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On Second Order Weakly Hyperbolic Equations and the Gevrey Classes

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Summary. - *We study the Cauchy problem for a second order weakly hyperbolic operator with coefficients depending only on time. We consider the case of coefficients of the principal part belonging to an intermediate class between C^∞ and the real analytic class and we specify the function spaces in which the Cauchy problem is well posed. Moreover we show by a counter example that this results are in some sense optimal.*

1. Introduction

In this note we are concerned with the following Cauchy problem

$$\begin{cases} Pu = \partial_t^2 u - \sum_{i,j=1}^n a_{ij}(t)\partial_{x_i}\partial_{x_j} u + b(t)u = 0 \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) \end{cases} \tag{1}$$

where we assume

$$\sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j \geq 0, \quad \forall t \in [0,T], \quad \forall \xi \in \mathbf{R}^n.$$

As for the Cauchy problem (1), if $a_{ij}(t) \in C^\omega([0,T])$ then (1) is C^∞ well posed for any $b(t) \in C^0([0,T])$ and if $a_{ij}(t) \in C^k([0,T])$ then (1) is $\gamma^{(1+k/2)}$ well posed for any $b(t) \in C^0([0,T])$ (see [2]), where $\gamma^{(s)}$ stands for the Gevrey class of order s. On the other hand there

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is a $a(t) \in C^\infty([0,T])$ which is positive apart from $t = 0$ such that the Cauchy problem (1) for

$$P = \partial_t^2 - a(t)\partial_x^2 \quad \text{in} \quad \mathbf{R}^2$$

is not C^∞ well posed ([4]). Thus the general picture would be stated as: the smoother coefficients the wider class of well posedness. Our main concern is to study this picture when the coefficients belong to an intermediate class between C^∞ and the real analytic class and to specify function spaces in which the Cauchy problem is well posed.

To study this question we first introduce some function spaces between C^∞ and the real analytic class. Let $M(x) \in C^1([0,\infty))$ such that $M(x) \geq 1$ and

$$M(x)^{1/x} \geq cx \tag{2}$$

with some $c > 0$.

Definition 1.1. *We say that $a(t) \in \Gamma(M)([0,T])$, if we have*

$$|a^{(n)}(t)| \leq CA^n M(n), \quad n = 0, 1, 2, ..., \quad t \in [0,T]$$

with some $C > 0$ and $A \geq 1$.

If we take $M(n) = n^{sn}$, $s > 1$ then $\Gamma(M)([0,T])$ coincides with the usual Gevrey class $\gamma^{(s)}([0,T])$. From (2) it is easy to see that for any closed interval $I \subset (0,\infty)$ there are $c > 0$ and N such that

$$nM(n)^{1/n}\delta^{-1/n} \geq cn, \quad \delta \in I, \quad n \geq N. \tag{3}$$

Then the minimum of the set $\{nM(n)^{1/n}\delta^{-1/n} \mid n = 1, 2, ...\}$ is attained. Let us set

$$\phi(M)(\delta) = \min_{n=1,2,...} \{nM(n)^{1/n}\delta^{-1/n}\}. \tag{4}$$

Then we see that $\phi(M)(\delta)$ is continuous in $\delta > 0$. From (2) again we have

$$nM(n)^{1/n}\delta^{-1/n} \geq c(\log \delta)^2 \tag{5}$$

with some $c > 0$ for any $n = 1, 2, ...$ and hence $\phi(\delta) \uparrow \infty$ if $\delta \downarrow 0$. Then we define $\Phi(\xi)$ by

$$\Phi(\xi) = \min_{\delta>0} \max\{\phi(M)(\delta), \sqrt{\delta}|\xi|\}. \tag{6}$$

Since $\phi(M)(\delta)$ is strictly decreasing there is a unique $\delta = \delta(\xi) > 0$ so that $\Phi(\xi) = \phi(M)(\delta(\xi)) = \sqrt{\delta(\xi)}|\xi|$. It is clear that $\delta(\xi) \downarrow 0$ as $|\xi| \to \infty$ and $\Phi(\xi) \geq 1$ for large $|\xi|$.

DEFINITION 1.2. *Let $\Phi(\xi)$ be a non negative function on $\mathbf{R}^n$. Then we say that $u(x) \in \mathcal{S}'(\mathbf{R}^n)$, a tempered distribution, belongs to $\hat{\Gamma}(\Phi)$ if for any $C > 0$ there is $C_1 > 0$ such that*

$$|\hat{u}(\xi)| \leq C_1 e^{-C\Phi(\xi)}$$

for large ξ where $\hat{u}(\xi)$ stands for the Fourier transform of $u(x)$.

REMARK 1.3. *Let $\Phi(\xi)$ and $\Phi_A(\xi)$ be given by (6) with $M(n)$ and $\tilde{M}(n) = A^n M(n)$ $(A \geq 1)$ respectively. Then it is easy to see that*

$$\Phi(\xi) \leq \Phi_A(\xi) \leq A\Phi(\xi)$$

and this shows that the class $\hat{\Gamma}(\Phi)$ is well defined by the class $\Gamma(M)$. It is also easy to check that

$$C\Phi(\xi) \geq (\log|\xi|)^2$$

with some $C > 0$. Hence $u \in C^\infty(\mathbf{R}^n)$ if $u \in \mathcal{S}' \cap \hat{\Gamma}(\Phi)$.

