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Life-Span of Solutions to Nonlinear Dissipative Evolution Equations: a Singular Perturbation Approach

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Summary. - *We investigate the large time behavior of solutions to nonlinear dissipative wave equations of the general form*

$$\varepsilon u_{tt} + u_t - \Delta u = F(x,t,u,D_x u,D_x^2 u);$$

in particular, we study the dependence of the solutions $u = u^\varepsilon$ and of their life span T_ε on the (small) parameter ε. We are interested in the behavior of u^ε and T_ε as $\varepsilon \to 0$, and in their relations with the solution v, and its life span T_p, of the corresponding limit equation when $\varepsilon = 0$, which is of parabolic type. We look for conditions under which either $T_\varepsilon = +\infty$, or $T_\varepsilon \to T_p \leq +\infty$ as $\varepsilon \to 0$.

1. Introduction.

1.1 This survey paper presents some problems concerning the large time behavior of nonlinear dissipative evolution equations of parabolic and hyperbolic type, which are related when the latter are a perturbation of the former. More precisely, we consider a family of nonlinear dissipative hyperbolic equations of general form

$$\varepsilon u_{tt} + u_t - \Delta u = F(x,t,u,D_x u,D_x^2 u), \qquad \varepsilon > 0, \tag{1}$$

which we see as a perturbation of the nonlinear parabolic equation

$$v_t - \Delta v = F(x,t,v,D_x v,D_x^2 v). \tag{2}$$

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We call equations (1) "weakly hyperbolic" when ε is small; we are interested in the asymptotic behavior, as $t \to +\infty$, of smooth solutions to (1) and (2) when both problems have global solutions, and in the life-span and blow-up mechanism for either problem, when at least one of the life spans is finite. We will formulate several questions related to these problems in a very general setting, and give some answers in some special cases.

We can consider these equations either in the whole space R^n, or in a bounded domain $\Omega \subset R^n$ with sufficiently smooth boundary $\partial\Omega$, or in the exterior of a bounded domain; in the latter cases, appropriate types of boundary conditions, such as Dirichlet, Neumann, or mixed ones, are of course added to the problem. The proper choice of conditions for each problem is extremely important, because it is essential to be able to compare solutions of (2) and (1) in one common function space. For classical solutions, as in example (5) below, one naturally considers spaces like $W^{k,\infty}(R^n)$, for some $k \in N$; later on, we shall present some results on the comparison of solutions to a class of initial-boundary value problems for a quasilinear version of (1) and (2) in suitable Sobolev spaces $H^s(\Omega)$.

The life span of regular solutions of (1) and (2) may very well be finite, and in general depends on the size of their initial values, i.e. respectively

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \tag{3}$$

for (1) and, for (2),

$$v(x,0) = v_0(x), \tag{4}$$

as well as on the particular form and the growth properties of the nonlinearity F. A simple example is given by the semilinear initial value problem in R^2

$$\begin{cases} \varepsilon u_{tt} + \alpha u_t - \Delta u = |u|^p, \\ u(x,0) = \eta\,\varphi(x), \\ u_t(x,0) = \eta\,\psi(x) : \end{cases} \tag{5}$$

combining the results of Li Ya-Chun, [24], with those of Li Ta-Tsien and Yi Zhou, [22], if $1 \le p \le 2$ and

$$\int_{R^2} (\varphi(x) + \psi(x))\, dx > 0,$$

the corresponding solution of (5) must blow up in finite time, and the life span $T = T_\eta$ of these solutions satisfies the estimates

$$T_\eta = \begin{cases} O\left(\eta^{(1-p)/(2-p)}\right) & \text{if} \quad 1 \le p < 2, \\ O\left(\exp(\eta^{1-p})\right) & \text{if} \quad p = 2, \end{cases} \qquad \text{as } \eta \to 0, \tag{6}$$

with explicit upper and lower bounds (compare to Linblad, [26], for the nondissipative case). In fact, if we define as usual the CRITICAL EXPONENT of problem (5) to be the smallest number p_c such that if $p > p_c$ then for all data $\{u_0, u_1\}$ there is $\eta_0 > 0$ such that (5) has a global solution for all $\eta \le \eta_0$, the following is known:

1. For the parabolic equation, i.e. $\varepsilon = 0$, $\alpha = 1$ (with, of course, no condition on $u_t(\cdot, 0)$), $p_c = 2$ (see Fujita, [5]); more generally, the critical exponent in R^n is $p_c = 1 + \frac{1}{n}$.

