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Some New Results on Global Nonexistence and Blow-up for Evolution Problems with Positive Initial Energy

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Summary. - *This paper deals with some new results on blow–up or global nonexistence for evolution equations with positive initial energy. The positive level of the energy which can be reached has a Mountain Pass type characterization, which is emphasized in the paper. We consider wave problems with source and damping in the interior or at the boundary of the domain and porous media equation with source, in both the slow diffusion and fast diffusion cases.*

1. Introduction

This paper is concerned with global nonexistence results for evolution equations which can be described by the abstract model

$$(P(u_t))_t + A(u) + Q(t, u_t) = F(u), \qquad t \in [0, \infty) \tag{1}$$

where A, F, and P are possibly nonlinear operators in appropriate Banach spaces, respectively having potentials $\mathcal{A}$, $\mathcal{F}$, and $\mathcal{P}$, understanding that A is a divergence type differential operator, P an evolution operator (which can also be zero) and F a driving force.

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The operator Q represents a damping term, that is $Q(t, u_t)u_t \geq 0$ in some appropriate sense. To the solutions of (1) is associated, at least formally, an energy function

$$E(t) = \mathcal{P}^*(u_t) + \mathcal{A}(u) - \mathcal{F}(u)$$

where $\mathcal{P}^*(u_t) = \mathcal{P}(u_t)u_t - P(u_t)$. There is a large literature on global nonexistence and blow–up for solutions, with negative initial energy, of equations which belongs to the class described by (1), from the classical papers [3], [10], [11], [14], [17], [18], [19], [20], [28], [25], [26], [27], [40], [41], [43], [44] when P and Q are linear, mainly obtained with the so called convexity method, to the more recent papers [13] (dealing with the case $Pu_t = u_t$, $A = -\Delta$), [32] (dealing with equation (1)), [29] (dealing with the nonlinear parabolic case), [30] (in which A and F can be also time–dependent).

Much less is known when the initial energy is positive. In [36] was studied the Cauchy–Dirichlet problem

$$\begin{cases} u_{tt} - \Delta u = |u|^{p-2}u, & \text{in } [0,\infty) \times \Omega, \\ u = 0, & \text{in } [0,\infty) \times \partial\Omega \\ u(0) = u_0, \qquad u_t(0) = u_1 & \end{cases} \tag{2}$$

where Ω is a bounded and smooth subset of $\mathbb{R}^n$, $n \geq 1$, $2 < p < r$ (here and in the sequel $r = 2n/(n-2)$ if $n \geq 3$, r is arbitrarily large if $n = 2$, $r = \infty$ if $n = 1$), $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. The authors proved that, if $E(0) < d$ and $J'(u_0)u_0 < 0$, the solution blows–up in finite time, while if $E(0) < d$ and $J'(u_0)u_0 > 0$ the solution is global, where (here and in the sequel $\|\cdot\|_p$ denotes the usual L^p norm),

$$d = \inf_{u \in H_0^1(\Omega), u \neq 0} \sup_{\lambda > 0} J(\lambda u), \tag{3}$$

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, \tag{4}$$

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + J(u_0). \tag{5}$$

The result can be usefully visualized in the following way: note that

$$E(0) \geq \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{B_1^p}{p}\|\nabla u_0\|_2^p := g(\|\nabla u_0\|_2),$$

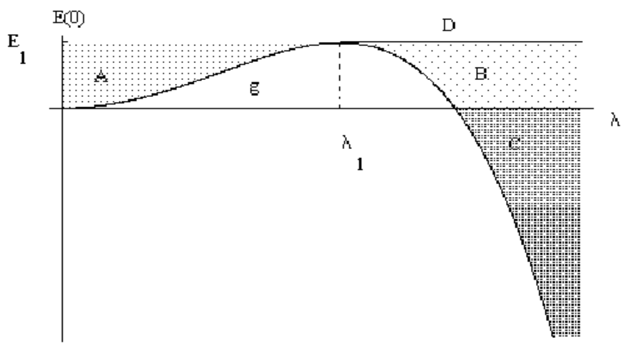

Figure 1: The four regions A, B, C and D in the plane $(\lambda, E(0))$, where $\lambda = \|\nabla u_0\|_2$.

where B_1 is the optimal constant of Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. Then the behavior of the solution u is known when $(\|\nabla u_0\|_2, E(0))$ lies in the regions A, B and C of the plane characterized by (see Figure 1)

$$A = \{(\lambda, E) \in [0,\infty) \times \mathbb{R} : g(\lambda) \leq E < E_1, \lambda < \lambda_1\},$$
$$B = \{(\lambda, E) \in [0,\infty) \times \mathbb{R} : \max\{g(\lambda), 0\} \leq E < E_1, \lambda > \lambda_1\},$$
$$C = \{(\lambda, E) \in [0,\infty) \times \mathbb{R} : g(\lambda) \leq E < 0\},$$

where λ_1 is the absolute maximum point of g and $E_1 = g(\lambda_1) > 0$. It is easy to see that $E_1 = d$ (see section 2 below) and that, under condition $E(0) < E_1$, $J'(u_0)u_u < 0 (> 0)$ if and only if $\|u_0\|_p > \lambda_1 (< \lambda_1)$. So the quoted result can be restated as follows: if $(\|u_0\|_p, E(0)) \in A$ the solution is global, while if $(\|u_0\|_p, E(0)) \in C$ blow–up in finite time occurs. In [16] a similar result is established for the parabolic problem

$$\begin{cases} u_t - \Delta_s u = |u|^{p-2}u, & \text{in } [0,\infty) \times \Omega, \\ u = 0, & \text{in } [0,\infty) \times \partial\Omega, \\ u(0) = u_0, \end{cases} \tag{6}$$

where $\Delta_s u = \operatorname{div}(|\nabla u|^{s-2}\nabla u)$, $s > 1$, and d is again given by (3) and (4), with the obvious modifications due to the presence of the s–Laplacian operator (see subsection 3.1 below).

More recently IKEHATA studied in [15] the case of wave equations with source, like (2), with the addition of a nonlinear damping (see (9) below), proving blow–up of the solutions when $E(0) << d$. Moreover PUCCI and SERRIN dealed in [37] with the case in which P and Q are linear, and A and F can be also time–dependent. Finally LEVINE and TODOROVA studied in [34] wave equations with nonlinear damping and source terms, proving blow–up for arbitrarily large initial positive energy, with a particular choice of initial data.

The same lack of knowledge in the positive energy case occurs when $A(u) = -\Delta(|u|^{m-1}u)$, $m > 0$, $m \neq 1$, $P = 0$ and $Q(t, u_t) = u_t$, i.e for the porous media equation with source, which is not of type (1). We refer to [12], [31] and [35] (see also the recent book [39]) in the so called *slow diffusion case* $m > 1$, and to [9] in the fast diffusion case $0 < m < 1$. See section 4 below for more information on known results for this problem.

In the first part of this paper (see section 2 below) we show some interesting and new applications of the global nonexistence result of [37] to some concrete evolution problems which were not studied by PUCCI and SERRIN. More precisely we study the situation in which the evolution process, developing in some bounded domain Ω of $\mathbb{R}^n$, is governed by the wave operator (i.e. $A = -\Delta$ and $Pu_t = u_t$ in (1)), in presence of a damping operator Q and a source operator F taking their origin from the boundary of Ω. This type of problems, known as wave equation with boundary stabilization, has been widely investigated in the framework of Control Theory when F is an attractive force or $F \equiv 0$. See [5], [6], [8], [7], [21], [22], [23], [24], [38], [48], and the more recent paper [4].