In this note we prove

THEOREM 1.4. *Assume that $a_{ij}(t) \in \Gamma(M)([0,T])$ and let $\Phi(\xi)$ be defined in (6). Then the Cauchy problem (1) has a unique solution $u \in C^2([0,T]; \hat{\Gamma}(\Phi))$ for any $u_i(x)$ with $u_i(x) \in \hat{\Gamma}(\Phi) \cap \mathcal{E}'(\mathbf{R}^n)$, $i = 0, 1$.*

On the other hand one can not improve this result much more. In fact we show

THEOREM 1.5. *Let $M(n)$ verify (2) and let $\Phi(\xi)$ be defined by (6). Then there exists a function $a(t) \in \Gamma(M(n)n^{2n}(\log(n+2))^{2n})([0,T])$ such that the Cauchy problem (1.1) is not well posed in $\hat{\Gamma}(\Phi/(\log\Phi)^2)$. More precisely there exist $u_i \in \hat{\Gamma}(\Phi/(\log\Phi)^2)$, $i=0,1$ for which the Cauchy problem (1) has no solution u in $C^2([0,T],\mathcal{D}')$.*

If we take $M(n) = n^{sn}$ we get

COROLLARY 1.6. *Assume that $a_{ij}(t) \in \gamma^{(s)}([0,T])$. Then the Cauchy problem (1) has a unique solution $u \in C^2([0,T];\hat{\Gamma}((\log|\xi|)^{s+1}))$ for any $u_0(x)$, $u_1(x) \in \hat{\Gamma}((\log|\xi|)^{s+1}) \cap \mathcal{E}'(\mathbf{R}^n)$. Conversely for $s > 2$ there exists a function $a(t) \in \cap_{r>s}\gamma^{(r)}([0,T])$ such that the Cauchy problem (1) is not well posed in $\hat{\Gamma}((\log|\xi|)^{s-1}/(\log\log|\xi|)^2)$.*

Proof. Let $M(n) = n^{sn}$ and take

$$\tilde{\delta}(\xi) = |\xi|^{-2}(\log|\xi|)^{2(s+1)}.$$

Since $nM(n)^{1/n}\tilde{\delta}(\xi)^{-1/n} \le C(\log|\xi|)^{s+1}$ with $n = [\log|\xi|]$ this shows

$$\phi(M)(\tilde{\delta}(\xi)) \le C(\log|\xi|)^{s+1}.$$

Noticing $\sqrt{\tilde{\delta}(\xi)}|\xi| = (\log|\xi|)^{s+1}$ one can apply Theorem 1.4 to get the assertion. □

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2. Energy inequality

To prove Theorem 1.4 we derive an energy estimate for u satisfying (1). After Fourier transform of (1) with respect to x we get

$$\begin{cases} \partial_t^2\hat{u}(t,\xi) - \sum_{i,j=1}^n a(t,\xi)|\xi|^2\hat{u}(t,\xi) + b(t)\hat{u}(t,\xi) = 0 \\ \hat{u}(0,\xi) = \hat{u}_0(\xi), \quad \partial_t\hat{u}(0,\xi) = \hat{u}_1(\xi) \end{cases} \tag{7}$$

where

$$a(t,\xi) = \sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j/|\xi|^2 \ge 0, \quad t \in [0,T], \quad \xi \in \mathbf{R}^n.$$

To simplify notations we put $v(t,\xi) = \hat{u}(t,\xi)$ and $\partial_t v = v'$. Let us set

$$a_\delta(t,\xi) = a(t,\xi) + \delta$$

where $\delta > 0$ will be determined later. We define the energy density $E_\delta(t,\xi)$

$$E_\delta(t,\xi) = F_\delta(t,\xi)e^{\Lambda_\delta(t,\xi)}$$

where

$$F_\delta(t,\xi) = |v'(t,\xi)|^2 + a_\delta(t,\xi)|\xi|^2|v(t,\xi)|^2 + \gamma|v(t,\xi)|^2,$$
$$\Lambda_\delta(t,\xi) = -\int_0^t \left(\frac{|a'(t,\xi)|}{a_\delta(t,\xi)} + \sqrt{\delta}|\xi| + \gamma\right) dt + \beta(\xi).$$

Here $\beta(\xi) > 0$ and $\gamma > 0$ will be determined later. Note that

$$E'_\delta(t,\xi) = (F'_\delta(t,\xi) + \Lambda'_\delta(t,\xi)F_\delta(t,\xi))e^{\Lambda_\delta(t,\xi)}$$

where

$$\begin{aligned} F'_\delta &= \delta|\xi|^2(v''\bar{v}' + v'\bar{v}'') + a_\delta|\xi|^2(v'\bar{v} + v\bar{v}') \\ &\quad + a'|\xi|^2|v|^2 + \gamma(v\bar{v}' + v'\bar{v}). \end{aligned} \tag{8}$$

Since $v'' = -a|\xi|^2 v - bv$ from (7) we plug this into (8) to get

$$\begin{aligned} F'_\delta &= \delta|\xi|^2(v\bar{v}' + v'\bar{v}) - (bv\bar{v}' + \bar{b}v'\bar{v}) + a'|\xi|^2|v|^2 + \gamma(v\bar{v}' + v'\bar{v}) \\ &\le 2\delta|\xi|^2|v||v'| + 2|b||v||v'| + 2\gamma|v||v'| + \frac{|a'|}{a_\delta}a_\delta|\xi|^2|v|^2. \end{aligned}$$

