Positive Solutions of Quasilinear Elliptic Systems with the Natural Growth in the Gradient

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Summary. - *We study the problem of existence and nonexistence of positive, spherically symmetric solutions of a quasilinear elliptic system involving p-Laplacians, with the natural growth in the gradient on the right-hand side. The existence proof is constructive, with solutions possessing explicit integral representation. We also obtain various qualitative results. The elliptic system is studied by relating it to the corresponding system of singular ODE's of the first order.*

1. Introduction

In the course of the preceding decade there has been a considerable progress in the study of existence and nonexistence of positive solutions of nonlinear elliptic systems involving two nonlinear elliptic equations, see a survey article by De Figueiredo [4], Clement, Manásevich, Mitidieri [3], and the references therein. In [3] a class of problems is considered involving systems with p-Laplace operators, also in [4, Lecture 4]. These articles deal with problems where the right-hand side of the system does not depend on the gradient of unknown functions. Here we study quasilinear elliptic systems

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which have the natural growth in the gradient on the right-hand side. Our approach is based on the reduction of such problem to a system of two singular ODE's of the first order, using a suitable integral representation of solutions. This method was introduced in Korkut, Pašić, Žubrinić [11] in the scalar case. Quasilinear elliptic problems for scalar equations and systems with gradient terms on right-hand sides are studied also in Serrin, Zou [14], Jie Jiang [8], Furusho, Takasi, Ogata [5], and Caristi, Mitidieri [2].

The paper is organized as follows: 1. Introduction; 2. A system of two singular ODE's of the first order; 3. Nonexistence of strong solutions; 4. Qualitative properties of solutions; 5. Approximation of solutions; 6. Weak solutions of quasilinear elliptic systems, singularities; 7. Appendix.

We study the problem of existence, nonexistence, and qualitative properties of solutions of the following quasilinear elliptic system:

$$\begin{cases} -\Delta_p u = \tilde{g}_0 |x|^{m_0} + \tilde{f}_0 |\nabla v|^q & \text{in } B \setminus \{0\}, \\ -\Delta_q v = \tilde{g}_1 |x|^{m_1} + \tilde{f}_1 |\nabla u|^p & \text{in } B \setminus \{0\}, \\ u > 0, \ v > 0 \text{ on } B, \text{ spherically symmetric, decreasing,} \\ u = v = 0 \text{ on } \partial B. \end{cases} \tag{1}$$

In Section 7 we consider a class of much more general problems, see (87). Here $1 < p < \infty$, $1 < q < \infty$, $B = B_R(0)$ is a ball of radius R in $\mathbf{R}^N$, $N \geq 1$, $m_i \in \mathbf{R}$, $\tilde{f}_i > 0$, $\tilde{g}_i > 0$, $i = 0, 1$, and Δ_p is p-Laplacian, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$. The Lebesgue measure (volume) of B will be denoted by $|B|$, and the volume of the unit ball in $\mathbf{R}^N$ by C_N. The conjugate exponent of p is defined by $p' = \frac{p}{p-1}$. We identify spherically symmetric functions $u(x)$ and $v(x)$ with $u(r)$ and $v(r)$ respectively, where $r = |x|$.

Here we study strong solutions of (1), which we define as pairs of functions $u, v \in C^2(B \setminus \{0\})$ satisfying (1) pointwise. Keeping $\tilde{f}_0$ and $\tilde{f}_1$ fixed, our goal is to show that problem (1) possesses at least one strong solution provided $\tilde{g}_0$ and $\tilde{g}_1$ are small enough (see Theorem 1.1), and has no strong solutions if parameters $\tilde{g}_0$ or $\tilde{g}_1$ are large enough (see Theorem 1.4). Also, if we keep $\tilde{f}_1$, $\tilde{g}_0$ and $\tilde{g}_1$ fixed, then for $\tilde{f}_0$ small enough there exists a strong solution, while for $\tilde{f}_0$ large enough there is no solution. Analogously for $\tilde{f}_1$, see Remarks 1.3 and 1.5. Our existence proof are constructive in the

sense that there exists a sequence (u_n, v_n) of monotone iterations converging to a solution (u, v) pointwise, sometimes even in $C^2(\overline{B}) \times C^2(\overline{B})$, see Theorem 5.1.

In Section 6 we shall deal with problem of finding weak solutions of (1), i.e./ spherically symmetric, decreasing functions $u \in W_0^{1,p}(B) \cap L^\infty(B)$, $v \in W_0^{1,q}(B) \cap L^\infty(B)$, satisfying both equations in (1) in the weak sense. We show that under certain conditions some strong solutions of (1) are also weak solutions, see Theorem 6.2.

It is easy to see that constants $\tilde{f}_0$ and $\tilde{f}_1$ can be eliminated from (1) by scaling $u \to \lambda u$, $v \to \mu v$, with a suitable choice of $\lambda > 0$ and $\mu > 0$:

$$\begin{cases} -\Delta_p u = \tilde{g}_0 |x|^{m_0} + |\nabla v|^q & \text{in } B \setminus \{0\}, \\ -\Delta_q v = \tilde{g}_1 |x|^{m_1} + |\nabla u|^p & \text{in } B \setminus \{0\}, \\ u > 0, \, v > 0 \text{ on } B, \text{ spherically symmetric, decreasing}, \\ u = v = 0 \text{ on } \partial B. \end{cases} \quad (2)$$

Indeed, assume that (u, v) is a solution of (1). Defining $\overline{u} = \lambda u$, $\overline{v} = \mu v$ we obtain that

$$\begin{cases} -\Delta_p \overline{u} = \lambda^{p-1} \tilde{g}_0 |x|^{m_0} + \frac{\lambda^{p-1} \tilde{f}_0}{\mu^q} |\nabla \overline{v}|^q, \\ -\Delta_q \overline{v} = \tilde{\mu}^{q-1} g_1 |x|^{m_1} + \frac{\mu^{q-1} \tilde{f}_1}{\lambda^p} |\nabla \overline{u}|^p. \end{cases} \quad (3)$$

If we take

$$\lambda = (\tilde{f}_0 \tilde{f}_1^{q'})^{\frac{q-1}{p+q-1}}, \quad \mu = (\tilde{f}_1 \tilde{f}_0^{p'})^{\frac{p-1}{q+p-1}}$$

then the pair of functions $(\overline{u}, \overline{v})$ satisfies elliptic system of the form (2) with $\lambda^{p-1} \tilde{g}_0$ and $\mu^{q-1} \tilde{g}_1$ instead of $\tilde{g}_0$ and $\tilde{g}_1$ respectively. Therefore it suffices to study elliptic systems of the form (2). We can assume without loss of generality that $m_0 \geq m_1$. The main existence result of this paper is the following (its proof is given in Section 2).

THEOREM 1.1. (existence result) Assume that $1 < p < \infty$, $1 < q < \infty$,

$$\begin{cases} m_0 \geq m_1 > \max\{-1 - \frac{N}{p'}, -N\}, \\ m_1 \geq \max\{-q, -\frac{(p'+1)q'}{p'q'}\}. \end{cases} \quad (4)$$

Then there exist two explicit positive constants $\tilde{a}$ and $\tilde{b}$, see Appendix (A), such that if

$$\tilde{a} \cdot \tilde{b}^{\delta'-1} \leq \frac{\delta-1}{\delta^{\delta'}}, \tag{5}$$

where $\delta = p'q' > 1$, then problem (2) possesses at least one strong solution. It can be obtained constructively, using method of monotone iterations.

Moreover, we shall prove existence of a strong solution (u, v) of elliptic system (2) which has a specific integral representation. It is described in Lemma 2.1 below, where a system of singular ODE's of the first order plays a crucial role, see (12). Strong solutions of system (2) obtained in this way will be called (ω, ρ)-solutions. Precise definition is given after Lemma 2.1.

REMARK 1.2. *Note that our solvability condition (5) in Theorem 1.1 can be written in the following form:*

$$\tilde{g}_0 \leq \tilde{C} - \tilde{D}\tilde{g}_1^{q'}, \tag{6}$$

where $\tilde{C}$ and $\tilde{D}$ are positive constants depending on p, q, N, $|B|$, m_0, m_1. This shows the geometrical picture of the solvability region in $(\tilde{g}_0, \tilde{g}_1)$-plane obtained in Theorem 1.1. As we see, our solvability region is obviously bounded. Moreover, from our nonexistence result below, see Theorem 1.4, we shall see that also the set of all $(\tilde{g}_0, \tilde{g}_1)$ for which (2) possesses a strong solution is bounded in $(0, \infty)^2$.

Solvability condition (5) is analogous to solvability condition for the following scalar quasilinear elliptic problem with the natural growth in the gradient:

$$\begin{cases} -\Delta_p v = \tilde{g}_0 |x|^m + \tilde{f}_0 |\nabla v|^p & \text{in } B \setminus \{0\}, \\ \quad\quad v = 0 \text{ on } \partial B, \\ v(x) \text{ spherically symmetric and decreasing,} \end{cases} \tag{7}$$

which is considered in [11, Theorem 8]. If we let $\tilde{f}_0 = 1$, then the main solvability result in [11] states that under suitable conditions the set of all $\tilde{g}_0 > 0$ for which there exists a strong solution of

(7) is bounded, and contains an interval of the form $(0, \tilde{g}_*)$ with $\tilde{g}_*$ expressed explicitly in terms of N, p, m_0 and $|B|$. Moreover, we were able to prove existence of the unique weak solution of (7) for all such $\tilde{g}_0$. We were not able to obtain any uniqueness result for elliptic system (2), either for weak or for strong solutions.

REMARK 1.3. *Theorem 1.1 can be reformulated to include the general elliptic system (1) instead of (2). It suffices to change $\tilde{g}_0$ and $\tilde{g}_1$ in the definition of $\tilde{a}$ in (91) by*

$$(\tilde{f}_0 \tilde{f}_1^{q'})^{\frac{(p-1)(q-1)}{p+q-1}} \tilde{g}_0, \quad (\tilde{f}_1 \tilde{f}_0^{p'})^{\frac{(q-1)(p-1)}{q+p-1}} \tilde{g}_1 \tag{8}$$

respectively. Our condition (46) for solvability of (1) then becomes

$$(\tilde{g}_0 + \tilde{C} \tilde{g}_1^{q'}) \cdot (\tilde{f}_0 \tilde{f}_1^{q'})^{\delta'-1} \le \tilde{D}, \tag{9}$$

with the same $\tilde{C}$ and $\tilde{D}$ as in (6), and $\delta = p'q'$.

Now we formulate a result about nonexistence of strong solutions of elliptic system (2). It will be convenient to introduce an auxilliary function:

$$S(p, q) = \frac{q' + 1}{(p' + 1)q'}. \tag{10}$$

Note that here we do not assume $m_0 \ge m_1$ as in Theorem 1.1.

THEOREM 1.4. *(nonexistence of strong solutions) Let $1 < p < \infty$, $1 < q < \infty$, $m_0 > -N$, $m_1 > -N$, and*

$$m_1 > -1 - \frac{N}{p'}, \quad m_1 \ge -\frac{(p'+1)q'}{p'q'-1}. \tag{11}$$

Let $\tilde{g}_i^*$, $\tilde{g}_i^+$, $i = 0, 1$ be four explicit positive constants defined in the Appendix (B). Assume that $\tilde{g}_0$ or $\tilde{g}_1$ are large enough, so that any of the following four conditions is satisfied:

(a) $S(p, q) \le 1 - \frac{1}{N}$, $p \le N$, and ($\tilde{g}_0 \ge \tilde{g}_0^*$ or $\tilde{g}_1 \ge \tilde{g}_1^*$),
(b) $S(p, q) \le 1 - \frac{1}{N}$, $p > N$, and ($\tilde{g}_0 > \tilde{g}_0^*$ or $\tilde{g}_1 > \tilde{g}_1^*$),
(c) $S(p, q) > 1 - \frac{1}{N}$, $p \le N$, and ($\tilde{g}_0 \ge \tilde{g}_0^+$ or $\tilde{g}_1 \ge \tilde{g}_1^+$),
(d) $S(p, q) > 1 - \frac{1}{N}$, $p > N$, and ($\tilde{g}_0 > \tilde{g}_0^+$ or $\tilde{g}_1 > \tilde{g}_1^+$).