The case in which F is a source term, but $Q \equiv 0$, has been investigated in [33], where the authors consider the problem

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } [0,\infty) \times \Omega, \\ u = 0 & \text{in } [0,\infty) \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} = f(u), & \text{in } [0,\infty) \times \Gamma_1, \\ u(0) = u_0, \qquad u_t(0) = u_1, \end{cases}$$

where Ω is a bounded regular domain of $\mathbb{R}^n$, roughly $\partial\Omega = \Gamma_0 \cup \Gamma_1$, and $f(u)u \geq 0$.

Here we consider the following two problems: first of all

$$\begin{cases} u_{tt} - \Delta u + a(x)u_t = |u|^{p-2}u, & \text{in } [0,\infty) \times \Omega, \\ u = 0 & \text{in } [0,\infty) \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} = -b(x)u_t, & \text{in } [0,\infty) \times \Gamma_1, \\ u(0) = u_0, \qquad u_t(0) = u_1, \end{cases} \tag{7}$$

where Ω (as before) is a bounded regular domain of $\mathbb{R}^n$ and $\partial\Omega$ is the union of two measurable subsets Γ_0 and Γ_1 having intersection of zero $(n-1)$–dimensional Lebesgue measure. Moreover we suppose that Γ_0 has positive $(n-1)$–dimensional Lebesgue measure, that $2 < p \leq r$, that $a \in L^\infty(\Omega)$, $b \in L^\infty(\Gamma_1)$ be nonnegative and that $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$.

Next we consider the problem

$$\begin{cases} u_{tt} - \Delta u + a(x)u_t = 0, & \text{in } [0,\infty) \times \Omega, \\ u = 0 & \text{in } [0,\infty) \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} = -b(x)u_t + |u|^{p-2}u, & \text{in } [0,\infty) \times \Gamma_1, \\ u(0) = u_0, \qquad u_t(0) = u_1, \end{cases} \tag{8}$$

where Ω, Γ_0, Γ_1, a, b, u_0 and u_1 are as before, and $2 < p \leq 1 + r/2$.

The first aim of this paper is to show that for the solutions of (7) and (8) a global nonexistence result for initial data $(\|\nabla u_0\|_2, E(0))$ in region $B \cup C$ can be proved, where E_1 has a variational characterization (which is d for problem (8) and a similarly characterized level for (7)). To prove these two results it is at first necessary to give a (formally) more general version of the abstract result of [37]. Moreover we extend these two results also for initial data in region D.

The second purpose of the paper is to illustrate as the abstract global nonexistence result of [37] for initial data in the region $B \cup C$ can be extended to the case of nonlinear operators Q and P, that is to problem (1), studied by Levine and Serrin in [32] when $E(0) < 0$ (i.e. $(\|\nabla u_0\|_2, E(0)) \in C$). The complete statement of the result for equation (1) and the proof of it are contained in the author's recent paper [46], so the aim of this part is essentially expository. For this reason we give in section 3 the result and a (self–contained) version of the proof in the case of the model equation, first introduced in [13].

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-2}u_t = |u|^{p-2}u, & \text{in } [0,\infty) \times \Omega, \\ u = 0 & \text{in } [0,\infty) \times \partial\Omega, \\ u(0) = u_0, \qquad u_t(0) = u_1, \end{cases} \tag{9}$$

where Ω, u_0 and u_1 are as before, $2 < p \le r$ and $1 < m < p$. In this way we show the exact generalization of the blow–up result of [36] to the case of nonlinear damping terms. Moreover we show some applications of the result of [46] to concrete evolution problems arising in the applications.

The last aim of the paper is to give a blow-up result for initial data

$$(\|\nabla(|u_0|^{m-1}u_0)\|_2, E(0)) \in B \cup C \cup D$$

(which sets have to be conveniently defined) for the non linear case $A(u) = -\Delta(|u|^{m-1}u)$, that is for the Cauchy–Dirichlet problem related to the porous media equation with source

$$\begin{cases} u_t = \Delta(|u|^{m-1}u) + |u|^{p-2}u, & \text{in } [0,\infty) \times \Omega, \\ u = 0 & \text{in } [0,\infty) \times \partial\Omega, \\ u(0) = u_0, \end{cases} \tag{10}$$

where Ω is as before, $m > 0$, $\max\{2, m+1\} < p < 1 + r(m-1)$, and $|u_0|^{m-1}u_0 \in H_0^1(\Omega)$. The proof is only sketched, as a complete one can be founded in the forthcoming paper of the author [45].

2. Problems with boundary damping and source

This section is devoted to apply the result of [37] to (7) and (8), and to generalize these applications to initial data in region D. To reach this goal it is necessary, at first, to slightly modify the abstract setting of [37]. For our purpose it is enough to consider the case in which A and F are autonomous. We consider the abstract equation

$$Pu_{tt} + Q(t)u_t + \beta^* R(t)(\beta u)_t + A(u) = F(u), \qquad \text{on } [0,\infty). \tag{11}$$

Our assumptions on P, Q, A and F are the same of the paper quoted above, and we recall them here for the reader convenience. There are Hilbert spaces V, Y, and Banach spaces W, X, respectively having duals V', Y' and W', X', and operators

$$P \in \mathcal{L}(V, V'), \qquad A \in C(W; W'),$$
$$F \in C(X; X'), \qquad Q \in C^1([0,\infty); \mathcal{L}(Y, Y'))$$

where $\mathcal{L}(A, B)$ denotes the space of bounded linear operators from the Banach space A to the Banach space B. The linear operator P is symmetric and nonnegative definite, while the (possibly) nonlinear operators A and F respectively possess C^1–potentials

$$\mathcal{A} : W \to \mathbb{R}, \qquad \mathcal{F} : X \to \mathbb{R},$$

normalized with the condition $\mathcal{A}(0) = \mathcal{F}(0) = 0$. Moreover $Q(t)$ is symmetric and nonnegative definite, and $Q_t(t)$ is non positive definite (and necessarily symmetric) for all $t \geq 0$.

Concerning the additional term $\beta^* R(t)(\beta u)_t$ introduced in (11), we suppose that $R \in C^1([0,\infty); \mathcal{L}(Z, Z'))$ for some Hilbert space Z (with dual Z'), and that $R(t)$ and $R_t(t)$ verify the same symmetry and sign assumptions verified by $Q(t)$ and $Q_t(t)$. Moreover β is a fixed bounded linear operator from W to Z, and β^* denotes the usual adjoint operator $\beta^* \in \mathcal{L}(Z'; W')$, defined by $\beta^* v = v \circ \beta$ for all $v \in Z'$.

As in [37] we suppose that there is a nontrivial subspace (non necessarily closed) G of V, W and Y. We define

$$\mathcal{K} = \{\phi : [0,\infty) \to G \quad \text{such that} \quad \beta\phi \in H^1_{\text{loc}}([0,\infty); Z) \quad \text{and}$$
$$\phi \in C([0,\infty); W) \cap C([0,\infty); X) \cap C^1([0,\infty); V) \cap H^1_{\text{loc}}([0,\infty); Y)\}.$$

Adapting the analogous definition of [37], and denoting $\langle \cdot, \cdot \rangle_A$ as the usual duality product in a Banach space A, we say that $u \in \mathcal{K}$ is a strong global solution of (11) if:

(a) the distribution identity

$$\langle Pu_t(s), \phi(s)\rangle_V \Big|_0^t = \int_0^t \langle Pu_t, \phi_t\rangle_V - \langle A(u)), \phi\rangle_W - \langle Qu_t, \phi\rangle_Y$$
$$-\langle R(\beta u)_t, \beta\phi\rangle_Z + \langle F(u), \phi\rangle_X \tag{12}$$

for all $t \geq 0$, $\phi \in \mathcal{K}$, is verified;

(b) the energy conservation law (in the weak form of inequality)

$$E(t) - E(0) \leq - \int_0^t [\langle Qu_t, \phi_t \rangle_Y + \langle R(\beta u)_t, (\beta\phi)_t \rangle_Z] \tag{13}$$

holds for $t \geq 0$, where

$$E(t) = \frac{1}{2} \langle Pu_t(t), u_t(t) \rangle_V + \mathcal{A}(u(t)) - \mathcal{F}(u(t)), \qquad t \in [0, \infty), \tag{14}$$

is the total energy associated to u.

