Rend. Istit. Mat. Univ. Trieste
Suppl. Vol. XXXI, 63–85 (2000)

1-$\mathcal{D}$ Relaxation from Hyperbolic to Parabolic Systems with Variable Coefficients

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Summary. - *In this paper we study the relaxation of semilinear hyperbolic systems to parabolic system. The singular limits are studied using Gérard's generalized compensated compactness.*

Contents

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Key words and phrases: hyperbolic system, parabolic systems, relaxation theory.

1. Introduction

This paper is concerned with the semilinear system of partial differential equations

$$W_s + \sum_{j=1}^{d} E_j(x)\partial_j W = B(x,W) + F(x,W)$$

where $W = W(x,t)$ takes values in $\mathbb{R}^N$, $x \in \mathbb{R}$, $t \geq 0$, $E_j(x)$, $j = 1,\dots,d$ is an $N\times N$ matrix for any $x \in \mathbb{R}$. We want to investigate the relaxation phenomena where the relaxed equilibria are described by means of an equation (or a system) of parabolic type. A study of this kind is usefull to understand the hydrodinamical limit of Boltzmann equation when the Mach number and the Knudsen number are of the same order and we deal with a discreet set of velocities. In fact in this case Boltzmann equation can be rewritten as a semilinear system.

In particular we extend the result of [23] and of [6]. In [23] they consider a semilinear system with constant coefficients and by energy estimate they obtain that it relaxes to a parabolic equation. In [6] they studied the following semilinear system

$$\begin{cases} \partial_t \mathcal{U} + K^{(1)}\partial_x \mathcal{V} = 0 \\ \partial_t \mathcal{V} + H^{(2)}\partial_x \mathcal{U} + K^{(2)}(x)\partial_x \mathcal{V}^\varepsilon = \dfrac{1}{\varepsilon} R(\mathcal{U})\,\mathcal{V}. \end{cases} \tag{1}$$

where $\varepsilon > 0$, $(x,t) \in \mathbb{R}\times\mathbb{R}_+$, $\mathcal{U} = \mathcal{U}(x,t) \in \mathbb{R}^k$, $\mathcal{V} = \mathcal{V}(x,t) \in \mathbb{R}^{N-k}$, $K^{(1)}, H^{(2)}, K^{(2)}(x), R(U)$ are matrices such that $K^{(1)} \in \mathcal{M}_{k\times(N-k)}$, $H^{(2)} \in \mathcal{M}_{(N-k)\times k}$, $K^{(2)}(x) \in \mathcal{M}_{(N-k)\times(N-k)}$, $R(U) \in \mathcal{M}_{(N-k)\times(N-k)}$. Using energy estimate they obtain also that (1) relaxes to a parabolic equation.

In our paper we consider (1) when all the coefficients are variable, namely

$$\begin{cases} \partial_t \mathcal{U} + K^{(1)}(x)\partial_x \mathcal{V} = F^{(1)}(x,\mathcal{U},\mathcal{V}) \\ \partial_t \mathcal{V} + H^{(2)}(x)\partial_x \mathcal{U} + K^{(2)}(x)\partial_x \mathcal{V}^\varepsilon = \dfrac{1}{\varepsilon} R(x,\mathcal{U})\,\mathcal{V} + F^{(2)}(x,\mathcal{U},\mathcal{V}). \end{cases} \tag{2}$$

In order to study the relaxation phenomena we will focus our attention on the investigation of the convergence problem. We will achieve

this aim using the same techniques like in [6] but with a difference. In [6] one of the main mathematical tools used is Tartar's Compensated Compactness ([26], [27]), since in this case all the coefficients are variable classical compensated compactness doesn't work and so we use a generalization of compensated compactness due to P.Gérard [9].
The plan of the paper is the following. In Section 2 we give definitions and we describe our scheme of investigation of this kind of problems. In Section 3 we show the complete theory. We provide a priori estimates on the sequences of solutions as $\varepsilon \downarrow 0$ and combining energy estimates with Gérard's Theorem, we get the convergence result. We investigate also the parabolic nature of the relaxed system. Finally we dedicate Section 4 to the reverse problem. We use the previous result to approximate a given parabolic system by means of a suitable hyperbolic system. To conclude we remark that we will not investigate the existence of solutions of the relaxing problem since in this paper we are interested to the convergence analysis only.

2. General Framework

2.1. Prerequisites

In this section we introduce the main notations and definitions used in the article and we recall the principal notions and results that will be used later. Therefore

(a) $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^q$, $(q = 1, 2, ...)$ and $|\cdot|$ the usual norm of $\mathbb{R}^q$ $(q = 1, 2, ...)$,

(b) $\|\cdot\|$ denotes the norm in $L^2(\mathbb{R} \times \mathbb{R}_+)$,

(c) $\mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$ denotes the space of test function $C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$, $\mathcal{D}'(\mathbb{R} \times \mathbb{R}_+)$ the Schwarz space of distributions and $\langle\cdot,\cdot\rangle$ the duality bracket in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}_+)$,

(d) $\mathcal{M}_{m\times n}$ denotes the linear space of $m \times n$ matrices,

(e) H is a separable Hilbert space, $\mathcal{L}(H)$ the space of bounded operators, $\mathcal{K}(H)$ the space of compact operators,

(f) we denote by $H^s_{loc}(\Omega, H)$ the classical local Sobolev space of order s, i.e. $u \in H^s_{loc}(\Omega, H) \iff \forall\, \varphi \in C_0^\infty,\ (\widehat{\varphi u}) \in L^2(\mathbb{R}^n, (1 + |\xi|^2)^s d\xi)$.

Since in our limit process we need to study the convergence of quadratic forms with variable coefficients, now we recall Tartar's and Gérard's generalization of classical Compensated Compactness Theorem ([28], [9]). So let as consider H, $H^\sharp$ separable Hilbert spaces, $\Omega \in \mathbb{R}$, an open set. Take $m \in \mathbb{N}$, and, for every $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq m$ take $a_\alpha \in C(\Omega, \mathcal{L}(H, H^\sharp))$ so that the formula

$$Pu(x) = \sum_{|\alpha| \leq m} \partial^\alpha (a_\alpha(x) u(x)) \tag{3}$$

defines a differential operator $P : L^2_{loc}(\Omega) \longrightarrow H^{-m}_{loc}(\Omega, H^\sharp)$. Finally we denote by p the principal symbol of P, given by

$$p(x, \xi) = \sum_{|\alpha| = m} \xi^\alpha a_\alpha(x).$$

We have the following theorem

THEOREM 2.1. **(Compensated Compactness)**
Let P defined by (3) and $\{u_k\}$ be a bounded sequence of $L^2_{loc}(\Omega, H)$, such that $u_k \rightharpoonup u$. Assume that there exists a dense subset $D \in H^\sharp$ such that, for any $h \in D$, the sequence $(\langle Pu_k, h\rangle)$ is relatively compact in $H^{-m}_{loc}(\Omega)$. Moreover, let $q \in C(\Omega, \mathcal{K}(H))$.
(i) *If $q = q^*$ and*

$$\forall\, (x, \xi, h) \in S^*\Omega \times H, \qquad (p(x, \xi)h = 0) \ \Rightarrow\ (\langle q(x)h, h\rangle \geq 0)$$