On the other hand plugging

$$\Lambda'_\delta = -\left(\frac{|a'|}{a_\delta} + \sqrt{\delta}|\xi| + \gamma\right)$$

into the above inequality we get

$$\begin{aligned} F'_\delta + \Lambda'_\delta F_\delta &\le 2\delta|\xi|^2|v||v'| - \sqrt{\delta}|\xi|F_\delta + 2|b||v||v'| \\ &\quad + 2\gamma|v||v'| - \gamma F_\delta + \frac{|a'|}{a_\delta}a_\delta|\xi|^2|v|^2 - \frac{|a'|}{a_\delta}F_\delta. \end{aligned}$$

Noticing $\delta|\xi|/\sqrt{a_\delta} \le \sqrt{\delta}|\xi|$ one has

$$2\delta|\xi|^2|v||v'| - \sqrt{\delta}|\xi|F_\delta \le \frac{\delta|\xi|}{\sqrt{a_\delta}}(a_\delta|\xi|^2|v|^2 + |v'|^2) - \sqrt{\delta}|\xi|F_\delta \le 0.$$

Since it is clear with some $c > 0$ that

$$2\gamma|v||v'| - \gamma F_\delta \le -c\gamma(|v'|^2 + \gamma|v|^2)$$

we get

$$F'_\delta + \Lambda'_\delta F_\delta \le 2|b||v||v'| - c\gamma(|v'|^2 + \gamma|v|^2).$$

Taking γ so that

$$\gamma^{-3/2} \sup_{t\in[0,T]} |b(t)| \le c$$

we obtain

$$E'_\delta(t,\xi) \le 0.$$

We summarize above observations.

PROPOSITION 2.1. *Let $\Phi(\xi)$, $\delta(\xi)$ be non negative and assume that*

$$C_1\Phi(\xi) \le \Lambda_{\delta(\xi)}(t,\xi) \le C_2\Phi(\xi), \quad 0 \le t \le T$$

with some $C_i > 0$. Then we have

$$\begin{aligned}&\left(|\partial_t\hat{u}(t,\xi)|^2 + \gamma|\hat{u}(t,\xi)|^2\right) e^{C_1\Phi(\xi)}\\ &\le C'\left(|\hat{u}_1(\xi)|^2 + (\gamma + |\xi|^2)|\hat{u}_0(\xi)|^2\right) e^{C_2\Phi(\xi)}\end{aligned}$$

for $0 \le t \le T$.

3. A lemma and proof of theorem

In this section we prove a key lemma, which generalizes Lemma 1 in [2] (see also [3], [5]), to establish an energy inequality and complete the proof of Theorem 1.4.

LEMMA 3.1. *Assume that $a_{ij}(t) \in \Gamma(M)([0,T])$. Then for any $n \in$ $\mathbf{N}$ we have*

$$\int_0^T \frac{|a'(t,\xi)|}{a(t,\xi)+\delta}dt \le C'n \max\left(TM(n)^{1/n}\delta^{-1/n}, \log\delta^{-1}\right)$$

for every $0 < \delta < 1/2$ with C' independent of n and δ.

Corollary 3.2. *Assume that* $a_{ij}(t) \in \Gamma(M)([0,T])$. *Then we have*

$$\int_0^t \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} \le CnM(n)^{1/n}\delta^{-1/n}, \quad 0 \le t \le T$$

for every $0 < \delta < 1/2$ *and* $n \in \mathbf{N}$.

Proof. Since we have

$$M(n)^{1/n}\delta^{-1/n} \ge ce \log \delta^{-1}$$

the result follows from Lemma 3.1 choosing C so that $C > (ce)^{-1}C'$, $C'T$. ☐

To prove this lemma we prepare several lemmas. Let $I = (s,t)(\subset (0,T))$ be an open interval. Set

$$F(I;\xi) = \max\left(\frac{a(t,\xi)+\delta}{a(s,\xi)+\delta}, \frac{a(s,\xi)+\delta}{a(t,\xi)+\delta}\right)$$

and note that $F(I;\xi) \ge 1$ by definition. We also note that if $a'(t,\xi) \ne 0$ in $I = (s,t)$ then

$$\int_s^t \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt = \log F(I;\xi). \tag{9}$$

The next lemma is found in [3]. We repeat the proof because, in the following, we need the proof rather than the result itself.

Lemma 3.3. *We have*

$$\int_0^T \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt = \sup_{\triangle} \sum_{I_i \in \triangle} \log F(I_i;\xi)$$

where the supremum is taken over all finite partitions $\triangle = \{I_i\}$ *of* $[0,T]$.

