The Fibering Method and Its Applications to Nonlinear Boundary Value Problem

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Summary. - *We consider the fibering method, proposed in [1] for investigating some variational problems, and its applications to nonlinear elliptic equations. Let X and Y be Banach spaces, and let A be an operator (nonlinear in general) acting from X to Y. We consider the equation*

$$A(u) = h\,. \tag{1}$$

The fibering method is based on representation of solutions of equation (1) *in the form*

$$u = tv\,. \tag{2}$$

Here t is a real parameter ($t \neq 0$ in some open $J \subseteq \mathbb{R}$), and v is a nonzero element of X satisfying the condition

$$H(t, v) = c\,. \tag{3}$$

Generally speaking, any functional satisfying a sufficiently general condition can be taken as the functional $H(t, v)$. In particular, the norm $H(t, v) = \|v\|$ can be taken as such a fibering functional $H(t, v)$. Then condition (3) *takes the form $\|v\| = 1$: in this case we get a so-called spherical fibering; here a solution $u \neq 0$ of* (1) *is sought in the form* (2)*, where $t \in \mathbb{R} \setminus \{0\}$ and $v \in S = \{w \in X : \|w\| = 1\}$. Thus, the essence of the fibering method consists in imbedding the space X of the original problem* (1) *in the larger space $\mathbb{R} \times X$ and investigating the new*

problem of conditional solvability in the space $\mathbb{R}\times X$ under condition (3)*. This method makes it possible to get both new existence and nonexistence theorems for solutions of nonlinear boundary value problems. Moreover, in the investigation of solvability of boundary value problems this method makes it possible to separate algebraic and topological factors of the problem, which affect the number of solutions. The origin of this approach is in the calculus of variations, where this fibration arises in a natural way in investigating variational problems on relative extrema involving a given normalizing parameter $t \neq 0$. A description of the method of spherical fibering and some of its applications are given in [18], [19], [20], [21], [22], [12]. We remark that the elements of this method were used as far back as in [18] in establishing the "Fredholm alternative" for nonlinear odd and homogeneous (mainly) strongly closed operators.*

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Lecture 1

General theory of the one-parametric fibering method

1.1. Introduction

We begin this lecture with an example. Consider the following BVP

$$\begin{cases} \Delta_p u + \lambda|u|^{p-2}u = h(x) & \text{in}\Omega \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

Here Δ_p is the p-Laplacian:

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

with $p > 1$, $\lambda \in \mathbb{R}$ and $h \in W_{p'}^{-1}(\Omega) \equiv \left(\overset{\circ}{W}{}_p^1(\Omega)\right)^*$, $\frac{1}{p} + \frac{1}{p'} = 1$, where Ω is a bounded domain in $\mathbb{R}^N$. Introduce the definition of the spectrum

$$\sigma_p := \left\{\lambda \in \mathbb{R} \,\middle|\, \exists u \in \overset{\circ}{W}{}_p^1(\Omega) \setminus \{0\} \quad \Delta_p u + \lambda|u|^{p-2}u = 0 \text{ in } \Omega\right\}.$$

For $p = 2$ we have the classical linear BVP and the classical definition of the spectrum of the Laplace operator. Note that, for λ greater than the first eigenvalue $\lambda_1 > 0$, the appropriate nonlinear operator

$$A_p(u) \equiv \Delta_p u + \lambda|u|^{p-2}u$$

is not coercive. Thus the classical theory of nonlinear monotone coercive operators, developed by M. Vishik, F. Browder, G. Minty and other mathematicians (see for instance [15]) is not applicable to BVP (1.1) with $\lambda > \lambda_1$. In order to overcome this lack in 1967 there was developed in [18] the "Nonlinear Fredholm Alternative".

Let X be a reflexive Banach space with basis, and denote by X^* the conjugate space. Let A and T be operators acting from X into X^*. Consider the abstract nonlinear equation

$$A(u) + \lambda T(u) = h$$

with a scalar parameter λ. The following assumptions are made:

($a1$) A and T are odd positive homogeneous (in principal part) continuous operators;

($a2$) A is a strictly closed operator;

($a3$) T is a compact operator.

Define the spectrum of the *pair* (A, T) as

$$\sigma(A,T) := \left\{ \lambda \in \mathbb{R} \,\middle|\, \exists v \neq 0 : A(v) + \lambda T(v) = 0 \right\};$$

then we have the following statement.

THEOREM 1.1.1 (NONLINEAR FREDHOLM ALTERNATIVE). *Let A and T satisfy assumptions* ($a1$)–($a3$). *Then the equation*

$$A(u) + \lambda T(u) = h$$

for any $h \in X^*$ admits a solution $u \in X$ if

$$\lambda \notin \sigma(A,T).$$

EXAMPLE 1.1.2: If we apply this general abstract result to BVP (1.1) we obtain that for any $\lambda \notin \sigma(\Delta_p)$ there exists $u \in \overset{\circ}{W}{}^1_p(\Omega)$ which is a solution of (1.1).

REMARK 1.1.3: If $p = 2$ then Δ_p reduces to the Laplacian; in this case the Nonlinear Fredholm Alternative gives the same result as the classical (linear) Fredholm Alternative.

EXAMPLE 1.1.4: Consider the problem

$$\begin{cases} A(u) = h(x) & \text{in } \Omega \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where

$$A(u) \equiv -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)^3 + c\sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}\right)^3 + u^3 + a\sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}\right)^2 u\,.$$

Then the Nonlinear Fredholm Alternative implies that for any $a, c \in \mathbb{R}$ there exists a solution $u \in \overset{\circ}{W}{}^1_4(\Omega)$.

REMARK 1.1.5: Note that the operator A is not coercive for a suitable choice of a and c. We can take for instance $c = 0$, and $a < 0$ sufficiently large such that for a fixed $\phi \in \overset{\circ}{W}{}^1_4(\Omega)$:

$$-a\int_\Omega |\nabla\phi|^2\,\phi^2 > \int_\Omega |\nabla\phi|^4 + \int_\Omega \phi^4;$$

in this case we have $\langle A(t\phi), t\phi\rangle \to -\infty$ as $t \to +\infty$; thus A is not coercive, and then the classical theory of nonlinear monotone coercive operators cannot be applied.

In order to develope this approach to noncoercive nonlinear equations we used the elements of the fibering method. Now we give a short description of the main underlying ideas.

1. The first idea is the *extension* of the nonlinear operator: that is, instead of equation

$$A(u) = h \tag{1.2}$$

where A is acting between Banach spaces X and Y, we consider wider spaces $\tilde{X}$ and $\tilde{Y}$ and

$$\tilde{A} : \tilde{X} \to \tilde{Y} \quad \text{with} \quad \tilde{A}\Big|_X = A;$$

then we get the extended nonlinear equation

$$\tilde{A}(\tilde{u}) = \tilde{h}\,.$$

2. The second idea is the equipment of $\tilde{X}$ with *nonlinear* structure associated with the nonlinear operator A.

A further development of this approach is to construct, for a given triple (X, A, Y), a corresponding triple (ξ, α, η), where ξ and η are fibrations of the spaces X and Y respectively, and α is a morphism of ξ into η; the correspondence

$$(X, A, Y) \mapsto (\xi, \alpha, \eta)$$

is determined by the initial triple and, for given spaces X and Y, by the operator A from X into Y. If we take

$$\tilde{X} = \mathbb{R}^k \times X$$

we obtain the k-parametric fibering method. We begin with the simplest case, namely when $k = 1$: in this case we get the so-called "one-parametric fibering method". The one-parametric fibering method is based on representation of solutions for equation (1.2) in the form

$$u = tv \tag{1.3}$$

where t is a real parameter ($t \neq 0$ in some open set $J \subseteq \mathbb{R}$), and v is a nonzero element of the Banach space X satisfying the <u>fibering constraint</u>

$$H(t, v) = c\,. \tag{1.4}$$

Roughly speaking, any functional satisfying a sufficiently general condition (see Section 1.2) can be taken as the "fibering functional" $H(t, v)$. In particular we can take the norm $H(t, v) \equiv \|v\|$; then condition (1.4) reduces to $\|v\| = 1$, realizing a so-called "spherical fibering". Here a solution $u \neq 0$ of (1.2) is sought in the form (1.3), where $t \in \mathbb{R}$ and $v \in S = \{w \in X : \|w\| = 1\}$.

Thus, the essence of the one-parametric fibering method consists in imbedding space X of the original problem (1.2) in the larger space $\tilde{X} = \mathbb{R} \times X$ and investigating the new problem of conditional solvability under condition (1.4). This method makes it possible to get both new solvability theorems and new theorems on the absence of solutions for nonlinear BVPs. Further, in the investigation of solvability of BVPs this method makes it possible to separate the algebraic and the topological factors of the problem, which affect the number of its solutions.

1.2. The one-parametric fibering method

Let X be a real Banach space with a norm $\|w\|_X$ which is differentiable for $w \neq 0$, and let f be a functional on X of class $C^1(X \setminus \{0\})$. We associate with f a functional $\tilde{f}$ defined on $\mathbb{R} \times X$ by

$$\tilde{f}(t, v) = f(tv) \tag{1.5}$$

where $(t, v) \in J \times S$; here J is an arbitrary nonempty set in $\mathbb{R}$, and S is the unit sphere in X.

THEOREM 1.2.1. *Let X be a real Banach space with norm differentiable on $X \setminus \{0\}$, and let $(t, v) \in (J \setminus \{0\}) \times S$ be a conditionally stationary point of the functional $\tilde{f}$, regarded on $J \times S$. Then the vector $u = tv$ is a stationary (critical) point of the functional f, that is, $f'(u) = 0$.*

Proof. At the conditionally stationary point (t, v) we have

$$\lambda \tilde{f}'_v(t, v) = \mu \|v\|' \tag{1.6}$$

$$\tilde{f}'_t(t, v) = 0 \tag{1.7}$$

with $\lambda^2 + \mu^2 \neq 0$. Here the prime and subscript mean the derivative with respect to the corresponding variable (the derivative with respect to v is understood as the values of the derivative with respect to v in the space X for $v \in S$). By (1.6) we have

$$\lambda \langle \tilde{f}'_v(t, v), v \rangle = \mu \langle \|v\|', v \rangle$$

where $\langle w^*, u \rangle$ is the value of a functional w^* in the dual space X^* on an element u in X. Then from this and the equalities

$$\langle \tilde{f}'_v(t, v), v \rangle = \langle f'(tv), v \rangle = t \tilde{f}'_t(t, v)$$

$$\langle \|v\|', v \rangle = 1 \quad \text{for } v \in S$$

we get $\lambda t \tilde{f}'_t(t, v) = \mu$; it follows from this equality and (1.7) that $\mu = 0$. Then $\lambda \neq 0$ and by (1.6)

$$t f'(u) = \tilde{f}'_v(t, v) = 0$$

for $u = tv$, $t \neq 0$. Consequently, $f'(u) = 0$ and the theorem is proved. □

Now we consider a more general fibering: for this we introduce a fibering functional $H(t,v)$ defined on $\mathbb{R}\times X$ and we consider the functional $\tilde{f}(t,v)$ under condition (1.4). As $H(t,v)$ we can in general take an arbitrary functional, differentiable under condition (1.4) and satisfying

$$\langle H'_v, v\rangle \neq tH'_t \qquad \text{for } H(t,v)=c\,; \tag{1.8}$$

we will call (1.8) the nondegeneracy condition.

THEOREM 1.2.2. *Let H be a functional of the indicated class. Let $(t,v)\in J\times X$ with $tv\neq 0$ be a conditionally critical point of the functional $\tilde{f}(t,v)$ under condition* (1.4). *Then the point $u=tv$ is a nonzero critical point of the original functional f, i.e. $f'(u)=0$ and $u\neq 0$.*

Proof. At the conditionally critical point (t,v) we have

$$\mu\tilde{f}'_v(t,v)=\lambda H'_v(t,v)\ ,\quad \mu\tilde{f}'_t(t,v)=\lambda H'_t(t,v) \tag{1.9}$$

with $\mu^2+\lambda^2\neq 0$; on the other hand,

$$\tilde{f}'_v(t,v)=tf'(tv)\ ,\quad \tilde{f}'_t(t,v)=\langle f'(tv),v\rangle\,.$$

Then from (1.9) we get

$$\mu tf'(tv)=\lambda H'_v(t,v)\ ,\quad \mu\langle f'(tv),v\rangle=\lambda H'_t(t,v)\,. \tag{1.10}$$

From this we get

$$\begin{aligned}\mu t\langle f'(tv),v\rangle &= \lambda\langle H'_v(t,v),v\rangle\\ \mu t\langle f'(tv),v\rangle &= \lambda tH'_t(t,v)\end{aligned}$$

and consequently

$$\lambda\langle H'_v(t,v),v\rangle=\lambda tH'_t(t,v)$$

for $t\neq 0$ and $H(t,v)=c$. Then, by condition (1.8), we get $\lambda=0$ and hence $\mu\neq 0$. As a result, the first equation in (1.10) takes the form $f'(u)=0$ with $u=tv\neq 0$. □

1.3. Comparison with the Ljapunov-Schmidt approach

Now, we compare the fibering method with the classical Ljapunov-Schmidt method. We restrict our comparison with the spherical fibering method: in this case

$$H(t,v) \equiv \|v\|$$

and condition (1.8) with $c=1$ takes the form

$$\langle H'_v, v\rangle = \|v\| = 1 \neq tH'_t \equiv 0\,.$$

Due to the Ljapunov-Schmidt approach we seek for a solution of (1.2) in the form $u = u_1 + u_2$, where u_1 is an element of a (usually finite dimensional) subspace of X, and u_2 is an element of a suitable "good" complement. Then from (1.2) we get the system of equations

$$\left\{\begin{array}{rcl} A_1(u_1,u_2) & = & h_1 \\ A_2(u_1,u_2) & = & h_2 \end{array}\right. \tag{1.11}$$

where the second equation for fixed $v_1 \in X_1$ is a correct, well-posed equation which has a unique solution

$$u_2 = T(u_1,h_2)\,.$$

By substituting this expression in the first equation of the system, we derive the so-called Ljapunov-Schmidt bifurcation equation

$$\mathcal{A}(u_1,h_2) = h_1 \tag{1.12}$$

where

$$\mathcal{A}(u_1,h_2) = A_1(u_1,T(u_1,h_2))\,.$$

Following the spherical fibering method we seek for a solution of the variational problem

$$f'(u) = 0$$

in the form $u = tv \neq 0$ with $(t,v) \in \mathbb{R}\times S$. Then the original variational problem is equivalent to the system

$$\left\{\begin{array}{l} \langle f'(tv), v\rangle = 0 \\ f'_\tau(tv) = 0 \qquad \text{for } v \in S\,. \end{array}\right. \tag{1.13}$$

(here f'_τ is the tangential derivative of f on the unit sphere S). The first equation of (1.13), namely,

$$\langle f'(tv), v\rangle = 0 \tag{1.14}$$

plays the same role as the bifurcation equation in the Ljapunov-Schmidt approach; therefore we will refer to it as to the bifurcation equation in the fibering method. Indeed, if we have a solution $t = t(v)$ of this equation then we get the induced functional

$$\hat{f}(v) := f(t(v)v).$$

The conditionally critical point $v_c \in S$ of $\hat{f}$ with $t_c = t(v_c) \neq 0$ generates a critical point $u_c = t_c v_c$ of the original functional f.