2. For the undamped wave equation, i.e. $\varepsilon = 1$, $\alpha = 0$, the critical exponent is *larger*: $p_c = (3 + \sqrt{17})/2$; more generally, the critical exponent in R^n is the positive solution of the quadratic equation $(n-1)p^2 = (n+1)p + 2$ (see Georgiev, Linblad and Sogge, [6]).

3. For the damped wave equation, i.e. $\varepsilon = 1$, $\alpha = 1$, we deduce from (6) that the critical exponent is again $p_c = 2$; the exact value of the critical exponent in R^n has only recently been settled by Todorova and Yordanov, [55], who proved that $p_c = 1 + \frac{2}{n}$ as for the parabolic case. This long standing conjecture was motivated also by the diffusion phenomenon of hyperbolic waves, which we shall describe at the end of §2.4 below.

1.2 As we have said, we are particularly interested in the dependence of the life span of solutions to (1) on the parameter ε, and its behavior as $\varepsilon \to 0$. Denoting respectively by T_ε and T_p the life span of solutions of the hyperbolic equations (1) and of the parabolic equation (2), roughly speaking we expect that, when ε is small, T_ε and T_p are comparable, in the following sense:

Q1: If T_p is finite, then so is T_ε, and their difference vanishes as $\varepsilon \to 0$.

Q2: If solutions of the parabolic equation (2) are globally defined (i.e. $T_p = +\infty$), then either the same is true for solutions of the perturbed hyperbolic equation (1) when ε is small (the smallness being in general determined by the data), or $T_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Conversely, if $T_\varepsilon = +\infty$ for all sufficiently small ε, then also $T_p = +\infty$.

Q3: When solutions to both (1) and (2) are globally defined, their asymptotic behavior as $t \to +\infty$ (for example, their growth rates, stability, convergence to stationary solutions, existence of attractors) should be similar for small values of ε, in the sense that a particular asymptotic behavior of the solutions of the parabolic equation should imply a similar behavior for the solutions of the hyperbolic equations when ε is small, and viceversa.

In particular, Q2 means that when ε is small the behavior of T_ε is essentially controlled by ε alone, in the sense that, for any given data of (1), one of the following possibilities should hold:

1. Either there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$, $T_\varepsilon = +\infty$,
2. Or $T_\varepsilon \to +\infty$ as $\varepsilon \to 0$, i.e.

$$\forall T > 0 \;\; \exists \varepsilon_T > 0 \quad \forall \varepsilon \le \varepsilon_T, \qquad T_\varepsilon > T. \tag{7}$$

As an example of the relationship between the life spans of both type of problems, we mention the comparison made by Sideris in [53] between the 3-dimensional compressible Euler equations and their incompressible limit; in particular, these would admit global solutions iff $\liminf_{\varepsilon\to 0} \varepsilon T_\varepsilon = +\infty$, ε being the viscosity parameter. However, it is generally believed that solutions to the incompressible problem exhibit blow-up in finite time: one way to show this would be to prove that the above limit is finite.

1.3. We conclude this introduction by mentioning some possible applications of results on these questions to several models of physical systems. A first example is given by Maxwell's equations in ferromagnetic media, when the constituent relations between the electromagnetic fields and inductions can be approximated by the conditions

$$H = \mu_0^{-1} B + \zeta(B), \qquad D = \varepsilon E, \qquad J = \sigma E,$$

where μ_0, ε and σ are (for simplicity) positive constants, and ζ is a nonlinear function simulating hysteresis; ε and σ are, respectively, a measure of the displacement and the eddy currents. Resorting to the usual electromagnetic potentials, i.e. setting $B = \operatorname{curl} A$, $E = -A_t + \nabla\phi$, with A, ϕ satisfying the gauge relations $\varepsilon\phi_t + \sigma\phi = \mu_0^{-1}\operatorname{div} A$, the complete nonhomogeneous Maxwell's equations

$$\left\{\begin{array}{rcl} D_t + J - \operatorname{curl} H & = & \Phi \\ B_t + \operatorname{curl} E & = & 0, \\ \operatorname{div} B = 0, \quad \operatorname{div} D & = & q \end{array}\right. \tag{8}$$

are transformed into the system of second order equations

$$\varepsilon A_{tt} + \sigma A_t - \mu_0^{-1}\Delta A = -\operatorname{curl}\zeta(\operatorname{curl} A) - \Phi, \tag{9}$$

which is of type (1). The interest of this model lies in the observation that in ferromagnetic media displacement currents are usually much weaker than eddy currents: that is, typically, $0 < \varepsilon << \sigma$, so the term εA_{tt} in (9) is commonly neglected in numerical simulations. It is therefore of importance in applications (for example, in the modelling of transformer cores) to have a control, in terms of ε, of the difference $A^\varepsilon - A^o$ between the exact solution A^ε of (9) and the solution A^o of its parabolic approximation

$$\sigma A_t - \mu_0\Delta A = -\operatorname{curl}\zeta(\operatorname{curl} A) - \Phi. \tag{10}$$