Proof. Denote

$$E_1(\xi) = \{t \in [0,T] \mid a'(t,\xi) = 0\}.$$

Since $(0,T) \setminus E_1(\xi)$ is open and hence a union of countable disjoint open intervals $I_p = (s_p, t_p)$:

$$(0,T) \setminus E_1(\xi) = \bigcup_{p=1}^{\infty} I_p. \tag{10}$$

Let $\epsilon > 0$ be given. We take m so that

$$\sum_{p=m+1}^{\infty} |I_p| < \epsilon, \quad |I_p| = t_p - s_p.$$

Let $\triangle_m$ be the partition of $[0,T]$ defined by the partition points

$$s_1, t_1, s_2, t_2, ..., s_m, t_m.$$

Note that

$$\int_0^T \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt = \sum_{p=1}^{\infty} \int_{I_p} \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt$$

$$= \sum_{p=1}^{m} \int_{I_p} \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt + \sum_{p=m+1}^{\infty} \int_{I_p} \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt. \tag{11}$$

From (9) the first term of the right-hand side of (11) is

$$\sum_{i=1}^{m} \log F(I_i;\xi)$$

which is bounded by $\sum_{I_i \in \triangle_m} \log F(I_i;\xi)$ since $F(I;\xi) \geq 1$ for any I. The second term of the right-hand side of (11) is estimated by

$$\epsilon \left(\sup_{t\in[0,T],\ \xi} |a'(t,\xi)| \right) \delta^{-1}.$$

Since $\epsilon > 0$ is arbitrary this proves that

$$\int_0^T \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt \leq \sup_{\triangle} \sum_{I_i \in \triangle} \log F(I_i;\xi).$$

□

Therefore to prove Lemma 3.3 it suffices to show

$$\sum_{I_i \in \triangle} \log F(I_i;\xi) \le \int_0^T \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt$$

for any partition $\triangle = \{I_i\}$. Thus it is enough to show the inequality

$$\log F(J;\xi) \le \int_J \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt \tag{12}$$

for any interval $J \subset [0,T]$. Let $J = (s,t)$ be an open interval. Denote

$$J \setminus E_1(\xi) = \bigcup_{p=1}^{\infty} J_p, \quad J_p = (s_p, t_p)$$

where $\{J_p\}$ are countable disjoint open intervals. Assume that $\epsilon > 0$ is given as before. Choose m so that

$$\sum_{p=m+1}^{\infty} |J_p| < \epsilon.$$

Take complementary disjoint open intervals $\{K_q\}_{q=1}^{r}$ such that $\{J_p\}_{p=1}^{m}$, $\{K_q\}_{q=1}^{r}$ make a partition of the interval J.

Here we apply the following remark: Let $\triangle = \{I_i\}$ be a partition of I. Then we have

$$\log F(I;\xi) \le \sum_{I_i \in \triangle} \log F(I_i;\xi).$$

To see this let $I = (\alpha,\beta)$ and $I_i = [t_{i-1}, t_i]$, $i = 1, ..., l$ where $t_0 = \alpha$, $t_l = \beta$. Then with $a_\delta(t) = a(t,\xi) + \delta$ we have

$$\frac{a_\delta(\beta)}{a_\delta(\alpha)} = \frac{a_\delta(t_1)}{a_\delta(\alpha)} \cdot \frac{a_\delta(t_2)}{a_\delta(t_1)} \cdots \frac{a_\delta(\beta)}{a_\delta(t_{l-1})} \le \prod_{i=1}^{l} F(I_i;\xi)$$

because

$$\frac{a_\delta(t_i)}{a_\delta(t_{i-1})} \le F(I_i;\xi), \quad i = 1, ..., l.$$

The same arguments give

$$\frac{a_\delta(\alpha)}{a_\delta(\beta)} \le \prod_{i=1}^{l} F(I_i;\xi)$$

and hence the assertion. Thus we get

$$\log F(J;\xi) \leq \sum_{p=1}^{m} \log F(J_p;\xi) + \sum_{q=1}^{r} \log F(K_q;\xi). \tag{13}$$

Since $a'(t,\xi) \neq 0$ in J_p, from (9) the first term of the right-hand side of (13) is bounded by

$$\sum_{p=1}^{m} \int_{J_p} \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt \leq \int_{J} \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt.$$

It remains to estimate the second term of the right-hand side of (13). Put

$$\phi(t,\xi) = \delta^{-1} a(t,\xi). \tag{14}$$

It is clear that

$$\phi(t,\xi) \leq \phi(s,\xi) + Cm(K_q;\xi)\delta^{-1}, \quad s,t \in K_q$$

where

$$C = \sup_{t\in[0,T],\ \xi} |a'(t,\xi)|,\ m(K_q;\xi) = |K_q \setminus E_1(\xi)|\,.$$

Here $|F|$ denotes the Lebesgue measure of F. From this inequality it follows that

$$(\phi(t,\xi)+1) \leq (\phi(s,\xi)+1)(1+Cm(K_q;\xi)\delta^{-1})$$

and hence we obtain

$$a(t,\xi)+\delta \leq (a(s,\xi)+\delta)(1+Cm(K_q;\xi)\delta^{-1}).$$

Thus we have

$$F(K_q;\xi) \leq (1+Cm(K_q;\xi)\delta^{-1})$$

and hence

$$\sum_{q=1}^{r} \log F(K_q;\xi) \leq C_1 C \delta^{-1} \sum_{q=1}^{r} m(K_q;\xi)$$

because $\log(1+x) \le C_1 x$ for $x \ge 1$. The right-hand side is estimated by

$$C_1 C \delta^{-1} \sum_{p=m+1}^{\infty} |J_p| \le \epsilon C_1 C \delta^{-1}.$$

Since $\epsilon > 0$ is arbitrary one obtains (12).

The next lemma is a key to the proof of Lemma 3.1.