Moreover we suppose that there are constants $p \geq q$ such that

$$\langle A(u), u \rangle_W - \langle F(u), u \rangle_X \leq q\mathcal{A}(u) - p\mathcal{F}(u)$$

for all $(t, u) \in [0, \infty) \times G$.

Remark 2.1. *It is clear that if the solution u satisfies the further regularity $u \in H^1_{\text{loc}}([0, \infty); W)$, the one can avoid to introduce the additional term $\beta^* R(t)(\beta u)_t$ in (11), subsuming it in $Q(t)u_t$, conveniently redefining Q as $Q + \beta^* R\beta$ on $Y \cap W$. However this further regularity is not known in the applications we consider later, motivating the generalization we made here.*

We can then state the generalization of [37, Theorem 1, (i)], which reads as

Theorem 2.2. *If $p > 2$ there is no strong global solution u of (11) such that*

$$\mathcal{A}(u(t)) \geq \lambda_0 > 0, \qquad t \in [0, \infty), \tag{15}$$

and

$$E(0) < (1 - q/p)\lambda_0 := D_0. \tag{16}$$

Proof. We generalize the proof of [37, Theorem1, (i)] by modifying the main auxiliary function which is needed in the proof, which is now defined by

$$\begin{aligned}\mathcal{I}(t) = \langle Pu(t), u(t)\rangle_V + \\ + \int_0^t [\langle Q(\tau)u(\tau), u(\tau)\rangle_Y + \langle R(\tau)\beta u(\tau), \beta u(\tau)\rangle_Z]\, d\tau \\ + \int_0^t \{(\tau - t)[\langle Q_t(\tau)u(\tau), u(\tau)\rangle_Y + \langle R_t(\tau)\beta u(\tau), \beta u(\tau)\rangle_Z]\, d\tau \\ +(T_0 - t)[\langle Q(0)u(0), u(0)\rangle_Y + \langle R(0)\beta u(0), \beta u(0)\rangle_Z] + \gamma(t+t_0)^2,\end{aligned}$$

where t_0, T_0, and γ are positive constants to be fixed. The proof can be completed arguing as in the quoted paper, and using the properties of R and R_t in addition to the assumptions we taken from [37]. □

2.1. Application to problem (7)

To handle with (7) we set

$$V = L^2(\Omega), \quad Pv = v, \tag{17}$$

$$X = L^p(\Omega), \quad F(u) = |u|^{p-2}u, \quad \mathcal{F}(u) = \frac{1}{p}\|u\|_p^p, \tag{18}$$

$$Y = V, \quad Qv = a(x)v, \quad Z = L^2(\Gamma_1), \quad Rv = b(x)v. \tag{19}$$

Next, using the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$ we can define the projection operators $\beta_{\Gamma_i} : H^1(\Omega) \to L^2(\Gamma_i)$ for $i = 1, 2$. Then we set

$$W = \{u \in H^1(\Omega) : \beta_{\Gamma_0} u = 0\}, \qquad Au = -\Delta u, \quad \mathcal{A}(u) = \frac{1}{2}\|\nabla u\|_2^2, \tag{20}$$

$G = W$ and $\beta = \beta_{\Gamma_1}|_W$. Clearly in this case

$$\begin{aligned}\mathcal{K} = \{\phi \in C([0,\infty); W) \cap C^1([0,\infty); L^2(\Omega)) : \\ (\beta\phi)_t \in L^2_{\text{loc}}([0,\infty), L^2(\Gamma_1))\},\end{aligned} \tag{21}$$

which is exactly the kind of regularity founded for the solutions of such type of problems when f is an attractive force. If the additional regularity $\nabla u_t \in L^2((0,T) \times \Omega)$ was known for the solutions of (7) one could use directly the abstract result of [37] without the generalization we made. However this kind of regularity cannot be expected, due to the nonlinear nature of (7) (see [21]).

The distributional identity (12) reduces to

$$\int_\Omega u_t(s)\phi(s)\Big|_0^t = \int_0^t \left(\int_\Omega u_t\phi_t - \nabla u \nabla \phi - a(x)u_t\phi + |u|^{p-2}u\phi \right. \\ \left. - \int_{\Gamma_1} b(x)u_t\phi \right)$$

which is exactly the (weak) formulation of (7) in $\mathcal{K}$.

As Γ_0 has positive $(n-1)$–dimensional Lebesgue measure, by Poincaré inequality (see [47, Corollary 4.5.3]) we can endow W with the equivalent norm $\|u\|_W = \|\nabla u\|_2$. Set B_1 to be the optimal constant of the embedding inequality

$$\|u\|_p \le B_1 \|\nabla u\|_2, \qquad u \in W, \tag{22}$$

that is

$$B_1^{-1} = \inf_{u \in W, u \neq 0} \frac{\|\nabla u\|_2}{\|u\|_p}, \tag{23}$$

and

$$\lambda_1 = B_1^{-p/(p-2)}, \qquad E_1 = (1/2 - 1/p) B_1^{-2p/(p-2)}. \tag{24}$$

We can now state the result concerning (7).

THEOREM 2.3. *When $2 < p \le r$ there are no strong global solutions of (7) on $[0, \infty)$ such that*

$$\|\nabla u_0\|_2 > \lambda_1, \qquad \text{and} \quad E(0) < E_1.$$

Moreover, if $a > 0$, the same is true for solutions such that

$$\|\nabla u_0\|_2 > \lambda_1, \qquad \text{and} \quad E(0) = E_1.$$

Proof. Let us denote $\bar\lambda = \|\nabla u_0\|_2$ and $E_0 = E(0)$. By (22)

$$\begin{aligned} E(t) =& \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p}\|u(t)\|_p^p \ge \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p}\|u(t)\|_p^p \\ \ge& \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{B_1^p}{p}\|\nabla u(t)\|_2^p := g(\|\nabla u(t)\|_2) \end{aligned} \tag{25}$$

where $g(\lambda) = \lambda^2/2 - B_1^p \lambda^p/p$ for $\lambda \geq 0$. It is easy to see that g takes its maximum for $\lambda = \lambda_1$, with $g(\lambda_1) = E_1$, being strictly decreasing for $\lambda \geq \lambda_1$, and that $g(\lambda) \to -\infty$ as $\lambda \to \infty$. Then, as $E_0 < E_1$, there is $\lambda_2 > \lambda_1$ such that $g(\lambda_2) = E_0$. Since, by (25), $g(\bar{\lambda}) \leq E_0 = g(\lambda_2)$, it follows that $\lambda_2 \leq \bar{\lambda}$.

It is clear from (13) that

$$E(t) \leq E_0 \qquad \text{for } t \geq 0. \tag{26}$$

We claim that

$$\|\nabla u(t)\|_2 \geq \lambda_2 \qquad \text{for } t \geq 0. \tag{27}$$

Suppose for contradiction that $\|\nabla u(t_0)\|_2 < \lambda_2$ for some $t_0 \in (0, \infty)$. By the continuity of $\|\nabla u(\cdot)\|_2$ we can suppose that $\lambda_1 < \|\nabla u(t_0)\|_2$. Then, using (25), $E(t_0) \geq g(\|\nabla u(t_0)\|_2) > g(\lambda_2) = E_0$, in contradiction with (26), so proving (27). Then

$$\mathcal{A}(u(t)) = \frac{1}{2}\|\nabla u(t)\|_2^2 \geq \frac{1}{2}\lambda_2^2 > \frac{1}{2}\lambda_1^2.$$

Setting $\lambda_0 = \frac{1}{2}\lambda_1^2$, one has $E_1 = (1-2/p)\lambda_0$, so we apply Theorem 2.2 and conclude the proof in the main case $E_1 < E_0$. When $E_1 = E_0$ we argue as follows. By the continuity of $\|\nabla u(\cdot)\|_2$ only two possibilities can occur:

(a) there is $t_0 \geq 0$ such that $E(t_0) < E_1$ and $\|\nabla u(t_0)\|_2 > \lambda_1$;

(b) there is $\varepsilon_0 > 0$ such that $E(t) = E_1$ on $[0, \varepsilon_0)$.