Then elliptic system (2) has no strong solutions.

The proof of this result is given in Section 3.

REMARK 1.5. *Let us fix* p, q, N, $|B|$ *in elliptic system (2). Theorem 1.4 shows that the set of all pairs* $(\tilde{g}_0, \tilde{g}_1)$ *of positive real numbers for which system (2) possesses a strong solution is bounded, contained in the rectangle* $(0, \tilde{g}_0^*] \times (0, \tilde{g}_1^*]$. *Analogous nonexistence result can be stated for more general system (1), using the change indicated in Remark 1.3.*

2. A system of two singular ODE's of the first order

To our quasilinear system (2) we assign a system of two singular ODE's of the first order:

$$
\begin{cases}
\dfrac{d\omega}{dt} = g_0 \gamma_0 t^{\gamma_0 - 1} + f_0 \dfrac{\rho(t)^{\delta_0}}{t^{\varepsilon_0}}, & t \in (0, T] \\
\dfrac{d\rho}{dt} = g_1 \gamma_1 t^{\gamma_1 - 1} + f_1 \dfrac{\omega(t)^{\delta_1}}{t^{\varepsilon_1}}, & t \in (0, T], \\
\omega(0) = \rho(0) = 0,
\end{cases}
\tag{12}
$$

with the coefficients defined in Lemma 2.1. We study solvability of (12) using the following system:

$$
\omega(t) = K_0 \rho(t), \quad \rho(t) = K_1 \omega(t),
\tag{13}
$$

where we define

$$
K_i \varphi(t) = g_i t^{\gamma_i} + f_i \int_0^t \frac{\varphi(s)^{\delta_i}}{s^{\varepsilon_i}} \, ds, \quad i = 0, 1,
\tag{14}
$$

with suitable domains. We can obtain solutions of (12) via the fixed points of the composition operator $K_0 K_1 : D \subset C([0, T]) \to C([0, T])$ with its domain D defined by (34) below. More precisely, any solution of the fixed point equation

$$
\omega \in D, \quad K_0 K_1 \omega = \omega,
\tag{15}
$$

gives rise to $\rho := K_1 \omega$, and the pair (ω, ρ) is a solution of (2).

The following lemma shows how solutions of quasilinear system (2) can be generated using solutions of the system of singular ODE's (12) and a suitable integral reprezentation.

LEMMA 2.1. *Let* $1 < p < \infty$, $1 < q < \infty$, N, $|B|$, $\tilde{m}_0 > -N$, $\tilde{m}_1 > -N$ *be given. Let us denote* $p_0 = p$, $p_1 = q$, $T = |B|$, *and introduce the following constants:*

$$\gamma_i = 1 + \tfrac{m_i}{N}, \quad \delta_i = p'_{i+1}, \quad \varepsilon_i = p'_{i+1}(1 - \tfrac{1}{N}), \qquad (16)$$

$$f_i = N^{p_{i+1}-p_i} C_N^{\frac{p_{i+1}-p_i}{N}}, \quad g_i = \frac{\tilde{g}_i}{C_N^{\frac{m_i+p_i}{N}} N^{p_i-1}(m_i + N)}, \qquad (17)$$

for $i = 0, 1$, *where we define* $p_2 = p_0$. *Assume that* (ω, ρ) *is a solution of (12) obtained via (15), such that* $0 \le \omega(t) \le Mt^{\gamma_1}$ *for some* $M > 0$. *Let us define*

$$u(x) = V_0(C_N|x|^N), \quad v(x) = V_1(C_N|x|^N), \qquad (18)$$

where the functions $V_i : (0, T] \to \mathbf{R}$ *are given by*

$$V_0(t) = \int_t^T \frac{\omega(s)^{1/(p-1)}}{s^{p'(1-\frac{1}{N})}}\, ds, \quad V_1(t) = \int_t^T \frac{\rho(s)^{1/(q-1)}}{s^{q'(1-\frac{1}{N})}}\, ds, \qquad (19)$$

Then the pair (u, v) *is a strong solution of (2). Furthermore, we have*

$$u'(r) = -\frac{N}{C_N^{\frac{1}{p-1}(1-\frac{p}{N})}} r^{-\frac{N-1}{p-1}} \omega(C_N r^N)^{1/(p-1)}, \qquad (20)$$

$$v'(r) = -\frac{N}{C_N^{\frac{1}{q-1}(1-\frac{q}{N})}} r^{-\frac{N-1}{q-1}} \rho(C_N r^N)^{1/(q-1)}. \qquad (21)$$

and

$$|u'(r)| \le C \cdot r^{\frac{m_1+1}{p-1}}, \qquad (22)$$

where $C = N \cdot C_N^{\frac{m_1+1}{N(p-1)}} M^{1/(p-1)}$. *If* $m_1 \ge -p$ *then also*

$$|v'(r)| \le C_1 \cdot r^{\frac{m_1+1}{q-1}}, \qquad (23)$$

where $C_1 = N \cdot C_N^{\frac{m_1+1}{N(q-1)}} M^{1/(q-1)}$, *with* M_1 *defined in Proposition 4.1(a) below (see Section 4).*

Proof. (*sketch*; see Appendix (D); also compare with [11, Lemma 1]). Using (18) and (19) we obtain that

$$\omega(C_N|x|^N) \;=\; \frac{C_N^{1-\frac{p}{N}}}{N^{p-1}}|x|^{N-1}|\nabla u|^{p-1}, \tag{24}$$

$$\rho(C_N|x|^N) \;=\; \frac{C_N^{1-\frac{q}{N}}}{N^{q-1}}|x|^{N-1}|\nabla v|^{q-1}, \tag{25}$$

which proves (20) and (21). We also have

$$-\Delta_p u = C_N^{p/N} N^p \frac{d\omega}{dt}, \quad -\Delta_q v = C_N^{q/N} N^q \frac{d\rho}{dt}, \tag{26}$$

where we define $t = C_N|x|^N$, $t \in (0, T]$, $T = |B|$. Now from (12) and (24) we obtain

$$\begin{aligned}
-\Delta_p u &= C_N^{p/N} N^p [g_0 \gamma_0 t^{\gamma_0 - 1} + f_0 \frac{\rho(t)^{\delta_0}}{t^{\varepsilon_0}}] \\
&= C_N^{p/N} N^p [g_0 \gamma_0 (C_N|x|^N)^{\gamma_0 - 1} + f_0 \frac{\rho(C_N|x|^N)^{\delta_0}}{(C_N|x|^N)^{\varepsilon_0}}] \\
&= \tilde{g}_0 |x|^{m_0} + |\nabla v|^q,
\end{aligned} \tag{27}$$

and similarly $-\Delta_q v = \tilde{g}_1 |x|^{m_1} + |\nabla u|^p$. Estimate (22) follows easily from $\omega(t) \le M t^{\gamma_1}$ and (20). Estimate (23) follows analogously from (62) below, see Proposition 4.1(a). □

DEFINITION 2.2. *Following the terminology introduced in [11], we say that (u, v) is an (ω, ρ)-solution of (1) if it is a strong solution of (1) and possesses integral representation (18), (19), such that the condition $0 \le \omega(t) \le M t^{\gamma_1}$ is satisfied.*

Below we show that under some additional conditions any (ω, ρ)-solution (u, v) of elliptic system (2) is also a weak solution, see Theorem 6.2.

It is interesting that a partial converse of Lemma 2.1 also holds, that is, if (u, v) is a strong solution of (2) then it is representable in the form (18) via (19), with (ω, ρ) satisfying (12). This is described in the following lemma.

LEMMA 2.3. *Let $m_0 > -N$ and $m_1 > -N$. Let (u, v) be any strong solution of system (2), and let us define the functions $V_i : (0, T] \to \mathbf{R}$, $T = |B|$, $i = 1, 2$, by*

$$V_0(t) = u((tC_N^{-1})^{1/N}), \quad V_1(t) = v((tC_N^{-1})^{1/N}),$$

Then the functions $\omega(t)$ and $\rho(t)$ defined on $(0, T]$ by

$$\omega(t) = t^{p(1-\frac{1}{N})} \left| \frac{dV_0}{dt} \right|^{p-1}, \quad \rho(t) = t^{q(1-\frac{1}{N})} \left| \frac{dV_1}{dt} \right|^{q-1}, \tag{28}$$

satisfy the following system of singular ODE's:

$$\begin{cases} \dfrac{d\omega}{dt} = g_0 \gamma_0 t^{\gamma_0 - 1} + f_0 \dfrac{\rho(t)^{\delta_0}}{t^{\varepsilon_0}}, & t \in (0, T] \\ \dfrac{d\rho}{dt} = g_1 \gamma_1 t^{\gamma_1 - 1} + f_1 \dfrac{\omega(t)^{\delta_1}}{t^{\varepsilon_1}}, & t \in (0, T], \\ \omega, \rho \in D^+, \end{cases} \tag{29}$$

where the coefficients f_i, g_i, γ_i, ε_i, δ_i are the same as in Lemma 2.1, and

$$D^+ = \{\varphi \in C([0, T]) : \varphi(t) \geq 0, \ \forall t \in [0, T], \ \varphi(t) \ nondecreasing\}. \tag{30}$$

Also, relations (19) hold for any $t \in (0, T]$.

The proof of this fact is analogous to the proof of [11, Lemma 2], and therefore we omit it. Note that we do not claim that $\omega(t)$ can be dominated by Mt^{γ_1} for some M, but only $\omega \in D^+$. In other words, we do not claim that any strong solution (u, v) of (2) is necessarily (ω, ρ)-solution.

In order to prove Theorem 1.1, in the remaining part of this section we study some properties of operator $K_0 K_1$.

LEMMA 2.4. *Assume that*

$$\delta_0 > 1, \quad \delta_1 > 0, \quad \gamma_0 \geq \gamma_1 > 0, \tag{31}$$

$$\gamma_1(\delta_0 - 1) - \varepsilon_0 + 1 \geq 0, \quad \gamma_1 \delta_1 - \varepsilon_1 + 1 > 0, \tag{32}$$

$$\delta_0(\gamma_1 \delta_1 - \varepsilon_1 + 1) - \varepsilon_0 + 1 \geq \gamma_1. \tag{33}$$

Let the operators K_0 and K_1 be defined by (14). Then the composition operator $K_0 K_1 : D \subset C([0,T]) \to C([0,T])$ is well defined on

$$D = \{ \, \varphi \in C([0,T]) : 0 \leq \varphi(t) \leq M t^{\gamma_1} \, \}, \qquad (34)$$

where $M > 0$ is fixed. Moreover, we have that

$$0 \leq K_0 K_1 \varphi(t) \leq a + b M^{\delta_0 \delta_1}, \quad \forall t \in [0,T], \qquad (35)$$

where

$$a = g_0 T^{\gamma_0 - \gamma_1} + 2^{\delta_0 - 1} f_0 g_1^{\delta_0} \frac{T^{\gamma_1(\delta_0 - 1) - \varepsilon_0 + 1}}{\gamma_1 \delta_0 - \varepsilon_0 + 1}, \qquad (36)$$

$$b = \frac{2^{\delta_0 - 1} f_0 f_1^{\delta_0} T^{\delta_0(\gamma_1 \delta_1 - \varepsilon_1 + 1) - \varepsilon_0 + 1 - \gamma_1}}{(\gamma_1 \delta_1 - \varepsilon_1 + 1)^{\delta_0} [\delta_0(\gamma_1 \delta_1 - \varepsilon_1 + 1) - \varepsilon_0 + 1]}, \qquad (37)$$

and the operator $K_0 K_1$ is compact.