From a geometrical point of view:

- in the Ljapunov-Schmidt approach, the representation $u = u_1 + u_2$ corresponds to introduction of Cartesian coordinates;
- in the spherical fibering method, the representation $u = tv$ with $\|v\| = 1$ corresponds to introduction of curvilinear (spherical) coordinates.

Lecture 2

Simple examples

Simple examples of known problems. In these examples the bifurcation equation (1.14) admits an explicit smooth solution $t = t(v)$ for $v \in S$: this makes it possible to use a parameter-free realization of the spherical fibering method. In all examples below, Ω is a bounded domain in $\mathbb{R}^N$ with locally Lipschitz boundary $\partial\Omega$. The solutions of the problems are considered in the Sobolev space $\overset{\circ}{W}{}^1_2(\Omega)$, the dual space of which is denoted by $W_2^{-1}(\Omega)$.

EXAMPLE 2.0.1: Consider the eigenfunction problem

$$\begin{cases} \Delta u + |u|^{p-2}u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.1}$$

Here $2 < p < 2^*$, where

$$\begin{aligned} 2^* &:= \frac{2N}{N-2} && \text{for} N > 2, \\ 2^* &:= \infty && \text{for } N = 2\,. \end{aligned}$$

The Euler functional f has the form

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p \ ;$$

according to the fibering method, we set $u = tv$; then the functional f takes the form

$$f(tv) = \frac{t^2}{2}\int_\Omega |\nabla v|^2 - \frac{|t|^p}{p}\int_\Omega |v|^p\,.$$

In the spherical fibering

$$\|v\| = \left(\int_\Omega |\nabla v|^2\right)^{1/2} = 1$$

the functional f reduces to

$$\tilde{f}(t,v) = \frac{t^2}{2} - \frac{|t|^p}{p}\int_\Omega |v|^p\,.$$

From the bifurcation equation $\tilde{f}'_t(t,v) = 0$, i.e.

$$t - |t|^{p-2}t\int_\Omega |v|^p = 0,$$

we find explicitly the real nonzero solutions

$$t = \pm\left(\int_\Omega |v|^p\right)^{\frac{1}{2-p}};$$

then the functional $\hat{f}(v) = \tilde{f}(t(v),v)$ takes the form

$$\hat{f}(v) = \frac{p-2}{2p}\left(\int_\Omega |v|^p\right)^{-\frac{2}{p-2}}.$$

To this functional, regarded on the unit sphere $S \subset \overset{\circ}{W}{}^1_2(\Omega)$, we can apply the well-known Lyusternik-Shnirel'man theory, in view of which $\hat{f}$ has a countable set of geometrically different conditionally critical points $v_1, v_2, v_3, \dots$ on S, with $\hat{f}(v_m) \to \infty$ (and then $\int_\Omega |v_m|^p \to 0$) as $m \to \infty$. Hence we obtain problem (2.1) has a countable set of geometrically different solutions $\pm u_1, \pm u_2, \dots, \pm u_m, \dots$ $\blacksquare$ with

$$u_m(x) = \frac{v_m(x)}{\left(\int_\Omega |v_m|^p\right)^{\frac{1}{p-2}}}$$

and $\|u_m\| \to \infty$ as $m \to \infty$.

REMARK 2.0.2: If we start from the astrophysical meaning of the Emden-Fowler equation (2.1), and consider a solution u_m as a "star" in the Sobolev space $\overset{\circ}{W}{}^1_2(\Omega)$, then the set of all solutions of (2.1) looks like an "expanding Universe": indeed, since $\|u_m\| \to \infty$ as $m \to \infty$, for any $R > 0$ there exists a "star" u_m such that $\|u_m\| > R$. From the mathematical point of view it means that the BVP

$$\begin{cases} \Delta u + |u|^{p-2}u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

for $2 < p < 2^*$ doesn't admit a priori estimates: consequently the Leray-Schauder method cannot be applied to this problem.

EXAMPLE 2.0.3: Consider the linear Dirichlet problem for the Poisson equation

$$\begin{cases} \Delta u = h & \text{in } \Omega \\ u = 0 & \text{on} \partial\Omega \end{cases} \tag{2.2}$$

with $h \in W_2^{-1}(\Omega) \setminus \{0\}$. The functional associated with (2.2) is

$$f(u) = -\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega hu\,.$$

Following the fibering method,we set $u = tv$; then the functional f takes the form

$$f(tv) = -\frac{t^2}{2}\int_\Omega |\nabla v|^2 - t\int_\Omega hv\,.$$

In the spherical fibering

$$\|v\|^2 = \int_\Omega |\nabla v|^2 = 1$$

the functional $\tilde{f}$ equals to

$$\tilde{f}(t,v) = -\frac{t^2}{2} - t\int_\Omega hv \tag{2.3}$$

and then from the bifurcation equation

$$\tilde{f}'_t(t,v) = -t - \int_\Omega hv = 0$$

we find $t = -\int_\Omega hv$, and then

$$\hat{f}(v) := \tilde{f}(t(v), v) = \frac{1}{2}\left(\int_\Omega hv\right)^2 \tag{2.4}$$

(note that in this case the minimax realization of the fibering method – see Section 3.1 – would give rise to the same functional $\hat{f}$). We now consider the critical points of this even functional $\hat{f}$ on the unit sphere S. Obviously, there exists an infinite set of conditionally critical points of $\hat{f}$ on the unit sphere. In this set there are only two *regular* conditionally critical points v_1 and $v_2 = -v_1$, i.e. conditionally critical points such that $t_1 = t(v_1) \neq 0$ and $t_2 = t(v_2) \neq 0$: these are the points at which $\hat{f}(v)$ attains the maximum on the closed unit ball B (v_1 and v_2 cannot lie in the interior part of B because $\hat{f}$ is homogeneous on v; see also the maximum principle expressed by Corollary 3.4.4). Then $u_1 = t_1 v_1$ and $u_2 = t_2 v_2$ are solutions of the Dirichlet problem (2.2); note in particular that, since $t_1 = -t_2$ and $v_1 = -v_2$, we actually obtain $u_1 = u_2$, that is, the two nonzero solutions coincide.

Remark 2.0.4: Example 2 can be regarded as an application of Theorems 3.3.1 and 3.4.3, which will be stated in the next lecture; to clear up the essence of the fibering method we verify assumptions of these theorems in this example. We know $v_1 \in S$ is a maximum point of $\hat{f}$ on the unit sphere S; then, by Lagrange rule, at this point it is

$$h\int_\Omega hv_1 = -\nu\Delta v_1 \ , \qquad v_1 \in \overset{\circ}{W}{}^1_2(\Omega).$$

From this we find for $\int_\Omega |\nabla v_1|^2 = 1$ that $\nu = \left(\int_\Omega hv_1\right)^2$, and $\nu \neq 0$, because $\max\limits_{v\in S} \hat{f}(v) > 0$ for $h \neq 0$. Then

$$h\int_\Omega hv_1 = -\left(\int_\Omega hv_1\right)^2 \Delta v_1$$

or, setting $t_1 = -\int_\Omega hv_1 \neq 0$ and $u_1 = t_1 v_1$,

$$\Delta u_1 = h \ , \qquad u_1 \in \overset{\circ}{W}{}^1_2(\Omega),$$

i.e., u_1 is a solution of problem (2.2). We see similarly that u_2 is a solution of this problem, and $u_1 = u_2$.

EXAMPLE 2.0.5: In the above we have considered some applications to the global analysis of certain nonlinear BVPs. It is clear that the fibering method can be applied to the local analysis of certain nonlinear variational problems: we restrict ourselves to a simple example. Let us consider the following BVP

$$\begin{cases} \Delta u + u^3 = h(x) & \text{in } \Omega \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

with $N < 4$. To this BVP there corresponds the Euler functional

$$E(u) = -\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4}\int_\Omega u^4 - \int_\Omega hu\,.$$

Due to spherical fibering we have $u(x) = tv(x)$ with

$$\|v\|^2 = \int_\Omega |\nabla u|^2 = 1 \quad \text{for } v \in \overset{\circ}{W}{}^1_2(\Omega).$$

Then the Euler functional E generates

$$\tilde{E}(t,v) = -\frac{t^2}{2} + \frac{t^4}{4}\int_\Omega v^4 - t\int_\Omega hv$$

and the bifurcation equation takes the form

$$\frac{d\tilde{E}}{dt} \equiv -t + t^3\int_\Omega v^4 - \int_\Omega hv = 0\,.$$

An elementary calculation shows that if the inequality

$$\sup_{v\in S}\left\{\left|\int_\Omega hv\right|\left(\int_\Omega |v|^4\right)^{1/2}\right\} < \frac{2}{3\sqrt{3}}$$

is satisfied, then the bifurcation equation possesses three isolated smooth branches of solutions: $t_1 = t_1(v,h)$, $t_2 = t_2(v,h)$ and $t_3 = t_3(v,h)$. By substituting them in $\tilde{E}$ we get three induced functionals

$$\begin{aligned} \hat{E}_i(v) &= \tilde{E}(t_i(v,h),v) = \\ &= -\frac{1}{2}t_i^2(v,h) + \frac{1}{4}t_i^4(v,h)\int_\Omega v^4 - t_i(v,h)\int_\Omega hv \end{aligned}$$

for $i = 1, 2, 3$; these functionals are distinct and smooth on S. Every functional $\hat{E}_i(v)$ has a critical point v_i on S; hence the original Euler functional E under condition on h possesses three distinct critical points, i.e. solutions of (2.5),

$$u_1(x) = t_1 v_1(x), \quad u_2(x) = t_2 v_2(x), \quad u_3(x) = t_3 v_3(x)$$

in $\overset{\circ}{W}{}_2^1(\Omega)$ such that

$$\int_\Omega h u_1 \leq 0, \quad \int_\Omega h u_2 \leq 0, \quad \int_\Omega h u_3 \geq 0,$$

and $\|u_1\| < \|u_2\|$.

REMARK 2.0.6: For sufficiently small h the existence of a first solution u_1 for (2.5) can be proved via contraction mapping principle. The existence of a second solution u_2 can be obtained from

$$\begin{cases} \Delta w + (w + u_1)^3 - u_1^3 = 0 \\ w = 0 \quad \text{on} \partial\Omega \end{cases}$$

by means of the theory of eigenfunctions for nonlinear elliptic problems. However, I don't know how is it possible to prove the existence of a third distinct solution u_3 without using the fibering method.

Lecture 3

Some realizations of the fibering method in variational problems

The fibering method admits various manners of realization; here we consider some of these ones.

3.1. Minimax realization

Let X be a real Banach space with norm differentiable on $X \setminus \{0\}$, let f be a functional on X belonging to the class $C^1(X \setminus \{0\})$, denote by S the unit sphere of X, and let J be a nonempty open subset of $\mathbb{R}$. Then the following result holds.

THEOREM 3.1.1. *Suppose that for any $v \in S$ the quantity*

$$\hat{f}(v) = \max_{t \in J} f(tv) \tag{3.1}$$

exists, and $\hat{f}(v) > f(0)$ if $0 \in J$. Assume that $\hat{f}$ is differentiable on the unit sphere S. Then to each conditionally stationary point v_c of the functional $\hat{f}$, regarded on S, there corresponds a stationary point $u_c = t_c v_c$ of f with $t_c \in J \setminus \{0\}$ such that $f(u_c) = \hat{f}(v_c)$.

