\right)^n \\ \Delta^*(n,1), & \Delta[n,\beta_q], & \end{matrix}\right) = \sum_{m=0}^{\infty}\frac{(\lambda+2m)\Gamma(\lambda+m)}{m!}\,J_{\lambda+2m}(2x) \cdot {}_{n(p+2)}F_{nq+n-1}\left(\begin{matrix}\Delta(n,-m), & \Delta(n,\lambda+m), & \Delta[n,\alpha_p], & \\ & & & \left(\frac{u}{n^{1-p-2+q}}\right)^n \\ & \Delta^*(n,1), & \Delta[n,\beta_q], & \end{matrix}\right), \quad (23)$$

which one can also deduce from Srivastava-Shreshtha identity (cf. [32], p. 455, Eq. (3.6)):

$$\left(\frac{z}{2}\right)^{\lambda} {}_rF_s\begin{pmatrix}(a_r), & \\ & \left(\frac{\omega z}{2n}\right)^{2n} \\ (b_s), & \end{pmatrix} = \sum_{m=0}^{\infty}(-1)^m \frac{(\lambda+2m)\Gamma(\lambda+m)}{m!} I_{\lambda+2m}(z) \cdot {}_{2n+r}F_s\begin{pmatrix}\Delta(n,-m), \Delta(n,\lambda+m), (a_r), & \\ & \omega^{2n} \\ (b_s), & \end{pmatrix}. \quad (24)$$

6.2. Generalized Manocha identities

It's easy to establish that if a family $\{\varphi_m\}_{m\in\mathbb{N}}$ is defined by a generating function $G(x,t)$, the families $\{\varphi_{mn+k}\}_{m\in\mathbb{N}}$, $k \in \mathbb{N}_n$, are generated by

$$(x,t) \to t^{-\frac{k}{n}} \cdot \Big(t \to G(x,t)\Big)_{[n,k]} \left(x, t^{\frac{1}{n}}\right). \quad (25)$$

If we apply this result to the Brafman polynomials generated by (14), we obtain that the family $\left\{\frac{1}{(nm+k)!}\mathcal{B}^1_{nm+k}[(a_p);(b_q);x]\right\}_{m\in\mathbb{N}}$ is generated by

$$t^{-\frac{k}{n}} \sum_{r+s=k \text{ ou } k+n} h_{n,s}\left(t^{\frac{1}{n}}\right) \cdot \frac{(a_1)_r \cdots (a_p)_r}{(b_1)_r \cdots (b_q)_r} \cdot \frac{\left(-xt^{\frac{1}{n}}\right)^r}{r!} \cdot {}_{np}F_{nq+n-1}\begin{pmatrix} & \Delta(n,a_1+r), \ \dots \ , \Delta(n,a_p+r), & \\ & & \frac{(-x)^n t}{n^{(1-p+q)n}} \\ \Delta^*(n,r+1), & \Delta(n,b_1+r), \ \dots \ , \Delta(n,b_q+r), & \end{pmatrix}, \quad (26)$$

where the functions $h_{n,s}$ are the generalized hyperbolic functions of order n and of k-th kind.

For $n = 2$, $p = 0$, $q = 1$, and $b_1 = \alpha + 1$, the identity (26) is reduced the Manocha decomposition (cf. [16]):

$$\sum_{m=0}^{\infty} \frac{t^m}{(\alpha+1)_{2m}} L_{2m}^{(\alpha)}(x)$$

$$= \cosh\sqrt{t}\cdot {}_0F_3\left(\begin{matrix} -, \\ \frac{1}{2}\,,\ \frac{1}{2}\alpha+\frac{1}{2}\,,\ \frac{1}{2}\alpha+1, \end{matrix}\ \frac{x^2t}{16}\right)$$

$$+ \frac{(-x\sqrt{t})}{\alpha+1}\sinh\sqrt{t}\cdot {}_0F_3\left(\begin{matrix} -, \\ \frac{3}{2}\,,\ \frac{1}{2}\alpha+1\,,\ \frac{1}{2}\alpha+\frac{3}{2}, \end{matrix}\ \frac{x^2t}{16}\right),$$

and

$$\sum_{m=0}^{\infty} \frac{t^m}{(\alpha+1)_{2m+1}} L_{2m+1}^{(\alpha)}(x)$$

$$= \frac{\sinh\sqrt{t}}{\sqrt{t}}\cdot {}_0F_3\left(\begin{matrix} -, \\ \frac{1}{2}\,,\ \frac{1}{2}\alpha+\frac{1}{2}\,,\ \frac{1}{2}\alpha+1, \end{matrix}\ \frac{x^2t}{16}\right)$$

$$- \frac{x}{\alpha+1}\cosh\sqrt{t}\cdot {}_0F_3\left(\begin{matrix} -, \\ \frac{3}{2}\,,\ \frac{1}{2}\alpha+1\,,\ \frac{1}{2}\alpha+\frac{3}{2}, \end{matrix}\ \frac{x^2t}{16}\right).$$

