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Regularity Aspects of Fractional Evolution Equations

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Summary. - *A brief review of results on evolution equation with fractional derivatives is presented. We emphasize regularity results and consider both linear and non linear equations.*

1. Introduction

In this survey we review some recent progress concerning regularity results on solutions of equations with fractional derivatives. We consider both fractional differential equations in Banach spaces and also certain partial differential equations with fractional derivatives. Linear, as well as nonlinear results are considered.

Let $u : [0, T] \to X$, where X is a Banach space. Assume (at least) that u is continuous and satisfies $u(0) = 0$. We define the fractional derivative of u of order $\alpha \in (0, 1)$ by

$$(D_t^\alpha u)(t) = \frac{d}{dt} \int_0^t g_{1-\alpha}(t-s) u(s)\, ds, \quad t > 0,$$

and

$$(D_t^\alpha u)(0) = \lim_{h \downarrow 0} \frac{1}{h} \int_0^h g_{1-\alpha}(h-s) u(s)\, ds,$$

where

$$g_\beta(t) = \frac{1}{\Gamma(\beta)} t^{\beta - 1}, \quad t > 0, \quad \beta \in (0, 1).$$

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For $\alpha \in (1,2)$, the fractional derivative of u is given by

$$D_t^\alpha u = D_t^{\alpha-1} u_t,$$

where u_t is the usual first order derivative.

Our work on the regularity of fractional evolution equations has partially been motivated by problems connected with the fractional conservation law

$$D_t^\alpha (u - u_0) + \sigma(u)_x = f, \quad 0 < \alpha < 1, \tag{1}$$

where $u = u(t,x)$; $t \geq 0$, $x \in R$; $u(0,x) = u_0(x)$, and where the nonlinear function σ is sufficiently smooth. Equations of type (1) can be used to approach nonlinear conservation laws. In fact, if u_α is the entropy solution of (1), then (under certain assumptions on σ and u_0), $u_\alpha \to u_e$, as $\alpha \uparrow 1$, where u_e is the entropy solution of

$$u_t + \sigma(u)_x = f, \quad u(0,x) = u_0(x). \tag{2}$$

(A more precise statement will be given below).

Another challenging motivation can be formulated as follows.

Consider the equation

$$D_t^\alpha (u_t - u_1) = \sigma(u_x)_x + f, \tag{3}$$

with $0 < \alpha < 1$; $u = u(t,x)$, $t > 0$, $x \in (0,1)$; $u(0,x) = u_0(x)$, $u_t(0,x) = u_1(x)$; $u(t,0) = u(t,1) = 0$, $t > 0$, and with σ smooth, satisfying

$$0 < m \leq \sigma'(y) \leq M < \infty, \quad \sigma(0) = 0, \tag{4}$$

for some constants m, M.

Obviously, (3) is (under assumption (4)) an equation intermediate to the nonlinear heat equation and the nonlinear wave equation. While the behavior of (3) in the linear case $\sigma(y) = y$ is reasonably well understood (it is, in fact, essentially parabolic - see [12]), only very partial results exist for the nonlinear case. Thus it is known, see [5], that if $\alpha \in (0, \frac{1}{2}]$, then a solution u of (3) with $u_{xx} \in L^2((0,T); L^2(0,1))$ exists. For larger α-values however, only results on weak solutions are available, i.e., only $u \in L^\infty((0,T); H_0^1(0,1))$ or

$u_x \in L^\infty_{loc}((0,T); L^\infty(0,1))$, depending on whether (4) is assumed or certain convexity assumptions are made on σ, see [6], [9].

Formally, (3) can obviously be written

$$D_t^\alpha(u_t - u_1) = a(t,x)u_{xx} + f, \tag{5}$$

with $a(t,x) = \sigma'(u_x(t,x))$. This may be viewed as a version of

$$D_t^\alpha(u_t - u_1) - u_{xx} = f, \tag{6}$$

with non-constant coefficients. Thus there is a definite motivation for a maximal regularity analysis of (5) and (6); the ultimate goal being to arrive at regularity results on (3) via these maximal regularity statements.

2. Linear problems

We first review some results on the linear abstract fractional evolution equation

$$D_t^\alpha(u - u_0) + Bu = f. \tag{7}$$

Here u takes values in a complex Banach space X; B is a positive operator mapping $\mathcal{D}(B) \subset X$ into X, $u_0 \in X$, and $f \in C([0,T];X)$, for some $T > 0$. A function u: $[0,T] \to X$ is said to be a strict solution of (7) on $[0,T]$ if $u \in C([0,T];\mathcal{D}(B))$; $g_{1-\alpha} * (u - u_0) \in C^1([0,T];X)$ and (7) holds for all $t \in [0,T]$.