Then, for any nonnegative $\varphi \in C_0^\infty(\Omega)$

$$\liminf_{k \to \infty} \int_\Omega \varphi \langle q(x) u_k, u_k\rangle dx \geq \int_\Omega \varphi \langle q(x) u, u\rangle dx$$

(ii) *If*

$$\forall\, (x, \xi, h) \in S^*\Omega \times H, \qquad (p(x, \xi)h = 0) \ \Rightarrow\ (\langle q(x)h, h\rangle = 0)$$

Then

$$\langle q(x) u_k, u_k\rangle \quad \textit{converges to} \quad \langle q(x) u, u\rangle \qquad \textit{in } \mathcal{D}'(\Omega)$$

We will also make use of the notion of parabolicity for systems of equations in various way (see Taylor [30] volume III, [29], Eidel'man [7], Kreiss and Lorenz [10]). Let us consider the system

$$u_t + \sum_{j,k} A^{j,k}(t,x,D'_x u)\partial_j\partial_k u + B(t,x,D'_x u) = 0, \tag{4}$$

where $u \in \mathbb{R}^p$, $A^{j,k}(t,x,D'_x u) \in \mathcal{M}_{p\times p}$, $B(t,x,D'_x u) \in \mathbb{R}^p$ and D'_x is a differential operator of order not greater than two. The system is said *strongly parabolic* if there exists $c_0 > 0$ such that for all $\xi \in \mathbb{R}^d$ one has

$$\sum_{j,k} A^{j,k}(t,x,D'_x u)\xi_j\xi_k \leq -c_0|\xi|^2 I.$$

Namely, if we denote $L(t,x,D'_x u,\xi) = -\sum_{j,k} A^{j,k}(t,x,D'_x u)\xi_j\xi_k$ this condition is equivalent to say $L + L^T$ is a positive definite matrix. Unfortunately this condition is often difficult to be verified then we formulate now a more general notion of parabolicity (which in the book of Taylor [30] volume III, [29] is referred to as *Petrowski parabolicity*).

We say that the system (4) is *parabolic* if, denoted by $\lambda_k(t,x,D'_x u,\xi)$ the eigenvalues of the matrix $L(t,x,D'_x u,\xi)$, one has there exists $\alpha_0 > 0$ such that, for all $\xi \in \mathbb{R}^d$,

$$Re\lambda_k(t,x,D'_x u,\xi) \geq \alpha_0|\xi|^2.$$

The latter notion of parabolicity is equivalent to ask the existence of a matrix $P_0(t,x,D'_x u,\xi)$, homogeneous of degree 0 in ξ, such that

$$P_0 L + L^* P_0 \geq \alpha_0|\xi|^2 I.$$

2.2. Formal limit analysis

Here we will give, at a formal level the basic ideas that we will use to study the relaxation phenomena. Let us consider the following semilinear problem

$$W_s + \sum_{j=1}^{d} E_j\partial_j W = \frac{1}{\varepsilon}B(x,W) + F(x,W) \tag{5}$$

where $s \geq 0$, $W \in \mathbb{R}^N$. Moreover we assume the following hypotheses hold.

(A.1) there exists an open set $\mathcal{O} \subset \mathbb{R}^N$ such that $B \in C^1\left(\mathcal{O}, \mathbb{R}^N\right)$ and $F \in C^1\left(\mathcal{O}, \mathbb{R}^N\right)$

(A.2) $E_j \in \mathcal{M}_{N\times N}$, $j = 1, \ldots, d$,

(A.3) setting $\mathcal{A}(W)\,\xi = \sum_{j=1}^{d} \xi_j E_j W$, the system (5) is hyperbolic, namely for all nonzero vector $\xi \in \mathbb{R}^d$, the $N \times N$ matrix $D_W \mathcal{A}(W) \cdot \xi$ has real eigenvalues and is diagonalizable,

(A.4) there exists a matrix $P \in \mathcal{M}_{k\times N}$, $1 \leq k < N$, such that $PB(W) = 0$, for all $W \in \mathcal{O}$ and $rank\, P = k$.

Let us consider $[\sigma_1, \ldots, \sigma_{N-k}]$ a vector basis for the subspace $ker\, P$ and denote by Q the $(N-k) \times N$ matrix having σ_i as row vectors. Then we set

$$U = PW, \qquad V = QW, \qquad M = \begin{bmatrix} P \\ Q \end{bmatrix}^{-1},$$

Then there exists matrices $H_j^{(1)} \in \mathcal{M}_{k\times k}$, $K_j^{(1)} \in \mathcal{M}_{k\times(N-k)}$, $H_j^{(2)} \in \mathcal{M}_{(N-k)\times k}$, $K_j^{(2)} \in \mathcal{M}_{(N-k)\times(N-k)}$ and $F^{(1)} \in \mathbb{R}^k$, $F^{(2)} \in \mathbb{R}^{N-k}$ such that

$$H_j^{(1)} U + K_j^{(1)} V = PE_j M\,(U, V) \quad F^{(1)}(x, U, V) = PF(x, M(U, V))$$
$$H_j^{(2)} U + K_{(j)}^{2} V = QE_j M\,(U, V) \quad F^{(2)}(x, U, V) = QF(x, M(U, V)).$$

Hence the system (5) becomes

$$\begin{cases} \partial_s U + \sum_{j=1}^{d} H_j^{(1)} \partial_j U + \sum_{j=1}^{d} K_j^{(1)} \partial_j V = F^{(1)}(x, U, V) \\ \partial_s V + \sum_{j=1}^{d} H_j^{(2)} \partial_j U + \sum_{j=1}^{d} K_j^{(2)} \partial_j V = \frac{1}{\varepsilon} R\,(x, U)\, V + F^{(2)}(x, U, V), \end{cases} \tag{6}$$

where $R\,(x, U,)\, V = QB\,(M\,(U, V))$. The most important assumption which is needed to develope our theory is given by

(A.5) $H_j^{(1)} \equiv 0$ for all $j = 1, \dots, d$.