LEMMA 3.4. *Let $\triangle = \{I_i\}$, $I_i = [t_{i-1}, t_i]$, $i = 1, ..., N$ be a partition of $[0, T]$ given by zeros of $a'(t,\xi)$, that is $a'(t_i, \xi) = 0$, $i = 1, ..., N-1$. Assume $N \ge 2n-2$. Then we have*

$$F(I_k;\xi) \le (1 + CA^n M(n)(n!)^{-1} |\tilde{I}_k|^n \delta^{-1})$$

with C independent of the partition, where

$$|\tilde{I}_k| = |I_{k_*+1}| + \cdots + |I_k| + \cdots + |I_{k^*}|$$

with $k_* = \max(k-n-1, 1)$, $k^* = \min(k+n-2, N)$.

Proof. By the assumption $N \ge 2n-2$ we have either $k+n-2 < N$ or $k-n-1 > 0$. We first study the case $k+n-2 < N$. Since $\phi'(t,\xi)$ has at least $n-1$ zeros in $[t_k, t_{k^*}]$ then $\phi^{(2)}(t,\xi)$ has at least $n-2$ zeros in the same interval. Take a zero α_2 of $\phi^{(2)}(t,\xi)$ so that in $[\alpha_2, t_{k^*}]$, $\phi^{(2)}(t,\xi)$ has at least $n-2$ zeros. Then $\phi^{(3)}(t,\xi)$ has at least $n-3$ zeros in $[\alpha_2, t_{k^*}]$. Choose $\alpha_3 \in [\alpha_2, t_{k^*}]$ so that $\phi^{(3)}(\alpha_3, \xi) = 0$ and $\phi^{(3)}(t,\xi)$ has at least $n-3$ zeros in $[\alpha_3, t_{k^*}]$. Repeating this arguments we can take α_i so that

$$\phi^{(i)}(\alpha_i, \xi) = 0, \quad \alpha_1 = t_k \le \alpha_2 \le \alpha_3 \le \cdots \le \alpha_{n-1} (\le t_{k^*}).$$

Write

$$\phi^{(i)}(t,\xi) = -\int_t^{\alpha_i} \phi^{(i+1)}(s,\xi) ds, \quad 1 \le i \le n-1$$

and assume that

$$|\phi^{(n-i)}(t,\xi)| \le CA^n M(n) \delta^{-1} \frac{(\alpha_{n-1} - t)^i}{i!}, \quad t_{k-1} \le t \le \alpha_{n-i}. \quad (15)$$

When $i = 0$, (15) follows from

$$\sup_{t\in[0,T],\xi} |a^{(n)}(t,\xi)| \le CA^n M(n)\delta^{-1}, \quad n = 1, 2, ...$$

which results from the assumption $a_{ij}(t) \in \Gamma(M)([0,T])$. Since

$$|\phi^{(n-i-1)}(t,\xi)| \le \int_t^{\alpha_{n-i-1}} |\phi^{(n-i)}(s,\xi)|ds, \quad t_{k-1} \le t \le \alpha_{n-i-1}$$

and $\alpha_{n-i-1} \le \alpha_{n-i} \le \alpha_{n-1}$, applying (15) the right-hand side is estimated by

$$\int_t^{\alpha_{n-i-1}} CA^n M(n)\delta^{-1}\frac{(\alpha_{n-1}-s)^i}{i!}ds \le CA^n M(n)\delta^{-1}\frac{(\alpha_{n-1}-t)^{i+1}}{(i+1)!}$$

for $t_{k-1} \le t \le \alpha_{n-i-1}$. By induction we get (15) for every $1 \le i \le n-1$. This shows that

$$\phi(t_k,\xi) \le \phi(t_{k-1},\xi) + CA^n M(n)\delta^{-1}\frac{|\alpha_{n-1}-t_{k-1}|^n}{n!}$$

and hence

$$\phi(t_k,\xi) + 1 \le (\phi(t_{k-1},\xi)+1)(1 + CA^n M(n)(n!)^{-1}\delta^{-1}|t_{k^*} - t_{k-1}|^n).$$

This gives that

$$a(t_k,\xi) + \delta \le (a(t_{k-1},\xi)+\delta)(1 + CA^n M(n)(n!)^{-1}\delta^{-1}|t_{k^*} - t_{k-1}|^n). \tag{16}$$

Similarly one gets

$$a(t_{k-1},\xi) + \delta \le (a(t_k,\xi)+\delta)(1 + CA^n M(n)(n!)^{-1}\delta^{-1}|t_{k^*} - t_{k-1}|^n). \tag{17}$$

From (16) and (17) we have

$$F(I_k;\xi) \le (1 + CA^n M(n)(n!)^{-1}\delta^{-1}|\tilde{I}_k|^n) \tag{18}$$

because $|t_{k^*} - t_{k-1}| \le |\tilde{I}_k|$.