Actually, we should remark that the electromagnetic fields vary periodically in time, with a (low) frequency ω, and displacement currents are negligible precisely when $\varepsilon\omega << \sigma$; hence, the importance of determining conditions under which solutions of (9) are guaranteed to exist in possibly large time intervals $[0, T]$, $T = \frac{1}{\omega}$, and to estimate $A^\varepsilon - A^o$ also in terms of a suitable norm of Φ_t, which would allow us to account for the periodicity in an implicit way (a first result along this line was established in [40]).

Interesting other examples include models of random walk systems (see e.g. Hadeler, [7]), where the parameter ε is related to the reciprocal of the turning rates of the moving particles; models of traffic flow patterns (see e.g. Schochet, [50]), where ε would be a measure of the drivers' response time (hopefully very short!) to sudden disturbancies; also, some simple models of laser optic equations (see e.g. Haus' book [8]), where ε is related to measures of (low) frequencies of the electromagnetic field. Other examples that are more readily described by an equation like (1) include Cattaneo's model for the heat equation, the heat equation with delay, and a model for the flow of polytropic gases, which we are going to briefly recall.

In [4], Cattaneo argued that heat conduction in a one-dimensional nonlinear homogeneous medium should be modelled by the perturbed equation

$$\varepsilon u_{tt} + u_t - \left(\sigma(u_x)\right)_x = f, \tag{11}$$

where ε is a measure of the "thermal relaxation" properties of the medium, and σ is a C^2 function satisfying $\sigma''(r)r > 0 \;\; \forall r \in R - \{0\}$. Typically, the thermal relaxation is quite small, but not negligible, and this corrects the inconsistencies of "instant propagation with infinite speed" of the heat flow, that one is forced to deduce from the standard model of the heat equation. In this model, the basic equations relating the temperature $u = u(x,t)$ and the flux $q = q(x,t)$ are

$$\begin{cases} u_t + q_x &= f \\ q + k\,u_x &= 0, \end{cases} \tag{12}$$

the second of which expresses Fourier's law. If there is a delay $\tau > 0$, the second of (12) is replaced by

$$q(x, t+\tau) = -k\,u_x(x,t); \tag{13}$$

approximating the left side of (13) by means of Taylor's expansion, and neglecting higher order terms, we have

$$q(x, t+\tau) = q(x,t) + \tau\,q_t(x,t) = -k\,u_x(x,t),$$

from which

$$q_x(x,t) + \tau\,q_{xt}(x,t) = -k\,u_{xx}(x,t).$$

Replacing this in the first of (12), differentiated with respect to t, we obtain the heat equation with delay

$$\tau\,u_{tt} + u_t - k\,u_{xx} = f + \tau\,f_t, \tag{14}$$

which was extensively studied by Li Ta-Tsien in [20]. An analogous model can be obtained for nonlinear heat conduction with delay (i.e.

when k is a function of u_x in the second of equations (12)); in both cases, one is interested in small values of the delay parameter τ. A similar analysis can also be carried out for more general reaction-diffusion processes with delays. Finally, we recall that with the usual substitution $v = u_x$, $w = u_t$, equation (11) (with $f \equiv 0$) is transformed into the first order dissipative system

$$\begin{cases} \varepsilon w_t - (\sigma(v))_x & = & -w, \\ v_t - w_x & = & 0; \end{cases} \tag{15}$$

when $\sigma(r) = -k\,r^{-\gamma}$, $k > 0$, $1 < \gamma < 3$, (15) is a model for a polytropic gas in Lagrangean coordinates, where ε is a measure of the inertial forces. The convergence process for weak solutions of (15) as $\varepsilon \to 0$ was studied in [35], using monotone operators and compensated compactness techniques; note that when $\varepsilon = 0$ it is possible to deduce from the corresponding limit system the porous media equation

$$v_t - (\sigma(v))_{xx} = 0 \tag{16}$$

(for more on the connection between porous media equation and conservation laws, see e.g. the monograph [31], as well as Vazquez, [56], or T.P. Liu, [30], and Marcati, [33], [34]).