In particular we shall confine ourselves to the case $d = 1$ with the assumption that all the coefficients of the system depends on the variable x, namely

$$K^{(1)} = K^{(1)}(x) \qquad K^{(2)} = K^{(2)}(x) \qquad H^{(2)} = H^{(2)}(x).$$

Hence we are going to study the following semilinear system in 1-$\mathcal{D}$.

$$\begin{cases} \partial_s \mathcal{U}(y,s) + K^{(1)}(y)\partial_y \mathcal{V}(y,s) = F^{(1)}(x,\mathcal{U}(y,s),\mathcal{V}(y,s)) \\ \\ \partial_s \mathcal{V}(y,s) + H^{(2)}(y)\partial_y \mathcal{U}(y,s) + K^{(2)}(y)\partial_y \mathcal{V}(y,s) = \\ \varepsilon^{-1} R(y,\mathcal{U}(y,s))\mathcal{V}(y,s) + F^{(2)}(x,\mathcal{U}(y,s),\mathcal{V}(y,s)) \end{cases} \tag{7}$$

where $(y,s) \in \mathbb{R} \times \mathbb{R}_+$, $\mathcal{U} = \mathcal{U}(y,s) \in \mathbb{R}^k, \mathcal{V} = \mathcal{V}(y,s) \in \mathbb{R}^{N-k}$, $K^{(1)}(y), H^{(2)}(y), K^{(2)}(y), R(y,\mathcal{U})$ are matrices such that $K^{(1)}(y) \in \mathcal{M}_{k\times(N-k)}$, $H^{(2)}(y) \in \mathcal{M}_{(N-k)\times k}$, $K^{(2)}(y) \in \mathcal{M}_{(N-k)\times(N-k)}$, $R(y,\mathcal{U}) \in \mathcal{M}_{(N-k)\times(N-k)}$, $F^{(1)}(y,\mathcal{U},\mathcal{V}) \in \mathbb{R}^k$, $F^{(2)}(y,\mathcal{U},\mathcal{V}) \in \mathbb{R}^{N-k}$. The hypotheses in order to perform a rigorous analysis will be given at the beginning of next section. Since we are interested in the asymptotic behaviour as $s \to \infty$ for solution of system (7) we are going to explain the construction of the scaling that is needed. For any $\varepsilon > 0$ we set

$$\begin{aligned} y &= x, & s &= \frac{t}{\varepsilon}, \\ U^\varepsilon(x,t) &= \mathcal{U}\left(x, \frac{t}{\varepsilon}\right), & V^\varepsilon(x,t) &= \frac{1}{\varepsilon}\mathcal{V}\left(x, \frac{t}{\varepsilon}\right), \end{aligned} \tag{8}$$

then we have

$$\begin{aligned} \partial_x U^\varepsilon(x,t) &= \partial_y \mathcal{U}\left(x, \frac{t}{\varepsilon}\right), & \partial_t U^\varepsilon(x,t) &= \frac{1}{\varepsilon}\partial_s \mathcal{U}\left(x, \frac{t}{\varepsilon}\right), \\ \partial_x V^\varepsilon(x,t) &= \frac{1}{\varepsilon}\partial_y \mathcal{V}\left(x, \frac{t}{\varepsilon}\right), & \partial_t V^\varepsilon(x,t) &= \frac{1}{\varepsilon^2}\partial_s \mathcal{V}\left(x, \frac{t}{\varepsilon}\right). \end{aligned}$$

With previous position the system (7) transforms into

$$\begin{cases} \partial_t U^\varepsilon(x,t) + K^{(1)}(x)\partial_x V^\varepsilon(x,t) = F^{(1)}(x,U^\varepsilon(x,t),\varepsilon V^\varepsilon(x,t)) \\ \\ \varepsilon^2 \partial_t V^\varepsilon(x,t) + H^{(2)}(x)\partial_x U^\varepsilon(x,t) + \varepsilon K^{(2)}(x)\partial_x V^\varepsilon(x,t) = \\ R(x,U^\varepsilon(x,t))V^\varepsilon(x,t) + F^{(2)}(x,U^\varepsilon(x,t),\varepsilon V^\varepsilon(x,t)) \end{cases} \tag{9}$$

If we denote by (U^0, V^0) the limit profile as $\varepsilon \downarrow 0$, formally we obtain that the system (9) relaxes to the system

$$\begin{cases} \partial_t U^0(x,t) + K^{(1)}(x)\partial_x V^0(x,t) = F^{(1)}(x, U^0(x,t), 0) \\ \\ H^{(2)}(x)\partial_x U^0(x,t) = R(x, U^0(x,t))V^0(x,t) + F^{(2)}(x, U^0(x,t), 0) \end{cases} \tag{10}$$

where U^0 satisfies formally the resulting system

$$U_t^0 + K^{(1)}(x)\left(R(x,U^0)^{-1}H^{(2)}(x)U_x^0 - R(x,U^0)^{-1}F^{(2)}(x,U^0,0)\right)_x = F^{(1)}(x,U^0,0). \tag{11}$$

In the next section we will find sufficient conditions in order to justify rigorously this formal analysis. In particular this will be done when (11) is parabolic.

3. Estimates and Convergence

3.1. A priori estimates

Let us consider the system (7)

$$\begin{cases} \partial_s \mathcal{U} + K^{(1)}(y)\partial_y \mathcal{V} = F^{(1)}(y, \mathcal{U}, \mathcal{V}) \\ \partial_s \mathcal{V} + H^{(2)}(y)\partial_y \mathcal{U} + K^{(2)}(y)\partial_y \mathcal{V} = \varepsilon^{-1} R(y, \mathcal{U})\mathcal{V} + F^{(2)}(y, \mathcal{U}, \mathcal{V}). \end{cases} \tag{12}$$

By applying the rescaling (8) the system assumes the form

$$\begin{cases} \partial_t U^\varepsilon + K^{(1)}(x)\partial_x V^\varepsilon = F^{(1)}(x, U^\varepsilon, \varepsilon V^\varepsilon) \\ \varepsilon^2 \partial_t V^\varepsilon + H^{(2)}(x)\partial_x U^\varepsilon + \varepsilon K^{(2)}(x)\partial_x V^\varepsilon = \begin{array}{l} R(x, U^\varepsilon)V^\varepsilon + \\ F^{(2)}(x, U^\varepsilon, \varepsilon V^\varepsilon). \end{array} \end{cases} \tag{13}$$

We want to show that, as $\varepsilon \downarrow 0$, the weak solutions of the rescaled system satisfy

$$\begin{aligned} U^\varepsilon &\longrightarrow U^0 && \text{a.e. in } \mathbb{R} \times \mathbb{R}_+, \\ V^\varepsilon &\rightharpoonup V^0 && \text{weakly in } L^2(\mathbb{R} \times \mathbb{R}_+), \\ \varepsilon V^\varepsilon &\longrightarrow 0 && \text{strongly in } L^2_{loc}(\mathbb{R} \times \mathbb{R}_+), \\ \varepsilon^2 V^\varepsilon_t &\longrightarrow 0 && \text{in } H^{-1}_{loc}(\mathbb{R} \times \mathbb{R}_+)\ . \end{aligned}$$

To this purpose this section is devoted to establish a priori estimates, independent of ε, for the solution of the system (9). We make on (9) the following hypotheses

(B.1) $\mathcal{U}(x,0)=\mathcal{U}_0(x)\in\left[L^2(\mathbb{R})\right]^k$, $\mathcal{V}(x,0)=\mathcal{V}_0(x)\in\left[L^2(\mathbb{R})\right]^{N-k}$,

(B.2) there exist symmetric positive definite matrices $B_0(x)\in\mathcal{M}_{k\times k}$, $D_0(x)\in\mathcal{M}_{(N-k)\times(N-k)}$ such that $(K^{(1)}(x))^T B_0(x)=D_0(x)H^{(2)}(x)$, $\forall x\in\mathbb{R}$, $|B_0(x)|\le\gamma$, $|D_0(x)|\le\gamma$, $\text{meas}\{x \mid detB_0(x)=0\}=0$,