Proof. We have

$$K_0 K_1 \varphi(t) = g_0 t^{\gamma_0} + f_0 \int_0^t \frac{K_1 \varphi(s)^{\delta_0}}{s^{\varepsilon_0}} \, ds.$$

Using the definition of K_1, and inequality $(x+y)^{\delta_0} \leq 2^{\delta_0 - 1}(x^{\delta_0} + y^{\delta_0})$, for $x, y \geq 0$, $\delta_0 > 1$, we have

$$K_1 \varphi(s)^{\delta_0} \leq 2^{\delta_0 - 1} \left[g_1^{\delta_0} s^{\gamma_1 \delta_0} + f_1^{\delta_0} \left(\int_0^s \frac{\varphi(\sigma)^{\delta_1}}{\sigma^{\varepsilon_1}} \, d\sigma \right)^{\delta_0} \right].$$

Now from $0 \leq \varphi(\sigma) \leq M \sigma^{\gamma_1}$, and conditions on the coefficients in the lemma, we obtain after a short computation that

$$
\begin{aligned}
K_0 K_1 \varphi(t) \quad \leq \quad & g_0 t^{\gamma_0} + 2^{\delta_0 - 1} f_0 \left[g_0^{\delta_0} \frac{t^{\gamma_1 \delta_0 - \gamma_0 + 1}}{\gamma_1 \delta_0 - \varepsilon_0 + 1} + \right. \qquad (38) \\
& \left. + \frac{f_1^{\delta_0} t^{\delta_0(\gamma_1 \delta_1 - \varepsilon_1 + 1) - \varepsilon_0 + 1}}{(\gamma_1 \delta_1 - \varepsilon_1 + 1)^{\delta_0} [\delta_0(\gamma_1 \delta_1 - \varepsilon_1 + 1) - \varepsilon_0 + 1]} M^{\delta_0 \delta_1} \right].
\end{aligned}
$$

which shows that $K_0 K_1$ is well defined on D. Since $\gamma_0 \geq \gamma_1$ and $0 \leq t/T \leq 1$ we have that $(\frac{t}{T})^{\gamma_0} \leq (\frac{t}{T})^{\gamma_1}$, that is

$$t^{\gamma_0} \leq T^{\gamma_0 - \gamma_1} t^{\gamma_1},$$

and similarly

$$t^{\gamma_1\delta_0-\varepsilon_0+1} \le T^{\gamma_1(\delta_0-1)-\varepsilon_0+1}t^{\gamma_1},$$
$$t^{\delta_0(\gamma_1\delta_1-\varepsilon_1+1)-\varepsilon_0+1} \le T^{\delta_0(\gamma_1\delta_1-\varepsilon_1+1)-\varepsilon_0+1-\gamma_1}t^{\gamma_1}.$$

This together with (38) implies estimate (35). It shows that the family of functions $\{K_0K_1\varphi, \ \varphi \in D\}$ is uniformly bounded. To show that the operator K_0K_1 compact, it suffices to apply Ascoli's theorem. It remains to check uniform equicontinuity. Similarly as above we obtain that for all $a, b \in [0, T]$, $\varphi \in D$, we have

$$|K_0K_1\varphi(a) - K_0K_1\varphi(b)| \le g_0|a^{\gamma_0} - b^{\gamma_0}| + 2^{\delta_0-1}f_0|F(a) - F(b)|, \tag{39}$$

where

$$F(s) = \frac{s^{\gamma_1\delta_0-\varepsilon_0+1}}{\gamma_1\delta_0 - \varepsilon_0 + 1} + \frac{f_1^{\delta_0}M^{\delta_0\delta_1}}{(\gamma_1\delta_1 - \varepsilon_1 + 1)^{\delta_0}} \cdot \frac{s^{\delta_0(\gamma_1\delta_1-\varepsilon_1+1)-\varepsilon_1+1}}{\delta_0(\gamma_1\delta_1 - \varepsilon_1 + 1) - \varepsilon_1 + 1}.$$

$\square$

LEMMA 2.5. *If in addition to the hypotheses of the preceding lemma we assume that*

$$\gamma_1(\delta_1 - 1) - \varepsilon_1 + 1 > (\varepsilon_0 - 1)^+, \tag{40}$$

where we define $x^+ = \max\{0, x\}$, *then the operator* K_0K_1 *is Lipschitzian on D:*

$$\|K_0K_1\varphi - K_0K_1\|_\infty \le k \cdot \|\varphi - \psi\|_\infty, \quad \forall \varphi, \psi \in D. \tag{41}$$

Here

$$
\begin{aligned}
k &= \frac{f_0f_1\,\delta_0\delta_1\,M^{\delta_1}}{\gamma_1(\delta_1-1) - \varepsilon_1 + 1} \times \tag{42}\\
&\quad \times \int_0^T \left(g_1s^{\gamma_1} + f_1M^{\delta_1}\frac{s^{\gamma_1\delta_1-\varepsilon_1+1}}{\gamma_1\delta_1 - \varepsilon_1 + 1}\right)^{\delta_0-1} s^{\gamma_1(\delta_1-1)-\varepsilon_1+1-\varepsilon_0}ds.
\end{aligned}
$$

Proof. We have

$$|K_0 K_1 \varphi(t) - K_0 K_1 \psi(t)| \leq f_0 \int_0^t \frac{|K_1 \varphi(s)^{\delta_0} - K_1 \psi(s)^{\delta_0}|}{s^{\varepsilon_0}}\, ds. \quad (43)$$

Now recall the following elementary inequality:

$$|x^\delta - y^\delta| \leq \delta \cdot \max\{x^{\delta-1}, y^{\delta-1}\}|x - y|, \quad \forall x, y \geq 0, \ \delta > 1, \quad (44)$$

which is an immediate consequence of the mean value theorem applied to $F(x) = x^\delta$, $x \geq 0$. Using this inequality twice, together with $0 \leq \varphi(s) \leq M t^{\gamma_1}$, $0 \leq \psi(s) \leq M t^{\gamma_1}$, we obtain

$$|K_1 \varphi(s)^{\delta_0} - K_1 \psi(s)^{\delta_0}|$$

$$\leq \delta_0 f_1 \left(g_1 s^{\gamma_1} + f_1 M^{\delta_1} \frac{s^{\gamma_1 \delta_1 - \varepsilon_1 + 1}}{\gamma_1 \delta_1 - \varepsilon_1 + 1} \right)^{\delta_0 - 1} \int_0^s \frac{|\varphi(\sigma)^{\delta_1} - \psi(\sigma)^{\delta_1}|}{\sigma^{\varepsilon_1}}\, d\sigma$$

$$\leq \delta_0 f_1 \left(g s^{\gamma_1} + f_1 M^{\delta_1} \frac{s^{\gamma_1 \delta_1 - \varepsilon_1 + 1}}{\gamma_1 \delta_1 - \varepsilon_1 + 1} \right)^{\delta_0 - 1} \cdot$$

$$\cdot \delta_1 M^{\delta_1 - 1} \frac{s^{\gamma_1 (\delta_1 - 1) - \varepsilon_1 + 1}}{\gamma_1 (\delta_1 - 1) - \varepsilon_1 + 1} \|\varphi - \psi\|_\infty.$$

The claim follows from (43).   □

The following elementary lemma will be useful in obtaining explicit a priori bounds.

LEMMA 2.6. *Let a and b be positive real numbers and $\delta > 1$. Then the condition*

$$\exists M > 0, \ a + b \cdot M^\delta \leq M \quad (45)$$

holds if and only if

$$a \cdot b^{\delta' - 1} \leq \frac{\delta - 1}{\delta^{\delta'}}. \quad (46)$$

Under condition (46) property (45) is fulfilled with

$$M_0 = \left[\frac{a}{b(\delta - 1)} \right]^{1/\delta}. \quad (47)$$

Proof. (sketch) Let us consider the function $f(x) = a + bx^\delta$, $x > 0$, and take any point $T_0(x_0, y_0)$ on its graph. Using the fact that $y_0' = b\delta x_0^{\delta-1}$ we obtain that the tangent drawn from T_0 passes through the origin if and only if $x_0 = M_0$. The necessary and sufficient condition for existence of $M > 0$ having property (45) is obviously $y_0' \leq 1$, and the claim follows. $\square$

It is easy to see that if we have strict inequality in (46) then the value of M_0 is not the smallest possible among those satisfying (45). The smallest possible M_0 in (45) is the smaller of two positive roots of $a + bM^\delta = M$. Now we formulate our basic solvability result for system (12) of singular ODE's. We provide two proofs, one nonconstructive, and one constructive, which will enable some generalizations Section 7.

THEOREM 2.7. *Let the conditions of Lemma 2.4 be satisfied.*

(a) Assume that $\delta_0\delta_1 > 1$ and let (46) be fulfilled. Then there exists a solution (ω, ρ) of the fixed point equation (15) such that $0 \leq \omega(t) \leq Mt^{\gamma_1}$ with some M satisfying (45). In particular, our system of singular ODE's (12) is solvable.

(b) If $\delta_1\delta_1 < 1$, then system of singular ODE's (12) is solvable for all positive g_0 and g_1.

Proof. (first, nonconstructive). (a) By the preceding lemma applied to $\delta = \delta_0\delta_1$ there exists $M > 0$ such that $a + bM^{\delta_0\delta_1} \leq M$. Defining domain D in (34) with such M we then have that $K_0K_1(D) \subset D$. Since K_0K_1 is compact, the claim follows using Schauder's fixed point theorem. (b) If $\delta_0\delta_1 < 1$, it is clear that $a + bM^{\delta_0\delta_1} \leq M$ for M large enough. $\square$

Proof. (second, constructive). We use the fact that the operator $K_0K_1 : D \to D$ is well defined and monotone in the sense that $\varphi \leq \psi$ in D implies that $K_0K_1\varphi \leq K_0K_1\psi$. Let us define a sequence (ω_n) in D inductively by

$$\omega_0 = 0, \quad \omega_n = K_0K_1\omega_{n-1}, \quad n = 1, 2, \ldots$$

Using $R(K_0K_1) \subseteq D$ and monotonicity of K_0K_1 it is easy to see that ω_n is well defined and nondecreasing in D. Since $0 \leq \omega_n(t) \leq Mt^{\gamma_1}$,

there exists

$$\omega(t) = \lim_{n \to \infty} \omega_n(t).$$

For the same reason, it is clear that $\omega \in L^\infty([0, T])$. Taking the limit in $\omega_n = K_0 K_1 \omega_{n-1}$ as $n \to \infty$ in $L^\infty([0, T])$, and using Levi's theorem, we obtain that $\omega = K_0 K_1 \omega$. This also proves that ω is continuous, that is, $\omega \in D$. □

We shall use the above monotone iterations ω_n in Section 5. It is worth noting that if we fix f_0 and g_0, then condition (46) for solvability of the system of singular ODE's (12), see Theorem 2.7, can be written in the form

$$g_0 \leq C - D g_1^{\delta_0},$$

where C and D are positive constants depending on γ_i, δ_i, ε_i and f_i, and with $g_0 > 0$, $g_1 > 0$. This shows that the corresponding solvability region in (g_0, g_1)-plane is bounded.

We do not know whether in Theorem 2.7(a) we also have uniqueness of solutions. It is not possible to apply comparison principle in [11, Theorem 1] since the operator $K_0 K_1$ is not generated by a Carathéodory function, i.e. there is no any function $k(t, \eta) : [0, T] \times \mathbf{R} \to \mathbf{R}$, measurable with respect to t and continuous with respect to η, such that $K_0 K_1 \varphi(t) = \int_0^t k(s, \varphi(s)) \, ds$. However, if we impose additional condition that $k < 1$ in (41), see (42), then due to Banach's fixed point theorem we have a unique solution in D defined by (34) and $M = M_0$.