In the first case, shifting the time origin to t_0 and applying the previous case, we conclude the proof. In the latter, by (13),

$$\int_0^t \int_\Omega a(x)|u_t|^2 = 0 \qquad \text{for } t \in [0, \varepsilon_0),$$

and then, as $a > 0$, $u_t = 0$ and hence $u(t) = u_0$ on $[0, \varepsilon_0)$. Then, putting $\phi = u$ in (12), we obtain that $-\|\nabla u_0\|_2^2 + \|u_0\|_p^p = 0$. Hence, using (27),

$$E_1 = E_0 \geq \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p}\|\nabla u_0\|_2^2 \geq \left(\frac{1}{2} - \frac{1}{p}\right)\lambda_2^2 > E_1,$$

a contradiction. □

It is easy to see that E_1 has the variational characterization

$$E_1 = \inf_{u\in W, u\neq 0} \sup_{\lambda>0} J(\lambda u) \tag{28}$$

where $J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p$ for $u \in W$ (argue as in [45, Final remarks]) and, moreover that, when $p < r$, E_1 is the Mountain Pass Level of the functional J on W, i.e.

$$E_1 = \inf_{\gamma\in\Lambda} \sup_{t\in[0,1]} J(\gamma(t))$$

where

$$\Lambda = \{\gamma \in C([0,1]; W) : \gamma(0) = 0, J(\gamma(1)) < 0\}, \tag{29}$$

which is associated to the mixed boundary values problem

$$\begin{cases} -\Delta u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0 & \text{in } \Gamma_0, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{in } \Gamma_1. \end{cases}$$

This second characterization can be proved as in [46, Final remarks], first observing that [2, Proof of Lemma 7.2] shows that J satisfies the Palais–Smale condition.

Moreover it is also easy to see that, when $E(0) < E_1$, condition $J'(u_0)u_0 < 0$ used in [36] is equivalent to the condition $\|\nabla u_0\|_2 > \lambda_1$ used here. Indeed, if $J'(u_0)u_0 < 0$, then $\|\nabla u\|_2^2 - B_1^p\|\nabla u_0\|_2^p < 0$, so $\|\nabla u_0\|_2 > \lambda_1$. Conversely, if $\|\nabla u_0\|_2 > \lambda_1$ then

$$\begin{aligned} J'(u_0)u_0 =& \|\nabla u_0\|_2^2 - \|u_0\|_p^p \leq pE_1 - \left(\frac{p}{2} - 1\right)\|\nabla u_0\|_2^2 \\ &< pE_1 - \left(\frac{p}{2} - 1\right)\lambda_1^2 = 0. \end{aligned}$$

2.2. Application to problem (8)

When considering problem (8) we keep the settings (17), (19) and (20) we made in previous application, while (18) has to be modified as follows:

$$X = H^1(\Omega), \qquad F(u) = |\beta u|^{p-2}\beta u, \qquad \mathcal{F}(u) = \frac{1}{p}\|\beta u\|_p^p,$$

where we use again the trace operator $\beta = \beta_{\Gamma_1}$, which now, as $p \le 1 + r/2$) (see [1]), can be considered having values in $L^p(\Gamma_1)$.

The solution space $\mathcal{K}$ is exactly given by (21), and (12) reduces to

$$\int_\Omega u_t(\tau)\phi(\tau)\Big|_0^t = \int_0^t \Big[\int_\Omega u_t\phi_t - \nabla u \nabla \phi - a(x) u_t \phi + \int_{\Gamma_1} (|u|^{p-2}u - b(x)u_t)\phi\Big]$$

which is exactly the weak formulation of (8) in $\mathcal{K}$.

We set $\tilde{B}_1$ to be the optimal constant of the trace inequality

$$\|\beta u\|_p \le \tilde{B}_1 \|\nabla u\|_2, \qquad \text{for } u \in W, \tag{30}$$

that is

$$\tilde{B}_1^{-1} = \inf_{u \in W, \beta u \neq 0} \frac{\|\nabla u\|_2}{\|\beta u\|_p},$$

where we are using again Poincaré inequality. We consequently modify the definition of λ_1 and E_1, indicating the corresponding new values with as $\tilde{\lambda}_1$ and $\tilde{E}_1$, that is

$$\tilde{\lambda}_1 = \tilde{B}_1^{-p/(p-2)}, \qquad \tilde{E}_1 = (1/2 - 1/p)\tilde{B}_1^{-2p/(p-2)}.$$

We can now state

Theorem 2.4. *When $2 < p \le 1 + r/2$ there are no strong global solutions of (8) on $[0, \infty)$ such that*

$$\|\nabla u_0\|_2 > \tilde{\lambda}_1, \qquad \textit{and} \quad E(0) < \tilde{E}_1.$$

Moreover, if $a > 0$, the same is true for solutions such that

$$\|\nabla u_0\|_2 > \tilde{\lambda}_1, \qquad \textit{and} \quad E(0) = \tilde{E}_1.$$

Proof. Argue as in Theorem 2.3. □

Also in this case $\tilde{E}_1$ has the variational characterization

$$\tilde{E}_1 = \inf_{u \in W, \beta_{\Gamma_1} u \neq 0} \sup_{\lambda > 0} \tilde{J}(\lambda u)$$

where

$$\tilde{J}(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|\beta u\|_p^p,$$

and, as before, when $p < 1 + r/2$, $\tilde{E}_1$ can be recognized as the Mountain Pass level of $\tilde{J}$ on W, that is

$$\tilde{E}_1 = \inf_{\gamma \in \Lambda} \sup_{t \in [0,1]} \tilde{J}(\gamma(t)),$$

where Λ is given in (29).

To prove this characterization one can first adapt the proof of [2, Lemma 7.2] quoted before using compactness of the trace operator (see [1]) for sub critical levels of p, to prove that the Palais–Smale condition holds. Then one can repeat the proof of [45, Final remarks] with the further observation that, any critical point w_0 of $\tilde{J}$ on W such that

$$\tilde{J}(w_0) = \inf_{\gamma \in \Lambda_0} \sup_{t \in [0,1]} \tilde{J}(\gamma(t)) > 0,$$

is nonzero and such that $\frac{d}{d\lambda}\tilde{J}(\lambda w_0)|_{\lambda=1} = 0$. Hence $\|\nabla w_0\|_2^2 = \|\beta u\|_p^p$, then $\beta w_0 \neq 0$.

This level is clearly associated to the problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = 0 & \text{in } \Gamma_0, \\ \dfrac{\partial u}{\partial \nu} = |u|^{p-2}u, & \text{in } \Gamma_1. \end{cases}$$

3. Nonlinear damping problems

Let us now consider problem (9). The energy function naturally associated to any solution u is given by

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p}\|u(t)\|_p^p. \tag{31}$$

Let now B_1 the number introduced in (23), in the particular case $\Gamma_1 = \emptyset$, that is

$$B_1^{-1} = \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\|\nabla u\|_2}{\|u\|_p} \tag{32}$$

and E_1, λ_1 be the numbers given by (24).