(B.3) $R(x,U)\in C(\mathbb{R}\times\mathbb{R}^k,\mathcal{M}_{(N-k)\times(N-k)})$, $D_0(x)R(x,U)+R(x,U)^T D_0(x)$ is negative definite $\forall x\in\mathbb{R}$, namely there exists $\lambda\in\mathbb{R}$, $\lambda>0$
such that $D_0(x)R(x,U)+R(x,U)^T D_0(x)\le-\lambda I \quad \forall x\in\mathbb{R}$,

(B.4) $K^{(1)}\in C^1(\mathbb{R},\mathcal{M}_{k\times(N-k)})$, $K^{(1)}(x)$ is bounded $\forall x\in\mathbb{R}$ and $det\left[K^{(1)}(x)(K^{(1)}(x))^T\right]\ne 0$ $\forall x\in\mathbb{R}$,

(B.5) $H^{(2)}\in C^1(\mathbb{R},\mathcal{M}_{(N-k)\times k})$, $H^{(2)}(x)$ is bounded $\forall x\in\mathbb{R}$ and we set $M=\sup\limits_{x\in\mathbb{R}}\left(\left(D_0(x)H^{(2)}(x)\right)_x\right)$,

(B.6) $K^{(2)}\in C^1(\mathbb{R},\mathcal{M}_{(N-k)\times(N-k)})$, for every $x\in\mathbb{R}$, $D_0(x)K^{(2)}(x)=(K^{(2)}(x))^T D_0(x)$ and there exists $N\in\mathbb{R}$, such that $|\mu_j(x)|\le N$, $\forall j=1,...,m$, where $\mu_j(x)$ are the eigenvalues of $\left(D_0(x)K^{(2)}(x)\right)_x+\left(\left(D_0(x)K^{(2)}(x)\right)_x\right)^T$,

(B.7) $F(x,U,V)=\left(F^{(1)}(x,U,V),F^{(2)}(x,U,V)\right)\in\mathbb{R}^N$, is a α- lipschitz function of (U,V), $\alpha\in\mathbb{R}$ moreover $F(x,0,0)=0$ $\forall x\in\mathbb{R}$, $F^{(1)}(x,U,0)=0$ $\forall\, U\in\mathbb{R}^k$, $F^{(2)}(x,0,V)=0$ $\forall\, V\in\mathbb{R}^{N-k}$.

Most of the previous hypotheses can be obtained if we suppose the system strictly hyperbolic. Since $K^{(1)}\in\mathcal{M}_{k\times(N-k)}$, from elementary linear algebra we deduce condition (B.4) is violated whenever $k>N/2$. Now we can establish the following result

THEOREM 3.1. *Let us consider the solution $\{U^\varepsilon\}$,$\{V^\varepsilon\}$ of the Cauchy problem for system (9). Assume that the hypotheses (B.1), (B.2),* *(B.3), (B.4), (B.5), (B.6), (B.7) hold. Then for ε small enough, one has*

(i) *there exist $\overline{M} \in \mathbb{R}$, $\overline{M} > 0$, independent from ε, such that $\|V^\varepsilon\| \leq \overline{M}$ and $\sup_{t\geq 0} \varepsilon \|V^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R})} \leq \overline{M}$*

(ii) *$\{\varepsilon^2 V_t^\varepsilon\}$ relatively compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}_+)$*

(iii) *$\{U^\varepsilon\}$ is uniformely bounded, with respect to ε, in $L^\infty\big(\mathbb{R}_+, L^2(\mathbb{R})\big)$, namely there exists $\overline{M} \in \mathbb{R}$, $\overline{M} > 0$, independent from ε, such that $\sup_{t\geq 0} \|U^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R})} \leq \overline{M}$.*

Proof. Multiplying the first equation of (9) by $B_0(x)U^\varepsilon$, we obtain,

$$(U_t^\varepsilon, B_0(x)U^\varepsilon) + (K^{(1)}(x)V_x^\varepsilon, B_0(x)U^\varepsilon) = (F^{(1)}(x, U^\varepsilon, \varepsilon V^\varepsilon), B_0(x)U^\varepsilon).$$

Multiplying the second equation by $D_0(x)V^\varepsilon$, we have

$$\begin{aligned}(\varepsilon^2 V_t^\varepsilon, D_0(x)V^\varepsilon) + (H^{(2)}(x)U_x^\varepsilon, D_0(x)V^\varepsilon) + \varepsilon(K^{(2)}(x)V_x^\varepsilon, D_0(x)V^\varepsilon) = \\ (R(x,U^\varepsilon)V^\varepsilon, D_0(x)V^\varepsilon) + (F^{(2)}(x,U^\varepsilon,\varepsilon V^\varepsilon), D_0(x)V^\varepsilon),\end{aligned}$$

if we sum the two relations, using hypotheses (B.2) and (B.6) we get the following energy identity

$$\begin{aligned}
&\partial_t \left\{ \frac{\varepsilon^2}{2} \left| D_0^{1/2}(x)V^\varepsilon \right|^2 + \frac{1}{2}\left| B_0^{1/2}(x)U^\varepsilon \right|^2 \right\} + \\
&\partial_x \left\{ (H^{(2)}(x)U^\varepsilon, D_0(x)V^\varepsilon) + \frac{1}{2}(\varepsilon K^{(2)}(x)V^\varepsilon, D_0(x)V^\varepsilon) \right\} \\
= \;& (D_0(x)R(x,U^\varepsilon)V^\varepsilon, V^\varepsilon) + \frac{1}{2}\left(\varepsilon \left(D_0(x)K^{(2)}(x)\right)_x V^\varepsilon, V^\varepsilon\right) + \\
&\left(V^\varepsilon, \left(D_0(x)H^{(2)}(x)\right)_x U^\varepsilon\right) + (F^{(2)}(x,U^\varepsilon,\varepsilon V^\varepsilon), D_0(x)V^\varepsilon) + \\
&(F^{(1)}(x,U^\varepsilon,\varepsilon V^\varepsilon), B_0(x)U^\varepsilon).
\end{aligned}$$

Taking into account the hypotheses (B.2), (B.3), (B.5), (B.6) and (B.7) we have

$$
\begin{aligned}
&\partial_t \left\{ \frac{\varepsilon^2}{2} \left| D_0^{1/2}(x) V^\varepsilon \right|^2 + \frac{1}{2} \left| B_0^{1/2}(x) U^\varepsilon \right|^2 \right\} + \\
&\partial_x \left\{ (H^{(2)}(x) U^\varepsilon, D_0(x) V^\varepsilon) + \frac{1}{2} (\varepsilon K^{(2)}(x) V^\varepsilon, D_0(x) V^\varepsilon) \right\} \\
&\qquad \le \left(-\lambda + \varepsilon \left(\frac{N}{2} + \alpha\gamma \right) \right) |V^\varepsilon|^2 + M\,(U^\varepsilon, V^\varepsilon) + \alpha\gamma |U^\varepsilon|^2.
\end{aligned}
$$

