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Remark on Subharmonic Solutions of Periodic Planar Systems

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Summary. - *Let W be a periodic isolating segment for the periodic planar system. Assume that time 0 section W_0 is a topological manifold with boundary and $H(W_0, W_0^-) \neq 0$, where W^- is the exit set of W. Then there is a subharmonic solution.*

1. Introduction

We are interested in the two-dimensional system of the form

$$\dot{x} = F(t, x), \tag{1}$$

where $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is $T > 0$ periodic in t and the associated Cauchy problem has a unique solution. We are concerned with the existence of kT-periodic (subharmonic) solutions of (1) or, equivalently periodic points of the Poincaré map $\varphi_{(0,T)}$. The famous Massera theorem (see [1], [8], [12]) says that if the Poincaré map is defined on all $\mathbb{R}^2$ and there is a future bounded solution of (1), then there is also a T-periodic solution. The Massera theorem follows immediately by applying the Brouwer Asymptotic Fixed Point Theorem to the Poincaré map. The Brouwer Asymptotic FPT says that if $g : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation preserving embedding with a future bounded trajectory, then g has a fixed point (for more details and extensive bibliographies, we refer the reader to [11]). The main

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result in this paper is based on the notion of periodic isolating segments introduced in [17] (see also [16], [15], [13], [18]). The basic property of the segment is that in any boundary point of the segment the vector field is directed outward or inward with respect to the segment. In all practical applications the segments are manifolds with corners. It was observed in [16] that the fixed point index of the Poincaré map inside the segment is equal to the Lefschetz number of the homeomorphism $\tilde{h}$ given by the segment. We prove that if W is a periodic isolating segment over $[0,T]$ for (1) (see definition in the next section), W_0 is a compact, connected topological manifold with boundary and $H(W_0, W_0^-) \neq 0$, then (1) has a subharmonic solution. Note that by the Ważewski Retract Theorem assumption $H(W_0, W_0^-) \neq 0$ implies existence of a future bounded solution of (1). On the other hand we do not assume that the Poincaré map is defined globally, which is a rather restrictive assumption from the point of view of applications. Our main result follows by Lemma 1 which is a generalization of Cor.7.5 in [16].

2. Periodic isolating segments

Let M be a smooth (i.e. of the class C^∞) manifold and let

$$f : \mathbb{R} \times M \longrightarrow TM$$

be a continuous time-dependent vector field. We assume that for every $(t_0, x_0) \in \mathbb{R} \times M$ the Cauchy problem

$$\dot{x} = f(t,x), \tag{2}$$

$$x(t_0) = x_0 \tag{3}$$

has a unique solution. By φ we denote the local process generated by (2) i.e. $\varphi_{(t_0,\tau)}(x_0) \in M$ is the value of the solution of the Cauchy problem (2), (3) at time $t_0 + \tau$. In the sequel we assume that f is T-periodic with respect to t, hence for all $\sigma, t \in \mathbb{R}$

$$\varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)}$$

and in order to determine all T-periodic solutions of the equation (2) it suffices to look for fixed points of $\varphi_{(0,T)}$ (called the Poincaré map).

Now we introduce the notion of periodic isolating segment. To this aim we use the following notation: by $\pi_1 : [0,T]\times M \to [0,T]$ and $\pi_2 : [0,T]\times M \to M$ we denote the projections and for a subset $Z \subset \mathbb{R}\times M$ and $t\in\mathbb{R}$ we put

$$Z_t = \{x \in M : (t,x) \in Z\}.$$

Let (W, W^-) be a pair of subsets of $[0,T]\times M$ (i.e. $W^- \subset W$). We call W a periodic isolating segment over $[0,T]$ (for the equation (2)) and W^- the exit set of W if:

(i) W and W^- are compact ENR's, $W_0 = W_T$ and $W_0^- = W_T^-$,

(ii) there exists a homeomorphism of pairs

$$h : ([0,T]\times W_0, [0,T]\times W_0^-) \longrightarrow (W, W^-)$$

such that $\pi_1 = \pi_1 \circ h$,

(iii) for every $\sigma \in [0,T)$ and $x \in \partial W_\sigma$ there exists a $\delta > 0$ such that for every $t \in (0,\delta)$ either $\varphi_{(\sigma,t)}(x) \notin W_{\sigma+t}$ or $\varphi_{(\sigma,t)}(x) \in \operatorname{int} W_{\sigma+t}$.

(iv) $W^- \cap ([0,T)\times M) = \{(\sigma,x) \in W : \sigma < T,\ \exists\delta > 0\ \forall t \in (0,\delta) : \varphi_{(\sigma,t)}(x) \notin W_{\sigma+t}\}$.