The interaction between the damping term $|u_t|^{m-2}u_t$ and the source $|u|^{p-2}u$ has been studied by LEVINE in [26] and [27] in the *linear* case $m = 2$ and, recently, for $m > 2$ by GEORGIEV and TODOROVA in [13]. In the last paper the authors determined suitable domains for the parameters p, m where there is global existence for any initial data, or alternatively, blow–up in finite time for the solutions of (9). In particular they proved that, when $m < p$ and $p \leq 1 + r/2$, then all solutions for which $E(0)$ is sufficiently negative blow–up in finite time. This result has been recently extended in the quoted paper [32] to initial data with negative energy.

These results do not extend completely the result of [36] to the case with damping, there still remaining the case $0 \leq E(0) < E_1$. As quoted before PUCCI and SERRIN handled this case in [37] but only for linear damping terms ($m = 2$). IKEHATA considered in [15] positive values of $E(0)$, with $m > 1$, but for a damping term $b|u_t|^{m-2}u_t$ and only for sufficiently small values of b, and assumed that $E(0) < E_b$, where E_b is always strictly less than E_1. The aim of this section is to show that the arguments of [13], as refined in [32], can be conveniently modified to study the case $\|\nabla u_0\|_2 > \lambda_1$, $E(0) < E_1$ in the general case $m > 1$. In this way we extend the results of [37] to nonlinear damping terms. Moreover we show that the presence of the damping term also allow us to consider the case $\|\nabla u_0\|_2 > \lambda_1$, $E(0) = E_1$, as in previous section.

We consider *global strong distribution* solutions of (9), i.e. functions

$$u \in \mathcal{K} = C(J; [W_0^{1,q}(\Omega)]^N) \cap C^1(J; [L^l(\Omega)]^N),$$

satisfying

$$\int_\Omega u_t(s)\phi(s)\Big|_0^t = \int_0^t \int_\Omega u_t\phi_t - \nabla u \nabla \phi - |u_t|^{m-2}u_t\phi + |u|^{p-2}u\phi \tag{33}$$

for all $t \geq 0$ and $\phi \in C_c^\infty([0,\infty)\times\Omega)$, with the natural energy identity

$$E(t) + \int_0^t \|u_t\|_m^m s = E(0). \tag{34}$$

A global strong distribution solution in this sense is a global solution of (9) in the sense introduced in section 2, that is in (33) we can take test functions ϕ in $\mathcal{K}$ instead that in $C_c^\infty([0,\infty) \times \Omega)$. The proof of this fact, based of convolution argument, here is omitted (see [46]).

Our first result concerning (9) is the following global nonexistence theorem

THEOREM 3.1. *When* $2 < p \le r$ *and* $1 < m < p$ *there are no global strong distribution solutions of* (9) *such that*

$$\|\nabla u_0\|_2 > \lambda_1, \qquad \textit{and} \quad E(0) \le E_1.$$

Theorem 3.1 is special case of [46, Theorem 2], where we essentially use the technique of [13] as refined in [32]. For the expository nature of this section we report a self–contained proof of Theorem 3.1.

One could also assume that (34) be verified in the weak form of inequality as in (13). Here we choose to present the result assuming classical energy identity in order to make more clear the idea of the proof.

Proof. The proof is done by contradiction, so we assume that there is a global solution u of (9). We set $E_0 = E(0)$. We first consider the case $E_0 < E_1$, where the proof essentially differs from the proofs of Theorems 2.3–2.4. As in these we first obtain that there are $\lambda_2 > B_1\lambda_1$ and $\lambda_3 > \lambda_1$ such that

$$E(t) \le E_0, \qquad \|u(t)\|_p \ge \lambda_2, \qquad \text{and} \quad \|\nabla u(t)\|_2 \ge \lambda_3 \tag{35}$$

for $t \ge 0$. We set $\mathcal{H}(t) = E_1 - E(t)$ which, by (34), is an increasing function. Then

$$\mathcal{H}(t) \ge \mathcal{H}_0 := \mathcal{H}(0) = E_1 - E_0 > 0 \qquad \text{for } t \ge 0. \tag{36}$$

Next, by (31)

$$\mathcal{H}(t) \le E_1 - \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{p}\|u(t)\|_p^p \qquad \text{for } t \ge 0. \tag{37}$$

By $(35)_3$

$$E_1 - \frac{1}{2}\|\nabla u(t)\|_2^2 \leq E_1 - \frac{1}{2}\lambda_1^2 = -\frac{1}{p}\lambda_1^2 < 0 \qquad \text{for } t \geq 0,$$

hence

$$\mathcal{H}_0 \leq \mathcal{H}(t) \leq \frac{1}{p}\|u(t)\|_p^p \qquad \text{for } t \geq 0. \tag{38}$$

Next, putting $\phi = u$ in (33), using (31) and (37)

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_t u =&\|u_t\|_2^2 - \|\nabla u\|_2^2 + \|u\|_p^p - \int_\Omega |u_t|^{m-2}u_t u\\ =&2\|u_t\|_2^2 + (1-2/p)\|u\|_p^p - 2E - \int_\Omega |u_t|^{m-2}u_t u\\ =&2\|u_t\|_2^2 + (1-2/p)\|u\|_p^p + 2\mathcal{H} - 2E_1 - \int_\Omega |u_t|^{m-2}u_t u.\end{aligned}$$

Then, using $(35)_2$,

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_t u \geq&2\|u_t\|_2^2 + \left(1 - \frac{2}{p} - 2E_1\lambda_2^{-p}\right)\|u\|_p^p + 2\mathcal{H} - \int_\Omega |u_t|^{m-2}u_t u\\ =&2\|u_t\|_2^2 + c_0\|u\|_p^p + 2\mathcal{H} - \int_\Omega |u_t|^{m-2}u_t u\end{aligned} \tag{39}$$

where $c_0 = 1 - \frac{2}{p} - 2E_1\lambda_2^{-p} > 0$ because $\lambda_2 > B_1\lambda_1$.

Now, to estimate the last term in (39), by applying Hőlder's inequality, we obtain

$$\left|\int_\Omega |u_t|^{m-2}u_t u\right| \leq \|u_t\|_m^{m-1}\|u\|_m = \|u\|_m^{1-p/m}\|u\|_m^{p/m}\|u_t\|_m^{m-1},$$

and then, by (38), Hőlder's inequality again, Young's inequality, and the fact that $\mathcal{H}' = \|u_t\|_m^m$, we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}\left|\int_\Omega |u_t|^{m-2}u_t u\right| \leq& c_1\|u\|_p^{1-p/m}\|u\|_p^{p/m}\|u_t\|_m^{m-1}\\ \leq& c_2\|u\|_p^{p/m}\mathcal{H}^{\frac{1}{p}-\frac{1}{m}}\|u_t\|_m^{m-1}\\ \leq& c_2(\varepsilon^m\|u\|_p^p + \varepsilon^{-m'}\|u_t\|_m^m)\mathcal{H}^{-\bar{\alpha}}\\ \leq& c_2(\varepsilon^m\|u\|_p^p + \varepsilon^{-m'}\mathcal{H}')\mathcal{H}^{-\bar{\alpha}},\end{aligned} \tag{40}$$

where $\bar{\alpha} = \frac{1}{m} - \frac{1}{p} > 0$, and we denote by $c_1, c_2, \dots$, suitable positive constants. Now let $0 < \alpha < \bar{\alpha}$. By (36)

$$\left| \int_\Omega |u_t|^{m-2} u_t u \right| \leq c_2 (\varepsilon^m \mathcal{H}_0^{-\bar{\alpha}} \|u\|_p^p + \varepsilon^{-m'} \mathcal{H}_0^{\alpha - \bar{\alpha}} \mathcal{H}^{-\alpha} \mathcal{H}'). \tag{41}$$