7. Examples

7.1. Brafman polynomials

Next, we list some polynomial families which may be expressed by Brafman polynomials.
Laguerre polynomials (cf. [33], p. 99, Eq. (5.3.3)):

$$L_m^{(\alpha)}(x) = \frac{(\alpha+1)_m}{m!}\,{}_1F_1\left(\begin{matrix} -m, \\ \alpha+1, \end{matrix}\ x\right) = \frac{(\alpha+1)_m}{m!}\,\mathcal{B}_m^1[-;\alpha+1;x].$$

Konhauser polynomials (cf. [13], p. 304, Eq. (5)):

$$Z_m^{\alpha}(x,k) \quad = \quad \frac{\Gamma(km+\alpha+1)}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{x^{kj}}{\Gamma(kj+\alpha+1)}$$

$$= \frac{(\alpha+1)_{km}}{m!}\, {}_1F_k \left(\begin{matrix} -m, \\ \Delta(k,\alpha+1), \end{matrix} \; \left(\frac{x}{k}\right)^k \right)$$

$$= \frac{(\alpha+1)_{km}}{m!}\, \mathcal{B}^1_m \left[-; \Delta(k,\alpha+1); \left(\frac{x}{k}\right)^k \right].$$

In particular, we have $Z^{\alpha}_m(x,1) = L^{(\alpha)}_m(x)$.
Notice that these polynomials, which are attribued in the literature to Konhauser, were introduced by Toscano [34] (cf. also MR # 18 (1957) p.390)
Koekoek polynomials : R. Koekoek [12] proved that the polynomials

$$L^{\alpha,M_0,M_1,\dots,M_N}_m(x) = \sum_{k=0}^{N+1} A_k \cdot D^k L^{(\alpha)}_m(x),$$

for certain real coefficients $\{A_k\}_{k=0}^{N+1}$, are orthogonal with respect to scalar product:

$$< f, g >= \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} x^{\alpha} e^{-x} f(x) g(x) dx + \sum_{\nu=0}^{N} M_{\nu} f^{(\nu)}(0) g^{(\nu)}(0),$$

where $\alpha > -1$, $N \in \mathbb{N}$ and $M_{\nu} \geq 0$ for all $\nu \in \{0, 1, \dots, N\}$, and he stated the hypergeometric reperesentation (cf. [12], p. 587, Eq. (6.1)):

$$L^{\alpha,M_0,M_1,\dots,M_N}_m(x)$$

$$= C(m,\alpha,M_0,\dots,M_N)\, {}_{N+2}F_{N+2} \left(\begin{matrix} -m, a_0, a_1, \dots, a_N, \\ \alpha+N+2, b_0, b_1, \dots, b_N, \end{matrix} \; x \right),$$

$$= C(m,\alpha,M_0,...,M_N) \mathcal{B}^1_m[a_0,a_1,...,a_N; \alpha+N+2, b_0, b_1, ..., b_N; x],$$

where $C(m,\alpha,M_0,M_1,\dots,M_N)$ is a coefficient independent of x and $a_0, a_1, \dots, a_N$, $b_0, b_1, \dots, b_N$ are parameters independent of x and m.
Batman polynomials (cf. [1], p. 575):

$$J^{(\nu,\sigma)}_m(x) = \binom{\frac{1}{2}\nu + \sigma + m}{m} \frac{x^{\nu}}{\Gamma(\nu+1)}\, {}_1F_2 \left(\begin{matrix} -m, \\ \nu+1, \frac{1}{2}\nu+\sigma+1, \end{matrix} \; x^2 \right)$$

$$= \binom{\frac{1}{2}\nu+\sigma+m}{m} \frac{x^\nu}{\Gamma(\nu+1)} \mathcal{B}_m^1\left[-;\nu+1;\tfrac{1}{2}\nu+\sigma+1;x^2\right].$$

Hermite polynomials (cf. [33], p. 106, Eq. (5.5.4)):

$$H_m(x) = (2x)^m\, {}_2F_0\left(\begin{matrix}-\frac{1}{2}m, \frac{1}{2}-\frac{1}{2}m & \\ & -\frac{1}{x^2} \\ -, & \end{matrix}\right)$$
$$= (2x)^m\, \mathcal{B}_m^2\left[-;-;-\frac{1}{x^2}\right].$$