Proof. Assume the theorem is false, and hence $t_c f'(u_c) \neq 0$. Then there exists $w_0 \in X$ such that

$$t_c \langle f'(u_c), w_0 \rangle > 0\,. \tag{3.2}$$

Since $v_c \in S$ is a conditionally stationary point of the functional f, which is differentiable on S, it follows that

$$\hat{f}\left(\frac{v_c + \zeta w_0}{\|v_c + \zeta w_0\|}\right) = \hat{f}(v_c) + \zeta\epsilon(\zeta) = f(t_c v_c) + \zeta\epsilon(\zeta) \tag{3.3}$$

for sufficiently small ζ, with $\epsilon(\zeta) \to 0$ as $\zeta \to 0$. On the other hand, by (3.1),

$$f\left(t\frac{v_c + \zeta w_0}{\|v_c + \zeta w_0\|}\right) \le \hat{f}\left(\frac{v_c + \zeta w_0}{\|v_c + \zeta w_0\|}\right) \qquad \forall t \in J\,. \tag{3.4}$$

By a condition of the theorem, $\max\limits_{t \in J} f(tv_c) = f(t_c v_c)$ is attained on the open set $J \setminus \{0\}$; hence $t_c \|v_c + \zeta w_0\| \in J \setminus \{0\}$ for sufficiently small ζ, because $\|v_c\| = 1$. Then, by setting $t = t_c \|v_c + \zeta w_0\|$ in (3.4) for sufficiently small ζ, we get by (3.3) that

$$f(t_c v_c + \zeta t_c w_0) \le f(t_c v_c) + \zeta\epsilon(\zeta). \tag{3.5}$$

Since f is differentiable,

$$f(t_c v_c + \zeta t_c w_0) = f(u_c) + \zeta t_c \langle f'(u_c), w_0 \rangle + \zeta\epsilon_1(\zeta),$$
$$\epsilon_1(\zeta) \to 0 \text{ as } \zeta \to 0.$$

Then it follows from (3.5) that

$$\zeta t_c \langle f'(u_c), w_0 \rangle \le \zeta\epsilon_2(\zeta)\ , \qquad \epsilon_2(\zeta) \to 0 \text{ as } \zeta \to 0.$$

From this last inequality, for sufficiently small $\zeta > 0$, we get

$$t_c \langle f'(u_c), w_0 \rangle \le 0\,;$$

in view of (3.2) this contradicts the assumption. The theorem is proved. □

Remark 3.1.2: Let J be a nonempty open set in $\mathbb{R}$, symmetric with respect to zero. If the functional $\hat{f}$ defined by (3.1) exists, then it is *even*: this makes it possible to use the Lyusternik-Shnirel'man theory for certain functionals that are not even, and obtain theorems on the existence of many geometrically different stationary points.

EXAMPLE 3.1.3: Consider again the linear Dirichlet problem (2.2):

$$\begin{cases} \Delta u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The Euler functional associated to this problem is

$$f(u) = -\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega hu\,.$$

In the spherical fibering

$$u = tv\ , \quad \|v\|^2 = \int_\Omega |\nabla v|^2 = 1$$

the functional f reduces to

$$\tilde{f}(t,v) = -\frac{t^2}{2} - t\int_\Omega hv\,.$$

Then, the minimax realization gives rise to the functional

$$\hat{f}(v) = \max_{t\in\mathbb{R}} \tilde{f}(t,v) = \frac{1}{2}\left(\int_\Omega hv\right)^2\ , \quad t_{\max} = -\int_\Omega hv\,.$$

which is the same as the $\hat{f}$ defined by (2.4). So: the original *not even* Euler functional f generates by the minimax realization of the fibering method the *even* functional $\hat{f}$.

REMARK 3.1.4 (TO EXAMPLE 3.4.5): From Lyusternik-Shnirel'man theory we know the even weakly continuous functional $\hat{f}$ possesses at least a countable set of critical points on S. Thanks to the fibering method we have to expect a countable set of solutions for BVP (2.2), but we know this problem has only one solution; what's the matter? Let us consider this "contradiction" in more detail. The even functional

$$\hat{f} = \frac{1}{2}\left(\int_\Omega hv\right)^2$$

has actually a *continuous* set of critical points on S; indeed, any $v \in S$, such that $\int_\Omega hv = 0$, is a critical point of $\hat{f}$, because in this case

$$\hat{f}'(v) = v\int_\Omega hv = 0;$$

thus the equator

$$\mathcal{E}_0 := S \cap \{h\}^{\perp} = \left\{ v \in S : \int_\Omega hv = 0 \right\}$$

is the critical set for $\hat{f}$. But for $v \in \mathcal{E}_0$ we have

$$t = t(v) = -\int_\Omega hv = 0,$$

and consequently $u = tv = 0$: that is, these critical points are **invisible** with respect to f. On the other hand, as we have already seen, $\hat{f}$ admits also a pair of conditionally critical points

$$\begin{aligned} v_+ \quad &: \quad -t(v_+) = \int_\Omega hv_+ = \max_{v \in S} \int_\Omega hv > 0 \\ v_- \quad &: \quad -t(v_-) = \int_\Omega hv_- = \min_{v \in S} \int_\Omega hv < 0 \end{aligned}$$

which give rise to a **visible** "double" solution $u_+ = u_-$. Thus, we can reasonably expect that a perturbed nonlinear BVP may admit a countable set of visible solutions; indeed, the BVP

$$\begin{cases} \Delta u + \epsilon |u|^\delta u = h(x) \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

for any sufficiently small $\epsilon, \delta > 0$ possesses a countable set of solutions $u_1, u_2, \ldots, u_k, \ldots$ with $\|u_k\| \to +\infty$ as $k \to \infty$.

Before the next example we point out an immediate corollary of Theorem 3.1.1:

THEOREM 3.1.5. *Let X be an infinite-dimensional reflexive Banach space. Let the functional*

$$\hat{f}(v) = \max_{t \in J} f(tv) > 0$$

(or $1/\hat{f}$) satisfy the Lyusternik-Shnirel'man conditions (in any version of this theory). Then the functional f admits at least a countable set of distinct critical points.

EXAMPLE 3.1.6: Consider the BVP

$$\begin{cases} -\Delta u - u^3 + \left(\int_\Omega hv\right)^2 h = 0 \quad \text{in } \Omega \subset \mathbb{R}^N,\ N \leq 3 \\ \\ u = 0 \quad \text{on } \partial\Omega \end{cases} . \tag{3.6}$$

Though the Euler functional

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{4}\int_\Omega u^4 + \frac{1}{3}\left(\int_\Omega hu\right)^3$$

is not even, by means of minimax realization we obtain the even functional

$$\hat{f}(v) = \max_{t\in\mathbb{R}} \tilde{f}(t,v),$$

where

$$\tilde{f}(t,v) = \frac{t^2}{2} - \frac{t^4}{4}\int_\Omega v^4 + \frac{t^3}{3}\left(\int_\Omega hv\right)^3;$$

then by Theorem 3.1.5 we get problem (3.6) admits a countable set of solutions in $\overset{\circ}{W}{}^1_2(\Omega)$ for each $h \in W_2^{-1}(\Omega)$, since:

- $\hat{f} \in C^1(S)$;
- $\hat{f}(-v) = \hat{f}(v)$;
- $\hat{f} > 0$ on S;
- $\hat{f}$ is weakly continuous on S.

3.2. The choice of the fibering functional

Realization of the fibering method depends evidently on the choice of the fibering functional $H(t,v)$ satisfying condition (1.8). As the simplest of such functionals we can take the norm of the Banach space, under the condition that it shall be differentiable away from zero; but this choice is not unique: here we propose, as a fibering functional, the functional naturally generated by the problem itself (i.e. generated by the Euler functional f). We remark, however, that in many cases this choice is not necessary. The scalar equation

$$\langle f'(tv), v\rangle = 0 \tag{3.7}$$

in the scalar parameter $t = t(v)$ is the defining bifurcation equation in the fibering method. To separate algebraically different solutions of this equation, it seems natural to take the following as a fibering functional H in the case of a functional f of class $C^3(X \setminus \{0\})$:

$$H(t, v) \equiv \langle f''(tv)v, v\rangle. \tag{3.8}$$

In this case H satisfies the relation

$$\langle H'_v, v\rangle - tH'_t = 2H\,; \tag{3.9}$$

in fact, by the equalities

$$\begin{aligned} \langle H'_v, v\rangle &= \frac{d}{d\zeta}H(t,\zeta v)\Big|_{\zeta=1} \\ tH'_t &= \frac{d}{d\zeta}H(\zeta t, v)\Big|_{\zeta=1} \\ H(t,\zeta v) &= \zeta^2\langle f''(\zeta tv)v, v\rangle \end{aligned}$$

we get

$$\begin{aligned} \langle H'_v, v\rangle &= \frac{d}{d\zeta}H(t,\zeta v)\Big|_{\zeta=1} = \\ &= 2\langle f''(tv)v, v\rangle + \frac{d}{d\zeta}\langle f''(tv)v, v\rangle\Big|_{\zeta=1} = 2H + tH'_t\,. \end{aligned}$$

By (3.9) the nondegeneracy condition (1.8), for the functional H defined by (3.8), turns out to hold always. When such a functional is chosen, the solution $t = t(v)$ of (3.7) a priori inherits the smoothness of the original functional, with loss of one derivative. Accordingly, the problem of finding critical points for $f \in C^3(X \setminus \{0\})$ reduces to the problem of finding conditionally critical points for $f(tv)$ under the condition

$$\langle f''(tv)v, v\rangle = c \neq 0\,, \quad t \in J\,.$$

Set $k := \sqrt{|c|} \neq 0$; then by substituting t/k and kv in place of t and v respectively, the last equality can be re-written as

$$\langle f''(tv)v, v\rangle = c_0\ , \quad t \in kJ\,, \tag{3.10}$$

where c_0 is either $+1$ or -1, and $kJ := \{kt : t \in J\}$.

THEOREM 3.2.1. *Let f be a functional on X of the class $C^3(X\backslash\{0\})$, and let $(t, v) \in kJ \times X$ with $tv \neq 0$ be a conditionally critical point of the functional $f(tv)$ under condition* (3.10)*. Then the point $u = tv$ is a nonzero critical point of f.*

The proof follows immediately from Theorem 1.2.2.

REMARK 3.2.2: If we apply the general formula (3.8) to the linear problem

$$\begin{cases} \Delta u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

we obtain

$$\begin{aligned} H(t,v) &= \frac{d^2}{dt^2} f(tv) = \\ &= \frac{d^2}{dt^2}\left(-\frac{t^2}{2}\int_\Omega |\nabla v|^2 - t\int_\Omega hv\right) = \\ &= -\int_\Omega |\nabla v|^2 \end{aligned}$$

and then our general condition (3.10) takes the form

$$\int_\Omega |\nabla v|^2 = 1\,;$$

that is, in this case the general constructive formula (3.8) leads to *spherical fibering.*

REMARK 3.2.3: The fibering functional H defined by (3.8) enables us to separate convex from concave nonlinearities, since formula (3.8) involves the second derivative of f:

- convex nonlinearity corresponds to $H(t, v) = +1$;
- concave nonlinearity corresponds to $H(t, v) = -1$.

3.3. A parameter-free realization of the fibering method

In the general case the fibering method reduces the original variational problem to a parametric variational problem and to the investigation of its conditionally critical points. However, when the fibering functional H is defined by (3.8) it is possible to eliminate the parameter t in the new variational problem. Indeed, condition (3.10) for a functional f of class $C^3(X \setminus \{0\})$ means that

$$\frac{d}{dt}\langle f'(tv), v\rangle = \langle f''(tv)v, v\rangle \neq 0$$

on the set defined by (3.10). Thus, condition (3.10) for a functional f of class $C^3(X \setminus \{0\})$ enables us to single out in the bifurcation equation (3.7) the algebraically different smooth solutions $t_i(v)$ (for $i = 1, \dots, m$) when they exist, and to eliminate the scalar parameter t. The problem of finding the nonzero critical points of $f \in C^3(X \setminus \{0\})$ reduces to the problem of finding conditionally critical points of the functionals

$$\begin{cases} F_i(v) := f\,(t_i(v)v) \\ \text{under condition} \\ H_i(v) := \langle f''(t_i(v)v)v, v\rangle = c_0 \end{cases} \tag{3.11}$$

with $c_0 = \pm 1$; here $t_i(v)$ is the corresponding solution of (3.7) for $i = 1, \dots, m$.

THEOREM 3.3.1. *Let f be a functional defined on X, $f \in C^3(X \setminus \{0\})$; let $t_i(v)$ be the solution of* (3.7) *under condition* (3.10)*, and let v_i be a conditionally critical point of problem* (3.11) *with $t_i(v_i) \neq 0$. Then the point $u_i = t_i(v_i)v_i$ is a nonzero critical point of f.*

The proof follows from Theorem 3.2.1, since the pair $(t_i, v_i) \in kJ \times X$ satisfies the conditions of that theorem.

3.4. Fibering functionals of norm type

In studying a variational problem it is sometimes convenient, when the principal part of the original Euler functional f is of norm type, to take this principal part as the fibering functional H. Then, if there is a complementary weakly continuous functional, the original problem reduces to the problem of investigating conditionally critical points of a continuous functional on a sphere-type surface (it is simpler to investigate a variational problem in a closed ball, because it is a convex set). We present a class of variational problems in which this approach can be implemented. Suppose that on a Banach space X, with norm differentiable away from zero, the given functional $f \in C^1(X)$ has the form $f(u) = f_0(u) + f_1(u)$, where $f_0(u)$ generates the norm of X; for definiteness we assume $f_0(u) = \|u\|^p$ for some $p > 1$, and $f_1 \in C^1(X)$. Then we choose the fibering functional $H(v) = f_0(v) = \|v\|^p$, so that condition (1.4) takes the form $\|v\| = 1$ (i.e., we use the method of spherical fibering). The functional $f(tv)$ takes the form

$$\tilde{f}(t, v) = |t|^p + f_1(tv) \qquad \text{for } v \in S$$

and the problem

$$f_0'(u) + f_1'(u) = 0 \tag{3.12}$$

is then equivalent to the system

$$p|t|^{p-2}t + \langle f_1'(tv), v\rangle = 0\,, \quad t \neq 0 \tag{3.13}$$

$$tf_1'(tv) = \nu\|v\|'\,, \quad v \in S. \tag{3.14}$$

Since the functional $\langle f_1'(tv), v\rangle$ is defined for all $v \in X$, the first scalar equation (3.13) in t can be considered for $v \in B = \{w \in X : \|w\| \leq 1\}$. Suppose that this equation has solutions $t_i(w)$ for $i = 1, \dots, N$; let

$$F_i(w) = |t_i|^p + f_1\left(t_i(w)w\right)$$

and consider these functionals on the closed unit ball B.