REMARK 2.8. *Condition $k < 1$ in the above theorem is fulfilled if for example f_0 and f_1 are fixed and g_0 and g_1 are sufficiently small. This can easily be seen by taking into account that M has the form (47).*

Proof. (of Theorem 1.1) It suffices to exploit Theorem 2.7 and Lemma 2.1. The corresponding conditions on N, p, q, m_i and $\tilde{g}_i$ in Theorem 1.1 are obtained after easy algebraic manipulations from conditions on γ_i, δ_i, ε_i and T in Theorem 2.7 using (16) and (17). □

The question of uniqueness of solutions for elliptic system (2) seems to be difficult. Here is a partial result.

PROPOSITION 2.9. *Assume that all conditions of Theorem 1.1 hold, and let*

$$m_1 > -p + (p-1)N[q'(1 - \frac{1}{N}) - 1]^+. \tag{48}$$

Let the coefficients $\tilde{g}_0$ and $\tilde{g}_1$ be small enough, so that $k < 1$, see (42), with M given by (47), and taking into account relations (16) and (17). Then there exists a unique (ω, ρ)-solution of (2).

Proof. Since the conditions of Lemma 2.5 hold with $k < 1$, and with M given by (47), then there exists a unique solution (ω, ρ) of system (12) such that $\omega \in D$. The claim follows from Lemma 2.7. $\quad\square$

3. Nonexistence of strong solutions

The aim of this section is to prove Theorem 1.4. To this end we first study the problem of nonexistence of solutions for the system of singular ODE's described by (29), with $\omega, \rho \in D^+$. Here we assume that both equations in (29) hold a.e. in $(0, T]$ (recall that ω and ρ are nondecreasing by the definition of D^+).

THEOREM 3.1. *(nonexistence result for system of singular ODE's) Assume that $\delta_0 \geq 1$, $\delta_1 \geq 1$, $\delta_0 \delta_1 > 1$, $\gamma_0 > 0$, $\gamma_1 > 0$,*

$$\gamma_1 \delta_1 - \varepsilon_1 + 1 > 0, \quad \delta_0(\gamma_1 \delta_1 - \varepsilon_1 + 1) - \varepsilon_1 + 1 \geq \gamma_1,$$

and let g_i^*, g_i^+, $i = 0, 1$, be four explicit positive constants defined in Appendix (C).

Assume that g_0 or g_1 are large enough, so that any of the following four conditions is satisfied:

  (a) $E \leq 0$, $\varepsilon_1 \geq 1$, and ($g_0 \geq g_0^*$ or $g_1 \geq g_1^*$),
  (b) $E \leq 0$, $\varepsilon_1 < 1$, and ($g_0 > g_0^*$ or $g_1 > g_1^*$),
  (c) $E > 0$, $\varepsilon_1 \geq 1$, and ($g_0 \geq g_0^+$ or $g_1 \geq g_1^+$),
  (d) $E > 0$, $\varepsilon_1 < 1$, and ($g_0 > g_0^+$ or $g_1 > g_1^+$).

Then the system of singular ODE's (29) has no solutions.

Let us describe briefly the idea of the proof, which has appeared for the first time in Pašić [13], see also [11] for improved version. We proceed by contradiction, that is, assume that there exists a solution (ω, ρ) of (29). First we start with zero function $\omega_0 = 0 \in D^+$, and then we show that:

$$[(K_0 K_1)^n \omega_0](t) \le \omega(t), \quad n = 1, 2, \ldots$$

One then proves that under conditions of Theorem 3.1 we have $(K_0 K_1)^n \omega_0(T) \to \infty$ as $n \to \infty$, which is a contradiction, since $\omega(T) < \infty$.

The following two lemmas will be the main tool in the proof.

LEMMA 3.2. *Let $\delta_0 \ge 1$, $\delta_1 \ge 1$, $\gamma_0 > 0$, $\gamma_1 > 0$, $\gamma_1 \delta_0 - \varepsilon_0 + 1 > 0$, and assume that (ω, ρ) is a solution of (29).*
(a) If we define

$$z(t) = g_0 t^{\gamma_0} + \frac{f_1^{\delta_0} g_1^{\delta_0}}{\gamma_1 \delta_0 - \varepsilon_0 + 1} t^{\gamma_1 \delta_0 - \varepsilon_0 + 1},$$

then $z(t) \le \omega(t)$.
(b) Let $z_0 \in D^+$ be such that $z_0(t) \le z(t)$, with $z(t)$ from (a). Let us define a sequence of functions z_m inductively:

$$z_{m+1}(t) = f_0 f_1^{\delta_0} \int_0^t s^{-\varepsilon_0} \left(\int_0^s \frac{z_m(\sigma)^{\delta_1}}{\sigma^{\varepsilon_1}} \right)^{\delta_0} ds, \quad m = 0, 1, 2, \ldots$$

$$\tag{49}$$

Then for all $t \in [0, T]$,

$$\omega(t) \ge \sum_{m=0}^{\infty} z_m(t). \tag{50}$$

Proof. (a) We use a well known fact that if ω is nondecreasing on $[0, T]$, then $\omega(t) - \omega(0) \ge \int_0^t \frac{d\omega}{ds} ds$. Therefore for $\omega, \rho \in D^+$ we have that $\omega(t) \ge \int_0^t \frac{d\omega}{ds} ds$ and $\rho(t) \ge \int_0^t \frac{d\rho}{ds} ds$. Using this and (29) we obtain

$$\begin{aligned}
\omega(t) &\ge g_0 t^{\gamma_0} + f_0 \int_0^t s^{-\varepsilon_0} \left(g_1 s^{\gamma_1} + f_1 \frac{\int_0^s \omega(\sigma)^{\delta_1}}{\sigma^{\varepsilon_1}} d\sigma \right)^{\delta_0} ds \quad (51) \\
&\ge g_0 t^{\gamma_0} + f_0 \int_0^t s^{-\varepsilon_0} (g_1 s^{\gamma_1})^{\delta_0} ds = z(t).
\end{aligned}$$

(b) Denoting $s_n(t) = z_0(t) + z_1(t) + \cdots + z_n(t)$, we must prove that $\omega(t) \geq s_n(t)$ for all n. We proceed by induction. For $n = 0$ we have $\omega(t) \geq z_0(t) = s_0(t)$. Assume that $\omega(t) \geq s_n(t)$ for some n. Then since (ω, ρ) is a solution of system (29) in $D^+ \times D^+$, $\delta_0 > 1$, $\delta_1 > 1$, we have that

$$
\begin{aligned}
\omega(t) \;\geq\; & g_0 t^{\gamma_0} + f_0 \int_0^t s^{-\varepsilon_0} \left(g_1 s^{\gamma_1} + f_1 \frac{\int_0^s \omega(\sigma)^{\delta_1}}{\sigma^{\varepsilon_1}} d\sigma \right)^{\delta_0} ds \\
\geq\; & g_0 t^{\gamma_0} + f_0 \int_0^t s^{-\varepsilon_0} \left[(g_1 s^{\gamma_1})^{\delta_0} + f_1^{\delta_0} \left(\frac{\int_0^s \omega(\sigma)^{\delta_1}}{\sigma^{\varepsilon_1}} d\sigma \right)^{\delta_0} \right] ds \\
\geq\; & z_0(t) + f_0 f_1^{\delta_0} \int_0^t s^{-\varepsilon_0} \left(\frac{\int_0^s s_n(\sigma)^{\delta_1}}{\sigma^{\varepsilon_1}} d\sigma \right)^{\delta_0} ds \\
\geq\; & z_0(t) + f_0 f_1^{\delta_0} \int_0^t s^{-\varepsilon_0} \left(\sum_{m=0}^n \frac{\int_0^s z_m(\sigma)^{\delta_1}}{\sigma^{\varepsilon_1}} d\sigma \right)^{\delta_0} ds \\
\geq\; & z_0(t) + f_0 f_1^{\delta_0} \sum_{m=0}^n \int_0^t s^{-\varepsilon_0} \left(\frac{\int_0^s z_m(\sigma)^{\delta_1}}{\sigma^{\varepsilon_1}} d\sigma \right)^{\delta_0} ds \\
=\; & s_{n+1}(t).
\end{aligned}
$$

$\square$

Since we assume that $z_0(t) \leq z(t)$, in what follows we shall take either $z_0(t) = g_0 t^{\gamma_0}$ or $z_0(t) = \frac{f_1^{\delta_0} g_1^{\delta_0}}{\gamma_1 \delta_0 - \varepsilon_0 + 1} t^{\gamma_1 \delta_0 - \varepsilon_0 + 1}$. This will enable us to find appropriate lower bounds on g_0 and g_1 respectively that will guarantee nonexistence of solutions of singular system (29).

LEMMA 3.3. *Assume that $\delta_0 \geq 1$, $\delta_1 \geq 1$, $\delta_0 \delta_1 > 1$, $\gamma_0 > 0$, $\gamma_1 > 0$, and*

$$
\delta_0(\gamma_1 \delta_1 - \varepsilon_1 + 1) - \varepsilon_0 + 1 \geq \gamma_1, \quad \gamma_1 \delta_1 - \varepsilon_1 + 1 > 0.
$$

Let a_0 be a given positive constant and let us define a sequence of functions z_m inductively by $z_0(t) = a_0 t^{b_0}$ and (49), with either $b_0 = \gamma_1$ or $b_0 = \gamma \delta_1 - \varepsilon_1 + 1$. Then

$$
z_m(t) = a_m t^{b_m}, \quad m = 0, 1, 2, \ldots
$$

where

$$a_m = a_0^{\delta^m} \prod_{i=1}^{m} A_{m-i}^{\delta^{i-1}}, \quad \delta = \delta_0 \delta_1,$$

and

$$A_m = \frac{f_0 f_1^{\delta_0}}{(b_m \delta_1 - \varepsilon_1 + 1)^{\delta_0} (\delta b_m + E)}, \tag{52}$$

and

$$b_m = \delta^m b_0 + \frac{\delta^m - 1}{\delta - 1} E. \tag{53}$$

We have $b_m \geq \gamma_1$ and $b_m \delta_1 - \varepsilon_1 + 1 > 0$ for all m.

Proof. Using (49) and $z_m(t) = a_m t^{b_m}$, we obtain after a short computation that

$$z_{m+1}(t) = A_m a_m^{\delta} t^{\delta b_m + E}. \tag{54}$$

Therefore,

$$a_{m+1} = A_m a_m^{\delta}, \quad b_{m+1} = \delta b_m + E.$$

The claim follows easily by induction.

Note that two integrations that we have performed in proving (54) are justified since $b_m \geq \gamma_1$, which follows by induction. For $m = 0$, this is clear since by the assumption, either $b_0 = \gamma_1$ or $b_0 = \gamma_1 \delta_1 - \varepsilon_1 + 1$. Now if $b_m \geq \gamma_1$ then $b_{m+1} = \delta b_m + E = \delta_0(\delta_1 \gamma_1 - \varepsilon_1 + 1) - \varepsilon_0 + 1 \geq \gamma_1$.

Inequality $b_m \geq \gamma_1$ implies that also $b_m \delta_1 - \varepsilon + 1 \geq \gamma_1 \delta_1 - \varepsilon_1 + 1 > 0$ for all m. $\qquad \square$

Proof. (Theorem 3.1) Let us assume, contrary to the claim, that there exists a solution (ω, ρ) of singular system (29). In what follows we assume that either

$$(a_0, b_0) = (g_0, \gamma_0), \tag{55}$$

or

$$(a_0, b_0) = (\frac{f_0 g_1^{\delta_0}}{\gamma_1 \delta_0 - \varepsilon_0 + 1}, \gamma_1 \delta_0 - \varepsilon_0 + 1), \tag{56}$$

which yields the corresponding two values for initial iteration $z_0(t) = a_0 t^{b_0}$ of the sequence z_m in Lemma 3.2 and Lemma 3.3.