Let $\gamma \in (0,1]$. We write

$$D_B(\gamma,\infty) = (X,\mathcal{D}(B))_{\gamma,\infty} \qquad D_B(\gamma) = (X,\mathcal{D}(B))_\gamma \quad ,$$

where the right sides denote the usual real interpolation spaces, see, e.g., [10]. One then has, [1],

THEOREM 2.1. *Suppose $\alpha \in (0,1)$ and let B be a positive operator in X with spectral angle $\phi_B < \Pi(1-\frac{\alpha}{2})$. Take $u_0 \in \mathcal{D}(B)$ and $f \in C([0,T];X)$. Then*

(a) For $\gamma \in (0,\alpha]$ and $f \in C^\gamma([0,T];X)$ there exists a unique strict solution u of (7) such that $Bu(t) \in C^\gamma([0,T];X)$ iff $Bu_0 -$

$f(0) \in D_B(\frac{\gamma}{\alpha}, \infty)$. Moreover, in this case there exists a constant M; depending on γ, α, B, T; such that

$$\|Bu(t) - f(0)\|_{C^\gamma([0,T];X)} \leq M\Big(\|Bu_0 - f(0)\|_{D_B(\frac{\gamma}{\alpha},\infty)} + \|f(t) - f(0)\|_{C^\gamma([0,T];X)}\Big). \quad (8)$$

(b) Let $\gamma \in (0,1)$ and $f \in B([0,T]; D_B(\gamma,\infty))$. Then there exists a unique strict solution u of (7) such that

$$Bu(t) \in C([0,T];X) \cap B([0,T]; D_B(\gamma,\infty))$$

iff $Bu_0 - f(0) \in D_B(\gamma,\infty)$. Moreover, an estimate analogous to (8) holds.

This Theorem is proved by writing $u = v + w + B^{-1}f(0)$, where v, w satisfy, respectively,

$$D_t^\alpha(v - v_0)(t) + Bv(t) = 0; \qquad v(0) = v_0 = u_0 - B^{-1}f(0);$$

$$D_t^\alpha w(t) + Bw(t) = f(t) - f(0); \qquad w(0) = 0,$$

and applying resolvent techniques to the v-equation, and the Da Prato-Grisvard Method of Sums to the w-equation. For details and further results, see [1].

The statements above can be applied to yield maximal regularity results on, e.g., the partial differential equation with fractional derivatives:

$$D_t^\alpha(u - h_1) + D_x^\beta(u - h_2) = f. \quad (9)$$

Here $u = u(t,x)$, $(t,x) \in R^+ \times R^+$; $\alpha, \beta \in (0,1)$; $u(0,x) = h_1(x)$; $u(t,0) = h_2(t)$; $h_1(0) = h_2(0) = f(0,0) = 0$. In particular, one has that if f is Hölder-continuous in both variables, i.e., $f \in C^{\mu,\nu}([0,T] \times [0,1])$, with $\mu \in (0,\alpha)$, $\nu \in (0,\beta)$, then

$$D_t^\alpha(u - h_1), \quad D_x^\beta(u - h_2) \in C^{\mu,\nu},$$

and the appropriate Schauder-estimates hold.

By analogous techniques one can prove results for the higher order equation

$$D_t^{\alpha}(u_t - u_1)(t) + Bu(t) = f(t), \quad u(0) = u_0, \quad t \geq 0, \tag{10}$$

with u_0, u_1, f given. Then one has, [3],

THEOREM 2.2. *Suppose $\alpha \in (0,1)$ and let B be a positive operator in a complex Banach space X with spectral angle $\phi_B < \Pi(\frac{1}{2} - \frac{\alpha}{2})$. Assume $u_0 \in \mathcal{D}(B)$, with $u_1 \in D_B(\frac{\alpha}{1+\alpha})$ and take f continuous on $[0,T]$ with values in X. Then*

(a) With $\gamma \in (0,1)$ and $f \in C^{\gamma}([0,T];X)$ there exists a unique strict solution of (10) such that $Bu(t) \in C^{\gamma}([0,T];X)$ provided

$$Bu_0 - f(0) \in D_B\Big(\frac{\gamma}{1+\alpha}, \infty\Big) \quad , \quad u_1 \in D_B\Big(\frac{\alpha+\gamma}{1+\alpha}, \infty\Big) \tag{11}$$

hold. Moreover, an apriori estimate for $\|Bu(t) - f(0)\|_{C^{\gamma}([0,T];X)}$ in terms of the norms of the quantities in (11) and $\|f\|_{C^{\gamma}}$ holds.