Definition 3.4.1. *A point $w \in B$ is a critical point of the differentiable functional $F_i(w)$ in the closed unit ball B if one of the following conditions holds:*

1. *w lies in the interior part of B and it is an ordinary critical point of F_i;*

2. w lies on the boundary $\partial B = S$ and it is a conditionally critical point of F_i on the sphere S.

DEFINITION 3.4.2. *A critical point $w_i \in B$ of the differentiable functional $F_i(w)$ is a regular critical point of F_i if $w_i \neq 0$, $t_i(w_i) \neq 0$, and the functional t_i is differentiable at w_i. Here $t_i(w)$ is a solution of* (3.13) *for $v = w \in B$.*

THEOREM 3.4.3. *Let $w_i \in B$ be a regular critical point of $F_i(w)$. Then $w_i \in \partial B$, and $u_i = t_i(w_i)w_i$ is a nonzero solution of* (3.12).

Proof. Suppose by contradiction that the regular critical point $w_i \in B$ of F_i is in the interior part of the unit ball B. We study the behaviour of F_i along the ray ζw_i as $\zeta \to 1$. By differentiability of f and differentiability of t_i at w_i, it is

$$\left.\frac{dF_i(\zeta w_i)}{d\zeta}\right|_{\zeta=1} = \left[p|t_i|^{p-2}t_i + \langle f_1'(t_i w_i), w_i\rangle\right] \left.\frac{dt_i}{d\zeta}\right|_{\zeta=1} + \langle f_1'(t_i w_i), w_i\rangle t_i$$

where $t_i = t_i(w_i)$. Hence, in view of (3.13) with $v = w_i$,

$$\left.\frac{dF_i(\zeta w_i)}{d\zeta}\right|_{\zeta=1} = t_i\langle f_1'(t_i w_i), w_i\rangle = -p|t_i|^p \neq 0,$$

which contradicts the fact that w_i is a (regular) critical point: therefore $w_i \in \partial B$. Moreover, by Theorem 1.2.1 the point $u_i = t_i w_i$ is a nonzero solution of (3.12). □

Following Definitions 3.4.1 and 3.4.2, we introduce the concept of a regular extremal point $w \in B$ for the functional F_i by replacing in those definitions the word “critical” by the word “extremal”. Then from Theorem 3.4.3 we get

COROLLARY 3.4.4 (THE MAXIMUM PRINCIPLE). *Let $w_i \in B$ be a regular extremal point of $F_i(w)$. Then $w_i \in \partial B$, and $u_i = t_i(w_i)w_i$ is a nonzero solution of* (3.12).

EXAMPLE 3.4.5: Here we demonstrate the above “maximum principle” in the simplest situation. Consider the linear problem (2.2)

$$\Delta u = h, \quad u = 0 \text{ on } \partial\Omega$$

and the correspondent Euler functional

$$f(u) = -\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega hu.$$

Then the reduced functional is

$$\hat{f}(v) = \frac{1}{2}\left(\int_\Omega hv\right)^2, \quad v \in \overset{\circ}{W}{}^1_2(\Omega), \ \|v\| = 1.$$

Now we consider B instead of S, i.e. $\|v\| \leq 1$ instead of $\|v\| = 1$; in this case we know from classical results that the functional $\hat{f}$ admits a maximum point v_0 in the convex bounded domain B. Actually, v_0 is on the boundary $S = \partial B$: in fact, if we suppose $\|v_0\| < 1$ then for sufficiently small $\epsilon > 0$ it is still $(1+\epsilon)v_0 \in B$ and

$$\hat{f}((1+\epsilon)v_0) = (1+\epsilon)^2 \hat{f}(v_0) > \hat{f}(v_0)\,,$$

which contradicts the fact that v_0 is a maximum point. Therefore $v_0 \in S$.

3.5. On the connection between critical points and conditionally critical points

In the preceding sections we used the fibering method to establish a connection between critical points and conditionally critical points of functionals; we now consider this connection from a somewhat different point of view. Let l be a differentiable mapping (which can also be a nonzero constant) from a real Banach space X into X itself. With any functional f twice differentiable (in the Gâteaux sense) on X we associate the functional f_l defined by

$$f_l(u) := \langle f'(u), l(u)\rangle.$$

Clearly, every critical point u of f satisfies $f_l(u) = 0$ and hence it is a conditionally critical point of f, considered under the condition $f_l(u) = 0$; the following simple theorem gives a condition for the validity of the converse.

THEOREM 3.5.1. *Let f be a twice differentiable (in the Gâteaux sense) functional on a real Banach space X. Suppose that there exists a differentiable mapping l from X to X such that at a conditionally critical point u_0 of $f(u)$, considered under the constraint $f_l(u) = 0$, it is*

$$\langle f_l'(u_0), l(u_0)\rangle \neq 0. \tag{3.15}$$

Then the conditionally critical point u_0 is actually an unconditionally critical point of f.

Proof. Indeed, at u_0 it is

$$\lambda f'(u_0) = \mu f_l'(u_0)\ , \qquad \lambda^2 + \mu^2 \neq 0;$$

by this equality and (3.15) we get $\mu = 0$, and then $\lambda \neq 0$, since

$$\mu\langle f_l'(u_0), l(u_0)\rangle = \lambda\langle f'(u_0), l(u_0)\rangle = \lambda f_l(u_0) = 0;$$

therefore, $f'(u_0) = 0$. The theorem is proved. □

Lecture 4

Applications of the bifurcation equations

In this lecture we'll demonstrate the application of the bifurcation equations to various nonlinear BVPs in the simplest cases.

4.1. The algebraic factor

The bifurcation equation enables us to extract the algebraic factor of nonlinearities. Consider the bifurcation-fibering equation

$$\langle f'(tv), v\rangle = 0\,. \tag{4.1}$$

Let $t_i(v)$ (for $i = 1, 2, \ldots, k$) be the algebraic solutions of (4.1) under condition

$$\|v\| = 1\,;$$

then we obtain k functionals $f_1, \ldots, f_k$ defined by

$$\begin{aligned} f_1(v) &:= f(t_1(v)v) \\ f_2(v) &:= f(t_2(v)v) \\ &\vdots \\ f_k(v) &:= f(t_k(v)v)\,, \end{aligned}$$

for which the following result holds (cf. Theorem 3.11).

THEOREM 4.1.1. *Let $f_1, \dots, f_k \in C^1(S)$. Let v_i be a conditionally critical point of $f_i(v)$ with $t_i(v_i) \neq 0$. Then the point*

$$u_i = t_i(v_i)v_i$$

is a nonzero critical point of $f(u)$.

4.2. The problem of nontrivial solutions

Let Ω be a bounded domain in $\mathbb{R}^N$ with locally Lipschitz continuous boundary $\partial\Omega$. We consider the question of existence of nontrivial solutions for the boundary value problem

$$\begin{cases} \Delta u + g_1(x,u)u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

The conditions on the function g_1 are as follows.

(C1) $g_1(x,0) \equiv 0$, and g_1 is a Carathéodory function on $\Omega \times \mathbb{R}$, i.e. it is measurable with respect to x for all $u \in \mathbb{R}$ and it is continuous with respect to u for almost all $x \in \Omega$.

(C2) For $N \geq 2$ there exist positive constants A and B such that, for all $x \in \Omega$ and all $u \in \mathbb{R}$:

- for $N > 2$, $|g_1(x,u)| \leq A + B|u|^m$ where $0 \leq m < \frac{4}{N-2}$;
- for $N = 2$, $|g_1(x,u)| \leq A + B\, e^{|u|^\alpha}$ where $0 \leq \alpha < 2$.

(C3) For any function $v \in \overset{\circ}{W}{}_2^1(\Omega)$ with $\int_\Omega |\nabla v|^2 = 1$, i.e. for any v in the unit sphere S, the equation

$$\int_\Omega g_1\left(x, tv(x)\right) v^2(x)dx = 1 \tag{4.3}$$

in $t \in \mathbb{R}$ has a solution $t = t(v)$, and $t(v) \in C^1(S)$.

Let $t = t(v)$ be a solution of class $C^1(S)$; we consider the functional

$$F(v) = -\frac{t^2(v)}{2} + \int_\Omega G\left(x, t(v)v\right) dx\,, \tag{4.4}$$

where $G(x,s) = \int_0^s g_1(x,y)y\,dy\,.$

THEOREM 4.2.1. *Assume conditions **(C1)**, **(C2)** and **(C3)**. Suppose that the weakly continuous functional F defined in* (4.4) *admits a conditionally critical point v on the sphere S. Then $u = t(v)v$ is a nontrivial solution of problem* (4.2).

Proof. The proof follows directly from Theorem 3.4.3, since the regularity of a conditionally critical point of F on S follows from conditions **(C1)** and **(C3)**.

EXAMPLE 4.2.2: Consider for $N \leq 3$ the BVP

$$\begin{cases} \Delta u + a(x)u^2 + u^3 = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.5}$$

with $a \in L^q(\Omega)$, where $q = 1$ for $N = 1$, $q > 1$ for $N = 2$, $q > 2$ for $N = 3$. Then by Theorem 4.2.1 this BVP has a nontrivial solution $u \in \overset{\circ}{W}{}^1_2(\Omega)$.

□

Notice that the Euler functional associated to (4.5)

$$E(u) = -\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{3}\int_\Omega a \cdot u^3 + \frac{1}{4}\int_\Omega u^4$$

is not even. On the other hand, from (4.3) it follows that F is even *anyhow*: in fact, if to any $v_1 \in S$ there corresponds the solution $t_1 = t(v_1)$, then to $v_2 = -v_1$ there corresponds $t_2 = t(-v_1) = -t_1$; if this even functional F is smooth, then the Lyusternik-Shnirel'man theory can be applied to it, under appropriate conditions.

THEOREM 4.2.3. *Assume conditions **(C1)**, **(C2)** and **(C3)**. Suppose that the even functional F defined by* (4.4) *satisfies on S the Lyusternik-Shnirel'man conditions, in any version of this theory. Then the boundary value problem* (4.2) *has a countable set of geometrically different solutions.*

Proof. The existence of a countable set of geometrically different conditionally critical points for the even weakly continuous functional F on the unit sphere S follows from the Lyusternik-Shnirel'man theory. The regularity of each conditionally critical point of F on S follows from conditions **(C1)** and **(C3)**, since a solution of (4.3) at a conditionally critical point of F on S is nonzero and differentiable at this point. Then we get Theorem 4.2.3 from Theorem 3.4.3. □

EXAMPLE 4.2.4: Consider the BVP

$$\begin{cases} \Delta u + \lambda(x)u + \mu(x)u^{m-1}u = 0 & \text{in}\Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $1 < m < \frac{N+2}{N-2}$ for $N > 2$ and $m > 1$ for $N = 1, 2$. Denote by λ_1 the first eigenvalue of the Laplace operator in the domain Ω with Dirichlet boundary condition. We get by Theorem 4.2.3 that for any functions $\lambda, \mu \in C(\overline{\Omega})$, with $\lambda(x) < \lambda_1$ and $\mu(x) > 0$ in $\overline{\Omega}$, this problem has a countable set of geometrically different solutions in the Sobolev space $\overset{\circ}{W}{}^1_2(\Omega)$.

EXAMPLE 4.2.5: We consider for $N \leq 3$ the BVP

$$\begin{cases} \Delta u + a(x)|u|^{\alpha-1}u + u^3 = 0 & \text{in}\Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $1 < \alpha < 3$ and $a \in L^q(\Omega)$ (without any assumption on the *sign* of $a(x)$), where $q = 1$ for $N = 1$, $q > 1$ for $N = 2$, $q > \frac{6}{5-\alpha}$ for $N = 3$. Then by Theorem 4.2.3 this problem admits a countable set of geometrically different solutions in $\overset{\circ}{W}{}^1_2(\Omega)$.

REMARK 4.2.6 (TO EXAMPLE 4.2.5): We note that the above problem with $0 < \alpha < 1$ was already considered, by using another method called "linking method", by [7].