(a) Let us assume that $g_0 \geq g_0^*$. Condition $E \leq 0$ in (53) implies that $b_m \leq \delta^m b_0$. Using this together with $\varepsilon_1 \geq 1$ in (52), we obtain that

$$A_m \geq \frac{f_0 f_1^{\delta_0}}{(b_m \delta_1^{\delta_0}) \delta b_m} \geq \frac{f_0 f_1^{\delta_0}}{\delta^{(\delta_0+1)m+1} b_0^{\delta_0+1} \delta_1^{\delta_0}}.$$

Now from Lemma 3.3 we have

$$
\begin{aligned}
z_m(T) &= a_m T^{b_m} = a_0^{\delta^m} \left(\prod_{i=1}^{m} A_{m-i}^{\delta^{i-1}} \right) T^{\delta^m b_0 + \frac{\delta^m-1}{\delta-1} E} \\
&\geq a_0^{\delta^m} \left(\frac{f_0 f_1^{\delta_0}}{b_0^{\delta_0+1} \delta_1^{\delta_0}} \right)^{\frac{\delta^m}{\delta-1}} \frac{T^{\delta^m b_0 + \frac{\delta^m-1}{\delta-1} E}}{\delta^{(\delta_0+1)S_m + \frac{\delta^m-1}{\delta-1}}},
\end{aligned}
$$

where

$$S_m = \sum_{i=1}^{m-1} (m-i)\delta^{i-1} = \frac{1}{\delta^2} \left[\frac{2\delta-1}{(\delta-1)^2}(\delta^m - 1) - \frac{m}{\delta-1} + \right.$$

$$\left. + \delta^m - (m+1) - m\delta \right].$$

Therefore

$$z_m(T) \geq C \cdot \left(\frac{a_0^{\delta-1} f_0 f_1^{\delta_0} T^{(\delta-1)b_0+E}}{b_0^{\delta_0+1} \delta_1^{\delta_0} \delta^{\frac{\delta_0+\delta}{\delta-1}}} \right)^{\frac{\delta^m}{\delta-1}}, \qquad (57)$$

where

$$C = T^{-\frac{E}{\delta-1}} \delta^{\frac{-(2\delta-1)(\delta_0+1)}{\delta^2(\delta-1)^2} + \frac{1}{\delta-1}} \left(\frac{f_0 f_1^{\delta_0}}{b_0^{\delta_0+1} \delta_1^{\delta_0}} \right)^{\frac{-1}{\delta-1}}$$

does not depend on m. Here we have also used the fact that

$$\delta^{(\frac{m}{\delta-1}+m+1+m\delta)\frac{\delta_0+1}{\delta^2}} > 1,$$

which follows immediately from $\delta > 1$. Since we have assumed that $g_0 \geq g_0^*$, we now take (a_0, b_0) as defined by (55), with the corresponding $z_0(t)$. This together with (57) implies that $z_m(T) \geq C > 0$ for

all m. We conclude that

$$\sum_{m=1}^{\infty} z_m(T) = \infty, \tag{58}$$

and using Lemma 3.2 we obtain that $\omega(T) = \infty$, which is a contradiction. The case $g_1 \geq g_1^*$ is treated in exactly the same way, with (a_0, b_0) defined by (56).

(b) The case $E \leq 0$ and $\varepsilon_1 < 1$ is treated using a slight modification of the proof in (a). We still have $b_m \leq \delta^m b_0$. To estimate $b_m \delta_1 - \varepsilon_1 + 1$ in the denominator of A_m in (52), let us fix any constant $\alpha > 1$. Since $\delta^m \to \infty$ as $m \to \infty$, there exists m_0 such that for $m \geq m_0$ we have

$$b_m \delta_1 - \varepsilon_1 + 1 \leq \delta^m b_0 \delta_1 + (-\varepsilon_1 + 1) \leq \alpha(\delta^m b_0 \delta_1).$$

As in (a) we obtain

$$A_m \geq \frac{f_0 f_1^{\delta_0}}{\delta^{(\delta_0+1)m+1} b_\alpha^{\delta_0+1} \delta_1^{\delta_0}},$$

where $b_\alpha = \alpha^{\frac{\delta_0}{\delta+1}} b_0$. We obtain estimate (57) with b_α instead of b_0.

Now assume that $\tilde{g}_0 > \tilde{g}_0^*$. We take (a_0, b_0) as in (55). Since here we have strict inequality in $\tilde{g}_0 > \tilde{g}_0^*$, we can find $\alpha > 1$ sufficiently close to 1 such that $z_m(T) \geq C$ for all $m \geq m_0$. Therefore the series (58) is again divergent, and we obtain the desired contradiction in the same way as in (a).

In cases (c) and (d) we have $E > 0$. We start with the following estimate:

$$b_m = \delta^m b_0 + \frac{\delta^m - 1}{\delta - 1} E \leq \delta^m \left(b_0 + \frac{E}{\delta - 1} \right).$$

Therefore we can proceed in the same way as in (a) and (b), using $b_0 + \frac{E}{\delta-1}$ instead of b_0.   □

4. Qualitative properties of solutions

The aim of this section is to study regularity of (ω, ρ)-solutions of quasilinear elliptic system (2), and to obtain a priori estimates at the

origin and gradient estimates on the boundary. The proofs use analogous methods as in [11] for the scalar case, see Section 8 there. We start our consideration with some qualitative properties of solutions of the system of singular ODE's (12) first.

PROPOSITION 4.1. *Let the conditions of Theorem 2.7 be fulfilled. Let (ω, ρ) be a solution of (2) obtained in the same theorem. Then*

$$0 \leq \omega(t) \leq Mt^{\gamma_1}, \tag{59}$$

$$|\omega'(t)| \leq g_0 \gamma_0 t^{\gamma_0 - 1} + 2^{\delta_0 - 1}[t^{\gamma_1 \delta_0 - \varepsilon_1} +$$

$$\frac{f_1^{\delta_0} M^{\delta_0 \delta_1}}{(\gamma_1 \delta_1 - \gamma_1 + 1)^{\delta_0}} t^{\delta_0(\gamma_1 \delta_1 - \varepsilon_1 + 1) - \varepsilon_0}], \tag{60}$$

and $\omega, \rho \in C^\infty((0, T])$.

(a) If in addition to the above hypotheses we assume that

$$\gamma_1(\delta_1 - 1) - \varepsilon_1 + 1 \geq 0, \tag{61}$$

then also

$$0 \leq \rho(t) \leq M_1 t^{\gamma_1}, \tag{62}$$

where $M_1 = g_1 + M^{\delta_1} \frac{T^{\gamma_1(\delta_1 - 1) - \varepsilon_1 + 1}}{\gamma_1 \delta_1 - \varepsilon_1 + 1}$. If

$$\gamma_1(\delta_1 - 1) - \varepsilon_1 + 1 > 0, \tag{63}$$

then

$$\lim_{t \to 0} \frac{\rho(t)}{t^{\gamma_1}} = \lim_{t \to 0} \frac{\rho'(t)}{\gamma_1 t^{\gamma_1 - 1}} = g_1. \tag{64}$$

(b) If in addition to the hypotheses of Theorem 2.7 we assume that

$$\gamma_1 \delta_0 - \varepsilon_0 + 1 > \gamma_0, \quad \gamma_1(\delta_1 - 1) - \varepsilon_1 + 1 \geq 0, \tag{65}$$

then

$$\lim_{t \to 0} \frac{\omega(t)}{t^{\gamma_0}} = \lim_{t \to 0} \frac{\omega'(t)}{\gamma_0 t^{\gamma_0 - 1}} = g_0. \tag{66}$$

Proof. Estimate (59) follows from $\omega \in D$, while estimate (60) follows easily from (39) by dividing by $b - a$ and letting $b \to a = t$. The fact that $\omega, \rho \in C^\infty((0, T])$ follows easily from $\omega = K_0 \rho$ and $\rho = K_1 \omega$.

(a) To prove (62), note that $\gamma_1 \delta_1 - \varepsilon_1 + 1 \geq \gamma_1$ implies $t^{\gamma_1 \delta_1 - \varepsilon_1 + 1} \leq T^{\gamma_1(\delta-1) - \varepsilon_1 + 1} t^{\gamma_1}$, and therefore

$$\rho(t) \leq K_1 \omega(t) \leq g_1 t^{\gamma_1} + M^{\delta_1} \frac{t^{\gamma_1 \delta_1 - \varepsilon_1 + 1}}{\gamma_1 \delta_1 - \varepsilon_1 + 1} \leq M_1 t^{\gamma_1}.$$

If $\gamma_1 \delta_1 - \varepsilon_1 + 1 > \gamma_1$ then

$$\frac{\omega(t)^{\delta_1}}{t^{\varepsilon_1 + \gamma_1 - 1}} \leq M^{\delta_1} t^{\gamma_1(\delta_1 - 1) - \varepsilon_1 + 1} \to 0, \quad \text{as } t \to 0.$$

Hence, (64) follows immediately from $\frac{\rho'(t)}{\gamma_1 t^{\gamma_1 - 1}} = g_1 + \frac{\omega(t)^{\delta_1}}{t^{\varepsilon_1 + \gamma_1 - 1}}$, and L'Hospital's rule.

(b) We have that

$$\frac{\rho(t)^{\delta_0}}{t^{\varepsilon_0 + \gamma_0 - 1}} \leq M^{\delta_0} t^{\gamma_1 \delta_0 - \gamma_0 + 1 - \varepsilon_0} \to 0, \quad \text{as } t \to 0.$$

and the claim follows from $\frac{\omega'(t)}{\gamma_0 t^{\gamma_0 - 1}} = g_0 + \frac{\rho(t)^{\delta_0}}{t^{\varepsilon_0 + \gamma_0 - 1}}$, using L'Hospital's rule. $\square$

Now we are ready to study qualitative properties of (ω, ρ)-solutions of quasilinear elliptic system (2). Recall that by (ω, ρ)-solution we mean a strong solution obtained via solving the fixed point equation $K_0 K_1 \omega = \omega$, such that $0 \leq \omega(t) \leq M t^{\gamma_1}$, as described in Lemma 2.1.

THEOREM 4.2. *(behaviour of solutions at the origin) Assume that conditions of Theorem 1.1 are fulfilled. Let (u, v) be an (ω, ρ)-solution of elliptic system (2) obtained in Theorem 1.1.*

(a) Then $u, v \in C^\infty(\overline{B} \setminus \{0\})$. Furthermore, if $m_1 > -p$ then $u \in C(\overline{B})$. If $m_1 \geq -p$ and $m_1 > -q$, then $v \in C(\overline{B})$.

(b) If $m_1 > -p$, then

$$\lim_{r \to 0} \frac{v'(r)}{r^{\frac{m_1 + 1}{q - 1}}} = -\left(\frac{\tilde{g}_1}{m_1 + N}\right)^{1/(q-1)} \tag{67}$$

$$\lim_{r \to 0} \frac{v''(r)}{r^{\frac{m_1 - q + 2}{q - 1}}} = -\frac{m_1 + 1}{q - 1}\left(\frac{\tilde{g}_1}{m_1 + N}\right)^{1/(q-1)}, \tag{68}$$

(c) If $m_1 \geq -p$ and $q'(m_1 + 1) > m_0$, then

$$\lim_{r \to 0} \frac{u'(r)}{r^{\frac{m_0+1}{p-1}}} = -\left(\frac{\tilde{g}_0}{m_0 + N}\right)^{1/(p-1)} \tag{69}$$

$$\lim_{r \to 0} \frac{u''(r)}{r^{\frac{m_0-p+2}{p-1}}} = -\frac{m_0 + 1}{p - 1}\left(\frac{\tilde{g}_0}{m_0 + N}\right)^{1/(p-1)}. \tag{70}$$

Proof. (a) To prove that $u, v \in C^\infty(B \setminus \{0\})$, it suffices to use $W^{\in,C}(\,)\infty((0,T])$, (18) and (19).