(b) Let $\gamma \in (1, 1+\alpha]$ and $f \in C^{\gamma}([0,T];X)$. Assume the first part of (11) and, in addition, that

$$u_1 - B^{-1}f'(0) \in D_{B^2}\Big(\frac{\alpha+\gamma}{2(1+\alpha)}, \infty\Big).$$

Then there exists a unique strict solution u of (10) satisfying $Bu(t) \in C^{\gamma}([0,T];X)$. Moreover, there is an M depending on γ, α, B, T such that

$$\|Bu(\underline{t}) - f(0) - \underline{t}f'(0)\|_{C^{\gamma}([0,T];X)} \leq M\Bigg(\|Bu_0 - f(0)\|_{\mathcal{D}_B(\frac{\gamma}{1+\alpha},\infty)} +$$

$$\|u_1 - B^{-1}f'(0)\|_{\mathcal{D}_{B^2}(\frac{\alpha+\gamma}{2(1+\alpha)},\infty)} + \|f(\underline{t}) - f(0) - \underline{t}f'(0)\|_{C^{\gamma}([0,T];X)}\Bigg).$$

For details and additional results, see [3]. Clearly this Theorem 2.2 may be applied to partial differential equations with fractional derivatives.

Instead of making this application explicit, we consider - in view of the application below to fractional conservation laws - an extension of (9), namely an extension to the nonconstant coefficient case with $\beta = 1$. Thus, consider

$$D_t^\alpha(u-u_0)(t,x) + c(t,x)u_x(t,x) = f(t,x), \qquad (12)$$
$$0 \le t \le T, \quad 0 \le x \le 1,$$

with initial and boundary conditions $u(0,x) = u_0(x)$, $u(t,0) = u_1(t)$.

For brevity, below we only look at Hölder-continuity in t.

THEOREM 2.3. *Assume that* $\alpha \in (0,1)$, $\mu \in (0,\alpha)$, $c(t,x) > 0$ *on* $[0,T]\times[0,1]$, *and that*

$$c \in C([0,1]; C^\mu([0,T])),$$

$$f \in C^\mu([0,T]; C([0,1])), \quad f(0,0) = 0,$$

$$u_0 \in C^1([0,1]), \quad u_0(0) = u_0'(0) = 0,$$

$$c(0,x)u_0'(x) - f(0,x) \in C^{\frac{\mu}{\alpha}}([0,1]),$$

$$u_1 \in C([0,T]), \quad D_t^\alpha u_1 \in C^\mu([0,T]),$$

with $u_1(0) = (D_t^\alpha u_1)(0) = 0$.

Then there exists a unique continuous solution u *of (12) such that* $u_x \in C^\mu([0,T]; C([0,1]))$.

By (12), and by the conclusion on u_x, one may easily obtain regularity statements on u. For Theorem 2.3, and for further results in the same vein, in particular for Hölder-continuity in x, see [2].

3. Results on nonlinear problems

As was indicated earlier, in the nonlinear case we are at present quite restricted as to the order of the equation.

Let A be a nonlinear (possibly multivalued) operator mapping $\mathcal{D}(A) \subset X$ into X. We recall that A is said to be m-accretive if $(I+\lambda A)^{-1}$ is nonexpansive and $R(I+\lambda A) = X$, both for any $\lambda > 0$. The Yosida-approximations A_λ are defined by $A_\lambda = \lambda^{-1}(I - J_\lambda)$, where $J_\lambda = (I+\lambda A)^{-1}$.

The following result by Gripenberg [7] is fundamental.

THEOREM 3.1. *Let X be a real Banach space. Assume that*

$k \in L^1_{loc}(R^+; R)$ *is positive and nonincreasing,*
$\lim_{t\downarrow 0} k(t) = \infty$, *and* $\log(k(t))$ *is convex,*
A *is an m-accretive operator on* X,
$y \in \hat{D}(A)$, *i.e.,* $y \in X$ *and* $\sup_{\lambda>0} \|A_\lambda y\|_X < \infty$,
$f \in C(R^+; X)$ *is such that* $\int_0^T \omega_{f,T}(s)|k'(s)|\, ds < \infty$, *for each* $T > 0$, *where* $\omega_{f,T}(s) = \sup_{t_1,t_2\in[0,T];|t_1-t_2|\le s} \|f(t_1) - f(t_2)\|_X$.