4.3. A problem with even nonlinearity

We consider an application of Theorem 3.4.3 to the following boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \leq 5$ and with smooth boundary $\partial\Omega$:

$$\begin{cases} \Delta\Phi + \Phi^2 = \phi(x) & \text{in } \Omega \\ \Phi = h_0(x) & \text{on } \partial\Omega \end{cases} \tag{4.6}$$

where $\phi \in W_2^{-1}(\Omega) = \left(\overset{\circ}{W}{}^1_2(\Omega)\right)^*$ and $h_0 \in W_2^{\frac{1}{2}}(\partial\Omega)$. Let h be a harmonic function in $W_2^1(\Omega)$ such that $\Delta h = 0$ in Ω and $h = h_0$ on $\partial\Omega$; then the original BVP is equivalent to the following one, by simply letting $\Phi = u + h$:

$$\begin{cases} \Delta u + (u + h)^2 = \phi & \text{in}\Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.7}$$

The Euler functional associated with (4.7) on the Sobolev space $\overset{\circ}{W}\ {}^1_2(\Omega)$ is $E = \frac{1}{2}H + G$, where

$$\begin{aligned} H(u) &:= \int_\Omega |\nabla u|^2 = \|u\|^2 \\ G(u) &:= \int_\Omega \left(-\frac{1}{3}(u+h)^3 + \phi u + \frac{1}{3}h^3 \right) \end{aligned}$$

(we choose H as the fibering functional); the bifurcation equation for $t(v)$ in this case takes the form

$$a(v)t^2 - b(v)t - c(v) = 0$$

where

$$\begin{aligned} a(v) &= \int_\Omega v^3 \\ b(v) &= 1 - 2\int_\Omega hv^2 \\ c(v) &= \int_\Omega (\phi - h^2)v\,. \end{aligned}$$

From this we get, for $a(v) = \int_\Omega v^3 \neq 0$,

$$\begin{aligned} t_\pm(v) &= \frac{b(v) \pm \sqrt{b(v)^2 + 4a(v)c(v)}}{2a(v)} \\ &= \left\{ 1 - 2\int_\Omega hv^2 \pm \left[\left(1 - 2\int_\Omega hv^2 \right)^2 + \right.\right. \\ &\qquad \left.\left. + 4\left(\int_\Omega v^3 \right) \int_\Omega (\phi - h^2)v \right]^{\frac{1}{2}} \right\} \left(2\int_\Omega v^3 \right)^{-1} \end{aligned} \tag{4.8}$$

and, accordingly,

$$\begin{aligned} F_\pm(v) &= E\left(t_\pm(v)v\right) \\ &= \frac{b(v)}{6}t_\pm^2(v) + \frac{2c(v)}{3}t_\pm(v) \\ &= \frac{1}{12a^2(v)}\Big(b^3(v) + 6a(v)b(v)c(v) \pm \\ &\qquad \pm (b^2(v) + 4a(v)c(v))^{3/2} \Big) \end{aligned} \tag{4.9}$$

We assume $b(v) > 0$; then the functional F_- is defined for all v in the closed unit ball B of the sapce $\overset{\circ}{W}{}_2^1(\Omega)$ (we obtain $F_-(v) = -c^2(v)/2b(v)$ as $a(v) = 0$), while F_+ is defined for all $v \in B$ such that $a(v) \neq 0$. Let us consider the behaviour of these functionals in the unit ball B as $\|v\| \to 0$ for $a(v) \neq 0$. We have

$$a(v) \to 0\,, \qquad b(v) \to 1\,, \qquad c(v) \to 0\,;$$

by Taylor's formula, up to the second order in $\zeta := \frac{a(v)c(v)}{b^2(v)}$, it is

$$F_\pm = \left(\frac{b^3}{12a^2} + \frac{bc}{2a}\right)(1 \pm 1) \pm \left[\frac{c^2}{2b} - \frac{ac^3}{3b^3}\,(1 + \sigma(\zeta))\right]$$

for $a(v), b(v) \neq 0$, with $\sigma(\zeta) \to 0$ as $\zeta \to 0$. Concerning the boundary function $h_0 \in W_2^{\frac{1}{2}}(\partial\Omega)$ we assume also that there exists a constant $C_0 > 0$ such that the corresponding harmonic function h satisfies for any $v \in B$

$$b(v) = 1 - 2\int_\Omega hv^2 \geq C_0\,;$$

this holds, in particular, if $h(x) \leq 0$. Concerning the function $\phi \in W_2^{-1}(\Omega)$ we assume that there exists a constant $C_1 > 0$ such that for any $v \in B$

$$b(v)^2 + 4a(v)c(v) = \left(1 - 2\int_\Omega hv^2\right)^2 + 4\left(\int_\Omega v^3\right)\int_\Omega (\phi - h^2)v \geq C_1\,;$$

this holds, in particular, if $\phi - h^2$ is sufficiently small in the norm of the dual space $W_2^{-1}(\Omega)$. Under our assumptions,

$$\sup_{v\in B} F_+(v) = \sup_{v \in S} F_+(v) = +\infty\,,$$

$$\inf_{v\in B} F_+(v) > -\infty\,, \qquad \inf_{v \in B} F_-(v) > -\infty\,.$$

We mention that in the case when $\phi = h^2$ a.e. in Ω the trivial solution $u \equiv 0$ is one of the solutions of problem (4.7); therefore, it is assumed below that

$$\|\phi - h^2\|_{W_2^{-1}} \neq 0$$

and then we get

$$\inf_{v\in B} F_-(v) < 0.$$

Further, the corresponding minimum points v_- and v_+ exist for the functionals F_- and F_+ in the unit ball $B \subset \overset{\circ}{W}{}_2^1(\Omega)$. For the functional F_- at the point $v_- \in B$ we have

$$F_-(v_-) = \inf_{v\in B} F_-(v) < 0;$$

then from representation (4.9) we get $v_- \neq 0$ and $t_-(v_-) \neq 0$. The functional F_- is differentiable at the point v_-; thus, the conditions of Theorem 3.4.3 are satisfied for F_-, and hence $u_- = t_-(v_-)v_-$ is a solution of the BVP (4.7) under the conditions on ϕ and h indicated above. Now, let us consider the functional F_+. For this functional at the point $v_+ \in B$ we have

$$F_+(v_+) = \inf_{v\in B} F_+(v) > -\infty;$$

then we get from representation (4.8), (4.9) for F_+ that $t_+v_+ \neq 0$. The functional F_+ is differentiable at the point v_+, and $a(v_+) = \int_\Omega v_+^3 \neq 0$; thus, the conditions of Theorem 3.4.3 hold for F_+, and hence $u_+ = t_+(v_+)v_+$ is a solution of (4.7) under the indicated conditions on ϕ and h. We notice that the solutions u_- and u_+ are different. Indeed, if $u_- = u_+$ then $t_-v_- = t_+v_+$; for $v_-, v_+ \in S$ it would follow that $v_- = \pm v_+$ and $|t_-| = |t_+|$. The last equalities contradict (4.8) under our assumptions on ϕ and h.

4.4. A test for the absence of solutions

We continue to demonstrate applications of the fibering method to nonlinear BVPs. Now we outline an application to the nonexistence problem: we first present a scheme for getting sufficient conditions for the absence of solutions. Let us consider the variational problem in the situation of Section 3.4, that is, we consider a Banach space X with norm differentiable away from zero and a functional $f(u) = f_0(u) + f_1(u)$ with $f_0(u) = \|u\|^p$ and $f_1 \in C^1(X)$. Then the problem

$$f_0'(u) + f_1'(u) = 0 \tag{4.10}$$

is equivalent to the system

$$\begin{cases} p|t|^{p-2}t + \langle f_1'(tv), v\rangle = 0, & t \neq 0 \\ |t|^{p-2}t \cdot (\|v\|^p)' + f_1'(tv) = 0, & v \in S. \end{cases}$$

From this system we obtain, for any $w \in X$, the following system of two scalar equations:

$$\begin{cases} p|t|^{p-2}t + \langle f_1'(tv), v\rangle = 0 \\ |t|^{p-2}t\langle (\|v\|^p)', w\rangle + \langle f_1'(tv), w\rangle = 0. \end{cases} \tag{4.11}$$

This gives us the following test for the absence in X of nonzero solutions for equation (4.10).

Theorem 4.4.1. *Let f_0 and f_1 be the functionals defined above, and suppose that there exists an element $x \in X$ such that system* (4.11) *is inconsistent for any value of $t \neq 0$ and $v \in S$. Then equation* (4.10) *doesn't admit nontrivial solutions in X.*

Obviously, the zero solution of (4.10) doesn't exist if

$$f_0'(0) + f_1'(0) \neq 0\,.$$

Remark 4.4.2: Consider again the BVP with quadratic nonlinearity (4.7). In this case system (4.11) takes the form

$$\begin{cases} t - \int_\Omega (tv+h)^2 v + \int_\Omega \phi v = 0 \\ -t\int_\Omega v\Delta\psi - \int_\Omega (tv+h)^2\psi + \int_\Omega \phi\psi = 0 \end{cases}$$

where ψ is an arbitrary function in $\overset{\circ}{W}{}_2^1(\Omega)$. Notice that the first equation in this system can be obtained from the second one by setting $\psi = v$; therefore, we now consider the second scalar equation with respect to t, namely:

$$t^2\int_\Omega \psi v^2 + t\int_\Omega (\Delta\psi + 2h\psi)v + \int_\Omega (h^2-\phi)\psi = 0.$$

This equation clearly doesn't admit any real solution if there exists a function $\psi \in \overset{\circ}{W}{}_2^1(\Omega)$ such that, for all $v \in S$:

$$\left(\int_\Omega (\Delta\psi + 2h\psi)v\right)^2 < 4\int_\Omega (h^2-\phi)\psi\int_\Omega \psi v^2. \tag{4.12}$$

On the other hand, if $\psi(x) \geq 0$ in Ω, then

$$\begin{aligned}\left(\int_\Omega (\Delta\psi + 2h\psi)v\right)^2 &= \left(\int_\Omega \frac{\Delta\psi + 2h\psi}{\sqrt{\psi}}\sqrt{\psi}v\right)^2 \leq \\ &\leq \int_\Omega \frac{(\Delta\psi + 2h\psi)^2}{\psi}\int_\Omega \psi v^2\,;\end{aligned}$$

hence (4.12) holds if there exists a function $\psi \geq 0$ in $\overset{\circ}{W}{}_2^1(\Omega)$ such that

$$\int_\Omega \frac{(\Delta\psi + 2h\psi)^2}{\psi} < 4\int_\Omega (h^2 - \phi)\psi\,,$$

or, equivalently,

$$\int_\Omega \left(\frac{(\Delta\psi)^2}{\psi} + 4h\Delta\psi + 4\phi\psi\right) < 0. \tag{4.13}$$

Accordingly, we get the following result:

PROPOSITION 4.4.3. *Suppose that there exists a function $\psi \geq 0$ in $\overset{\circ}{W}{}_2^1(\Omega)$ such that (4.13) holds. Then the boundary value problem (4.6) doesn't admit solutions in $\overset{\circ}{W}{}_2^1(\Omega)$.*

EXAMPLE 4.4.4: Consider problem (4.6) with $h_0 \equiv 0$, namely, the Ovsjannikov problem

$$\begin{cases} \Delta\Phi + \Phi^2 = \phi(x) & \text{in } \Omega \\ \Phi = 0 & \text{on } \partial\Omega \end{cases}$$

Then from Proposition 4.4.3 we obtain absence of solutions if we are able to find a $\psi \in \overset{\circ}{W}{}_2^1(\Omega)$ such that:

1. $\psi > 0$ in Ω and $\psi \geq 0$ on $\partial\Omega$;
2. $\displaystyle\int_\Omega \psi\phi < -\frac{1}{4}\int_\Omega \frac{(\Delta\psi)^2}{\psi}$.

For instance, if we take ψ such that

$$\begin{cases} \Delta\psi + \lambda_1\psi = 0, \;\; \psi > 0 \text{ in } \Omega \\ \psi = 0 \text{ on } \partial\Omega \end{cases}$$

then we obtain that the Ovsjannikov problem doesn't admit any solution if

$$\int_\Omega \psi\phi < -\frac{\lambda_1^2}{4}\int_\Omega \psi\,;$$

in particular, we obtain absence of solutions if $\phi(x) < -\lambda_1^2/4$.

We remark that the general nonexistence test (4.13), in contrast with traditional tests for quasilinear elliptic equations of second order, is not a pointwise test but an integral test: we explain this feature by the following special example.

EXAMPLE 4.4.5: Consider the BVP (4.6) where Ω is the open unit disk $D \subset \mathbb{R}^2$, $\phi \equiv 0$ and the boundary function h_0 is equal to $A\cos\theta$ in polar coordinates:

$$\begin{cases} \Delta\Phi + \Phi^2 = 0 & \text{in } D \\ \Phi = A\cos\theta & \text{on } \partial D \end{cases} \tag{4.14}$$

where A is an arbitrary real parameter. The choice of this particular example is due to two circumstances. First, this problem is given without analysis in a number of books. Second (and this is the main thing) the traditional tests for the absence of real solutions are not applicable to problem (4.14), since the mean of the boundary values is equal to zero:

$$\int_0^{2\pi} A\cos\theta\, d\theta = 0.$$

For problem (4.14) inequality (4.13) takes the form

$$\int_0^{2\pi} d\theta \int_0^1 dr\left(\frac{(\Delta\psi)^2}{\psi} + 4\arccos\theta\cdot\Delta\psi\right) r < 0. \tag{4.15}$$

We now choose ψ to be a solution of the following problem with parameter $\tau > 0$:

$$\begin{cases} \Delta\psi = -(\tau + r\cos\theta)(1-r^2) & \text{in} D \\ \psi = 0 & \text{on } \partial D. \end{cases}$$

This solution can be written explicitly, and $\psi \geq 0$ for $\tau \geq 1/3$. We substitute this function ψ (which depends on $\tau \geq 1/3$) into (4.15);

then we get a parametric inequality for A with $\tau \geq 1/3$, and it yields the following estimate for $|A|$ when $\tau = (1 + \sqrt{5/2})/3$:

$$|A| > 20.65\,.$$

If A satisfies this last inequality, then the BVP (4.14) doesn't admit any solution in $\overset{\circ}{W}{}^1_2(D)$.

Lecture 5

Application of the fibering method to the p-Laplacian

In this lecture we apply the fibering method to the problem of existence of positive solutions for equations involving the p-Laplacian

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

in a bounded domain $\Omega \subset \mathbb{R}^N$. The particular equation we'll consider in this lecture was also studied in [10], [11] for $\Omega = \mathbb{R}^N$. Essentially the same result as here was proved in [11] by using the so-called "bifurcation argument" [8] combined with the critical point theory; however, it appears that our approach based on the fibering method yields the existence and multiplicity of positive solutions in a more explicit and constructive way. We first discuss an example with $p = 2$ (hence $\Delta_p = \Delta$).