Define a homeomorphism

$$\tilde{h} : (W_0, W_0^-) \longrightarrow (W_T, W_T^-) = (W_0, W_0^-)$$

by $\tilde{h}(x) = \pi_2(h(T, \pi_2 h^{-1}(0,x)))$ for $x \in W_0$. Geometrically, $\tilde{h}$ moves a point $x \in W_0$ to $W_T = W_0$ along the arc $h([0,T]\times\{\pi_2 h^{-1}(0,x)\})$. A different choice of the homeomorphism h in (ii) leads to a map which is homotopic to $\tilde{h}$ (compare [16]), hence the automorphism

$$\mu_W = \tilde{h}_* : H(W_0, W_0^-) \longrightarrow H(W_0, W_0^-)$$

induced by $\tilde{h}$ in singular homology, is an invariant of the segment W. Recall that its Lefschetz number is defined as

$$\operatorname{Lef}(\mu_W) = \sum_{n=0}^{\infty} (-1)^n \operatorname{tr}\, \tilde{h}_{*n}.$$

In particular, if $\mu_W = \mathrm{id}_{H(W_0,W_0^-)}$ then $\mathrm{Lef}(\mu_W)$ is equal to the Euler characteristic $\chi(W_0, W_0^-)$. In the sequel we will use the following theorem which, up to slightly different notation, was proved in [16]:

THEOREM 2.1. *If W is a periodic isolating segment over $[0,T]$ then the set of T-periodic solutions which are contained in the segment W:*

$$F_W = \{x \in M : \varphi_{(0,T)}(x) = x,\ \forall t \in [0,T] : \varphi_{(0,t)}(x) \in W_t\}$$

is compact and open in the set of fixed points of $\varphi_{(0,T)}$ and the fixed point index of $\varphi_{(0,T)}$ in F_W is given by

$$\mathrm{ind}(\varphi_{(0,T)}, F_W) = \mathrm{Lef}(\mu_W),$$

(See [3] for the definition and properties of the fixed point index; here we use a different notation from the one in that book.)

The Lefschetz zeta function for a continuous map of a compact ANR into itself was introduced by Smale in [14]. From this time many Lefschetz type zeta functions have been introduced in the theory of discrete dynamical systems. For example: the reduced *mod* 2 zeta function ([4]), twisted zeta function, which have coefficients in the group rings $\mathbb{Z}H$ or $\mathbb{Z}_2H$ of an abelian group H ([6], [5]), zeta function in the Nielsen index theory ([7]), zeta function for an isolated invariant set ([10], [9]).

Assume that W is a periodic isolating segment over $[0,T]$. The zeta function of a T-periodic isolating segment W is defined by

$$Z_W(t) = \exp[\sum_{n=1}^{\infty} \frac{\mathrm{ind}(\varphi_{(0,nT)}, F_{W^n})}{n} t^n],$$

where

$$F_{W^n} = \{x \in M : \varphi_{(0,nT)}(x) = x,\ \ \forall t \in [0,nT]\ \ \varphi_{(0,t)}(x) \in W_{t \bmod T}\}.$$

This is a formal power series in t. We show that this is always a rational function of the variable t which depends only on the homology class of μ_W.

LEMMA 2.2. *(1) We have*

$$Z_W(t) = \prod_{k=0}^{\infty} [\det(Id - (\mu_W)_k t)]^{(-1)^{k+1}},$$

where H is the singular homology functor with coefficients in $\mathbb{Q}$, $(\mu_W)_k : H_k(W_0, W_0^-) \to H_k(W_0, W_0^-)$ and $\mathrm{Id} = \mathrm{Id}_{H(W_0,W_0^-)}$.
(2) If $\chi(W_0, W_0^-) \neq 0$ then there exists $n \in \mathbb{N}$ such that

$$\mathrm{ind}(\varphi_{(0,nT)}, F_{W^n}) \neq 0.$$

Proof. Let $B_i = \dim H_i(W_0, W_0^-)$. Put

$$p = \sum_{i \text{ even}} B_i, \quad q = \sum_{i \text{ odd}} B_i.$$

(1) It follows by Th.1 and Lemma 5.2 in [4].
(2) Obviously if $Z_W(t) \neq 1$ then there exists $n \in \mathbb{N}$ such that

$$\mathrm{ind}(\varphi_{(0,nT)}, F_{W^n}) \neq 0.$$

Since $\widetilde{h}_W : (W_0, W_0^-) \to (W_0, W_0^-)$ is a homeomorphism and

$$q - p = -\chi(W_0, W_0^-) \neq 0,$$

then $Z_W(t) \neq 1$. □

The main result of this section is the following (compare Cor.7.5 in [16])