When $k - n - 1 > 0$, choosing β_i so that $(t_{k_*} \le)\beta_{n-1} \le \beta_{n-2} \le \cdots \le \beta_2 \le \beta_1 = t_{k-1}$, $\phi^{(i)}(\beta_i,\xi) = 0$, we get the desired assertion by the same arguments. □

Proof of Lemma 3.1. We first assume that the number of zeros of $a'(t,\xi)$ is less than $2n-2$ and let

$$0 \le t_1 < \cdots < t_{p-1} \le T$$

be zeros of $a'(t,\xi)$. From (9) we see that

$$\int_0^T \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt = \sum_{i=1}^{p} \log F(I_i;\xi) \tag{19}$$

where $I_i = [t_{i-1}, t_i]$, $i = 1, ..., p$, $t_0 = 0$, $t_p = T$. Since

$$\frac{a(s,\xi)+\delta}{a(t,\xi)+\delta} \le (\sup_{\tau\in[0,T],\,\xi} a(\tau,\xi)+1)\delta^{-1}$$

it is clear that $F(I_k;\xi) \le C\delta^{-1}$ with C independent of δ and the partition. Thus one has

$$\sum_{i=1}^{p} \log F(I_i;\xi) \le C \sum_{i=1}^{p} \log \delta^{-1} \le C' n \log \delta^{-1} \tag{20}$$

for $0 < \delta < 1/2$ which proves the assertion. We turn to the case when $a'(t,\xi)$ has more than $2n-2$ zeros in $[0,T]$. As we have seen in the proof of Lemma 3.3, there is a sequence of partitions $\triangle_k = \{I_j^{(k)}\}_{j=1}^{m_k}$ of $[0,T]$ of which partition points consist of zeros of $a'(t,\xi)$ such that

$$\int_0^T \frac{|a'(t,\xi)|}{a(t,\xi)+\delta} dt = \lim_{k\to\infty} \sum_{j=1}^{m_k} \log F(I_j^{(k)};\xi). \tag{21}$$

Note that $\log(1+x^n) \le nx$ for $x \ge 0$ and $[(n!)^{-1}]^{1/n} \le cn^{-1}$ with some $c > 0$ independent of $n \in \mathbf{N}$ by the Stirling's formula. Then applying Lemma 3.4 we get

$$\log F(I_j^{(k)};\xi) \le \log(1 + CA^n M(n)(n!)^{-1}\delta^{-1}|\tilde{I}_j^{(k)}|^n) \le C'AM(n)^{1/n}|\tilde{I}_j^{(k)}|\delta^{-1/n}$$

with C' independent of n. Taking the sum over $j = 1, ..., m_k$ we get

$$\sum_{j=1}^{m_k} \log F(I_j^{(k)};\xi) \le CAM(n)^{1/n}\delta^{-1/n} \textstyle\sum_{j=1}^{m_k} |\tilde{I}_j^{(k)}| \le CAM(n)^{1/n}\delta^{-1/n}(2nT). \tag{22}$$

Then (20) and (22) prove the assertion. $\square$

Proof of Theorem 1.4. Let $u_i(x) \in \hat{\Gamma}(\Phi) \cap \mathcal{E}'(\mathbf{R}^n)$, $i = 0, 1$ verify

$$|\xi||\hat{u}_0(\xi)|, \quad |\hat{u}_1(\xi)| \leq B_K e^{-K\Phi(\xi)}$$

for any $K > 0$. Let $\hat{u}(t, \xi)$ be a solution to the ordinary differential equation (2.1) with the parameter ξ. Let $\delta(\xi) > 0$ be such that $\Phi(\xi) = \phi(M)(\delta(\xi)) = \sqrt{\delta(\xi)}|\xi|$. From Corollary 3.2 it follows that

$$\int_0^t \frac{|a'(s,\xi)|}{a(s,\xi) + \delta(\xi)} ds \leq C'\phi(M)(\delta(\xi)), \quad 0 \leq t \leq T \tag{23}$$

with some $C' > 0$. From (23) it follows that

$$\beta(\xi) - C\Phi(\xi) \leq \Lambda_{\delta(\xi)}(t, \xi) \leq \beta(\xi), \quad 0 \leq t \leq T.$$

Taking $\beta(\xi) = \lambda\Phi(\xi)$ we have

$$(\lambda - C)\Phi(\xi) \leq \Lambda_{\delta(\xi)}(t, \xi) \leq \lambda\Phi(\xi), \quad 0 \leq t \leq T.$$

Now Proposition 2.1 proves that

$$|\hat{u}(t,\xi)|^2 e^{(\lambda - C)\Phi(\xi)} \leq C' \left(|\hat{u}_1(\xi)|^2 + (\gamma + |\xi|^2)|\hat{u}_0(\xi)|^2 \right) e^{\lambda\Phi(\xi)} \leq C$$

for any $\lambda > 0$, $0 \leq t \leq T$ and hence $u(t, \cdot) \in \hat{\Gamma}(\Phi)$. □

4. Counter example

Our construction of counter examples in Theorem 1.5 is inspired by the example in [4] for a second order hyperbolic Cauchy problem which is not well posed in C^∞. We shall consider the following Cauchy problem

$$\begin{cases} \partial_t^2 u - a(t)\partial_x^2 u = 0 \\ u(0, x) = u_0(x), \quad \partial_t(0, x) = u_1(x) \end{cases} \tag{24}$$

Before defining $a(t)$ we need a definition.

DEFINITION 4.1. *Let $\mathcal{B}$ be the set of all $f(t) \in C^\infty(\mathbf{R})$ such that for any compact $K \subset \mathbf{R}$ there is a C_K such that*

$$|f^{(n)}(t)| \leq C_K^{n+1} n^n (\log(n+2))^{2n}, \quad \forall n \in \mathbf{N}, \ \forall t \in K.$$

We recall that $\mathcal{B}$ is stable under multiplication and under differentiation; moreover, due to Denjoi-Carleman theorem, $\mathcal{B}$ is a non quasi-analytic class, i.e. there exists a non trivial $f \in \mathcal{B}$ with compact support.