We can choose $\varepsilon < \dfrac{\lambda}{N + 2\alpha\gamma}$ and for all $\delta > 0$ it follows that

$$
\begin{aligned}
&\partial_t \left\{ \frac{\varepsilon^2}{2} \left| D_0^{1/2}(x) V^\varepsilon \right|^2 + \frac{1}{2} \left| B_0^{1/2}(x) U^\varepsilon \right|^2 \right\} \\
&+\partial_x \left\{ (H^{(2)}(x) U^\varepsilon, D_0(x) V^\varepsilon) + \frac{1}{2} (\varepsilon K^{(2)}(x) V^\varepsilon, D_0(x) V^\varepsilon) \right\} \\
&\le -\frac{\lambda}{2} |V^\varepsilon|^2 + \frac{M + \gamma}{\delta} |U^\varepsilon|^2 + \delta |V^\varepsilon|^2
\end{aligned} \tag{14}
$$

Now we set

$$
E(t) = \int_{-\infty}^{+\infty} \frac{\varepsilon^2}{2} \left| D_0^{1/2}(x) V^\varepsilon \right|^2 dx + \int_{-\infty}^{+\infty} \frac{1}{2} \left| B_0^{1/2}(x) U^\varepsilon \right|^2 dx
$$

Integrating (14) on $[0, t] \times \mathbb{R}$ we obtain the energy $E(t)$ satisfies for all $\delta > 0$

$$
\begin{aligned}
E(t) \le{}& E(0) - \frac{\lambda}{2} \int_0^t \int_{-\infty}^{+\infty} |V^\varepsilon|^2 dx ds + \delta \int_0^t \int_{-\infty}^{+\infty} |V^\varepsilon|^2 dx ds \\
&+ \frac{M + \alpha\gamma}{\delta} \int_0^t \int_{-\infty}^{+\infty} \left| B_0^{-1/2}(x) B_0^{1/2}(x) U^\varepsilon \right|^2 dx ds.
\end{aligned}
$$

Choosing $\delta < \dfrac{\lambda}{6}$ and using (B.2) we get

$$
E(t) \le E(0) - \frac{\lambda}{3} \int_0^t \int_{-\infty}^{+\infty} |V^\varepsilon|^2 dx ds + c \int_0^t E(s) ds
$$

with c constant, $c>0$. Applying Gronwall's lemma we obtain

$$E(t) \leq E(0)e^{ct} \tag{15}$$

and the following estimate

$$\int_0^t \int_{-\infty}^{+\infty} |V^\varepsilon|^2 dx ds \leq cE(0)\left(e^{ct}+1\right) \tag{16}$$

By using (B.2) and (16) we can conclude that there exists $\overline{M}>0$, indipendent from ε, such that

$$\|V^\varepsilon\| \leq \overline{M}, \quad \sup_{t\geq 0} \varepsilon \|V^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R})} \leq \overline{M} \quad \sup_{t\geq 0} \|U^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R})} \leq \overline{M}.$$

In this way we proved (i) and (iii). Let us consider ω relatively compact in $\mathbb{R}\times\mathbb{R}_+$, then

$$\begin{aligned} \|\varepsilon^2 V_t^\varepsilon\|_{H^{-1}(\omega)} &= \sup_{\|\phi\|_{H_0^1(\omega)}=1} |\langle \varepsilon^2 V_t^\varepsilon, \phi\rangle| = \sup_{\|\phi\|_{H_0^1(\omega)}=1} \left| \int\!\!\int \varepsilon^2 V^\varepsilon \phi_t dx dt \right| \\ &\leq \varepsilon^2 \sup_{\|\phi\|_{H_0^1(\omega)}=1} (\|V^\varepsilon\| \|\phi_t\|) \leq \overline{M}\varepsilon^2. \end{aligned}$$

□

3.2. Strong convergence

In this section we study the limiting behavior as $\varepsilon \downarrow 0$ of the solutions of (9). We begin with a simple consequence of (i) and (ii) of Theorem (3.1).

THEOREM 3.2. *Let us consider the solution $\{V^\varepsilon\}$ of the Cauchy problem for system (9). Assume the hypotheses (B.1), (B.2), (B.3), (B.4), (B.5), (B.6), (B.7) hold. Then there exists $V^0 \in [L^2(\mathbb{R}\times\mathbb{R}_+)]^{N-k}$, such that, as $\varepsilon \downarrow 0$, one has (extracting eventually subsequences)*

$$V^\varepsilon \rightharpoonup V^0 \qquad \textit{weakly in } L^2(\mathbb{R}\times\mathbb{R}_+) \tag{17}$$

$$\varepsilon V^\varepsilon \longrightarrow 0 \qquad \textit{strongly in } L^2_{loc}(\mathbb{R}\times\mathbb{R}_+) \tag{18}$$

$$\{\varepsilon^2 V_t^\varepsilon\} \longrightarrow 0 \qquad \textit{in } H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}_+). \tag{19}$$

Our next step is to prove convergence for the sequence $\{U^\varepsilon\}$. To this end we apply Gérard's compensated compactness theorem (2.1). We make now different assumptions on $R(x,U)$ which at the end will lead to similar relaxation results. The first result is devoted to obtain the relaxation limit when the sequences $\{U^\varepsilon\}$, $\{V^\varepsilon\}$ satisfy only the estimates given in the Theorem (3.1).

THEOREM 3.3. *Let us consider the solution* $\{U^\varepsilon\}$,$\{V^\varepsilon\}$ *of the Cauchy problem for system (9). Assume the hypotheses (B.1), (B.2), (B.3), (B.4), (B.5), (B.6), (B.7) and moreover*

(C.1) $R(x,U)$ *is bounded on* U.

Then there exists $U^0 \in [L^2(\mathbb{R}\times\mathbb{R}_+)]^k$*, such that, as* $\varepsilon \downarrow 0$*, one has (extracting eventually subsequences)*

$$U^\varepsilon \longrightarrow U^0 \qquad a.e \ in\ \mathbb{R}\times\mathbb{R}_+ \tag{20}$$

$$R(x,U^\varepsilon) \longrightarrow R(x,U^0) \qquad strongly\ in\ L^p_{loc}(\mathbb{R}\times\mathbb{R}_+) \tag{21}$$

$$F^{(1)}(x,U^\varepsilon,\varepsilon V^\varepsilon) \longrightarrow F^{(1)}(x,U^0,0) \qquad strongly\ in\ L^2_{loc}(\mathbb{R}\times\mathbb{R}_+) \tag{22}$$

$$F^{(2)}(x,U^\varepsilon,\varepsilon V^\varepsilon) \longrightarrow F^{(2)}(x,U^0,0) \qquad strongly\ in\ L^2_{loc}(\mathbb{R}\times\mathbb{R}_+)\ . \tag{23}$$