PROPOSITION 2.3. *(1) If $\chi(W_0, W_0^-)$ is positive (resp. negative) then there exists $k \in \{1, ..., p\}$ (resp. $k \in \{1, ..., q\}$) such that*

$$\mathrm{ind}(\varphi_{(0,kT)}, F_{W^k}) \neq 0.$$

(2) If $\chi(W_0, W_0^-) = 0$ and $Z_W(t) \neq 1$ then there exists $k \in \{1, ..., p = q\}$ such that

$$\mathrm{ind}(\varphi_{(0,kT)}, F_{W^k}) \neq 0.$$

Proof. (1) We give the proof only for the case $\chi(W_0, W_0^-) > 0$, because the case $\chi(W_0, W_0^-) < 0$ is dual. $Z_W(t)$ is a formal power series in t , so

$$Z_W(t) = 1 + \sum_{n=1}^{\infty} a_n t^n.$$

By induction on n it is easy to see that

$$a_1 = ... = a_n = 0 \text{ iff } \mathrm{Lef}(\mu_W) = ... = \mathrm{Lef}(\mu_W^n) = 0.$$

Suppose that for all $i \in \{1, ..., p\}$ $\mathrm{Lef}(\mu_W^i) = 0$. By Lemma 1

$$Z_W(t)P(t) = Q(t),$$

where

$$Q(t) = \prod_{i \text{ odd}} \det(Id - (\mu_W)_i t), \quad P(t) = \prod_{i \text{ even}} \det(Id - (\mu_W)_i t).$$

So we obtain

$$P(t) + P(t)(\sum_{k=p+1}^{\infty} a_k t^k) = Q(t).$$

The coefficient near t^p on the left side is non-zero and is zero on the right side. This shows that it must exist $k \in \{1, ..., p\}$ such that $\mathrm{Lef}(\mu_W^k) \neq 0$.
(2) Suppose that $a_1 = ... = a_p = 0$, so we have

$$P(t) + P(t)(\sum_{k=p+1}^{\infty} a_k t^k) = Q(t).$$

By easy induction this gives that $a_n = 0$ for all $n \in \mathbb{N}$, and finally $Z_W(t) = 1$. □

Corollary 2.4. *(1) If $Z_W(t) \neq 1$ then there exists $k \in \{1, ..., \max\{p, q\}\}$ such that*

$$\mathrm{ind}(\varphi_{(0,kT)}, F_{W^k}) \neq 0.$$

(2) If $\chi(W_0, W_0^-) = 0$ and $\mathrm{ind}(\varphi_{(0,kT)}, F_{W^k}) = 0$ for $k \in \{1, ..., p\}$ then

$$\mathrm{ind}(\varphi_{(0,nT)}, F_{W^n}) = 0,$$

for all $n \in \mathbb{N}$.

3. Main result

In this section we assume that F is a time dependent vector field on $\mathbb{R}^2$, $T>0$ periodic with respect to the time variable.

THEOREM 3.1. *Let W be a periodic isolating segment over $[0,T]$ for the equation (1). Assume that W_0 is a 2-dimensional compact, connected topological manifold with boundary and $H(W_0,W_0^-)\neq 0$. Then there is a subharmonic solution of (1) (i.e. periodic with period kT for some $k\in\mathbb{N}$).*

Proof. Since W_0 is a 2-dimensional, compact manifold with boundary, so ∂W_0 is a finite sum (say $k+1$, $k\geq 0$) of sets homeomorphic to S^1 (as a compact boundaryless 1-dimensional manifold) and the set $\mathbb{R}^2\setminus W_0$ has one unbounded component N. Since W_0 is connected, so $K=\mathrm{cl}\mathrm{N}\cap W_0$ is homeomorphic to S^1. By assumption $W_0^-\subset\partial W_0$ is a compact ENR, so it is a sum of finite number of sets homeomorphic to S^1 or to a closed interval. In particular $\chi(W_0^-)\geq 0$ and $\chi(W_0)=1-k$.

Suppose that $K\subset W_0^-$. It follows that there is a periodic isolating segment U over $[0,T]$ such that U_0 is homeomorphic to a compact ball and U_0^- is homeomorphic to S^1. One can check that $\mathrm{Lef}(\mu_U)=1$. Consequently, by Srzednicki's result (Th.1) there is a T-periodic solution of (1).

Let $K\cap W_0^-$ be a finite sum of sets (say m) homeomorphic to closed intervals. Assume first that $m\geq 2$ or $m=0$. One can check that there is a periodic isolating segment U over $[0,T]$ such that U_0 is a compact ball and $\chi(U_0^-)=m$. We have

$$\chi(U_0,U_0^-)=\chi(U_0)-\chi(U_0^-)=1-m\neq 0,$$

so there is subharmonic solution by Lemma 1. Consider the case $m=1$. If $k>0$ then

$$\chi(W_0,W_0^-)=1-k-\chi(W_0^-)\leq -m=-1,$$

so the result follows by Lemma 1. If $k=0$ then W_0 is homeomorphic to a compact ball and $W_0^-\subset\partial W_0$ is homeomorphic to a closed interval, so $H(W_0,W_0^-)=0$, a contradiction. □

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