If $m_1 > -p$ then using the definition of $V_0(t)$ in Lemma 2.1 and $\omega(t) \leq M t^{\gamma_1}$ we obtain that $V_0(t)$ is well defined and continuous on $[0,T]$. Therefore $u(x) = V_0(C_N|x|^N)$ is continous on $\overline{B}$.

If $m_1 \geq -p$ then $\rho(t) \leq M_1 t^{\gamma_1}$, and therefore $m_1 > -q$ implies that $V_1(t)$ is continuous on $[0,T]$, so that $v(x) = V_1(C_N|x|^N)$ is continous on $\overline{B}$.

(b) Relation (67) follows immediately from the fact that

$$\lim_{t \to 0} \frac{\rho(t)}{t^{\gamma_1}} = g_1.$$

Relation (68) is proved using (64) in the same way as in [11, Lemma 8(b)]. We reproduce the proof for the sake of completeness. Since $\frac{\rho(t)}{t^\gamma} \to g_1$ and $\frac{\rho'(t)}{\gamma_1 t^{\gamma_1-1}} \to g_1$ as $t \to 0$, denoting $c = -NC_N^{\frac{1}{q-1}(\frac{q}{N}-1)}$ and $\gamma_1 = 1 + \frac{m_1}{N}$ we obtain after an easy computation that

$$\frac{v''(r)}{c\,r^{\frac{m_1-q+2}{q-1}}} = -C_N^{\frac{\gamma_1 q'}{q}}\frac{N-1}{q-1}\left[\frac{\rho(C_N r^N)}{(C_N r^N)^{\gamma_1}}\right]^{1/(q-1)} +$$

$$+\gamma_1 N C_N^{\frac{q'\gamma_1}{q}}\frac{1}{q-1}\left[\frac{\rho(C_N r^N)}{(C_N r^N)^{\gamma_1}}\right]^{\frac{1}{q-1}-1}\frac{\rho'(C_N r^N)}{\gamma_1(C_N r^N)^{\gamma_1-1}}$$

$$\to -C_N^{\frac{\gamma_1 q'}{q}}\frac{N-1}{q-1}g_1^{1/(q-1)} + \gamma_1 N C_N^{\frac{q'\gamma_1}{q}}\frac{1}{q-1}g_1^{\frac{1}{q-1}-1}g_1$$

$$= \frac{1}{q-1}g_1^{1/(q-1)}C_N^{\gamma_1/(q-1)}(\gamma_1 N - N + 1) \quad \text{as } r \to 0.$$

Now we use the definition of g_1 in Lemma 2.1 to obtain the desired result.

(c) is proved in the same way as (b), using Proposition 4.1(b). $\square$

An immediate consequence of the above theorem is the following regularity result.

THEOREM 4.3. *(regularity of solutions) Let conditions of Theorem 1.1 be fulfilled, and assume that (u, v) is (ω, ρ)-solution of quasilinear elliptic system (2).*

(a) Assume that $m_1 > -p$.

(a1) If $m_1 < -1$ then $\lim_{r \to 0} v'(r) = -\infty$. In particular, $v \notin C^1(B)$.

(a2) If $m_1 = -1$ then $\lim_{r \to 0} v'(r) = - \left(\frac{\tilde{g}_1}{m_1 + N} \right)^{1/(q-1)}$. As in case (a1), we have $v \notin C^1(B)$.

(a3) If $-1 < m_1 < q-2$ then $\lim_{r \to 0} v'(r) = 0$ and $\lim_{r \to 0} v''(r) = -\infty$. In particular, $v \notin C^2(B)$.

(a4) If $m_1 \geq q - 2$ then $\lim_{r \to 0} v'(r) = 0$ and

$$\lim_{r \to 0} v''(r) = \begin{cases} -\frac{m_1 + 1}{q - 1} \left(\frac{\tilde{g}_1}{m_1 + N} \right)^{1/(q-1)} & \text{for } m_1 = q - 2, \\ 0 & \text{for } m_1 > q - 2. \end{cases} \tag{71}$$

In particular, $v \in C^2(\overline{B})$.

(b) Assume that $m_1 \geq -p$ and $q'(m_1 + 1) > m_0$.

(b1) If $m_0 < -1$ then $\lim_{r \to 0} u'(r) = -\infty$. In particular, $u \notin C^1(B)$.

(b2) If $m_0 = -1$ then $\lim_{r \to 0} u'(r) = - \left(\frac{\tilde{g}_0}{m_0 + N} \right)^{1/(p-1)}$. As in case (a1), we have $u \notin C^1(B)$.

(b3) If $-1 < m_0 < p-2$ then $\lim_{r \to 0} u'(r) = 0$ and $\lim_{r \to 0} u''(r) = -\infty$. In particular, $u \notin C^2(B)$.

(b4) If $m_0 \geq p - 2$ then $\lim_{r \to 0} u'(r) = 0$ and

$$\lim_{r \to 0} u''(r) = \begin{cases} -\frac{m_0 + 1}{p - 1} \left(\frac{\tilde{g}_0}{m_0 + N} \right)^{p/p} & \text{for } m_0 = p - 2, \\ 0 & \text{for } m_0 > p - 2. \end{cases} \tag{72}$$

In particular, $u \in C^2(\overline{B})$.

(c) Let $m_1 > -p$ and $q'(m_1 + 1) > m > 0$. Then the solution (u, v) is classical, i.e. $u, v \in C^2(\overline{B})$, if and only if $m_0 \geq p - 2$ and $m_1 \geq q - 2$.

Claim (c) is an immediate consequence of (a) and (b).   □

With higher values of m_0 and m_1 we can obtain more and more regular solutions.

Now we establish a priori estimates of (ω, ρ)-solutions of (2) at the origin. Compare with [11, Proposition 7].

PROPOSITION 4.4. *Let conditions of Theorem 1.1 be satisfied. Then we have*

$$u(0) \leq N \frac{p-1}{m_1+p} \cdot C_N^{\frac{m_1+p}{N(p-1)}} R^{\frac{m_1+p}{p-1}} M_0^{1/(p-1)}, \qquad (73)$$

with

$$M_0 = \left(\frac{\tilde{a}}{\tilde{b}(p'q'-1)} \right)^{1/p'q'}, \qquad (74)$$

where $\tilde{a}$ and $\tilde{b}$ are defined in Theorem 1.1. If also $m_1 \geq -p$, then

$$v(0) \leq N \frac{q-1}{m_1+q} \cdot C_N^{\frac{m_1+q}{N} \cdot \frac{1}{q-1}} R^{\frac{m_1+q}{q-1}} M_1^{1/(q-1)}. \qquad (75)$$

where

$$M_1 = \frac{\tilde{g}_1}{C_N^{m_1+q} N^{q-1}(m_1+N)} + \frac{N M_0^{p'}}{p'(m_1+1)+N} |B|^{\frac{p'(m_1+1)-m_1}{N}}. \qquad (76)$$

Proof. We use $u(0) \leq \int_0^R |u'(r)| \, dr$ with estimate (22), and $v(0) \leq \int_0^R |v'(r)| \, dr$ with estimate (23).   □

REMARK 4.5. *Estimates involving M_0 in the above theorem can obviously be improved so that instead of M_0 we take the smaller of two positive roots of equation $\tilde{a} + \tilde{b}M^{p'q'} = M$.*

Now we obtain a lower bound on the gradient of strong solutions of elliptic systems on the boundary of domain B.

PROPOSITION 4.6. *Let conditions of Theorem 1.1 be satisfied and let* (u, v) *be any strong solution of system (2).*

(a) *If* $m_0 > -1$, $m_1 \geq -p$ *and* $q'(m_1 + 1) > m_0$, *then*

$$|u'(R)| \geq \left(\frac{\tilde{g}_0 R^{m_0+1}}{m_0 + N} \right)^{1/(p-1)}. \tag{77}$$

(b) *If* $m_1 > -1$, *then*

$$|v'(R)| \geq \left(\frac{\tilde{g}_1 R^{m_1+1}}{m_1 + N} \right)^{1/(q-1)}. \tag{78}$$

Proof. (a) Using Theorem 4.2(b) we have that $u'(r) \to 0$ as $r \to 0$. The claim follows from in [11, Proposition 9] adapted to our sitation. (b) is proved in the same way using Theorem 4.2(a). $\square$

REMARK 4.7. *Let us consider quasilinear elliptic system (1), but without condition that u and v be decreasing. Let* (u, v) *be a strong solution of (1). It can be shown that if* $p > 2$ *then u is necessarily decreasing, see [11, Proposition 8(c)]. Analogously, if* $q > 2$ *then v is necessarily decreasing.*

5. Approximation of solutions

The proof of Theorem 1.1 is in fact constructive, since we have used monotone iterations (ω_n) in the second proof of Theorem 2.7. Let us define a sequence (u_n, v_n) of successive approximations of solution (u, v) by

$$u_n(x) = \int_{C_N |x|^N}^{|B|} \frac{\omega_n(s)^{1/(p-1)}}{s^{p'(1-\frac{1}{N})}} \, ds, \quad v_n(x) = \int_{C_N |x|^N}^{|B|} \frac{\rho_n(s)^{1/(q-1)}}{s^{q'(1-\frac{1}{N})}} \, ds, \tag{79}$$

where $\rho_n = K_1 \omega_n$, see (14). Then we have the following approximation result.

THEOREM 5.1. *Let all conditions of Theorem 1.1 be satisfied. Let* (ω_n) *be a sequence of monotone iterations obtained in the second*

proof of Theorem 2.7, converging to ω, and let (u,v) be an (ω,ρ)-solution of (2) in Theorem 1.1, $\rho_n = K_1\omega_{n-1}$. Let (u_n,v_n) be defined by (79). Then $u_n(x) \le u(x)$, $v_n \le v(x)$ on $B \setminus \{0\}$, and the sequences of functions $u_n(x)$ and $v_n(x)$ are nondecreasing.

(a1) If $m_1 > -p$ then $u_n \to u$ in $C(\overline{B})$.

(a2) If $m_1 > -p$ and $q'(m_1+1) > m_0 \ge -1$, then $u_n \to u$ in $C^1(\overline{B})$.

(a3) If $m_1 > -p$ and $q'(m_1+1) > m_0 \ge p-2$, then $u_n \to u$ in $C^2(\overline{B})$.

(b1) If $m_1 > -q$ and $m_1 \ge -p$, then $v_n \to v$ in $C(\overline{B})$.

(b2) If $m_1 > -q$ and $m_1 \ge -1$, then $v_n \to v$ in $C^1(\overline{B})$.

(b3) If $m_1 \ge q-2$, then $v_n \to v$ in $C^2(\overline{B})$.

Note that the sequence ω_n is nondecreasing, and $\omega_n \to \omega$ uniformly on $[0,T]$. The same for ρ_n and ρ. The proof of this theorem rests on Theorem 4.3 and on the following lemma.

LEMMA 5.2. *Functions ω_n and ρ_n in the above theorem, $n = 1, 2, \ldots$, possess the same properties as ω and ρ in Proposition 4.1(a) and (b).*

Proof. It suffices to use $\omega_n = K_0 K_1 \omega_{n-1} = K_0\rho_n$ and $\rho_n = K_1\omega_{n-1}$, to obtain

$$\omega_n'(t) = g_0\gamma_0 t^{\gamma_0-1} + \frac{\rho_n(t)^{\delta_0}}{t^{\varepsilon_0}}, \quad \rho_n'(t) = g_1\gamma_1 t^{\gamma_1-1} + \frac{\omega_n(t)^{\delta_1}}{t^{\varepsilon_1}}.$$

Since $\omega_n(t) \le Mt^{\gamma_1}$ and $\rho_n(t) \le M_1 t^{\gamma_1}$, we can proceed in the same way as in the proof of Proposition 4.1(a) and (b). $\qquad \square$

Proof. (Theorem 5.1) Let us denote $t = C_N|x|^N$, $r = |x|$, $T = |B|$. Note that $m_1 > -p$ implies $m_0 > -p$, so that $u, u_n \in C(\overline{B})$. Also if $m_1 > -q$, then $v, v_n \in C(\overline{B})$.