Proof. By using the hypothesis (C.1), $R(x,U^\varepsilon)$ is uniformely bounded in L^∞ then $R(x,U^\varepsilon)V^\varepsilon$ is uniformely bounded in L^2, therefore $R(x,U^\varepsilon)V^\varepsilon$ is relatively compact in H^{-1}_{loc}. Combining the hypothesis (B.3), (i) and (iii) of Theorem (3.1) we get also $F^{(1)}(x,U^\varepsilon,\varepsilon V^\varepsilon)$ and $F^{(2)}(x,U^\varepsilon,\varepsilon V^\varepsilon)$ are uniformely bounded in L^2 and so they are relatively compact in H^{-1}_{loc}. The distribution $\varepsilon(K^{(2)}(x))V^\varepsilon_x$ is relatively compact in H^{-1}_{loc}, indeed, for any $\phi \in H^1_0$ one has

$$\begin{aligned}\left|\langle \varepsilon(K^{(2)}(x))V^\varepsilon_x,\phi\rangle\right| &= \varepsilon\left|\iint V^\varepsilon\left((K^{(2)}(x))^T\phi\right)_x dxdt\right| \\ &\le \varepsilon\|V^\varepsilon\|\|(K^{(2)}(x))^T\phi\|_{H^1_0}.\end{aligned} \tag{24}$$

Hence from the second equation of (9) we obtain that

$H^{(2)}(x)U^\varepsilon_x = R(x, U^\varepsilon)V^\varepsilon + F^{(2)}(x, U^\varepsilon, \varepsilon V^\varepsilon) - \varepsilon K^{(2)}(x)V^\varepsilon_x - \varepsilon^2 V^\varepsilon_t$
is relatively compact in H^{-1}_{loc}.

Moreover

$U^\varepsilon_t + K^{(1)}(x)V^\varepsilon_x = F^{(1)}(x, U^\varepsilon, \varepsilon V^\varepsilon)$ is relatively compact in H^{-1}_{loc}.

In order to fit into the framework of Theorem (2.1) we set

$$P \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} I_{k\times k} & 0 \\ 0 & 0 \end{bmatrix} \partial_t \begin{bmatrix} U \\ V \end{bmatrix} + \begin{bmatrix} 0 & K^{(1)}(x) \\ H^{(2)}(x) & 0 \end{bmatrix} \partial_x \begin{bmatrix} U \\ V \end{bmatrix}$$

the principal symbol of P is given by

$$p(x,\xi) = \begin{bmatrix} I_{k\times k} & 0 \\ 0 & 0 \end{bmatrix} \xi_0 + \begin{bmatrix} 0 & K^{(1)}(x) \\ H^{(2)}(x) & 0 \end{bmatrix} \xi_1,$$

for $\xi = (\xi_0, \xi_1) \in \mathbb{R}^2$, $|\xi| = 1$. We notice that

$$p(x,\xi) \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = 0 \Longleftrightarrow \begin{cases} \xi_0\lambda + \xi_1 K^{(1)}(x)\mu = 0 \\ \xi_1 H^{(2)}(x)\lambda = 0 \end{cases}$$

for all $\lambda \in \mathbb{R}^k$, $\mu \in \mathbb{R}^{N-k}$.
Now if $\xi_1 = 0$ then $\xi_0 \neq 0$ and so $\lambda = 0$, otherwise if $\xi_1 \neq 0$ then $H^{(2)}(x)\lambda = 0$ and also $(K^{(1)}(x))^T B_0(x)\lambda = D_0(x)H^2(x)\lambda = 0$, this entails $K^{(1)}(x)(K^{(1)}(x))^T B_0(x)\lambda = 0$. Using hypotheses (B.4) we get $p(x,\xi)\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = 0$ implies $B_0(x)\lambda = 0$. We take now

$$q(x) = \begin{bmatrix} B_0(x) & 0 \\ 0 & 0 \end{bmatrix}$$

and for all $\xi \neq 0$, $\xi = (\xi_0, \xi_1)$ we have

$$p(x,\xi)\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = 0 \qquad \text{implies} \qquad \langle q(x)\begin{bmatrix} \lambda \\ \mu \end{bmatrix}, \begin{bmatrix} \lambda \\ \mu \end{bmatrix}\rangle = 0$$

for all $\lambda \in \mathbb{R}^k$, $\mu \in \mathbb{R}^{N-k}$.
Now we can apply Theorem (2.1) of Gérard and we conclude that for any $\varphi \in \mathcal{D}(\Omega)$

$$\iint \left|B_0^{1/2}(x)U^\varepsilon\right|^2 \varphi(x,t)dxdt \longrightarrow \iint \left|B_0^{1/2}(x)U^0\right|^2 \varphi(x,t)dxdt$$

where U^0 denotes, in view of Theorem (3.1) the weak limit of U^ε in $L^2(\mathbb{R}\times\mathbb{R}_+)$. Using the energy estimate we get also

$$\iint \left|B_0^{1/2}(x)(U^\varepsilon - U^0)\right|^2 \varphi(x,t)dxdt \longrightarrow 0$$

and applying hypotheses (B.2) we have

$$U^\varepsilon \longrightarrow U^0 \qquad \text{a.e. in } (x,t) \tag{25}$$

Now we prove (21). Since $R(x,U)$ is continous in U, then by using (25),

$$R(x,U^\varepsilon) \longrightarrow R(x,U^0) \qquad \text{a.e. in } (x,t)$$

so it follows that

$$|R(x,U^\varepsilon) - R(x,U^0)|^p \longrightarrow 0 \qquad \text{a.e. in } (x,t)$$

and thanks to (C.1), $|R(x,U^\varepsilon) - R(x,U^0)|^p$ is bounded (then locally integrable). By applying the Lebesgue dominated convergence theorem we conclude

$$R(x,U^\varepsilon) \longrightarrow R(x,U^0) \qquad \text{strongly in } L^p_{loc}(\mathbb{R}\times\mathbb{R}_+).$$

Finally we prove (22). Let ω be a compact subset of $\mathbb{R}\times\mathbb{R}_+$, then using (B.7) we have for i=1,2

$$\left(\iint_\omega \left|F^{(i)}(x,U^\varepsilon,\varepsilon V^\varepsilon) - F^{(i)}(x,U^0,0)\right|^2 dxdt\right)^{1/2} \leq \alpha\left(\iint_\omega \left|U^\varepsilon - U^0\right|^2 dxdt + \iint_\omega |\varepsilon V^\varepsilon|^2 dxdt\right)^{1/2}.$$

Combining (iii) of Theorem (3.1) and (18), (20) we conclude that

$$F^i(x,U^\varepsilon,\varepsilon V^\varepsilon) \longrightarrow F^i(x,U^0,0) \quad i=1,2 \quad \text{strongly in } L^2_{loc}(\mathbb{R}\times\mathbb{R}_+).$$

□

In the previous theorem we restricted ourselves to consider $R(x,U)$ bounded but we would like to extend our result to a larger class of function. To this goal we replace (C.1) with the following different assumption

(C.2) $\{U^\varepsilon\}$ is uniformly bounded in $L^\infty(\mathbb{R}\times\mathbb{R}_+)$,

(C.3) there exists $c\in\mathbb{R}$, $c>0$ such that for any $U_1,U_2\in\mathbb{R}^k$, $|R(x,U_1)-R(x,U_2)|\leq c\left(1+|U_1|^{p-1}+|U_2|^{p-1}\right)|U_1-U_2|$, for some $p\geq 1$.