Now we introduce, as in [13] and [32], the main auxiliary function which shows the blow–up properties of u, i.e.

$$\mathcal{Z}(t) = \mathcal{H}^{1-\alpha}(t) + \delta \int_\Omega u_t(t) u(t),$$

where δ is a (small) positive constant to be fixed later. By (39)(41)

$$\begin{aligned} \mathcal{Z}' \geq & (1-\alpha)\mathcal{H}^{-\alpha}\mathcal{H}' + \delta \left[2\|u_t\|_2^2 + c_0 \|u\|_p^p + 2\mathcal{H} - \int_\Omega |u_t|^{m-2} u_t u \right] \\ \geq & (1 - \alpha - \delta c_2 \varepsilon^{-m'} \mathcal{H}_0^{\alpha - \bar{\alpha}}) \mathcal{H}^{-\alpha} \mathcal{H}' + \delta (c_0 - c_2 \varepsilon^m \mathcal{H}_0^{-\bar{\alpha}}) \|u\|_p^p \\ & + 2\delta \|u_t\|_2^2 + 2\delta \mathcal{H}. \end{aligned} \tag{42}$$

Now let $\delta < (1-\alpha) c_2^{-1} \varepsilon^{m'} \mathcal{H}_0^{\bar{\alpha} - \alpha}$. Then we can drop the first term on the right hand side of (42). Moreover, choosing ε sufficiently small we can estimate $c_0 - c_2 \varepsilon^m \mathcal{H}_0^{-\bar{\alpha}} \geq \frac{1}{2} c_0$, so

$$\mathcal{Z}' \geq \frac{1}{2} c_0 \delta \|u\|_p^p + 2\delta \|u_t\|_2^2 + 2\delta \mathcal{H} \geq c_3 \delta (\|u\|_p^p + \|u_t\|_2^2 + \mathcal{H}). \tag{43}$$

Letting δ sufficiently small we have $\mathcal{Z}(0) > 0$, so $\mathcal{Z}(t) \geq \mathcal{Z}(0) >$ for $t \geq 0$.

Now set $r = 1/(1-\alpha)$. Since $\alpha < \bar{\alpha} < 1$ it is evident that $1 < r < \bar{r} := 1/(1-\bar{\alpha})$. Using Young's inequality again

$$\mathcal{Z}^r \leq 2^{r-1} (\mathcal{H} + \delta^r \|u_t\|_2^r \|u\|_2^r) \leq c_4 (\mathcal{H} + \|u_t\|_2^2 + \|u\|_2^{1/(1/2 - \alpha)}).$$

Now choose $\alpha \in (0, \min\{\bar{\alpha}, 1/2 - 1/p\})$. Then $\|u\|_2^{1/(1/2-\alpha)} \leq 1 + \|u\|_2^p$ because $1/2 - \alpha < 1/p$, so we have

$$\mathcal{Z}^r \leq c_5 (\mathcal{H} + \|u_t\|_2^2 + \|u\|_p^p),$$

which, combined with (43), as $r > 1$, concludes the proof when $E_0 < E_1$. When $E_0 = E_1$ we argue as in the proof of Theorem 2.3. □

The global nonexistence result given above can be combined with local existence and continuation results (and it is a logical continuation of these) to obtain blow–up for the solutions of (9). In particular, using the local existence result of [13], one obtains blow–up under the more restrictive assumption $2 < m < p \leq 1 + r/2$, as done in [46, Theorem 5]. On the other hand, is nowadays available the technique of [42]. Arguing as in this paper (which deals with the more difficult case $\Omega = \mathbb{R}^n$), it is possible to prove the following local existence theorem

THEOREM 3.2. *Suppose that*

$$m > 1,\ 2 < p < 1 + r/2, \quad \text{or} \quad m > r/(r+1-p),\ 1 + r/2 \leq p < r. \tag{44}$$

Then, for $T > 0$ small enough there is a strong a.e. solution of (9) *as an abstract differential equation in $H^{-1}(\Omega) + L^{m'}(\Omega)$ such that*

$$u \in C([0,T]; H_0^1(\Omega)), \quad u_t \in C([0,T]; L^2(\Omega)), \quad u_t \in L^m((0,T) \times \Omega) \tag{45}$$

and the energy identity

$$\frac{1}{2}\|u_t(s)\|_2^2 + \frac{1}{2}\|\nabla u(s)\|_2^2 \Big|_0^t + \int_0^t \|u_t\|_m^m = \int_0^t \int_\Omega |u|^{p-2} u u_t \tag{46}$$

holds for $t \in [0,T]$.

It is easy to see that solutions of (9) given in the last Theorem are strong distribution solutions. Then, using the standard continuation procedure, the fact that T depends as a decreasing function from $\|\nabla u_0\|_2^2 + \|u_1\|_2^2$ (see [42]), one can easily yields the following blow–up result.

THEOREM 3.3. *Suppose that* (44) *holds, and let u be the solution given in Theorem 3.2. Then there is $T_{max} > 0$ such that $\|u(t)\|_p \to \infty$ (and then $\|u(t)\|_\infty \to \infty$) as $t \to T_{max}^-$.*

Theorem 3.1 has been extended in [46] to problem (1), under some specific assumptions on P, A, F and Q. Here we show only some concrete evolution problems to which the abstract setting is applicable.

3.1. A canonical model

We consider the system

$$\begin{cases} (|u_t|^{l-2}u_t)_t - \Delta_q u + |u_t|^{m-2}u_t = |u|^{p-2}u, & \text{in } [0,\infty)\times\Omega, \\ u = 0, & \text{in } [0,\infty)\times\partial\Omega, \\ u(0,\cdot)\in [W_0^{1,q}(\Omega)]^N, u_t(0,\cdot)\in [L^l(\Omega)]^N, & \end{cases} \tag{47}$$

where Ω is a bounded regular open subset of $\mathbb{R}^n$, $u = u(t,x)$, $u : [0,\infty)\times\Omega\to\mathbb{R}^N$, with $N\geq 1$, $n\geq 1$.

$$1 < l < p, \qquad 1 < m < p, \qquad 1 < q < p \leq r_q, \tag{48}$$

where r_q is the Sobolev critical exponent of $W_0^{1,q}$, that is $r_q = nq/(n-q)$ if $n > q$, $q < r_q < \infty$ if $n = q$, $r_q = \infty$ if $n < q$. In this case we set

$$B_1^{-1} = \inf_{u\in W_0^{1,q}(\Omega): u\neq 0} \frac{\|\nabla u\|_q}{\|u\|_p}.$$

The energy function is given by

$$E(t) = \frac{l-1}{l}\|u_t\|_l^l + \frac{1}{q}\|\nabla u\|_q^q - \frac{1}{p}\|u\|_p^p.$$

Let

$$\lambda_1 = B_1^{-p/(p-q)}, E_1 = (1/q - 1/p)B_1^{-pq/(p-q)}.$$

Also in this case it is easy to verify that (28) holds in the space $W_0^{1,q}(\Omega)$, where now J is given by

$$J(u) = \frac{1}{q}\|\nabla u\|_q^q - \frac{1}{p}\|u\|_p^p.$$

Moreover also the Mountain Pass type characterization of E_1 is still true, arguing as in section 2, in connection with the Dirichlet problem

$$\begin{cases} -\Delta_q u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega. \end{cases}$$

The statement of Theorem 3.1 continues to hold for strong distribution solutions of(47), which are defined generalizing in the obvious way previous definition.