Gould-Hopper polynomials (cf. [10], p. 58, Eq. (6.2)):

$$\begin{aligned} g_m^\ell(x,h) &= \sum_{k=0}^{[\frac{m}{\ell}]} \frac{m!}{k!(m-\ell k)!} h^k x^{m-\ell k} \\ &= x^m\, {}_\ell F_0\left(\begin{matrix}\Delta(\ell,-m), & \\ & h\left(\frac{\ell}{x}\right)^\ell \\ -, & \end{matrix}\right) \\ &= x^m\, \mathcal{B}_m^\ell\left[-;-;h\left(\frac{\ell}{x}\right)^\ell\right]. \end{aligned}$$

In particular, we have $g_m^2(2x,-1) = H_m(x)$.
Gegenbauer polynomials (cf. [20], p. 280, Eq. (20)):

$$C_m^\alpha(x) = \frac{(2\alpha)_m}{m!} x^m\, {}_2F_1\left(\begin{matrix}\Delta(2,-m), & \\ & -\frac{x^2-1}{x^2} \\ \alpha+\frac{1}{2}, & \end{matrix}\right)$$
$$= \frac{(2\alpha)_m}{m!} x^m\, \mathcal{B}_m^2\left[-;\alpha+\frac{1}{2};-\frac{x^2-1}{x^2}\right].$$

7.2. Srivastava-Panda polynomials

We present in the sequel some polynomial families which are expressed by Srivastava-Panda polynomials.
Generalized Bessel polynomials (cf. [14], p. 108, Eq. (34)):

$$y_m(x,\alpha,\beta) = {}_2F_0\left(\begin{matrix}-m, \alpha+m-1, & \\ & \frac{-x}{\beta} \\ -, & \end{matrix}\right).$$

Jacobi polynomials (cf. [33], p. 68, Eq. (4.3.2.))

$$P_m^{(\alpha,\beta)}(x) \;=\; \binom{m+\alpha}{m}\; {}_2F_1\left(\begin{matrix}-m, m+\alpha+\beta+1, \\ \alpha+1,\end{matrix}\; \frac{1-x}{2}\right)$$

This family contains:

. **Gegenbauer polynomials** (or **ultraspheric polynomials**):

$$C_m^{\nu}(x) \;=\; \binom{m+\nu-\frac{1}{2}}{m}^{-1} \binom{m+2\nu-1}{m} P_m^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x),$$

. **Legendre polynomials** (or **spherical polynomials**):

$$P_m(x) \;=\; P_m^{(0,0)}(x),$$

. **Tchebycheff polynomials** of first and second kind:

$$T_m(x) = \frac{m!}{\left(\frac{1}{2}\right)_m} P_m^{(-\frac{1}{2},-\frac{1}{2})}(x), \quad \text{and} \quad U_m(x = \frac{(m+1)!}{\left(\frac{3}{2}\right)_m} P_m^{(\frac{1}{2},\frac{1}{2})}(x).$$

Batman polynomials (cf. [1], p. 574):

$$Z_m(x) \;=\; {}_2F_2\left(\begin{matrix}-m, m+1, \\ 1,1,\end{matrix}\; x\right).$$

Rice polynomials (cf. [22], p. 108):

$$H_m(\xi,p,v) \;=\; {}_3F_2\left(\begin{matrix}-m, m+1, \xi, \\ 1, p,\end{matrix}\; v\right).$$

Generalized Rice polynomials (cf. [11], p. 158, Eq. (2.3)):

$$H_m^{(\alpha,\beta)}(\xi,p,v) \;=\; \binom{m+\alpha}{m}\; {}_3F_2\left(\begin{matrix}-m, m+1+\alpha+\beta, \xi, \\ \alpha+1, p,\end{matrix}\; v\right)$$

Batman polynomials (cf. [1], p. 574):

$$Z_m(x) \;=\; {}_2F_2\left(\begin{matrix} -m, m+1, \\ & x \\ 1,1, \end{matrix}\right)$$

Srivastava-Pathan polynomials (cf. [31], p. 106, Eq. (1.3)):

$$\mathcal{F}_m^{(\lambda)}[(\alpha_p),(\beta_q),x] \;=\; {}_{p+2}F_q\left(\begin{matrix} -m, m+\lambda, (\alpha_p), \\ & x \\ (\beta_q), \end{matrix}\right).$$

Fasenmyer polynomials (cf. [8], p. 806, Eq. (1)):

$$P_m(x) \;=\; {}_{p+2}F_{q+2}\left(\begin{matrix} -m, m+1, (a_p), \\ & x \\ \frac{1}{2}, 1, (b_q), \end{matrix}\right)$$

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