Proof of Theorem 1.5. Let $\rho(\tau)$ be a function in $\mathcal{B}$, 2π periodic, non negative such that $\rho(\tau) \equiv 0$ for $|\tau| \leq \pi/3$, and

$$\int_0^{2\pi} \rho(s) \cos^2 s ds = \pi. \tag{25}$$

Les us define (cf. [4])

$$\alpha(\tau) = 1 + 4\epsilon\rho(\tau) \sin 2\tau - 2\epsilon\rho'(\tau) \cos^2 \tau - 4\epsilon^2\rho^2(\tau) \cos^4 \tau \tag{26}$$

and fix ϵ so that

$$1/2 \leq \alpha(\tau) \leq 3/2. \tag{27}$$

Let us put

$$L = \max |\alpha'(\tau)|. \tag{28}$$

Obviously $\alpha \in \mathcal{B}$. Let now W be the solution to the Cauchy problem

$$\begin{cases} W'' + \alpha W = 0 \\ W(0) = 1 \\ W'(0) = 0. \end{cases} \tag{29}$$

By a simple computation we see that

$$W(\tau) = \cos \tau \exp \left[2\epsilon \int_0^{\tau} \rho(s) \cos^2 s ds \right].$$

In particular, we have for $\nu = 1, 2, ...$

$$\begin{cases} W(\pm 2\pi\nu) = e^{\pm 2\epsilon\pi\nu} \\ W'(\pm 2\pi\nu) = 0 \end{cases}. \tag{30}$$

Let $\beta(\tau)$ be a non increasing function belonging to $\mathcal{B}$ such that $\beta(\tau) = 1$ for $\tau \leq 0$, $\beta(\tau) = 0$ for $\tau \geq 1$. Finally we introduce 3 sequences; for $k = 1, 2, ...$

$$\rho_k = k^{-3/2} \tag{31}$$

$$\nu_k = \mu^k \tag{32}$$

$$\delta_k = \phi^{-1}(M)(\frac{\nu_k}{\rho_k}) \tag{33}$$

for some integer $\mu \geq 2$ to be chosen later, where $\phi^{-1}(M)$ is the inverse of $\phi(M)$ defined by (4).

Now we can define the coefficient $a(t)$ in $[0, +\infty)$, by setting

$$a(t) = \delta_k \alpha \left(4\pi\nu_k \frac{t - t_k}{\rho_k} \right) \tag{34}$$

on I_k and

$$a(t) = \delta_{k+1} + (\delta_k - \delta_{k+1})\beta \left(\frac{t - t''_k}{t'_{k+1} - t''_k} \right) \tag{35}$$

on J_k and $a(t) = 0$ for $t \geq T$ where

$$t'_1 = 0, \quad t'_k = 2\sum_{j=1}^{k-1} \rho_j \quad (k = 2, 3, ...)$$

$$t_k = t'_k + \rho_k/2, \quad t''_k = t'_k + \rho_k, \quad T = 2\sum_{j=1}^{\infty} \rho_j$$

$$I_k = [t'_k, t''_k], \quad J_k = [t''_k, t'_{k+1}].$$

It is immediate that $a(t) \in C^\infty([0, T))$; moreover $a(t)$ tends to zero as $t \uparrow T$ since $\delta_k \to 0$. Now we want $a(t)$ to be C^∞ near $t = T$; it will be sufficient to show that all derivatives of $a(t)$ go to zero as $t \uparrow T$. On I_k we have

$$|a^{(n)}(t)| \leq \delta_k \left(\frac{4\pi\nu_k}{\rho_k} \right)^n A^{n+1} n^n \left(\log(2 + n)\right)^{2n} \tag{36}$$

and on J_k

$$|a^{(n)}(t)| \leq \delta_k \left(\frac{1}{\rho_k} \right)^n A^{n+1} n^n \left(\log(2 + n)\right)^{2n} \tag{37}$$

if

$$|\alpha^{(n)}(\tau)|, \quad |\beta^{(n)}(\tau)| \leq A^{n+1} n^n \left(\log(n + 2)\right)^{2n}.$$

By (33) and (4) we have

$$\delta_k \left(4\pi \frac{\nu_k}{\rho_k} \right)^n \leq (4\pi)^n M(n) n^n \tag{38}$$

for all k and n. In particular the right-hand side of (36) and (37) goes to zero as $k \to \infty$ for all n because

$$\delta_k \left(\frac{\nu_k}{\rho_k}\right)^n \leq M(n+1)(n+1)^{n+1}\frac{\rho_k}{\nu_k}$$

This shows that $a(t) \in C^\infty([0, +\infty))$. Moreover from (38) one can check that

$$|a^{(n)}(t)| \leq C^{n+1}M(n)n^{2n}(\log{(n+2)})^{2n}$$

that is $a \in \Gamma(M(n)n^{2n}(\log{(n+2)})^{2n})$.