3.2. A problem in elasticity

We consider the system

$$\begin{cases} \rho(x)u_{tt} - \operatorname{div}(C(x)\nabla u) + d(x)|u_t|^{m-2}u_t = f(x,u), & \text{in } [0,\infty)\times\Omega, \\ u = 0, & \text{in } [0,\infty)\times\partial\Omega, \\ u(0,\cdot) \in [H_0^1(\Omega)]^n, u_t(0,\cdot) \in [L^2(\Omega)]^n, & \end{cases} \tag{49}$$

where Ω is as before, $u : [0,\infty) \times \Omega \to \mathbb{R}^n$, $\rho \in L^\infty(\Omega; \mathbb{R}_0^+)$ and

$$C \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{2n}; \mathbb{R}^{2n})),$$

that is $C(x)$ is a linear operator from $\mathbb{R}^{2n}$ to itself for all $x \in \Omega$, i.e. a tensor of rank 4 which can be represented as $C(x) = (c_{ijkl}(x))$, $i, j, k, l = 1, \ldots, n$, and $c_{ijkl} \in L^\infty(\Omega)$. We assume that C is symmetric, i.e.

$$(C(x)y, z) = (C(x)z, y) \qquad \text{for all } x \in \Omega,\ y, z \in \mathbb{R}^{2n},$$

and that it is uniformly positive definite in Ω, that is

$$(C(x)y, y) \geq a_0|y|^2 \qquad \text{for all } x \in \Omega,\ y \in \mathbb{R}^{2n},$$

for some constant $a_0 > 0$. We also assume that $f(x,u) = \nabla_u\Phi(x,u)$ for some $\Phi \in C^1(\Omega \times \mathbb{R}^n)$, we assume that Φ be p–homogeneous in u, that

$$\inf\{\Phi(x,u) : |u| = 1, x \in \Omega\} > 0,$$

and that $2 < p \leq 2n/(n-2)$. Moreover $d \in L^{p/(p-m)}(\Omega)$, $d > 0$.

System (49), when $n = 3$, describes the effect of a nonlinear damping and forcing terms in the classical equations of linear elasticity, and has been studied by Levine (see [26, Example V, p. 15] when $Q \equiv 0$). Clearly the term $C(x)\nabla u$ represents the *engineering stress tensor*, or *first Piola–Kirchoff stress tensor*, and (49) express the Cauchy first law of motion, with boundary and initial conditions.

In this case the energy is given by

$$E(t) = \frac{1}{2}\int_\Omega [\rho(x)|u_t|^2 + (C(x)\nabla u, \nabla u) - 2\Phi(x,u)]\, dx,$$

B_1 is the constant given by

$$B_1^{-1} = \inf_{u\in H_0^1(\Omega),\ u\neq 0} \frac{\left[\int_\Omega (C(x)\nabla u, \nabla u)\, dx\right]^{1/2}}{\left[\int_\Omega p\Phi(x,u)\, dx\right]^{1/p}},$$

and λ_1, E_1 are the numbers given in (24). [1] With these modifications the statement of Theorem 3.1 continues to hold for solutions of (49). Moreover, it is also easy to give the variational characterization of E_1 as a Mountain Pass level associated to the problem

$$\begin{cases} -\operatorname{div}(C(x)\nabla u) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega. \end{cases}$$

3.3. The clamped plate equation

Consider the problem

$$\begin{cases} (2\rho(x)h_1 u_t/h_2)_t + (-\Delta)^2 u + d(x)|u_t|^{m-2}u_t & \\ = f(x,u), & \text{in } [0,\infty)\times\Omega, \\ u = 0, \nabla u = 0, & \text{in } [0,\infty)\times\partial\Omega, \\ u(0,\cdot) \in H_0^2(\Omega), u_t(0,\cdot) \in L^2(\Omega), & \end{cases} \tag{50}$$

where Ω is as before, $h_1, h_2 > 0$, $\rho \in L^\infty(\Omega; \mathbb{R}_0^+)$. We assume that f and d satisfy the assumptions of previous subsection. The problem describes the motion of a damped clamped plate with density $\rho \geq 0$, flexural rigidity $h_2 > 0$ and thickness $2h_1 > 0$, with a given loading function f acting vertically on the plate, which has been studied by Levine in the case $d = 0$ (see [26, Example IV, p. 15]).

In this case [2]

$$B_1^{-1} = \inf_{u\in H_0^2(\Omega), u\neq 0} \frac{\|\Delta u\|_2}{\left[\int_\Omega p\Phi(x,u)\right]^{1/p}},$$

---

[1] Actually the value of E_1 founded in [46] can be less that the value given above. However, it is not difficult to obtain the value we indicate here, conveniently modifying the proof

[2] Also in this case one has to slighty modify the proof of [46] to obtain the value of E_1 given above

the energy function has the form

$$E(t) = \frac{1}{2}\int_\Omega \left[\frac{2\rho(x)h_1}{h_2}|u_t|^2 + |\Delta u|^2 - 2\Phi(x,u)\right] dx,$$

and λ_1, E_1 are again given by (24). The Mountain Pass characterization of E_1 is still true, in connection with the problem

$$\begin{cases} (-\Delta)^2 u = f(x,u), & \text{in } \Omega, \\ u = 0, \nabla u = 0, & \text{in } \partial\Omega. \end{cases}$$

4. The porous media equation with source

We consider the Cauchy–Dirichlet problem for the porous media equation with source term (10), which (see [12] and [39]), when $u_0 \geq 0$, describes the propagation of thermal perturbations in a medium with a nonlinear heat–conduction coefficient and a heat source depending on the temperature. Local existence for the solutions of (10) has been proved when $m > 1$ (the so called *slow diffusion case*) in [12], [31] and [35] (see also the recent book [39]) and, when $0 < m < 1$ (the *fast diffusion case*) in [9]. More precisely, in the slow diffusion case, if $|u_0|^{m-1}u_0 \in H_0^1(\Omega)$ and

$$p < 1 + m + \frac{2(m+1)}{n}, \tag{51}$$

local existence of a solution u such that

$$\begin{gathered} |u|^{m-1}u \in L^\infty(0,T;H_0^1(\Omega)), \quad |u|^{(m-1)/2}u \in L^\infty(0,T;L^2(\Omega)), \\ (|u|^{(m-1)/2}u)_t \in L^2((0,T)\times\Omega), \end{gathered} \tag{52}$$

has been proved in [12] and [35]. If $u_0 \in L^\infty(\Omega)$ (see [31]), local existence of a solution u such that

$$\begin{gathered} u \in L^\infty((0,T)\times\Omega), \quad |u|^{m-1}u \in L^2(0,T;H_0^1(\Omega)), \\ (|u|^{(m-1)/2}u)_t \in L^2((0,T)\times\Omega), \end{gathered} \tag{53}$$

is known, without adding restrictions from above on p. In the fast diffusion case (see [9]) local existence of a weak solution, when solely

$u_0 \in L^\infty(\Omega)$, and of a strong solution, when also $|u_0|^{m-1}u_0 \in H^1_0(\Omega)$ is proved in the class of functions u such that

$$|u|^{m-1}u \in C([0,T];L^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega)) \cap L^\infty((0,T)\times\Omega),\\ |u|^{(m-1)/2}u \in H^1(0,T;L^2(\Omega)). \tag{54}$$

Global existence has been proved in the quoted papers when $p \le \max\{2, m+1\}$ (see [39] for a more precise statement in the delicate case $p = m+1$), while the blow–up of the solutions is proved (in the fast diffusion case blow–up is proved only for strong solutions), when

$$p > \max\{2, m+1\}, \tag{55}$$

$|u_0|^{m-1}u_0 \in H^1_0(\Omega)$, and the initial energy

$$E(0) = \frac{1}{2}\|\nabla(|u_0|^{m-1}u_0)\|_2^2 - \frac{m}{m+p-1}\|u_0\|_{m+p-1}^{m+p-1}$$

is negative, i.e the initial data are in region C (see Figure 1).