2. Life Span and Asymptotic Behavior of Smooth Solutions.

2.1. In the sequel, we shall refer to the hyperbolic problem (1)+(3) (together with appropriate boundary conditions if required), as "Problem (H_ε)", and to the parabolic problem (2)+(4) (also together with a boundary condition if required), as "Problem (P)". One of the first steps in determining the large time behavior of solutions to problems (H_ε) and (P) consists in obtaining sharp estimates, with explicit upper and lower bounds as in (6), on the life spans of regular solutions of both problems, in relation to the size of their initial values and the form and the growth properties of the nonlinearities of the equations; then, when global existence can be guaranteed, we can try to determine the asymptotic behavior of the solutions as $t \to +\infty$. In the present case, it is of particular interest to determine how the long time behavior results that are available for nonlinear hyperbolic problems are affected by the presence of the parameter ε and, when appropriate, by the dissipation term u_t; of course, we assume the data $\{u_0, u_1\}$ to be independent of ε.

For a general review of global existence and regularity results in absence of dissipation and when $\varepsilon = 1$, we refer in particular to the monographs of Li Ta-Tsien, [19] and [21], as well as to those of Strauss, [54], and Racke, [48]; see also the papers by Klainerman, [12], [14], [15], [16], and Tai-Ping Liu, [29]. For explicit blow-up results for hyperbolic equations, we refer to the book by Alinhac, [1], and, for specific examples, to the papers of F. John, [11] and [10], and Sideris, [52] and [51]. Corresponding results for the dissipative case are also available; the basic global existence result for quasilinear equations like (21) below is established in Matsumura's paper [36], and regularity and blow-up results for nonlinear dissipative equations can be found in the papers [22], [24], [23]. For general results on global existence, asymptotics and regularity for problem (P), we refer mainly to the books by Lunardi, [32], Amann, [2], Lieberman, [25], and Zheng, [58]; for corresponding results on finite time blow-up, see e.g. the monographs [49], or [3], and the references quoted therein.

2.2. We now present, in addition to the problems described in Q1, Q2 and Q3 above, a list, by no means exhaustive, of related questions, some of which we propose to study by means of the equivalence principle described below.

Q4: Determine whether there are cases for which solutions of problems (P) and (H_ε) share a common existence interval, at least if ε is small; that is, if

$$\exists\, \varepsilon_0 > 0,\ \exists\, T_c \in (0, +\infty] \ \mid \quad \forall\, \varepsilon \le \varepsilon_0, \quad T_\varepsilon \ge T_c, \quad T_p \ge T_c. \tag{17}$$

Q5: Whether there are situations for which T_ε is in fact independent of ε, at least if ε is small; that is, if (17) can be replaced by

$$\exists\, \varepsilon_0 > 0,\ \exists\, T_0 \in (0, +\infty] \ \mid \quad \forall\, \varepsilon \le \varepsilon_0, \quad T_\varepsilon = T_0. \tag{18}$$

Q6: Determine sufficient conditions on the data $\{u_0, u_1\}$, on ε and the nonlinearity F under which $T_\varepsilon = +\infty$; for instance, when

$\varepsilon = 1$ this is often true for equations with nonlinearities independent of u if the initial data (2) are small, but in general not for equation (1), even if the data are small, at least in the whole space case (see e.g. Linblad, [27]). Concurrently, characterize conditions on v_0 and the nonlinearity F which separately assure that $T_p = +\infty$ as well.

Q7: When (17) holds and u^ε converges to some function $\overline{u}$ in a suitable sense as $\varepsilon \to 0$, identify the problem (P) to which $\overline{u}$ is the solution in the interval $[0, T_c]$. If T_p is the life span of $\overline{u}$ and $T_0 \doteq \inf_{\varepsilon>0} T_\varepsilon$, determine whether $T_0 \leq T_p$, and if there are cases when $T_0 = T_p$. In this respect, note that the convergence $u = u^\varepsilon \to v$ as $\varepsilon \to 0$ is singular in time, due to the loss of the initial condition on u_t; in particular, there is in general an initial layer at $t = 0$.

Q8: When indeed $u^0 \doteq \lim_{\varepsilon\to 0} u^\varepsilon$ is a solution of problem (P) in $[0, T_c]$, to what extent can one generalize to the nonlinear case the asymptotic analysis by means of the expansion in ε

$$u^\varepsilon = u^0 + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots \tag{19}$$

that is sometimes possible in the linear case (see e.g. J.L. Lions, [28], and Zlamal, [59], [60]). In this case, establish suitable error estimates on the difference $u^\varepsilon - u^0$ (for a first step in this direction, see e.g. [41] for the case of the quasilinear initial value problem; for an application of this type of estimates to the time periodic problem in a bounded domain, which could be applied to the model (9) of Maxwell's equations, see [40]).