Then there is a unique strong solution u of

$$\frac{d}{dt}\int_0^t k(t-s)[u(s)-y]\, ds + A(u(t)) \ni f(t), \quad t > 0, \quad u(0) = y, \tag{13}$$

such that $u \in C(R^+; X)$; $u(0) = y$, *and there is a function* $w \in C((0,\infty); X)$ *such that for each* $T > 0$, $sup_{0<t<T}\|w(t)\|_X < \infty$, $w(t) \in A(u(t))$ *for* $t > 0$, *and*

$$\frac{d}{dt}\int_0^t k(t-s)[u(s)-y]\, ds + w(t) = f(t), \quad t > 0.$$

The key result here is, of course, the continuity and (local) boundedness of $w(t)$.

Our next goal is to apply Theorem 3.1 to the fractional conservation law (1). In particular, we wish to analyze the Riemann-problem connected with (1), i.e.,

$$D_t^\alpha(u - \chi_{(-\infty,0]}(x))(t,x) + \sigma(u)_x(t,x) = 0, \quad t > 0; \quad x \in R; \tag{14}$$

with $u(0,x) = \chi_{(-\infty,0]}(x)$.

First, observe that if one takes

$$u(t,x) = 1, \quad t \ge 0; \quad x \le 0;$$

then this function solves (14) (in the second quadrant). Thus, there remains

$$(D_t^\alpha u)(t,x) + \sigma(u)_x(t,x) = 0; \quad t > 0, \quad x > 0,$$

with $u(t,0) = 1$ for $t > 0$ and $u(0,x) = 0$ for $x > 0$.

Next, note that if one takes

$$\mathcal{D}(A) = \{\, u \in L^1(R^+; R) | \sigma(u) \in AC(R^+; R); \\ u(0) = 1,\, \sigma(u)' \in L^1(R^+; R)\,\},$$

and defines $Au = \sigma(u)'$ for $u \in \mathcal{D}(A)$, then A is closed and m-accretive in $L^1(R^+; R)$.

Combining these observations with Theorem 4 one can show the following concerning (14). (For the proof, see [8, Theorem 5], and [7, Theorem 2]). Observe that $k(t) = c_\alpha t^{-\alpha}$ obviously satisfies the assumption on k made in Theorem 3.1.

THEOREM 3.2. *Assume that*
$\sigma \in C^1(R; R)$ *is strictly increasing on* $(0,1)$ *and there are constants* C *and* $\gamma > 1$ *such that*

$$\frac{1}{C} r^\gamma \le \sigma(r) \le C r^\gamma, \quad r \in [0,1].$$

Then there is a solution u *of the Riemann problem (14) which is continuous for* $(t,x) \in R^+ \times R \backslash \{0,0\}$ *and is such that for each* $t > 0$ *the function* $x \to u(t,x)$ *is absolutely continuous and nonincreasing, for* $x \in R$ *the function* $t \to u(t,x)$ *is nondecreasing, and (14) holds a.e. on* $R^+ \times R$. *Moreover,*

$$u(t,x) = 0, \qquad x \ge \frac{1}{k(t)} \int_0^1 \frac{\sigma'(r)}{r}\, dr, \quad t > 0, \tag{15}$$

and the function $\phi(t) = \inf\{\, x > 0 | u(t,x) = 0\,\}$ *is continuous and strictly increasing.*

Note the somewhat surprising outcome (15) of the nonlinear problem: the x-support of the solution is compact. In the linear case $\sigma(u) = ku$, this does not hold. See [11].

Additional regularity results on solutions of

$$D_t^\alpha (u - u_0) + \sigma(u)_x = f \tag{16}$$

can be obtained in the case where $\sigma'(u) > 0$ (σ not merely strictly increasing). See [7] for details. These results are obtained by using both Theorem 2.3 (together with the corresponding result on

x-regularity) and Theorem 3.1, and also observing that (16) can formally be written

$$D_t^{\alpha}(u - u_0) + b(t,x)u_x = f,$$

with $b(t,x) = \sigma'(u(t,x))$. (Cf. (5),(6) above).