5.1. An interesting example

Consider the boundary value problem

$$\begin{cases} -\Delta u - \lambda u = f(x)|u|^{\gamma-2}u & \text{in}\Omega \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

where $2 < \gamma < 2^* := 2N/(N-2)$, and $f \in L^\infty(\Omega)$ satisfies the following assumptions:

$(f.1)$ f^+ (the positive part of f) is not identically zero;

$(f.2)$ $\displaystyle\int_\Omega f \cdot e_1^\gamma < 0$, where e_1 is "the" eigenfunction associated to the first eigenvalue λ_1 of $-\Delta$ ([1]).

THEOREM 5.1.1 (ALAMA & TARANTELLO, 1993). *The following results hold.*

1. *Let* $0 \le \lambda < \lambda_1$, *and assume* $(f.1)$. *Then* (5.1) *has a positive solution in* $W^2_\infty(\Omega)$.
2. *Let* $\lambda = \lambda_1$, *and assume* $(f.1)$ *and* $(f.2)$. *Then* (5.1) *has a positive solution in* $W^2_\infty(\Omega)$.
3. *Let* $\lambda > \lambda_1$, *and assume* $(f.1)$ *and* $(f.2)$. *Then there exists* $\delta > 0$ *such that for* $\lambda < \lambda_1 + \delta$ *problem* (5.1) *has* two *positive solutions in* $W^2_\infty(\Omega)$.

This result is of considerable interest: let us point out some features.

1. Solutions of (5.1) are *not* small. Indeed, let $\lambda = 0$; then we have
$$\begin{cases} -\Delta u = f(x)|u|^{\gamma-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$
By multiplying by u and integrating by parts we get
$$\begin{aligned} \int_\Omega |\nabla u|^2 &= \int_\Omega f|u|^\gamma \le \\ &\le \|f\|_\infty \int_\Omega |u|^\gamma \le \\ &\le \|f\|_\infty \,(C_\gamma \|u\|_{1,2})^\gamma \end{aligned}$$
where C_γ is (since $\gamma < 2^*$) the Sobolev constant
$$\|u\|_\gamma \le C_\gamma \|u\|_{1,2}\,.$$

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[1]Namely, $e_1 = e_1(x)$ is such that:
$$\begin{cases} \Delta e_1 + \lambda_1 e_1 = 0 & \text{in } \Omega \\ e_1 > 0 & \text{in } \Omega \\ e_1 = 0 & \text{on } \partial\Omega \end{cases}$$
See also Lemma 5.2.3 in the next section.

Since $u = 0$ on $\partial\Omega$ we obtain

$$\|u\|_{1,2}^2 \le C\|f\|_\infty \|u\|_{1,2}^\gamma \,;$$

from this last inequality we obtain, since u is not identically zero and $\gamma > 2$:

$$\|u\|_{1,2} \ge \left(\frac{1}{C\,\|f\|_\infty}\right)^{\frac{1}{\gamma-2}} \to +\infty \text{ as } \|f\|_\infty \to 0\,.$$

Therefore, Theorem 5.1.1 does *not* follow from the classical bifurcation theory.

2. Solutions of (5.1) are *positive* if λ is near to λ_1 (including $\lambda > \lambda_1$!).

3. If $\lambda > \lambda_1$, Theorem 5.1.1 states the existence of *two* positive solutions (an *even* number of solutions) for (5.1).

These interesting features stimulated further investigations by S. Alama, G. Tarantello, L. Nirenberg, H. Brézis, H. Berestycki, I. C. Dolcetto, and other mathematicians.

5.2. A problem involving the p-Laplacian

In this section and in the next ones we consider an application of the fibering method to the p-Laplacian. Here we follow the paper [12] by P. Drábek and S. Pohožaev, where additionally existence and nonexistence problems are examined in the whole $\mathbb{R}^N$. Let Ω be a bounded domain in $\mathbb{R}^N$; assume $p, \lambda, \gamma \in \mathbb{R}^N$, $1 < p < \gamma < p^*$, where $p^* := Np/(N-p)$ for $p < N$ and $p^* := \infty$ for $p \ge N$. We consider the equation

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u + f(x)|u|^{\gamma-2}u \tag{5.2}$$

for $x \in \Omega$, under Dirichlet boundary condition

$$u = 0 \quad \text{ on } \partial\Omega\,. \tag{5.3}$$

This problem is studied in connection with the corresponding eigenvalue problem

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u\,. \tag{5.4}$$

We concentrate ourselves on the existence and multiplicity of positive solutions for (5.2) when $0 \le \lambda < \lambda_1 + \epsilon$, where ϵ is a "small" positive number and λ_1 is the first eigenvalue of (5.4). In particular for $\lambda > \lambda_1$ we'll prove the existence of (at least) two solutions, similarly to the case $\Delta_p = \Delta$ that we discussed in the previous section.

Let us premise some definitions and notations. We work in the Sobolev space $W := \overset{\circ}{W}{}^1_p(\Omega)$ equipped with the usual norm

$$\|u\|_W = \left(\int_\Omega |\nabla u|^p\right)^{1/p}.$$

We assume $f, g \in L^\infty(\Omega)$, with $g \ge 0$ and g not identically zero.

DEFINITION 5.2.1. *A function $u \in W$ is a* <u>*weak solution*</u> *for problem* (5.2) *under condition* (5.3) *iff it satisfies the integral identity*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v = \lambda \int_\Omega g \cdot |u|^{p-2}uv + \int_\Omega f \cdot |u|^{\gamma-2}uv \tag{5.5}$$

for every $v \in W$.

DEFINITION 5.2.2. *A real number λ is an* <u>*eigenvalue*</u> *for problem* (5.4) *under condition* (5.3), *and $u \in W\backslash\{0\}$ is a corresponding* <u>*eigenfunction*</u>, *iff*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v = \lambda \int_\Omega g \cdot |u|^{p-2}uv \tag{5.6}$$

for every $v \in W$.

The following result is now well-known (see e.g. [3], [6], [14]).

LEMMA 5.2.3. *There exists the first positive eigenvalue λ_1 for problem* (5.4) *under condition* (5.3)*; λ_1 is characterized as the minimum of the Rayleigh quotient:*

$$\lambda_1 = \min\left\{ \frac{\int_\Omega |\nabla u|^p}{\int_\Omega g|u|^p} \,\middle|\, u \in W \;,\; \int_\Omega g|u|^p > 0 \right\}. \tag{5.7}$$

Moreover, λ_1 is simple (i.e. each associated eigenfunction can be obtained from any other by multiplying by a nonzero constant), isolated (i.e. there are no eigenvalues in a suitable neighborhood of λ_1), and it admits an eigenfunction $e_1 \in W$ which is positive in Ω.

We denote by $\langle\ ,\ \rangle_W$ the duality between W^* and W, so that the left-hand side of (5.5) and (5.6) can be written as

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v = \langle -\Delta_p u, v \rangle_W$$

Since $g \in L^\infty(\Omega)$ and $1 < \gamma < p^*$, it follows from the continuity of the Nemytskii operator [13] and the Sobolev Imbedding Theorem [1] that:

$(g.0)$ the functional

$$G(u) := \int_\Omega g \cdot |u|^p$$

is weakly continuous on W;

$(f.0)$ the functional

$$F(u) := \int_\Omega f \cdot |u|^\gamma$$

is weakly continuous on W.

Notice that $G(u)$ is p-homogeneous and $F(u)$ is γ-homogeneous.

5.3. The application of the fibering method

Let us consider the Euler functional

$$\begin{aligned} E_\lambda(u) &:= \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{p}\int_\Omega g|u|^p - \frac{1}{\gamma}\int_\Omega f|u|^\gamma \\ &= \frac{1}{p}\|u\|_W^p - \frac{\lambda}{p}G(u) - \frac{1}{\gamma}F(u) \end{aligned} \tag{5.8}$$

associated with (5.2), (5.3); according to (5.5), critical points of E_λ are the same as weak solutions of BVP (5.2), (5.3). Following the fibering method, we substitute $u = tv$ (with $t \in \mathbb{R} \setminus \{0\}$ and $v \in W$) into (5.8); we get

$$E_\lambda(tv) = \frac{|t|^p}{p}\|v\|_W^p - \frac{\lambda |t|^p}{p} G(v) - \frac{|t|^\gamma}{\gamma} F(v); \tag{5.9}$$

As the fibering functional H_λ we'll choose the principal part of E_λ, i.e.

$$\begin{aligned} H_\lambda(v) &:= \int_\Omega |\nabla v|^p - \lambda \int_\Omega g|v|^p \\ &= \|v\|_W^p - \lambda G(v) \end{aligned} \tag{5.10}$$

(notice that H_λ is independent from t). Then the bifurcation equation $\frac{\partial}{\partial t} E_\lambda(tv) = 0$ takes the form

$$|t|^{p-2} t H_\lambda(v) - |t|^{\gamma-2} t F(v) = 0,$$

i.e., since $t \neq 0$:

$$H_\lambda(v) - |t|^{\gamma-p} F(v) = 0.$$

From this we obtain

$$|t| = \left(\frac{H_\lambda(v)}{F(v)}\right)^{\frac{1}{\gamma-p}} > 0 \tag{5.11}$$

under the *necessary* conditions

$$F(v) \neq 0 \quad , \quad \frac{H_\lambda(v)}{F(v)} > 0 \tag{5.12}$$

By substituting (5.11) into (5.9) we define:

$$\begin{aligned} \hat{E}_\lambda(v) &:= E_\lambda(t(v)v) = \\ &= \left(\frac{1}{p} - \frac{1}{\gamma}\right)\left(\frac{H_\lambda(v)}{F(v)}\right)^{\frac{\gamma}{\gamma-p}} F(v). \end{aligned} \tag{5.13}$$

LEMMA 5.3.1. *The functional $\hat{E}_\lambda$ is 0-homogeneous, i.e. for every $\tau \in \mathbb{R} \setminus \{0\}$ and every $v \in W$ such that $F(v) \neq 0$ we have*

$$\hat{E}_\lambda(\tau v) = \hat{E}_\lambda(v).$$

In particular, $\hat{E}_\lambda$ is even and its Gâteaux derivative at v in the direction v is zero:

$$\langle \hat{E}_\lambda(v), v\rangle_W = 0.$$

Moreover, if $v_c \in W$ is a critical point of $\hat{E}_\lambda$ then also $|v_c|$ is a critical point of $\hat{E}_\lambda$.

The proof of Lemma 5.3.1 is obvious. It follows from here that whenever we find some critical point v_c of $\hat{E}_\lambda$, we can automatically assume that v_c is *nonnegative* in Ω.

The following sections are devoted to studying problem (5.2) in the three distinct cases:

- $0 \le \lambda < \lambda_1$;
- $\lambda = \lambda_1$;
- $\lambda_1 < \lambda < \lambda_1 + \epsilon$.

Here and in the following sections, λ_1 is the first positive eigenvalue of (5.4) under condition (5.3); by e_1 we denote the corresponding positive eigenfunction (see Lemma 5.2.3).

5.4. The case $0 \le \lambda < \lambda_1$

Let $0 \le \lambda < \lambda_1$; as the fibering functional we can take H_λ, as defined by (5.10). In fact, it follows from Lemma 5.2.3 that $H_\lambda(v) \ge 0$ for any $v \in W$, hence the fibering constraint becomes

$$H_\lambda(v) = 1$$

since H_λ is p-homogeneous. We still have to verify the nondegeneracy condition (cf. inequality (1.8)); indeed, it follows directly from (5.10) that

$$\langle H'_\lambda(v), v\rangle_W = p \cdot H_\lambda(v) \neq 0$$

(we recall that in the present case the derivative of the fibering functional with respect to t is zero). Since $H_\lambda(v) \ge 0$, it follows from (5.12) that we have to consider the conditionally critical points of $\hat{E}_\lambda(v)$ satisfying

$$F(v) = \int_\Omega f \cdot |v|^\gamma > 0\,;$$

so the following hypothesis is a natural one (cf. hypothesis $(f.1)$ in Section 5.1):

$(f.1)$ f^+ is not identically zero.

By (5.13), the functional $\hat{E}_\lambda(v)$ under constraint $H_\lambda(v) = 1$ assumes the form

$$\hat{E}_\lambda(v) = \left(\frac{1}{p} - \frac{1}{\gamma}\right) F(v)^{-\frac{p}{\gamma-p}};$$

therefore we consider the conditional variational problem:

(P_λ) "Find a maximizer $v_c \in W$ of the problem

$$0 < M_\lambda = \sup_{v \in W} \left\{ F(v) \,\Big|\, H_\lambda(v) = 1 \right\} \text{ ''}.$$

PROPOSITION 5.4.1. *Assume* $(g.0)$, $(f.0)$, $(f.1)$. *Then problem* (P_λ) *admits a nonnegative solution.*

Proof. Let us consider the set

$$W_\lambda := \{v \in W : H_\lambda(v) = 1\};$$

W_λ is nonempty since $H_\lambda(e_1) > 0$ and H_λ is homogeneous. Due to (5.10) and to the variational characterization (5.7) of λ_1, we get for any $v \in W_\lambda$

$$\int_\Omega |\nabla v|^p = 1 + \lambda \int_\Omega g|v|^p \leq 1 + \frac{\lambda}{\lambda_1} \int_\Omega |\nabla v|^p$$

and then

$$\|v\|_W^p = \int_\Omega |\nabla v|^p \leq \frac{\lambda_1}{\lambda_1 - \lambda};$$

hence, W_λ is bounded in W. Therefore, any maximizing sequence $(v_n)_{n=1}^\infty$ for problem (P_λ) is bounded in W; consequently we can assume

$$v_n \rightharpoonup v_c \text{ in } W.$$

By $(f.0)$ and $(f.1)$ it is

$$F(v_n) \leq F(v_c) = M_\lambda > 0; \tag{5.14}$$

then v_c is not identically zero, and we can assume $v_c \geq 0$ (cf. Lemma 5.3.1). We have only to prove that $v_c \in W_\lambda$. Due to $(g.0)$ and to the weak lower semicontinuity of the norm $\|.\|_W$, we get

$$H_\lambda(v_c) \leq \liminf_{n\to\infty} H_\lambda(v_n) = 1;$$

assume now by contradiction that $v_c \notin W_\lambda$, i.e.

$$H_\lambda(v_c) < 1.$$

Since H_λ is homogeneous, we can find $k_c > 1$ such that

$$H_\lambda(k_c v_c) = 1;$$

but then $k_c v_c \in W_\lambda$ and by (5.14)

$$F(k_c v_c) = k_c^\gamma F(v_c) = k_c^\gamma M_\lambda > M_\lambda$$

which contradicts the definition of M_λ. Hence $v_c \in W_\lambda$ is the desired solution of (P_λ). □

Thanks to the fibering method we can state the following result.