(a1) Using the fact that $\omega_n(s) \le \omega(s) \le Ms^{\gamma_1}$ and (79), we

obtain

$$
\begin{aligned}
\|u - u_n\|_\infty &\leq \int_0^T \frac{\omega(s)^{1/(p-1)} - \omega_n(s)^{1/(p-1)}}{s^{p'(1-\frac{1}{N})}} ds \\
&\leq \int_0^T \frac{\left(\frac{\omega(s)}{s^{\gamma_1}}\right)^{1/(p-1)} - \left(\frac{\omega_n(s)}{s^{\gamma_1}}\right)^{1/(p-1)}}{s^\alpha} ds = \int_0^a + \int_a^T \\
&\leq M^{1/p(p-1)} \int_0^a s^{-\alpha} ds + \\
&\quad + \max_{s\in[a,T]} \left| \left(\frac{\omega(s)}{s^{\gamma_1}}\right)^{1/(p-1)} - \left(\frac{\omega_n(s)}{s^{\gamma_1}}\right)^{1/(p-1)} \right| \cdot \\
&\quad \cdot \int_a^T s^{-\alpha} ds.
\end{aligned}
$$

Here $\alpha = p'(1 - \frac{1}{N}) - \frac{\gamma_1}{p-1}$, $\gamma_1 = 1 + \frac{m_1}{N}$, and $-\alpha + 1 > 0$ is equivalent to $m_0 > -p$. Therefore, for any given $\varepsilon > 0$ there exists $a > 0$ suficiently small, such that $M^{1/(p-1)} \int_0^a s^{-\alpha} ds < \frac{\varepsilon}{2}$. Also, since $\omega_n \to \omega$ uniformly on $[a, T]$, there exists n_0 large enough such that for $n \geq n_0$ the second term in the sum does not exceed $\frac{\varepsilon}{2}$. Hence, $\|u - u_n\|_\infty < \varepsilon$ for all $n \geq n_0$, which proves that $u_n \to u$ uniformly on $\overline{B}$.

(a2) Using (20) we have that

$$
|u'(r) - u_n'(r)| \leq C \cdot t^{-\frac{1}{p-1}(1-\frac{1}{N})} |\omega(t)^{1/(p-1)} - \omega_n(t)^{1/(p-1)}|.
$$

Therefore, $u_n'(r) \to u'(r)$ for all $r \in (0, R]$. Using Lemma 5.2 and Proposition 4.2, see (b2), (b3) and (b4), we have $u_n'(0) = u'(0)$ for all n. Hence, since u_n' and u' are continuous and $u_n'(r) \to u'(r)$ pointwise on compact interval $[0, T]$, then $u_n'(r) \to u'(r)$ uniformly on $[0, T]$.

(a3) Using (20) again, an easy computation shows that

$$
\begin{aligned}
u''(r) &= C \left(\frac{\omega(t)}{t^{1-\frac{1}{N}}}\right)^{p'-2} [\omega'(t) - (1 - \frac{1}{N})\frac{\omega(t)}{t}], \\
u_n''(r) &= C \left(\frac{\omega_n(t)}{t^{1-\frac{1}{N}}}\right)^{p'-2} [\omega_n'(t) - (1 - \frac{1}{N})\frac{\omega_n(t)}{t}].
\end{aligned}
$$

Therefore, since $\omega_n(t) \to \omega(t)$ and $\omega_n'(t) \to \omega'(t)$ for all $t \in (0, T]$, then also $u_n''(r) \to u''(r)$ for $r \in (0, R]$. Using Lemma 5.2 and Theorem 4.2(b4) we have that $u_n''(0) = u''(0)$, where we have denoted $u_n''(0) = \lim_{r \to 0} u_n''(r)$ and $u''(0) = \lim_{r \to 0} u''(r)$. Now since u_n'' and u'' are continuous on $[0, T]$, and $u_n''(r) \to u''(r)$ pointwise on compact interval $[0, T]$, then the convergence is uniform.

Cases (b1), (b2) and (b3) are treated in the same way.   □

6. Weak solutions of quasilinear elliptic systems, singularities

Here we want to find some sufficient conditions that guarantee that (ω, ρ)-solutions of system (2) are also weak solutions.

LEMMA 6.1. *Let conditions of Lemma 2.1 be fulfilled with $N \geq 2$. Let (u, v) be any (ω, ρ)-solution of system (2). If $m_1 > -1 - \frac{N}{p'}$, then $u \in W_0^{1,p}(B)$.*

Proof. (a1) Let us prove that the pointwise derivative $\frac{\partial u}{\partial x_i}$ is also weak derivative. If we denote $\Omega_\varepsilon = B \setminus B_\varepsilon(0)$, with $\varepsilon > 0$ small, and $S_\varepsilon = \partial B_\varepsilon(0)$, then for any test function $\varphi \in \mathcal{D}(B)$ we have

$$\int_{\Omega_\varepsilon} u \frac{\partial \varphi}{\partial x_i} \, dx = -\int_{\Omega_\varepsilon} \frac{\partial u}{\partial x_i} \varphi \, dx + \int_{S_\varepsilon} u \varphi \nu_i \, dS$$

Let us denote the last integral by $A(\varepsilon)$. It suffices to show that $A(\varepsilon) \to 0$ as $\varepsilon \to 0$, since then

$$\int_B u \frac{\partial \varphi}{\partial x_i} \, dx = -\int_B \frac{\partial u}{\partial x_i} \varphi \, dx.$$

We have

$$A(\varepsilon) \leq C \cdot u(\varepsilon) \cdot \varepsilon^{N-1}, \tag{80}$$

where $C > 0$ is a generic constant. We have three cases. (i) If $m_1 > -p$, then u is continuous, see Theorem 4.2(a), and hence bounded. Therefore $A(\varepsilon) \to 0$ as $\varepsilon \to 0$.

(ii) Assume that $m_1 < -p$. Using $\omega(t) \leq M t^{\gamma_1}$ in Lemma 2.1, we obtain that $V_0(s)$ has singularity of order $\frac{p'(m_1+1)-m_1}{N}$, so that $u(x)$,

see (19), has singularity of order which is at most $p'(m_1 + 1) - m_1$ at $x = 0$. The claim follows from (80) and the fact that $p'(m_1 + 1) - m_1 + N - 1 > 0$.

(iii) If $m_1 = -p$, then $u(x)$ has at most logarithmic singularity at $x = 0$, and the claim follows easily from (80).

(a2) Using (22) we have

$$\int_B |\nabla u|^p dx \leq C \int_0^R r^{(m_1+1)p'+N-1} dr.$$

The last integral is finite due to $m_1 > -1 - \frac{N}{p'}$. Therefore $u \in W_0^{1,p}(B)$. $\square$

THEOREM 6.2. *Let all conditions of Theorem 1.1 be fulfilled, and assume that*

$$m_1 \geq -p, \quad m_1 > -1 - \frac{N(p-1)}{q}. \tag{81}$$

Then any (ω, ρ)-solution (u, v) of quasilinear elliptic system (2) is weak solution.

Proof. We can proceed analogously as in [11], using estimates (22) and (23). We use Lemma 6.1 and the fact that

$$\int_B |\nabla v|^q dx \leq C \int_0^R r^{q\frac{m_1+1}{p-1}+N-1} dr < \infty,$$

which follows from $m_1 > -1 - \frac{N(p-1)}{q}$. Let us prove that system (2) is satisfied by (u, v) in the weak sense. Since $u, v \in C^\infty(B \setminus \{0\})$, see Theorem 4.2, both equations in (2) are satisfied by (u, v) in the classical sense on the set $\Omega_\varepsilon = B \setminus B_\varepsilon(0)$, for any $\varepsilon > 0$. Multiplying by $\varphi \in \mathcal{D}(B)$ and using Green's formula we obtain that:

$$\tilde{g}_0 \int_{\Omega_\varepsilon} |x|^{m_0} \varphi \, dx + \int_{\Omega_\varepsilon} |\nabla v|^q \varphi \, dx = -\int_{\Omega_\varepsilon} \text{div}(|\nabla u|^{p-2}\nabla u)\varphi \, dx$$
$$= \int_{\Omega_\varepsilon} |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, dx - A(\varepsilon),$$

where

$$A(\varepsilon) = \int_{S_\varepsilon} \sum_{i=1}^N |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \nu_i \, dS \leq C \int_{S_\varepsilon} |\nabla u|^{p-1} dS \leq C \cdot |u'(\varepsilon)|^{p-1} \varepsilon^{N-1},$$

and $S_\varepsilon = \partial B_\varepsilon(0)$. Using (22) we obtain that $A(\varepsilon) \leq C \cdot \varepsilon^{m_1+N} \to 0$ as $\varepsilon \to 0$. This proves that $\int_B |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = \tilde{g}_0 \int_B |x|^{m_0} \varphi \, dx + \int_B |\nabla v|^q \varphi \, dx$ for all $\varphi \in \mathcal{D}(B)$. Analogously for the second equation in (2). $\square$

As we have seen in Theorem 4.2(a), the component u of any (ω, ρ)-solution (u, v) is continuous at $x = 0$ provided $m_1 > -p$. Note that since $m_0 \geq m_1$, then also $m_0 > -p$. Now we find some sufficient conditions for (u, v) to have singularity at $x = 0$.

THEOREM 6.3. *Let the conditions of Theorem 6.2 be fulfilled, let $p < N$, and let (u, v) be an (ω, ρ)-solution.*
    (a) If $m_0 \leq -p$, then

$$u(x) \geq$$

$$\begin{cases} \left(\dfrac{\tilde{g}_0}{m_0 + N} \right)^{1/(p-1)} \dfrac{R^{(m_0+1)p'-m_0} - |x|^{(m_0+1)p'-m_0}}{(m_0+1)p'-m_0} & \text{for } m_0 < -p, \quad (82) \\ \left(\dfrac{\tilde{g}_0}{N-p} \right)^{1/(p-1)} \log \dfrac{R}{|x|} & \text{for } m_0 = -p. \end{cases}$$

In particular, u has singularity of order at least $\frac{|m_0|-p}{p-1}$ at $x_0 = 0$ if $m_0 < -p$, and at least logarithmic singularity if $m_0 = -p$.
    (b) If $m_1 = -q$ then v has at least logarithmic singularity at $x = 0$:

$$v(x) \geq \left(\frac{\tilde{g}_1}{N-q} \right)^{1/(q-1)} \log \frac{R}{|x|} \qquad (83)$$

Proof. (a) We have that $u \in W_0^{1,p}(B)$, see Lemma 6.1. Since $-\Delta_p u \geq \tilde{g}_0 |x|^{m_0}$ pointwise in $B \setminus \{0\}$, the same inequality holds also in the weak sense. The claim follows from [15, Theorem 3]. (b) This case is treated in the same way. $\square$

PROPOSITION 6.4. *Let (u, v) be any weak solution of (1). Then*

$$v(0) \geq \begin{cases} \dfrac{1}{(2p)^{p'}} \left(\dfrac{R^{m_0+p}\tilde{g}_0}{2^N-1} \right)^{1/(p-1)} & \text{for } m_0 < 0, \\ \left(\dfrac{c(m_0, p, N)R^{m_0+p}\tilde{g}_0}{p^p} \right)^{1/(p-1)} & \text{for } m_0 \geq 0, \end{cases} \qquad (84)$$

and

$$v(0) \geq \begin{cases} \dfrac{1}{(2q)^{q'}} \left(\dfrac{R^{m_1+q}\tilde{g}_1}{2^N-1} \right)^{1/(q-1)} & \textit{for } m_1 < 0, \\[4mm] \left(\dfrac{c(m_1,q,N)R^{m_1+q}\tilde{g}_1}{q^q} \right)^{1/(q-1)} & \textit{for } m_1 \geq 0, \end{cases} \tag{85}$$

where

$$c(m,p,N) = \sup_{t\in(0,\frac{1}{2})} \frac{t^p[(1-t)^N - t^N](1-t)^m}{1+t^N-(1-t)^N}. \tag{86}$$

In particular, these estimates hold for (ω,ρ)-solutions obtained in Theorem 6.2.