In this section we show how the blow-up result for initial data in region C of Figure 1, known for problem (6), can be extended to $m \neq 1$. Indeed, the global existence result for initial data in the region A has been extended in [12] to the slow diffusion case, while there are not extensions, in author's knowledge, of the blow–up theorem for initial data in the region B. The method used in the proof is inspired to the arguments of [37], where the classical convexity method is adapted to handle with positive initial energy for abstract evolution equation of hyperbolic type. The main idea in [29], in which (10) is treated by the change of variable $v = |u|^{m-1}u$, cannot be extended here to the important slow diffusion case, due to the singularity that appears in the transformed equation. However we adapt the method of [37] using the change of variable in a somewhat implicit way. In order to have an unified proof for the slow and fast diffusion cases, and to minimize the assumptions on p in the first one, when also $u_0 \in L^\infty(\Omega)$, we consider distributional solutions of (10), i.e. functions u defined on a suitable cylinder $Q_T := (0,T)\times\Omega$, such that

$$\int_\Omega u(\eta)\phi(\eta,x)\Big|_{\eta=0}^{\eta=t} = \int_0^t \int_\Omega \big[u\phi_t - |\nabla(|u|^{m-1}u)|^{s-2}\nabla(|u|^{m-1}u)\nabla\phi + |u|^{p-2}u\phi\big] \tag{56}$$

for all $t \in [0,T]$ and $\phi \in C_c^\infty([0,\infty)\times\Omega)$, such that

$$|u|^{m-1}u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^r(\Omega)), \tag{57}$$

$$|u|^{(m-1)/2}u \in H^1(0,T;L^2(\Omega)), \tag{58}$$

and u verifies the energy identity

$$E(t) = E(0) - \frac{4m}{(m+1)^2}\int_0^t \|(|u|^{(m-1)/2}u)_t\|_2^2 \tag{59}$$

for a.a. $t \in [0,T]$, where the energy function E, naturally associated to u, is given by

$$E(t) = \frac{1}{2}\|\nabla(|u(t)|^{m-1}u(t))\|_2^2 - \frac{m}{m+p-1}\|u(t)\|_{m+p-1}^{m+p-1}. \tag{60}$$

Then we prove a global nonexistence result for this type of solutions of (10), which includes as a particular case the solutions founded in [12], [31], [35], and the strong solutions founded in [9], while the weak solutions obtained in [9] are "too weak" to prove global nonexistence. More precisely, we prove

THEOREM 4.1. *Let λ_1 and E_1 be the number defined* (24) *and* (32), *with p replaced by $(m+p-1)/m$. If $|u_0|^{m-1}u_0 \in H_0^1(\Omega)$,*

$$(\|\nabla(|u_0|^{m-1}u_0)\|_2, E(0)) \in B \cup C \cup D,$$

then the corresponding solution u of (10) *is not global.*

This global nonexistence result can be conveniently applied to the local solution founded in [12], [31], [35], and the strong solutions founded in [9], to obtain a the following blow–up result.

THEOREM 4.2. *Let u be a solution of* (10) *whose existence is proved in one of the papers quoted above. Assume that* (55) *holds, that*

$$p < 1 + m(r-1) \quad (\text{when } n \geq 3), \tag{61}$$

and that $|u_0|^{m-1}u_0 \in H_0^1(\Omega)$, $(\|\nabla(|u_0|^{m-1}u_0)\|_2, E(0)) \in B \cup C \cup D$. *Then*

(i) *if* (51) *holds and* $m \geq 1$, *there is* $T_0 > 0$ *such that* $\|u(t)\|_{m+1} \to \infty$ *as* $t \to T_0^-$;

(ii) *if* $u_0 \in L^\infty(\Omega)$, *then there is* $T_1 > 0$ *such that* $\|u(t)\|_\infty \to \infty$ *as* $t \to T_1^-$.

The two different cases (i) and (ii) arise from local existence results available in the literature and quoted above.

Sketch of the proof. The proof, given by contradiction, is based on the inequality

$$\|v\|_{m+p-1}^m \leq B_1 \|\nabla(|v|^{m-1}v)\|_2$$

for $v \in H_0^1(\Omega)$ (which is an obvious consequence of (61) and of Sobolev embedding theorem) and on the identity

$$\begin{aligned} \int_0^t \int_\Omega \frac{2}{m+1}(|u|^{(m-1)/2}u)_t |u|^{(m-1)/2}u + |\nabla(|u|^{m-1}u)|^2 \\ - |u|^{m+p-1} = 0, \qquad t \geq 0, \end{aligned} \tag{62}$$

which is valid for the solutions of (10) satisfying (57)–(58) (see [45]). We sketch the proof in the main case $E_0 := E(0) < E_1$, as in the case $E_0 = E_1$ one can complete the proof as in Theorem 2.3. First of all we observe, as in Theorem 2.3 that, if $(\|\nabla(|u_0|^{m-1}u_0)\|_2, E(0)) \in B \cup C$, then there are $\lambda_2 > B_1\lambda_1$ and $\lambda_3 > \lambda_1$ such that, for all $t \geq 0$,

$$E(t) \leq E_0, \qquad \|u(t)\|_{m+p-1}^m \geq \lambda_2, \qquad \text{and} \quad \|\nabla(|u|^{m-1}u)\|_2 \geq \lambda_3.$$

The global nonexistence is then proved by the convexity method as in [37]. The main auxiliary function of [37] is replaced by

$$\mathcal{I}(t) = \int_0^t \|u\|_{m+1}^{m+1} + (T_0 - t)\|u_0\|_{m+1}^{m+1} + \gamma(t+t_0)^2, \tag{63}$$

where t_0, T_0 and γ are positive constants to be fixed later. Then

$$\mathcal{I}'(t) = 2\int_0^t \int_\Omega (|u|^{m-1)/2}u)_t |u|^{m-1)/2} u\, dx + 2\gamma(t+t_0),$$

and, using (62),

$$\frac{1}{2}\mathcal{I}'' = \frac{m+1}{2}[(q/2-1)\|\nabla(|u|^{m-1}u)\|_2^2 - qE] + \gamma.$$

Using the energy identity and the explicit values of λ_1 and E_1 one gets

$$\frac{1}{2}\mathcal{I}'' \geq \frac{(m+1)q}{2}\left[(E_1 - E_0) + \frac{4m}{(m+1)^2}\int_0^t \|(|u|^{(m-1)/2}u)_t\|_2^2\right] + \gamma,$$

so, choosing $\gamma = (m+1)^2(E_1 - E_0)/2m > 0$,

$$\mathcal{I}'' \geq 2\left(\frac{mq}{m+1} + 1\right)\gamma + \frac{4mq}{m+1}\int_0^t \|(|u|^{(m-1)/2}u)_t\|_2^2. \tag{64}$$

Clearly $\mathcal{I}'(0) = 2\gamma t_0 > 0$, $\mathcal{I}(0) = T_0\|u_0\|_{m+1}^{m+1} + \gamma t_0^2 > 0$. Moreover, $\mathcal{I}'' > 0$ by (64), so $\mathcal{I}'$ and $\mathcal{I}$ are both positive. The proof can then be finished as in [37] by proving that

$$\mathcal{I}\mathcal{I}'' - \alpha(\mathcal{I}')^2 \geq 0 \qquad \text{on } [0, T_0], \tag{65}$$

where $\alpha = [1 + mq/(m+1)]/2$, redefining the quantities $\mathbb{A}, \mathbb{B}, \mathbb{C}$ used in the quoted paper as

$$\mathbb{A} = \int_0^t \|u\|_{m+1}^{m+1} + \gamma(t+t_0)^2, \tag{66}$$

$$\mathbb{C} = \int_0^t \|(|u|^{(m-1)/2}u)_t\|_2^2 + \gamma, \tag{67}$$

and $\mathbb{B} = \mathcal{I}'/2$. □

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