2.3. We now introduce a sort of "equivalence principle" on the long time behavior of solutions to (H_ε) and (P), and show how we can answer some of the questions we have posed, when we can assume this equivalence principle. We shall then proceed to prove the equivalence principle in the special case of a class of initial-boundary value problems for the quasilinear version of (2) and (1). The equivalence principle is the following:

DEFINITION 2.1. *The global existence equivalence principle holds for problems* (H_ε) *and (P) if whenever the parabolic problem (P) is solvable for any choice of (compatible) data on any time interval* $[0, T]$, *the hyperbolic problem* (H_ε) *is also solvable for any choice of (compatible) data on the same time interval* $[0, T]$, *at least if* ε *is sufficiently small; and viceversa.*

In this definition, "solvable" means in a suitable common class of functions. In most cases, the smallness of ε will depend on the data of (H_ε) and (unfortunately!) on T.

To give an example of the results that can be obtained assuming the equivalence principle, we claim:

PROPOSITION 2.2. *Assume the equivalence principle holds in some class of suitably smooth functions. Set* $T_l \doteq \liminf_{\varepsilon\to 0} T_\varepsilon$ *and* $T_s \doteq \limsup_{\varepsilon\to 0} T_\varepsilon$*: then*

$$T_l = T_s = T_p. \tag{20}$$

Proof. We only prove that $T_l = T_p$; the argument for T_s is analogous. We show first that $T_l \le T_p$: otherwise, we choose $\lambda > 0$ such that $T_p < T_l - 2\lambda < T_l - \lambda < T_l$, and ε_0 such that $T_\varepsilon \ge T_l - \lambda$ for all $\varepsilon \le \varepsilon_0$. Then, since (H_ε) is solvable on $[0, T_l - 2\lambda]$ for all $\varepsilon \le \varepsilon_0$, by the equivalence principle (P) is also solvable on $[0, T_l - 2\lambda]$, and since this interval contains $[0, T_p]$, T_p cannot be the life span of (P). Thus, $T_l \le T_p$; if $T_l \ne T_p$, we choose $\delta > 0$ such that $T_l + \delta < T_p$: since (P) is solvable on $[0, T_l + \delta]$, by the equivalence principle we can also solve (H_ε) on $[0, T_l + \delta]$ for all ε less than some ε_0. But there is at least a sequence $\varepsilon_k \to 0$ such that $T_{\varepsilon_k} \le T_l + \delta/2$: this yields a contradiction, and we conclude that $T_l = T_p$. □

As a consequence of this Proposition, $\lim_{\varepsilon\to 0} T_\varepsilon$ exists, and equals T_p; thus, we have a positive answer to question Q1, and to the second part of question Q2. In contrast, the first part of Q2 remains open in general if, in the equivalence principle, ε_0 is not independent of T, because in this case it may happen that $\varepsilon_0(T) \to 0$ as $T \to +\infty$.

Of course, the second part of Q2 can be proved directly: given any $T > 0$, we solve (P) on $[0, T]$; by the equivalence principle, there is $\varepsilon_T > 0$ such that we can also solve (H_ε) on $[0, T]$ for all $\varepsilon \le \varepsilon_T$. This implies that $T_\varepsilon > T$: that is, (7) holds.

2.4. The equivalence principle has also the following consequences:

1) On Q5: since by Proposition 2.2 we have

$$\inf_{\varepsilon\to 0} T_\varepsilon \le T_l = T_p = T_s \le \sup_{\varepsilon\to 0} T_\varepsilon,$$

T_ε can be independent of ε only if $T_\varepsilon = T_p$ for all ε (sufficiently small, depending on T_p); in particular, this provides a positive answer also to the first part of question Q2, if $T_p = +\infty$.

2) A positive answer to the last part of question Q3: if $T_\varepsilon = +\infty$ for all sufficiently small ε, given any $T > 0$, there is $\varepsilon_T > 0$ such that (H_ε) is solvable on $[0, T]$ for all $\varepsilon \le \varepsilon_T$; by the equivalence principle, (P) is also solvable on $[0, T]$, and this implies that $T_p = +\infty$.

3) On Q7: in most cases, it is relatively easy to show that, as $\varepsilon \to 0$, smooth solutions of (H_ε) converge to a smooth solution of (P) (of course, one of the problems in this regard is exactly that of determining, in each case, the most suitable class of "smooth" functions): we can then show that $T_0 \le T_p$. Indeed, clearly $T_p > T_c$ and $T_0 \ge T_c$; proceeding as in the proof of Proposition 2.2, if $T_0 > T_p$, we could choose $\lambda > 0$ such that $T_p < T_p + \lambda < T_0$: then, since $T_\varepsilon \ge T_0$ for all ε, we can solve (H_ε) on $[0, T_p + \lambda]$ for all ε sufficiently small. By the equivalency principle, we could then solve (P) on $[0, T_p + \lambda]$, so T_p would not be the life span of (P). In contrast, whether $T_0 = T_p$ will in general depend on the particular problem under consideration.