In the next theorem in order to have a broader set of R we need the additional assumption (C.2), which does not come from the standard energy estimates.

THEOREM 3.4. *Let us consider the solution $\{U^\varepsilon\}$, $\{V^\varepsilon\}$ of the Cauchy problem for system (9). Assume that the hypotheses (B.1), (B.2), (B.3), (B.4), (B.5), (B.6), (B.7),(C.2), (C.3) hold. Then there exists $U^0\in[L^2(\mathbb{R}\times\mathbb{R}_+)]^k$, such that, as $\varepsilon\downarrow 0$, one has (extracting eventually subsequences)*

$$U^\varepsilon\longrightarrow U^0 \qquad a.e \text{ in } \mathbb{R}\times\mathbb{R}_+ \tag{26}$$

$$R(x,U^\varepsilon)\longrightarrow R(x,U^0) \qquad \text{strongly in } L^p_{loc}(\mathbb{R}\times\mathbb{R}_+) \tag{27}$$

$$F^{(1)}(x,U^\varepsilon,\varepsilon V^\varepsilon)\longrightarrow F^{(1)}(x,U^0,0) \qquad \text{strongly in } L^2_{loc}(\mathbb{R}\times\mathbb{R}_+) \tag{28}$$

$$F^{(2)}(x,U^\varepsilon,\varepsilon V^\varepsilon)\longrightarrow F^{(2)}(x,U^0,0) \qquad \text{strongly in } L^2_{loc}(\mathbb{R}\times\mathbb{R}_+)\ . \tag{29}$$

Proof. Since $R(x,U)$ is continous in U and by using the (C.2) we have $R(x,U^\varepsilon)V^\varepsilon$ is relatively compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}_+)$ for the same reason of the previous theorem. With the same technique used in Theorem (3.3) we can prove (26), (28), (29) . The proof of (27) is given by using the hypothesis (C.3). Indeed because of the growth conditions given therein, for any ω compact subset of $\mathbb{R}\times\mathbb{R}_+$, there exists $c_0>0$, c_0 depending on ω and $\sup\limits_{\varepsilon>0}\|U^\varepsilon\|_{L^\infty(\mathbb{R}\times\mathbb{R}_+)}$ (which is finite because of (C.2)) such that

$$\iint_\omega \left|R(x,U^\varepsilon) - R(x,U^0)\right| dxdt \le c_0 \left(\iint_\omega \left|U^\varepsilon - U^0\right|^2 dxdt\right)^{1/2}$$

Taking into account (26) and (C.2) we conclude that

$$R(x,U^\varepsilon) \longrightarrow R(x,U^0) \quad \text{strongly in } L^1_{loc}(\mathbb{R}\times\mathbb{R}_+)$$

and finally

$$R(x,U^\varepsilon) \longrightarrow R(x,U^0) \quad \text{strongly in } L^p_{loc}(\mathbb{R}\times\mathbb{R}_+) \text{ for all } p\in[1,\infty).$$

□

Corollary 3.5. *Assume that the hypotheses of Theorems (3.2), (3.3) or (3.4) hold, then* (U^0,V^0) *verifies, in the sense of distributions, the following system*

$$\begin{cases} \partial_t U^0(x,t) + K^{(1)}(x)\partial_x V^0(x,t) = F^{(1)}(x,U^0(x,t),0) \\ H^{(2)}(x)\partial_x U^0(x,t) = R(x,U^0(x,t))V^0(x,t) + F^{(2)}(x,U^0(x,t),0). \end{cases} \tag{30}$$

Proof. From (21) or (27) it follows that

$$R(x,U^\varepsilon)V^\varepsilon \rightharpoonup R(x,U^0)V^0 \quad \text{weakly in } L^2(\mathbb{R}\times\mathbb{R}_+) \tag{31}$$

Now let $\phi\in\mathcal{D}(\mathbb{R}\times\mathbb{R}_+)$, be a test function, using (24) we obtain

$$\langle \varepsilon K^{(2)}(x)V^\varepsilon_x, \phi\rangle \longrightarrow 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}\times\mathbb{R}_+). \tag{32}$$

Taking into account (17) and (20) or (26) we obtain that

$$\begin{aligned} &K^{(1)}(x)V^\varepsilon_x \longrightarrow K^{(1)}(x)V^0_x, \\ &H^{(2)}(x)U^\varepsilon_x \longrightarrow H^{(2)}(x)U^0_x \qquad \text{in } \mathcal{D}'(\mathbb{R}\times\mathbb{R}_+). \end{aligned} \tag{33}$$

Then by using (20) or (26), (17), (31), (32), (33), (22) or (28), (23) or (29) and passing to the limit in (9) we conclude that (U^0,V^0) verifies (30). □

An alternative formulation of the system (30) can be given in the following way

COROLLARY 3.6. *In the sense of distribution, U^0 satisfies the following second order equation*

$$U_t^0 + K^{(1)}(x)\left(R(x,U^0)^{-1}H^{(2)}(x)U_x^0 - R(x,U^0)^{-1}F^{(2)}(x,U^0,0)\right)_x = F^{(1)}(x,U^0,0). \tag{34}$$

Proof. By the second equation of (30)

$$V^0 = R(x,U^0)^{-1}H^{(2)}(x)U_x^0 - R(x,U^0)^{-1}F^{(2)}(x,U^0,0).$$

Let $\phi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}_+)$, be a test function, then

$$\iint \phi(x,t)_t U^0(x,t)dxdt + \iint \left(\phi(x,t)K^{(1)}(x)\right)_x V^0(x,t)dxdt = \int \phi(x,t)F^{(1)}(x,U^0(x,t),0)dxdt.$$

Because $V^0 \in L^2(\mathbb{R}\times\mathbb{R}_+)$ and $R(x,U^0) \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$, $R(x,U^0)V^0 \in L^2(\mathbb{R}\times\mathbb{R}_+)$, we have that $H^{(2)}(x)U_x^0 \in L^2(\mathbb{R}\times\mathbb{R}_+)$ and so

$$\begin{aligned}&\iint \phi(x,t)_t U^0 dxdt + \\ &\quad \iint \left((K^{(1)}(x))^T\phi(x,t)\right)_x \left(R(x,U^0)^{-1}H^{(2)}(x)U_x^0 dxdt\right) \\ &\quad - \iint \left((K^{(1)}(x))^T\phi(x,t)\right)_x \left(R(x,U^0)^{-1}F^{(1)}(x,U^0,0)\right) dxdt \\ &\qquad = \int \phi(x,t)F^{(1)}(x,U^0,0)dxdt.\end{aligned}$$