We now find a Cauchy data in $\hat{\Gamma}(\Phi/\log^2\Phi)$ such that the Cauchy problem (24) has no solution in $\mathcal{D}'$ for $t > T$. More precisely we construct a particular solution u to the equation $\partial_t^2 u - a(t)\partial_x^2 u = 0$ on $[0, T) \times \mathbf{R}_x$ such that

$$u \in C^\infty([0, T); \hat{\Gamma}(\Phi/\log^2\Phi)) \tag{39}$$

but

$$u(t, \cdot) \quad \text{is not bounded in} \quad \mathcal{D}' \quad \text{for} \quad t \uparrow T. \tag{40}$$

This solution will have the form

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t) \sin h_k x \tag{41}$$

with an increasing sequence h_k to be chosen later. We have then

$$u_k''(t) + h_k^2 a(t) u_k(t) = 0. \tag{42}$$

In particular for $t \in I_k$, (42) becomes

$$u_k''(t) + \delta_k h_k^2 \alpha\left(4\pi\nu_k \frac{t - t_k}{\rho_k}\right) u_k(t) = 0.$$

If we choose $h_k = 4\pi\nu_k/\sqrt{\delta_k}\rho_k$ and we impose

$$\begin{cases} u_k(t_k) = 1 \\ u_k'(t_k) = 0 \end{cases} \tag{43}$$

this shows that for $t \in I_k$

$$u_k(t) = W\left(4\pi\nu_k \frac{t - t_k}{\rho_k}\right)$$

where W is the solution to (4.6). In particular we get

$$\begin{cases} u_k(t_k') = W(-2\pi\nu_k) = e^{-2\pi\epsilon\nu_k} \\ u_k'(t_k') = W'(-2\pi\nu_k) = 0 \end{cases} \tag{44}$$

and

$$\begin{cases} u_k(t_k'') = W(2\pi\nu_k) = e^{2\pi\epsilon\nu_k} \\ u_k'(t_k'') = W'(2\pi\nu_k) = 0 \end{cases} \tag{45}$$

We now prove that the Fourier series in (41) are converging in $C([0,T),$ $\hat{\Gamma}(\Phi/\log^2\Phi))$.

Since $t_k' \to T$ as $k \to \infty$, it will be sufficient to prove that for all $\bar{k}$ and all C there exists C_1 such that for any $t \in [0, t_k']$ and for any $k \geq \bar{k}$ we have

$$|u_k(t)| + |u_k'(t)| \leq C_1 \exp\left(-C\Phi(h_k)/\log^2\Phi(h_k)\right).$$

This inequality and $a(t) \in C^\infty([0,\infty))$ prove (39).

Let us consider an energy

$$E_k(t) = |u_k'(t)|^2 + h_k^2 a(t)|u_k(t)|^2 \tag{46}$$

which verifies, from (44) and (45), that

$$E_k(t_k') = h_k^2\delta_k e^{-4\pi\epsilon\nu_k} \tag{47}$$

$$E_k(t_k'') = h_k^2\delta_k e^{4\pi\epsilon\nu_k} \tag{48}$$

By differentiating (46) and using (42) we get, for $t \leq t_k'$

$$E_k(t) \leq E_k(t_k') \exp\left(\int_0^{t_k'} \frac{|a'(s)|}{a(s)} ds\right). \tag{49}$$

But from (27), (28) and (34) we have

$$\int_{I_j} \frac{|a'(s)|}{a(s)} ds \leq 8\pi\nu_j L$$

while, from (35) one gets

$$\int_{J_j} \frac{|a'(s)|}{a(s)} ds = \log\frac{1}{\delta_{j+1}} - \log\frac{1}{\delta_j}.$$

Thus from (47) and (49) we get

$$\sup_{0\le t\le t_k'} E_k(t) \le$$
$$\exp[-4\pi\epsilon\nu_k + 2\log\tfrac{\nu_k}{\rho_k} + 8\pi L\textstyle\sum_{j=1}^{k-1}\nu_j + \log\tfrac{1}{\delta_k} - \log\tfrac{1}{\delta_1}].$$

Now we choose an integer μ in (32) so large that, for all k,

$$\nu_k > 8\frac{L}{\epsilon}\sum_{j=1}^{k-1}\nu_j$$

and

$$\nu_k > \frac{2}{\epsilon\pi}\log\frac{\nu_k}{\rho_k}.$$

Moreover from (5) we have $c(\log\delta_k)^2 \le \nu_k/\rho_k$ with some $c > 0$ and hence

$$\nu_k > \left(\frac{\nu_k}{\rho_k}\right)^{2/3} > c'\left(\log\frac{1}{\delta_k}\ right)^{1/3}\log\frac{1}{\delta_k}\right)$$

and finally

$$\nu_k > \frac{1}{\pi\epsilon}\log\frac{1}{\delta_k}$$

for large k. We have then

$$\sup_{0\le t\le t_k'} E_k'(t)\exp\left[C\Phi(h_k)/\log^2\Phi(h_k)\right] \le$$
$$C_1\exp\left[-\pi\epsilon\nu_k + C\Phi(h_k)/\log^2\Phi(h_k)\right].$$

But from (6) and (33) we obtain

$$\Phi(h_k) = \frac{\nu_k}{\rho_k}$$

and hence we conclude

$$\lim_{k\to\infty}\frac{\nu_k\log^2\Phi(h_k)}{\Phi(h_k)} = \infty \tag{50}$$

and then (39). On the other hand, from (48) we see

$$E_k(t_k'')e^{-\Phi(h_k)/\log^2\Phi(h_k)} \ge e^{4\pi\epsilon\nu_k - \Phi(h_k)/\log^2\Phi(h_k)}$$

and by (50), noticing $C\Phi(\xi) \ge \log^2|\xi|$ as remarked in section 1, we conclude the assertion (40). $\square$

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