4) For completeness' sake, we also report a result on question Q3, which is not directly a consequence of the equivalence principle, but does require the concurrent investigation of the asymptotic behavior of the difference $u^\varepsilon - v$ as $t \to +\infty$. Specifically, we consider the question of the asymptotic stability of the solutions, and the related diffusion phenomenon for nonlinear hyperbolic waves, established originally by L. Hsiao and Tai-Ping Liu (see [9]) and further studied by Li Ta-Tsien and Nishihara (see [20] and [43]). Roughly speaking, this means that if we compare the solutions to (H_ε) with that of the reduced problem (P), as $t \to +\infty$ the difference of these solutions should decay to 0 with a rate *faster* than that of either solution. In this direction, we have shown in [57] that for equations in divergence form, the norm $\|u(\cdot,t) - v(\cdot,t)\|_{L^\infty(R^n)}$ decays as $O(t^{-(n+1)/2})$, while u and v each decay, in $L^\infty(R^n)$, only as $O(t^{-n/2})$.

3. The Quasilinear Equations.

3.1. In this section we recall two results which show that the equivalence principle does hold for a class of quasilinear dissipative evolution equations, of the form

$$\varepsilon u_{tt} + u_t - a_{ij}(\nabla u)\partial_i\partial_j u = f(x,t), \tag{21}$$

subject to the initial conditions (3) and, if the problem is considered in a bounded open domain $\Omega \subset R^n$ with smooth boundary $\partial\Omega$, to homogeneous Dirichlet boundary conditions. The corresponding parabolic equation is then

$$v_t - a_{ij}(\nabla v)\partial_i\partial_j v = g(x,t); \tag{22}$$

in (21) and (22), as well as in the sequel, summation with respect to i, j from 1 to n is understood. We assume that the coefficients a_{ij} are sufficiently smooth functions defined, for simplicity, on all of R^n; they are symmetric (i.e. $a_{ij} = a_{ji}$), and satisfy the uniformly strong ellipticity condition

$$\exists\, \nu > 0 \ \ \forall\, p, q \in R^n, \quad a_{ij}(p)q^i q^j \geq \nu |q|^2. \tag{23}$$

Since we also need to apply the classical theory to the quasilinear parabolic problem (22) (see e.g. Ladyzensakya - Solonnikov - Ural'tzeva, [18], or Krylov, [17]), we shall also have to assume that the a_{ij}'s are bounded, that is,

$$\exists\, \mu > 0, \ \ \forall\, p, q \in R^n, \quad a_{ij}(p)q^i q^j \leq \mu |q|^2, \tag{24}$$

and that their derivatives satisfy some decay estimates. While these assumptions are not required for the classical solution theory for hyperbolic equations (as for instance in Kato, [13], or Okazawa, [44], Okazawa and Unai, [46], [45], [47]), they are satisfied in many applications; for instance, in Maxwell's model (9) if ζ is asymptotically linear, or for the minimal surface operator

$$-\Delta u - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

A more general model would allow the coefficients a_{ij} to depend on x, t and u as well; in the hyperbolic case (21), the coefficients could in principle depend also on u_t, but this doesn't seem to be feasible in the present context, given the parabolic nature of the limit equation (22), and the singular nature of the convergence $u = u^\varepsilon \to v$ as $\varepsilon \to 0$, with the consequent loss of regularity of u_t.

Under the assumptions stated above, local and global existence results for regular solutions of both problems (H_ε) and (P) are well known: for instance, we refer to Matsumura, [36], who provides global solutions to (H_ε) (with $\varepsilon = 1$) under smallness assumptions on the data u_0, u_1 and f, and to [37], where we prove a global existence result for regular solutions of problem (P) corresponding to data v_0 and g of arbitrary size.

3.2. The proof of the equivalence principle between problems (21) and (22) for the pure initial value problem, i.e. when $\Omega = R^n$, is given in [41]; here, we briefly describe the same result for the case of a bounded domain, with homogeneous Dirichlet boundary conditions.