□

3.3. Parabolicity

We have proved that U^0 satisfies this second order equation

$$U_t^0 + K^{(1)}(x)\left(R(x,U^0)^{-1}H^{(2)}(x)U_x^0 - R(x,U^0)^{-1}F^2(x,U^0,0)\right)_x = F^1(x,U^0,0),$$

using (B.2) this equation is equivalent to the the following

$$\begin{aligned} & U_t^0 + \left(K^{(1)}(x)R(x,U^0)^{-1}D_0^{-1}(x)(K^{(1)}(x))^T B_0(x)U_x^0\right)_x \\ & = (K^{(1)}(x))_x R(x,U^0)^{-1}H^{(2)}(x)U_x^0 \\ & + K^{(1)}(x)\left(R(x,U^0)^{-1}F^{(2)}(x,U^0,0)\right)_x + F^{(1)}(x,U^0,0). \end{aligned} \tag{35}$$

We want to prove that (35) is parabolic in the sense of Section 2.1. If we denote

$$C = K^{(1)}(x)R(x,U^0)^{-1}D_0^{-1}(x)(K^{(1)}(x))^T(B_0(x)$$

we are going to prove that C is a negative definite matrix. Since the notion of parabolicity is independent of similar transformation we prove there exist a matrix T such that $T^{-1}(C+C^T)T$ is negative definite. Taking $T = B_0^{-1/2}(x)$ we get

$$T^{-1}CT = B_0^{1/2}(x)K^{(1)}(x)R(x,U^0)^{-1}D_0^{-1}(x)(K^{(1)}(x))^T B_0^{1/2}(x)$$

and using (B.3) we have

$$\begin{aligned} & \left(\left(T^{-1}CT\right)\xi,\xi\right) = \\ & \left(\left(R(x,U^0)^{-1}D_0^{-1}(x)\right)(K^{(1)}(x))^T B_0^{1/2}(x)\xi,(K^{(1)}(x))^T B_0^{1/2}(x)\xi\right) \\ & \quad \le -\frac{1}{\lambda}\left|(K^{(1)}(x))^T B_0^{1/2}(x)\xi\right|^2. \end{aligned}$$

If $det\left[K^{(1)}(x)(K^{(1)}(x))^T\right] \ge c_1 > 0$ and remembering that $B_0(x)$ is positive definite it follows

$$\left(\left(T^{-1}CT\right)\xi,\xi\right) \le -\eta|\xi|^2,$$

$\eta \in \mathbb{R}$, $\eta > 0$.

In the same way, taking $T = B_0^{1/2}(x)$ we get $T^{-1}C^T T$ negative definite and so the parabolicity in the sense of Section 2.1.

4. Approximation of Parabolic Systems

In this section we follow a path which is somehow opposite to that one we followed in the previous part of the paper. We want to show here how, given a nonlinear parabolic system, we can construct a suitable larger semilinear hyperbolic system which relaxes on it. The advantage of such an approach is that we can apply this scheme to construct numerical approximation of a parabolic system. Consider the following system of k equations

$$U_t = (M(x,U)U_x)_x + (G(x,U))_x + H(x,U) \tag{36}$$

We make the following hypotheses:

(D.1) $M(x,U) \in C(\mathbb{R} \times \mathbb{R}^k, \mathcal{M}_{k\times k})$, M is an invertible matrix for any (x,U) and $M^{-1}(x,U)$ is bounded on U,

(D.2) there exists a symmetric positive definite matrix $B_0(x) \in \mathcal{M}_{k\times k}$, such that $M(x,U)^T B_0(x) + B_0(x)M(x,U)$ is positive definite $\forall x \in \mathbb{R}$ (parabolicity) moreover $|B_0(x)| \leq \gamma$ and meas$\{x|\ detB_0(x)\} = 0$,

(D.3) $G(x,U), H(x,U) \in \mathbb{R}^k$ are lipschitz function of U and $G(x,0) = H(x,0) = 0$, $\forall x \in \mathbb{R}$,

(D.4) $k = \frac{N}{2}$.

THEOREM 4.1. *Let us consider the system (36), suppose that hypotheses (D.1), (D.2), (D.3), (D.4) hold, then the solution of the system*

$$\begin{cases} \mathcal{U}_s + \mathcal{V}_y = H(y,\mathcal{U}) \\ \mathcal{V}_s + B_0(y)\mathcal{U}_y = -\dfrac{1}{\varepsilon}B_0(y)M(y,\mathcal{U})^{-1}\mathcal{V} + B_0(y)M(y,\mathcal{U})^{-1}G(y,\mathcal{U}) \end{cases} \tag{37}$$

where $(y,s) \in \mathbb{R} \times \mathbb{R}_+$, $\mathcal{U} = \mathcal{U}(y,s) \in \mathbb{R}^k$, $\mathcal{V} = \mathcal{V}(y,s) \in \mathbb{R}^{N-k}$, approximate the system (36) in the sense of the Theorem (3.3).

Proof. Rescaling the variables as in (8) system (37) transforms into

$$\begin{cases} U_t^\varepsilon + V_x^\varepsilon = H(x,U^\varepsilon) \\ \\ \varepsilon^2 V_t^\varepsilon + B_0(x)U_x^\varepsilon = -B_0(x)M(x,U^\varepsilon)^{-1}V^\varepsilon \\ \qquad\qquad\qquad\qquad + B_0(x)M(x,U^\varepsilon)^{-1}G(x,U^\varepsilon) \end{cases} \tag{38}$$

Let us denote by $K^1 = I_{k\times k}$, $D_0 = I_{k\times k}$, $H^{(2)}(x) = B_0(x)$, by hypothesis (D.2) condition (B.2) is satisfied. Now we set $R(x,U) = -B_0(x)M(x,U)^{-1}$ and we show that the condition (B.3) given in Section 3.1 is satisfied. Indeed since $D_0 = I_{k\times k}$,

$$R(U) + (R(U))^T = -\left(B_0(x)M(x,U)^{-1} + (M(x,U)^{-1})^T B_0(x)\right)$$

which is negative definite in view of condition (D.2), moreover, using (D.1), $R(x,U)$ is continous and bounded. Setting $F^{(1)}(x,U,V) = H(x,U)$ and $F^{(2)}(x,U,V) = -B_0(x)M(x,U)^{-1}G(x,U)$ and using (D.1), (D.3) we get (B.7). Finally conditions (B.4), (B.5), (B.7) follow easily from the previous positions. We can now apply Theorem (3.3) and Corollaries (3.5), (3.6) and we obtain that the solutions to (38) satisfy as $\varepsilon \downarrow 0$ the parabolic equation

$$\begin{aligned} U^0_t + (R(x,U^0)^{-1}H^{(2)}(x)U^0_x)_x &= (R(x,U^0)^{-1}F^{(2)}(x,U^0,0))_x \\ &\quad + F^{(1)}(x,U^0,0). \end{aligned}$$

Since

$$R(x,U^0)^{-1}H^{(2)}(x) = -M(x,U^0)B_0(x)^{-1}B_0(x) = -M(x,U^0)$$

and

$$\begin{aligned} R(x,U^0)^{-1}F^{(2)}(x,U,0) =& M(x,U^0)B_0(x)^{-1}B_0(x)M(x,U)^{-1}G(x,U) \\ =& G(x,U) \end{aligned}$$

the limit system coincide with our system. □

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Received February 14, 2000.