THEOREM 5.4.2. *Let $1 < p < \gamma < p^*$, $0 \leq \lambda < \lambda_1$, $f, g \in L^\infty(\Omega)$ and hypothesis $(f.1)$ be satisfied. Then the boundary value problem* (5.2) *under condition* (5.3) *has at least one positive weak solution $u \in W \cap L^\infty(\Omega)$. Moreover $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.*

Proof. Recall that $(g.0)$, $(f.0)$ hold under assumptions of the theorem; then by Proposition 5.4.1 there is a nonnegative solution v_c of problem (P_λ). Clearly, v_c is a conditionally critical point of $\hat{E}_\lambda(v)$ under the fibering constraint $H_\lambda(v) = 1$; then, by means of the fibering method, we can take

$$u_c := t_c v_c \geq 0$$

as a critical point for $E_\lambda(v)$ (here $t_c > 0$ is defined by (5.11)): that is, u_c is a weak solution of (5.2). Following the bootstrap argument (used e.g. in [9]) we can prove that $u \in L^\infty(\Omega)$; then by applying the Harnack inequality due to Trudinger [25] we get $u > 0$ in Ω (cf. [7]). It follows from the result of Tolksdorf [24] that $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ (cf. [9]). □

REMARK 5.4.3: If $f(x) > 0$ and $\lambda < \lambda_1$, then by Lyusternik-Shnirel'man theory it follows immediately the existence of a countable set of nontrivial (sign-changing) solutions of (5.2), (5.3).

5.5. The case $\lambda = \lambda_1$

Let $\lambda = \lambda_1$; keeping the notation of the previous section we consider the conditional variational problem

(P_{λ_1}) "Find a maximizer $v_c \in W$ of the problem

$$0 < M_{\lambda_1} = \sup_{v \in W} \left\{ \int_\Omega f|v|^\gamma \,\middle|\, H_{\lambda_1}(v) = 1 \right\} \text{ ''}.$$

We have $M_{\lambda_1} > 0$ by $(f.1)$, as in the previous section. In this case, however, the set

$$W_{\lambda_1} = \{v \in W : H_{\lambda_1}(v) = 1\}$$

is *unbounded* in W; so we are forced to require the following additional condition on f (cf. hypothesis $(f.2)$ in Section 5.1):

$(f.2)$ $\displaystyle\int_\Omega f \cdot e_1^\gamma < 0\,.$

PROPOSITION 5.5.1. *Assume $(g.0)$, $(f.0)$, $(f.1)$ and $(f.2)$. Then problem (P_{λ_1}) admits a nonnegative solution.*

Proof. Let $(v_n)_{n=1}^\infty$ be a maximizing sequence of (P_{λ_1}), i.e.

$$H_{\lambda_1}(v_n) = 1, \quad F(v_n) \to M_{\lambda_1} > 0. \tag{5.15}$$

Suppose by contradiction that (v_n) is unbounded; then we can assume $\|v_n\|_W \to \infty$. Set

$$v_n = r_n w_n \quad \text{with} \quad |r_n| = \|v_n\|_W\ , \quad \|w_n\|_W = 1$$

so that, by (5.10),

$$H_{\lambda_1}(v_n) = |r_n|^p H_{\lambda_1}(w_n) = 1;$$

due to (5.7) we have

$$0 \le \|w_n\|_W^p - \lambda_1 G(w_n) = H_{\lambda_1}(w_n) = |r_n|^{-p} \to 0 \tag{5.16}$$

and then, since $\|w_n\|_W = 1$,

$$\lim_{n\to\infty} G(w_n) = \frac{1}{\lambda_1}. \tag{5.17}$$

We can assume that $w_n \rightharpoonup \tilde{w}$ in W for some $\tilde{w} \in W$; then (5.17) and $(g.0)$ imply

$$\int_\Omega g \cdot |\tilde{w}|^p = G(\tilde{w}) = \frac{1}{\lambda_1}$$

and consequently $\tilde{w} \neq 0$. Moreover,

$$\|\tilde{w}\|_W^p \le \liminf_{n\to\infty} \|w_n\|_W^p = 1$$

and then by (5.16)

$$0 \le \|\tilde{w}\|_W^p - \lambda_1 G(\tilde{w}) \le 0\,,$$

that is, $H_{\lambda_1}(\tilde{w}) = 0$. By Lemma 5.2.3, $\tilde{w}$ is a multiple of the first eigenfunction e_1, i.e. $\tilde{w} = ke_1$ for a suitable $k \neq 0$. On the other hand, thanks to (5.15) we get

$$F(w_n) = |r_n|^{-\gamma} F(v_n) \ge 0\,;$$

then, by $(f.0)$, $F(\tilde{w}) \ge 0$ and consequently

$$F(e_1) \ge 0$$

which contradicts $(f.2)$. Hence *the maximizing sequence is bounded*, and we can assume

$$v_n \rightharpoonup v_c \text{ in } W$$

for some $v_c \in W$. By $(f.0)$ we have

$$F(v_n) \to F(v_c) = M_{\lambda_1} > 0,$$

hence $v_c \neq 0$; it follows from (5.15), (5.7) and $(g.0)$ that

$$0 \leq H_{\lambda_1}(v_c) \leq 1.$$

We still have to prove that actually $H_{\lambda_1}(v_c) = 1$. First, $H_{\lambda_1}(v_c) > 0$ because if $H_{\lambda_1}(v_c) = 0$ then Lemma 5.2.3 would yield the existence of $k \neq 0$ such that $v_c = ke_1$, and then

$$|k|^\gamma F(e_1) = F(v_c) = M_{\lambda_1} > 0,$$

in contradiction with $(f.2)$. Second, $H_{\lambda_1}(v_c) = 1$. Indeed, suppose $H_{\lambda_1}(v_c) < 1$; then there would exist $k_c > 1$ such that $H_{\lambda_1}(k_c v_c) = 1$. As in the proof of Proposition 5.4.1 we obtain the contradiction

$$F(k_c v_c) = k_c^\gamma \cdot M_{\lambda_1} > M_{\lambda_1}.$$

Thus, v_c is a maximizer of problem (P_{λ_1}); we can assume $v_c \geq 0$ in Ω due to Lemma 5.3.1. □

THEOREM 5.5.2. *Let $1 < p < \gamma < p^*$; let $f, g \in L^\infty(\Omega)$ and $(f.1)$, $(f.2)$ be satisfied. Then problem (5.2), with $\lambda = \lambda_1$, under Dirichlet condition (5.3), admits at least one positive weak solution $u \in W \cap L^\infty(\Omega)$. Moreover $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.*

Proof. The proof is based on Proposition 5.5.1 and follows the same ideas as the proof of Theorem 5.4.2. □

5.6. The case $\lambda > \lambda_1$

We consider again problem (5.2) under condition (5.3), with $\lambda > \lambda_1$ but close enough to λ_1. The main result of this section is formulated in the next theorem.

THEOREM 5.6.1. *Let $1 < p < \gamma < p^*$; let $f, g \in L^\infty(\Omega)$ and $(f.1)$, $(f.2)$ be satisfied. Then there exists $\epsilon > 0$ such that for $\lambda_1 < \lambda < \lambda_1 + \epsilon$ problem (5.2) under condition (5.3) admits* two *positive weak solutions $u_1, u_2 \in W \cap L^\infty(\Omega)$; both solutions belong to $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.*

To prove this multiplicity result, we will consider two variational problems:

(P^1_λ) "Find a maximizer $v_1 \in W$ of the problem

$$M_\lambda = \sup_{v \in W} \left\{ F(v) \,\middle|\, H_\lambda(v) = +1 \right\} \text{ ''}.$$

(P^2_λ) "Find a maximizer $v_2 \in W$ of the problem

$$m_\lambda = \inf_{v \in W} \left\{ H_\lambda(v) \,\middle|\, F(v) = -1 \right\} \text{ ''}.$$

Notice that in problem (P^2_λ) we consider $F(v)$, no longer H_λ, as the fibering functional.

In the next two subsections we consider separately problem (P^1_λ) and problem (P^2_λ).

5.6.1. Problem (P^1_λ)

First, we state an equivalence. Consider the problem

$(\tilde{P}^1_\lambda)$ "Find a maximizer $\tilde{v}_1 \in W$ of the problem

$$\tilde{M}_\lambda = \sup_{v \in W} \left\{ F(v) \,\middle|\, H_\lambda(v) \leq 1 \right\} \text{ ''}.$$

Note that $\tilde{M}_\lambda > 0$ if we assume $(f.1)$; then the following statement holds:

Lemma 5.6.2. *Assume $(f.1)$; then problem (P^1_λ) is equivalent to $(\tilde{P}^1_\lambda)$.* $\blacksquare$

Proof. Let $\tilde{v}_1 \in W$ be a maximizer of $(\tilde{P}^1_\lambda)$ and suppose by contradiction that

$$H_\lambda(\tilde{v}_1) < 1.$$

Then for a sufficiently small $k > 1$ it is

$$H_\lambda(k\tilde{v}_1) \leq 1,$$

and

$$F(k\tilde{v}_1) = k^\gamma M_\lambda > M_\lambda$$

since $M_\lambda > 0$, by $(f.1)$. But this is in contradiction with the fact that $\tilde{v}_1$ is a maximizer. $\square$

Proposition 5.6.3. *Assume* $(g.0)$*,* $(f.0)$*,* $(f.1)$ *and* $(f.2)$*. Then there exists* $\epsilon_1 > 0$ *such that for* $\lambda_1 < \lambda < \lambda_1 + \epsilon_1$ *the problem* (P^1_λ) *admits a nonnegative solution.*

Proof. Due to Lemma 5.6.2 it suffices to show that there exists $\epsilon_1 > 0$ such that problem $(\tilde{P}^1_\lambda)$ admits a nonnegative solution for any λ, $\lambda_1 < \lambda < \lambda_1 + \epsilon$. Assume by contradiction that there is a sequence $\epsilon_k \to 0^+$ such that for any $\lambda^k := \lambda_1 + \epsilon_k$ the problem $(\tilde{P}^1_{\lambda^k})$ has no (nonnegative) solution. For any $k \in \mathbb{N}$, let $\left(v^k_n\right)_{n=1}^\infty$ be a maximizing sequence of $(\tilde{P}^1_{\lambda^k})$, i.e.

$$H_{\lambda^k}(v^k_n) \leq 1 \quad \text{and} \quad F(v^k_n) \to M_{\lambda^k} > 0 \text{ as } n \to \infty.$$

Suppose that the sequence $\left(v^k_n\right)_{n=1}^\infty$ is bounded; we can assume $v^k_n \rightharpoonup \bar{v}^k$ in W as $n \to \infty$. By using $(g.0)$, $(f.0)$, $(f.1)$ and repeating arguments from the proof of Proposition 5.5.1, we obtain

$$H_{\lambda^k}(\bar{v}^k) \leq 1 \quad \text{and} \quad F(\bar{v}^k) = M_{\lambda^k} > 0;$$

hence, as a contradiction we get $\bar{v}^k$ is a solution of $(\tilde{P}^1_\lambda)$.

Thus, for any k the sequence $\left(v^k_n\right)_{n=1}^\infty$ must be *unbounded.* We assume

$$\|v^k_n\|_W \to \infty \quad \text{as } n \to \infty;$$

moreover, by setting $v^k_n = r^k_n w^k_n$ with $|r^k_n| = \|v^k_n\|_W$ and $\|w^k_n\|_W = 1$, we can assume

$$w^k_n \rightharpoonup \bar{w}^k \quad \text{in } W \quad \text{as } n \to \infty.$$

Since $\|w^k_n\|_W = 1$, we have $\|\bar{w}^k\|_W \leq 1$, and then we can assume

$$\bar{w}^k \rightharpoonup \bar{w} \quad \text{in } W \quad \text{as } k \to \infty\,;$$

we get obviously $\|\bar{w}\|_W \leq 1$.