Following Kato, [13], we consider solutions of (21) in the spaces

$$X_m(0,T) \doteq \cap_{j=0}^m C^j([0,T]; H^{m-j}(\Omega))$$

for sufficiently large integer m; more precisely, we fix integer $s \geq \left[\frac{n}{2}\right] + 2$, $[\alpha]$ denoting the integer part of α, and assume that

HA1) $f \in X_{s-1}(0,T)$, $\partial_t^s f \in L^2(Q)$, $u_0 \in H_*^{s+1}(\Omega)$, $u_1 \in H_*^s(\Omega)$,

HA2) $\{f, u_0, u_1\}$ satisfy the hyperbolic compatibility conditions (HCC in short) of order s at $\partial\Omega$ for $t = 0$, which are defined as follows: setting

$$u_k(x) \doteq (\partial_t^k u)(x,0), \qquad 0 \leq k \leq s+1, \tag{25}$$

as formally computed from (21) by means of an explicit formula of the type

$$\varepsilon u_{k+2} = (\partial_t^k f)(\cdot,0) - u_{k+1} + A_k[u_0,\ldots,u_k], \qquad 0 \leq k \leq s-1 \tag{26}$$

(for example,

$$\begin{aligned}
\varepsilon u_2 &= f(\cdot,0) - u_1 + a_{ij}(\nabla u_0)\partial_i\partial_j u_0, \\
\varepsilon u_3 &= f_t(\cdot,0) - u_2 + a_{ij}(\nabla u_0)\partial_i\partial_j u_1 + (\nabla a_{ij}(\nabla u_0)\cdot\nabla u_1)\partial_i\partial_j u_0,
\end{aligned}$$

etc.), we require that the $s+1$ conditions $u_k \in H_*^{s+1-k}(\Omega)$ for $0 \leq k \leq s$ are satisfied. These conditions make sense, since our assumptions on the data guarantee that $u_k \in H^{s+1-k}(\Omega)$ for $0 \le k \le s+1$, and are necessary for the solvability of (21) in $X_{s+1}(0,T)$, for if $u \in X_{s+1}(0,T)$ solves (21), then (25) does hold in $H^{s+1-k}(\Omega)$, so the traces of u_k on $\partial\Omega$ are defined at least for $0 \le k \le s$, and must therefore vanish.

Local in time solvability of (H_ε) is established e.g. in Kato, [13] (Theorem 14.3); thus, since global existence for (P) is known, the equivalence principle implies that these local solutions can be extended to intervals $[0, \tilde{T}_\varepsilon]$, with $u = u^\varepsilon \in X_{s+1}(0, \tilde{T}_\varepsilon)$ and $\tilde{T}_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

3.3. To this end, we need to recall some results on the parabolic initial-boundary value problem (in short, IBV) for (22) for given data g in Q and v_0 in Ω. Following [42], we consider solutions of (22) in the space

$$\begin{aligned} H_c^m(Q) &\doteq \left\{ u \in H^{m,m/2}(Q) \mid \partial_t^{m'} u \in C([0,T]; H^{m-2m'-1}(\Omega)) \right\}, \\ Y_\alpha^m(Q) &\doteq H_c^m(Q) \cap C^{\alpha,\alpha/2}(\overline{Q}), \end{aligned}$$

with $m' \doteq \left[\frac{m-1}{2}\right]$, and m a sufficiently large integer, so that solutions of (22) in $H_c^m(Q)$ are, by embedding, also classical ones; that is, they are in the parabolic Hölder spaces $C^{2+\alpha,1+\alpha/2}(\overline{Q})$ (see e.g. Krylov, [17], for their definition, and for the relevant results on the solvability of (22) in these spaces). Solvability of (22) in the spaces $Y_\alpha^m(Q)$ is proven in [42], by

THEOREM 3.1. *Let $m \in N$ and $\alpha \in (0,1)$, and assume that*

PA1) $g \in Y_\alpha^m(Q)$, $\quad v_0 \in H^{m+1}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$,

PA2) $\{g, v_0\}$ satisfy the parabolic compatibility conditions (PCC in short) of order m.

There exists a unique $v \in Y_{\alpha+2}^{m+2}(Q)$, solution of the IBV problem for (22).

(The definition of the PCC of order m in Theorem 3.1 is similar to the hyperbolic case: namely, we require that the functions $v_k \doteq (\partial_t^k v)(\cdot, 0)$, understood in the same sense as in (25) for the HCC, be in $H_*^{m+1-2k}(\Omega)$ for $0 \le k \le [m/2]$).