Let us finally look at the possible convergence of u_α as $\alpha \uparrow 1$; where u_α solves (16). One does of course hope that the functions u_α do converge to the unique entropy solution of

$$u_t + \sigma(u)_x = f, \quad u(0,x) = u_0(x).$$

This is indeed the case, under some technical assumptions, stated below.

We consider the somewhat more general equation

$$u_t + div\, g(u) = f; \quad u(0,x) = u_0(x), \tag{17}$$

together with the corresponding fractional conservation law

$$\frac{\partial}{\partial t}(k * [u - u_0]) + div\, g(u) = f. \tag{18}$$

Here $(t,x) \in R^+ \times R^n$. Observe that we do not require k to equal $g_{1-\alpha}$.

We need the concept of a weak solution and that of an entropy solution of (18), see [4].

A weak solution of (18) is a function $u \in L^1_{loc}(R^+ \times R^n)$ such that $g(u(t,x)) \in L^1_{loc}(R^+ \times R^n; R^n)$ and

$$\int_{R^+}\int_{R^n} \Big(\phi_t(t,x)\int_0^t k(t-s)[u(s,x) - u_0(x)]\,ds + \phi_x(t,x)\cdot g(u(t,x)) + \phi(t,x)f(t,x)\Big)\,dx\,dt = 0$$

for all $\phi \in C^\infty$ having compact support in $R \times R^n$.

To define an entropy solution of (18) we require the kernel k to be nonnegative and nonincreasing, and to satisfy $k(0+) = \infty$.

A function $u \in L^1_{loc}(R^+ \times R^n)$ such that $g(u(t,x)) \in L^1_{loc}(R^+ \times R^n; R^n)$ is an entropy solution of (18) if it is a weak solution of (18) and if for every $c \in R$ and for all nonnegative and nonincreasing functions k_1, k_2 such that $k(t) = k_1(t) + k_2(t)$ and $k_2(0+) < \infty$, the inequality

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\int_0^t & k_1(t-s)(|u(s,x)-c| - |u_0(x)-c|)\,ds \\ & + sign(u(t,x)-c)\big(k_2(0+)[u(t,x)-u_0(x)] \\ & + \int_{(0,t]} (u(t-s,x)-u_0(x))k_2'(ds)\big) \\ & + div\big(sign(u(t,x)-c)(g(u(t,x))-g(c)\big) \\ & \quad \leq sign(u(t,x)-c)f(t,x)\end{aligned}$$

holds in the sense of distributions.

We then have

THEOREM 3.3. *Let $g \in C^1(R;R^n)$, $f \in L^1_{loc}(R^+;L^1(R^n))$ and ess sup $_{x\in R}|f(t,x)| \in L^1_{loc}(R^+)$. Take $u_0 \in L^1(R^n) \cap L^\infty(R^n)$. Then there is an entropy solution u of (18) satisfying ess $sup_{x\in R^n}|u(t,x)| \in L^1_{loc}(R^+)$. If $f \in C(R^+;L^1(R^n))$, then $u \in C(R^+;L^1(R^n))$. This entropy solution is unique among all entropy solutions that satisfy*

$$\lim_{t\downarrow 0}\|u(t,\cdot) - u_0(\cdot)\|_{L^1(R^n)} = 0.$$

As to convergence there follows (see [4] for more generality):

THEOREM 3.4. *Take $f = 0$. Let $sup_u\|g'(u)\| < \infty$ and assume that $u_0 \in L^1(R^n) \cap BV(R^n)$. Let u_e be the entropy solution of (17) and let u_N be the entropy solution of (18), satisfying (19), with $k = k_N$; each k_N nonnegative and nonincreasing. Assume $\int_0^t k_N(s)\,ds \to 1$ for each $t > 0$ as $N \to \infty$. Then $\|u_e - u_N\|_{L^1(R^n)} \to 0$ as $N \to \infty$, uniformly on compact subsets of R^+.*

4. Concluding Remarks

As we have indicated above, significant results on fractional equations of type

$$D_t^\alpha(u - u_0) + Au = f,$$

with $\alpha \in (0,1)$ and A m-accretive (exemplified by $Au = \operatorname{div} g(u)$) are available.

It is equally true that results on nonlinear equations with $\alpha \in (1,2)$ are very scarce. One key reason is the lack of a result analogous to Theorem 3.1 above. Thus, in general, only the existence of weak solutions can be proved. Mostly this is done through energy estimates. However, the regularity of these weak solutions remains an open problem.

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