Proof. If $m_0 < 0$ we use our oscillation estimate of solutions formulated in [9, Theorem 7] (or in [10, Theorem 3]), noting that the right-hand side of (1) is $\geq \tilde{g}_0|x|^{m_0}$. If $m_0 \geq 0$, we use a more general version of oscillation estimate of solutions, formulated in [9, Theorem 9]. The same for $v(0)$. See the proof of [11, Proposition 7(b)] for details. $\square$

Nontrivial lower bounds for any (ω,ρ)-solutions (u,v) can be obtained using approximation functions u_n and v_n introduced in Section 6. Under conditions of Theorem 5.1 we have

$$u(0) \geq u_n(0), \quad v(0) \geq v_n(0).$$

Note that $u_n(0)$ and $v_n(0)$ can be effectively computed via ω_n and ρ_n that are defined recursively by $\omega_n = K_0 K_1 \omega_{n-1}$, $\omega_0 = 0$, $\rho_n = K_1 \omega_n$. Furthermore, by Theorem 5.1 we have $u_n(0) \to u(0)$, $v_n(0) \to v(0)$ monotonically as $n \to \infty$.

7. Generalizations

Existence and nonexistence results stated for system (2) in Theorem 1.1 and Theorem 1.4 respectively can be extended to much more

general quasilinear elliptic systems:

$$\begin{cases} -\Delta_p u = F_0(|x|, v, |\nabla v|), & \text{in } B \setminus \{0\}, \\ -\Delta_q v = F_1(|x|, u, |\nabla u|), & \text{in } B \setminus \{0\}, \\ \quad u > 0, \ v > 0 \text{ on } B, \text{ spherically symmetric, decreasing}, \\ \quad u = v = 0 \text{ on } \partial B, \end{cases} \qquad (87)$$

where we assume that $F_i : (0, R] \times \mathbf{R}^+ \to \mathbf{R}$ are continuous functions such that for all (r, η, ξ),

$$0 \le F_0(r, \eta, \xi) \le \tilde{g}_0 r^{m_0} + \tilde{f}_0 \xi^q, \qquad (88)$$

$$0 \le F_1(r, \eta, \xi) \le \tilde{g}_1 r^{m_1} + \tilde{f}_1 \xi^p. \qquad (89)$$

Recall that by a strong solution of (87) we mean a pair functions $u, v \in C^2(\overline{B} \setminus \{0\})$ which satisfy the elliptic system pointwise. We confine ourselves only to state the corresponding results. Their proofs use a combination of our existence and nonexistence results with a general fixed point theorem in partially ordered Banach spaces, see Amann [1, Theorem 6.1]

THEOREM 7.1. *(existence of strong solutions) Assume that the growth conditions (88) and (89) are fulfilled, and let*

$$\forall a > 0, \ \exists r \in (0, a), \ \forall \eta > 0, \ \forall \xi > 0, \ F_0(r, \eta, \xi) > 0 \ and \ F_1(r, 0, 0) > 0. \qquad (90)$$

We assume that conditions of Theorem 1.1 hold with $\tilde{g}_0$ and $\tilde{g}_1$ changed to (8), where $\tilde{g}_i$, $\tilde{f}_i$, $i = 0, 1$, are positive constants. Then there exists a strong solution of elliptic system (87). Moreover, if $F_i(r, \eta, \xi)$ is nondecreasing with respect to variables η and ξ, then there exists a strong solution which can be obtained constructively using method of monotone iterations.

Now we state a nonexistence result. Note that we do not impose any monotonicity assumption on $F_1(r, \eta, \xi)$.

THEOREM 7.2. *(nonexistence of strong solutions) Assume that all conditions of Theorem 1.4 are satisfied with $\tilde{g}_0$ and $\tilde{g}_1$ changed to (8), and*

$$F_0(r, \eta, \xi) \ge \tilde{g}_0 r^{m_0} + \tilde{f}_0 \xi^q,$$

$$F_1(r, \eta, \xi) \ge \tilde{g}_1 r^{m_1} + \tilde{f}_1 \xi^p,$$

Let $F_0(r, \eta, \xi)$ be nondecreasing in the variables η and ξ. Then quasi-linear elliptic system (87) has no strong solutions.

8. Appendix

In this section we provide precise values of the corresponding constants appearing in Theorems 1.1, 1.4 and 3.1. They can easily be computed using Lemma 2.1.

(A) The values of constants $\tilde{a}$ and $\tilde{b}$ in Theorem 1.1 are

$$
\tilde{a} = \frac{|B|^{\frac{m_0 - m_1}{N}} \tilde{g}_0}{C_N^{\frac{m_0 + p}{N}} N^{p-1}(m_0 + N)} + \frac{2^{1/(q-1)} |B|^{\frac{q'(m_1+1) - m_1}{N}}}{N^{p-1} C_N^{\frac{q'(m_1+q)+p-q}{N}}} \tilde{g}_1^{q'}, \quad (91)
$$

$$
\tilde{b} = \frac{2^{1/(q-1)} N^{\frac{p-q+1}{q-1}+2} C_N^{-\frac{(p-q)q'}{Nq}} |B|^{\frac{q'[p'(m_1+1)+1]-m_1}{N}}}{[p'(m_1 + 1) + N]^{q'} [q'(p'(m_1 + 1) + 1) + N]}.
$$

(B) In Theorem 1.4 we introduce constants $\tilde{g}_i^*$ and $\tilde{g}_i^+$, $i = 0, 1$, in the following way. First we introduce

$$
\tilde{A}(b) = \left(\tilde{C} \cdot \frac{b^{q'+1}}{|B|^{(p'q'-1)b}} \right)^{\frac{1}{p'q'-1}}, \quad b > 0, \quad (92)
$$

where

$$
\tilde{C} = \frac{(p')^{q'} (p'q')^{\frac{(p'+1)q'}{p'q'+1}}}{N^{\frac{p-q}{q-1}} C_N^{\frac{p-q}{q-1}} |B|^{1 - \frac{q'[p'(N-1)-1]}{N}}}.
$$

Then we define:

$$
\begin{cases}
\tilde{g}_0^* = d_0 \cdot \tilde{A}(1 + \frac{m_0}{N})^{1/q'}, \\[2mm]
\tilde{g}_1^* = d_1 \cdot \left[\frac{q'(m_1 + N) + N}{C_N^{\frac{q-p}{N}} N^{q-p+1}} \tilde{A} \left(\frac{q'(m_1 + 1) + N}{N} \right) \right]^{1/q'},
\end{cases} \quad (93)
$$

and

$$
\begin{cases}
\tilde{g}_0^+ = d_0 \cdot \tilde{A}(1 + \dfrac{m_0}{N} + \dfrac{E}{p'q'-1})^{1/q'}, \\[2mm]
\tilde{g}_1^+ = d_1 \cdot \left[\dfrac{q'(m_1+N)+N}{C_N^{\frac{q-p}{N}} N^{q-p+1}} \tilde{A} \left(\dfrac{q'(m_1+1)+N}{N^{q-p+1}C_N^{\frac{q-p}{N}}} + \dfrac{E}{p'q'-1} \right) \right]^{1/q'}.
\end{cases}
$$

$$(94)$$

where

$$
\begin{aligned}
d_0 &= C_N^{\frac{m_0+p}{N}} N^{p-1}(m_0+N), \\
d_1 &= C_N^{\frac{m_1+q}{N}} N^{q-1}(m_1+N), \\
E &= -(p'+1)q'(1-\frac{1}{N}) + q' + 1.
\end{aligned}
$$

(C) In order to define four positive constants g_i^* and g_i^+, $i = 0, 1$, appearing in Theorem 3.1, it will be convenient to introduce the following function:

$$
A(b) = \left(C \cdot \frac{b^{\delta_0+1}}{T^{(\delta-1)b}} \right)^{\frac{1}{\delta-1}}, \quad b > 0,
$$

where

$$
\delta = \delta_0\delta_1, \quad C = \frac{\delta_1^{\delta_0} \delta^{\frac{\delta_0+1}{\delta^2}\frac{2\delta-1}{\delta-1}+\frac{\delta-1}{\delta^2}+1}}{f_0 f_1^{\delta_0} T^E}, \quad E = \delta_0(-\varepsilon_1 + 1) - \varepsilon_0 + 1,
$$

We define:

$$
g_0^* = A(\gamma_0), \quad g_1^* = \left(\frac{\gamma_1\delta_0 - \varepsilon_0 + 1}{f_0} \cdot A(\gamma_1\delta_0 - \varepsilon_0 + 1) \right)^{1/\delta_0}, \quad (95)
$$

and

$$
\begin{cases}
g_0^+ = A(\gamma_0 + \frac{E}{\delta-1}), \\
g_1^+ = \left(\frac{\gamma_1\delta_0 - \varepsilon_0 + 1}{\delta_0} \cdot A(\gamma_1\delta_0 - \varepsilon_0 + 1 + \frac{E}{\delta-1}) \right)^{1/\delta_0},
\end{cases}
\qquad (96)
$$

(D) As pointed out by the referee, system (1) can be studied using functions defined by $\phi(r) = r^{N-1}|u'(r)|^{p-1}$ and $\psi(r) = r^{N-1}|v'(r)|^{q-1}$. It is easy to see that if we deal with solutions (u, v) of elliptic system (1) with components in $C^2(B \setminus \{0\})$, then both approaches are equivalent, since we obviously have (here we need that u and v are decreasing and $u(R) = v(R) = 0$):

$$u(r) = \int_r^R \frac{\phi(\sigma)^{p'-1}}{\sigma^{\frac{N-1}{p-1}}} d\sigma \quad v(r) = \int_r^R \frac{\psi(\sigma)^{p'-1}}{\sigma^{\frac{p-1}{N-1}}} d\sigma, \qquad (97)$$

Relations (18), (19) follow from (97) by a change of variable $s = C_N \sigma^N$, and conversely. It is easy to see that $\phi(r)$ and $\omega(t)$ are related as follows:

$$\omega(t) = C_N^{1-\frac{p}{N}} N^{1-p} \phi(r), \quad t = C_N r^N, \qquad (98)$$

and analogously for $\rho(t)$ and $\psi(r)$. Note that first order systems corresponding to (ω, ρ) and (ϕ, ψ) are of the same nature. Indeed, if a solution (u, v) of elliptic system (1) is represented by (18), (19), then the system reduces to (12), while for (97) we have

$$\phi'(r) = \tilde{g}_0 r^{N-1+m_0} + \tilde{f}_0 \frac{\psi(r)^{q'}}{r^{\frac{N-1}{q-1}}}, \quad \psi'(r) = \tilde{g}_1 r^{N-1+m_1} + \tilde{f}_1 \frac{\phi(r)^{p'}}{r^{\frac{N-1}{p-1}}}.$$
$$(99)$$

REMARK 8.1. *It is possible to study solvability of elliptic systems like in (1) having arbitrary positive growth rate in the gradient. Using different methods, we can also treat diagonal quasilinear elliptic systems with right-hand sides depending on all unknown functions and their gradients, and with arbitrarily many equations. Also, polyharmonic equations can be studied. This will be a subject of forthcoming papers.*

Acknowledgement

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