3.4. Assume now that the source terms f of (21) and g of (22) are defined on all of $[0, +\infty)$, and satisfy the corresponding parts of (HA1), (PA1) for all $T > 0$. The equivalency principle for the IBV problems for (21) and (22) is contained in the following result, proven in [38]:

THEOREM 3.2. *Let $T > 0$ and $\{f, u_0, u_1\}$ be given, satisfying assumptions (HA1) and (HA2). Assume that the IBV problem for (22) has, for all $m \in N$ and all choices of compatible data $\{g, v_0\}$ satisfying (PA1), a solution $v \in Y^{m+2}_{\alpha+2}(Q)$. There exists $\varepsilon_0 > 0$, depending on T and $\{f, u_0, u_1\}$, such that if $\varepsilon \leq \varepsilon_0$, the local solutions $u \in X_{s+1}(0, \tau)$ of the IBV problem for (21) can be extended to global ones in $X_{s+1}(0, T)$.*

Conversely: Let $T > 0$ and $\{g, v_0\}$ be given, satisfying assumptions (PA1) and (PA2) of Theorem 3.1, with $m = s \geq [n/2] + 2$. Assume that the IBV problem for (21) has, for all choices of compatible data $\{f, u_0, u_1\}$ satisfying (HA1), and for all correspondingly small ε, a solution $u = u^\varepsilon \in X_{s+1}(0, T)$. Then the IBV problem for (22) has a unique solution $v \in H^{s+2}_c(Q)$.

Clearly, Theorems 3.1 and 3.2 together allow us to deduce that solutions of (21) are defined on arbitrary intervals $[0, T]$, for any (compatible) data $\{f, u_0, u_1\}$ independent of ε, provided ε is sufficiently small (depending on T). That is, given arbitrary $T > 0$ we can determine $\varepsilon_T > 0$ such that we can solve (21) on $[0, T]$ for all $\varepsilon \leq \varepsilon_T$: thus, $T_\varepsilon > T$ for all such ε, which is exactly (7). Note, however, that the spaces $X_{s+1}(0, T)$ and $H^{s+2}_c(Q)$ are not the best suited for the desired comparison of the solutions to (21) and (22), as we can only guarantee that both these spaces are embedded in $C([0, T]; H^{s+1}(\Omega))$. Nevertheless, we realize that this is only due to the effect of the initial-boundary layer, and as soon as $t > 0$, the smoothing effect of the parabolic operator takes over, as we would expect. Indeed, we have the following regularity result for the IBV problem for the parabolic equation (22), which is proven in [39]:

THEOREM 3.3. *Let $m \geq 1$, and assume, in addition to (PA1) and (PA2), that g satisfies*

PA3) $g \in X_{m-1}(\rho, T)$, $\partial_t^m g \in L^2(Q_\rho)$ for some $\rho \in (0, T/4)$:

then the solution v provided by Theorem 3.1 is such that $\nabla v \in X_m(4\rho, T)$, $\partial_t^{m+1} v \in L^2(Q_{4\rho})$.

This result allows us to compare solutions of (H_ε) and (P) in $X_{s+1}(4\rho, T)$, which is of course sufficient for the study of their long time behavior.

3.5. We conclude by reporting the main idea of the proof of the first part of Theorem 3.2 (the proof of the second part is based on a singular perturbation argument); we present a rather formal argument, neglecting issues of regularity that would otherwise need to be taken into account. Assuming that problem (P) has a global solution for all choices of data $\{v_0, g\}$, we choose $v_0 = u_0$ and $g = f - \varphi^\varepsilon$, φ^ε being a "small" corrector that takes care of the initial and boundary layers; in particular, we can choose φ^ε so that also $v_t(\cdot, 0) = u_1$, by imposing that $\varphi^\varepsilon(\cdot, 0) = \varepsilon u_2$. We consider then the change of variables $y = u - v$, so that u is globally defined iff so is y. Now, y solves the equation

$$\begin{aligned} \varepsilon y_{tt} + y_t \quad &- \quad a_{ij}(\nabla v + \nabla y)\partial_i\partial_j y = \\ &= \quad [a_{ij}(\nabla v + \nabla y) - a_{ij}(\nabla v)]\,\partial_i\partial_j v - \varepsilon v_{tt} + \varphi^\varepsilon, \end{aligned} \tag{27}$$

in which the right side is $O(|\nabla y|) + O(\varepsilon)$ (by choosing $\varphi^\varepsilon = O(\varepsilon)$; all this in an appropriate norm), and $y(\cdot, 0) = 0$, $y_t(\cdot, 0) = 0$: thus, we can apply the mentioned global existence result of Matsumura, [36], and deduce that y, and therefore u, is globally defined if ε is sufficiently small. This argument can be carried out both when $\Omega = R^n$ and when Ω is bounded; in the latter case, the major technical difficulty lies in the different type of compatibility conditions (HCC versus PCC), which gives rise to the mentioned boundary layer at $\partial\Omega$.

Finally, we mention that, since Matsumura's estimates would imply that y is uniformly bounded as $t \to +\infty$, the asymptotic behavior of u is indeed, consistently with the spirit of this discussion, essentially related to that of v, and these two questions can be studied together, as described in Q3 above.

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