By definitions of v^k_n and w^k_n it is

$$1 - \lambda^k G(w^k_n) = |r^k_n|^{-p} H_{\lambda^k}(v^k_n) \leq |r^k_n|^{-p}$$

and then, by twice applying $(g.0)$,

$$1 - \lambda_1 G(\bar{w}) \leq 0.$$

From this we immediately obtain $\bar{w} \neq 0$ and

$$H_{\lambda_1}(\bar{w}) = \|\bar{w}\|_W - \lambda_1 G(\bar{w}) \leq 0;$$

since in general $H_\lambda \geq 0$ for $\lambda \leq \lambda_1$, we have

$$H_{\lambda_1}(\bar{w}) = 0.$$

Then, by Lemma 5.2.3, for a suitable $k_1 \neq 0$ it is

$$\bar{w} = k_1 \cdot e_1 \,. \tag{5.18}$$

From $F(v_n^k) \to M_{\lambda^k} > 0$ we obtain, by twice applying $(f.0)$ as $n \to \infty$ and $k \to \infty$,

$$F(\bar{w}) \geq 0$$

and then by (5.18) we get

$$|k_1|^\gamma F(e_1) \geq 0$$

which contradicts $(f.2)$. Hence for some $\epsilon_1 > 0$ the problem $(\tilde{P}_\lambda^1)$ admits at least one (nonnegative) solution for $\lambda_1 < \lambda < \lambda_1 + \epsilon_1$. □

5.6.2. Problem (P_λ^2)

Now we choose $F(v)$ as the fibering functional, and

$$F(v) = -1$$

as the fibering constraint; the nondegeneracy condition

$$\langle F'(v), v\rangle \neq 0 \quad \text{as } F(v) = -1$$

can be proved the same way as for H_λ in the previous cases. By (5.13), the functional $\hat{E}_\lambda$ takes the form

$$\hat{E}_\lambda(v) = -\left(\frac{1}{p} - \frac{1}{\gamma}\right)(-H_\lambda(v))^{\frac{\gamma}{\gamma-p}}\,;$$

we have to search for a conditionally critical point of $\hat{E}_\lambda$ satisfying (5.12), i.e. satisfying

$$H_\lambda(v) < 0\,.$$

Thus, solving problem (P^2_λ) is a natural way to find such a critical point.

First, we prove that problem (P^2_λ) makes sense under hypothesis $(f.2)$.

LEMMA 5.6.4. *Assume $(f.2)$; then the set*

$$W^- := \left\{ v \in W \,\middle|\, F(v) = -1 \right\}$$

is nonempty, and $m_\lambda < 0$ for any $\lambda > \lambda_1$.

Proof. By $(f.2)$ it is $F(e_1) < 0$; since F is homogeneous, we can find t_1 such that

$$F(t_1 e_1) = -1$$

and then W^- is nonempty. We have also (see Lemma 5.2.3)

$$H_\lambda(t_1 e_1) = |t_1|^p (\lambda_1 - \lambda) G(e_1) < 0$$

for any $\lambda > \lambda_1$: then the infimum of H_λ in W^- is negative. □

PROPOSITION 5.6.5. *Assume $(g.0)$, $(f.0)$, $(f.1)$ and $(f.2)$. Then there exists $\epsilon_2 > 0$ such that for $\lambda_1 < \lambda < \lambda_1 + \epsilon_2$ the problem (P^2_λ) admits a nonnegative solution.*

Proof. Assume by contradiction that there is a sequence $\epsilon_k \to 0^+$ such that for any $\lambda^k := \lambda_1 + \epsilon_k$ the problem $(P^2_{\lambda^k})$ has no (nonnegative) solution. For any $k \in \mathbb{N}$, let $\left(v^k_n\right)_{n=1}^\infty$ be a minimizing sequence of $(P^2_{\lambda^k})$, i.e.

$$F(v^k_n) = -1 \quad \text{and} \quad H_{\lambda^k}(v^k_n) \to m_{\lambda^k} < 0 \text{ as } n \to \infty.$$

If $\left(v^k_n\right)_{n=1}^\infty$ is bounded we obtain a solution $\bar{v}^k$ of problem $(P^2_{\lambda^k})$, similarly as in the proof of Proposition 5.6.3; thus, for any k the sequence $\left(v^k_n\right)_{n=1}^\infty$ must be unbounded. We set $v^k_n = r^k_n w^k_n$ with $|r^k_n| \to \infty$ and $\|w^k_n\|_W = 1$; as in the proof of Proposition 5.6.3 we can assume

$$w^k_n \rightharpoonup \bar{w}^k \quad \text{in } W \quad \text{as } n \to \infty$$

and

$$\bar{w}^k \rightharpoonup \bar{w} \quad \text{in } W \quad \text{as } k \to \infty\,;$$

we get obviously $\|\bar{w}\|_W \leq 1$.

By definitions of v_n^k and w_n^k it is

$$|r_n^k|^p \left(1 - \lambda^k G(w_n^k)\right) = H_{\lambda^k}(v_n^k) \to m_{\lambda^k} < 0$$

and then, by twice applying $(g.0)$,

$$1 - \lambda_1 G(\bar{w}) \leq 0;$$

as in the proof of Proposition 5.6.3, by Lemma 5.2.3 we get, for a suitable $k_1 \neq 0$:

$$\bar{w} = k_1 \cdot e_1\,. \tag{5.19}$$

Since

$$F(w_n^k) = |r_n^k|^{-\gamma} F(v_n^k) = -|r_n^k|^{-\gamma},$$

as $n \to \infty$ and $k \to \infty$ we obtain, by twice applying $(f.0)$,

$$F(\bar{w}) = 0$$

and then by (5.19) we get

$$|k_1|^\gamma F(e_1) = 0,$$

which contradicts $(f.2)$. Hence for some $\epsilon_2 > 0$ the problem (P_λ^2) admits at least one (nonnegative) solution for $\lambda_1 < \lambda < \lambda_1 + \epsilon_2$. $\square$

5.6.3. Proof of Theorem 5.6.1

Set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and consider $\lambda_1 < \lambda < \lambda_1 + \epsilon$. By applying the fibering method we obtain weak solutions of (5.2) under condition (5.3), as in the previous sections. Namely, let $t_1, t_2 > 0$ be determined by (5.11) for $v = v_1, v_2$; then

$$u_1 := t_1 \cdot v_1 \quad , \quad u_2 := t_2 \cdot v_2$$

are nonnegative weak solutions for the problem under consideration. Clearly, we have

$$H_\lambda(u_1) = t_1^p H_\lambda(v_1) = t_1^p > 0$$

and

$$H_\lambda(u_2) = t_2^p H_\lambda(v_2) = t_2^p m_\lambda \leq 0\,;$$

therefore, u_1 and u_2 are distinct solutions. Other properties of u_1, u_2 (such as positivity, L^∞-boundedness and $C^{1,\alpha}_{\mathrm{loc}}$-regularity) can be derived the same way as in the proofs of the previous theorems.$\square$

Lecture 6

Positive solutions for Neumann problems

In this lecture we consider the application of the fibering method to a problem with Neumann boundary conditions. Here we follow the paper by A. Tesei and S. Pohožaev [23], where this problem is considered in a more general setting. Nevertheless, results stated in this lecture generalize some results from [5]. We begin with the case of a linear differential operator.

6.1. The semilinear case

Let Ω be a bounded domain in $\mathbb{R}^N$; we consider the BVP

$$\begin{cases} \Delta u + (\nabla\psi, \nabla u) + a(x)|u|^{p-2}u = 0, \quad u \geq 0, & \text{in}\Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on}\partial\Omega \end{cases} \tag{6.1}$$

with $2 < p < 2^*$ and $\psi \in C^1(\overline{\Omega})$, under the following assumptions:

$(A1)$ $\quad a \in L^\infty(\Omega)$;

$(A2)$ $\quad a^+(x) := \max\{a(x), 0\}$ is not identically zero;

$(A3)$ $\quad \displaystyle\int_\Omega a(x)\rho(x)\,dx < 0$, with $\rho(x) := e^{\psi(x)}$.

THEOREM 6.1.1. *Assume* $(A1)$–$(A3)$ *and* $2 < p < 2^*$. *Then there exists a nonnegative (nontrivial) solution of* (6.1).

The proof is divided into some steps.

6.1.1. Step 1

We consider the functional E defined by

$$E(u) := -\frac{1}{2}\int_\Omega |\nabla u(x)|^2 \rho(x)\,dx + \frac{1}{p}\int_\Omega a(x)|u(x)|^p \rho(x)\,dx\,.$$

Thanks to $(A1)$ and $2 < p < 2^*$, this functional is well-posed on the Sobolev space $W := {}_\rho W_2^1(\Omega)$, defined by

$$W = W_2^1(\Omega)\ , \quad \|u\|_W = \left(\int_\Omega |\nabla u|^2 \rho\right)^{1/2}.$$

Moreover, we have

$$E'(u) = 0 \Leftrightarrow u \text{ satisfies } (6.1)\,.$$

Indeed,

$$\begin{aligned}
\langle E'(u), v\rangle &= -\int_\Omega \nabla u \nabla v \rho + \int_\Omega a|u|^{p-2}uv\rho = \\
&= -\int_{\partial\Omega} v\frac{\partial u}{\partial \nu}\rho + \int_\Omega v\,\mathrm{div}(\rho\nabla u) + \int_\Omega a|u|^{p-2}uv\rho = \\
&= \int_\Omega \big(\Delta u\, v\rho + v(\nabla u, \nabla\rho) + a|u|^{p-2}uv\rho\big) = \\
&= \int_\Omega \big(\Delta u + (\nabla u, \nabla\psi) + a|u|^{p-2}u\big)\,\rho v.
\end{aligned}$$

Following the fibering method we set

$$u(x) = tv(x)\,, \quad t \in \mathbb{R}\setminus\{0\}\,,\ v \in W$$

and take the norm-type fibering functional

$$H(v) := \|v\|_W^2 = \int_\Omega |\nabla v|^2\rho\,;$$

accordingly, under the fibering constraint

$$H(v) = 1$$

the Euler functional $E(u)$ reduces to

$$\tilde{E}(t, v) = -\frac{1}{2}t^2 + \frac{1}{p}|t|^p E_1(v) \tag{6.2}$$

with

$$E_1(v) := \int_\Omega a(x)|v(x)|^p \rho(x)\, dx. \tag{6.3}$$

From the bifurcation equation $\tilde{E}_t' = 0$, i.e.

$$-t + |t|^{p-2} t\, E_1(v) = 0\,,$$

we obtain for $t \neq 0$:

$$|t(v)| = \Big[E_1(v)\Big]^{-\frac{1}{p-2}}; \tag{6.4}$$

thus, we define the functional $\hat{E}$ as

$$\hat{E}(v) := \tilde{E}(t(v), v) = \left(\frac{1}{p} - \frac{1}{2}\right) \Big[E_1(v)\Big]^{-\frac{2}{p-2}}. \tag{6.5}$$

6.1.2. Step 2

Now we search for an extremal point of E_1, i.e. of $\hat{E}$, under the fibering constraint $H(v) = 1$.

LEMMA 6.1.2. *Let*

$$M_0 := \sup_{v \in W} \left\{ E_1(v) \,\middle|\, v \in W,\ H(v) = 1 \right\}. \tag{6.6}$$

Then $0 < M_0 < \infty$, *and any maximizing sequence of* (6.6) *is bounded in* W.

Proof. From $(A2)$ and (6.3) it follows immediately that $M_0 > 0$. Thus, we have only to prove the boundedness of an arbitrary maximizing sequence (v_n), because from this we obtain also $M_0 < \infty$. Let

$$H(v_n) = 1\ , \quad E_1(v_n) \to M_0$$

and put

$$v_n(x) =: C_n + \bar{v}_n(x) \quad \text{with} \int_\Omega \bar{v}_n \rho = 0 \tag{6.7}$$

where C_n are constants. Then $\nabla v_n = \nabla \bar{v}_n$, and by virtue of the Sobolev imbedding theorem (the Poincaré inequality) we have

$$|E_1(v_n)| \leq \tilde{M} < \infty \tag{6.8}$$

thanks to (6.7); here $\tilde{M}$ doesn't depend on $\bar{v}_n$. Suppose by contradiction that $\|v_n\|_W \to \infty$. By (6.7) and (6.8) this implies

$$C_n \to \infty\,; \tag{6.9}$$

further, we have

$$E_1(v_n) = E_1(C_n + \bar{v}_n) = |C_n|^p \int_\Omega a(x) \left|1 + \frac{\bar{v}_n(x)}{C_n}\right|^p \rho(x)\,dx.$$

Then by (6.8), (6.9), (6.4) and $(A3)$ we get

$$|C_n|^{-p} E_1(v_n) = \int_\Omega a(x) \left|1 + \frac{\bar{v}_n(x)}{C_n}\right|^p \rho(x)\,dx \to \int_\Omega a(x)\rho(x)\,dx < 0\,,$$

which contradicts the fact that $M_0 > 0$. Hence, (v_n) is bounded in W. □

Lemma 6.1.3. *There exists in W a maximizer $\bar{v} \geq 0$ of* (6.6).

Proof. By $2 < p < 2^*$ and $(A1)$ we can use the Kondrashev imbedding theorem

$$W = {}_\rho W_2^1(\Omega) \subset\subset {}_\rho L^p(\Omega)$$

where

$$\|v\|_{{}_\rho L^p} = \int |v|^p \rho\,.$$

Thus, by Lemma 6.1.2 we can take a maximizing sequence that weakly converges to a $\bar{v}$ in W, so that

$$E_1(\bar{v}) = M_0 \ , \quad H(\bar{v}) = \|\bar{v}\|_W^2 \leq 1 \, ;$$

we need to prove that, actually, $H(\bar{v}) = 1$. First, notice that $H(\bar{v}) = \|\bar{v}\|_W^2 \neq 0$, because otherwise it would be $\bar{v}(x) \equiv \bar{C}$, and then by $(A3)$

$$0 < M_0 = E_1(\bar{v}) = |\bar{C}|^p \int_\Omega a(x)\rho(x)\, dx \leq 0.$$

Second, suppose $0 < H(\bar{v}) < 1$; hence, for a suitable $k > 1$:

$$H(k\bar{v}) = |k|^2 H(\bar{v}) = 1$$

and

$$E_1(k\bar{v}) = |k|^p E_1(\bar{v}) = |k|^p M_0 > M_0,$$

which is in contradiction with (6.6). Finally, from the general properties of the Sobolev space W, we have $|\bar{v}| \in W$; we have also

$$H(|\bar{v}|) = H(\bar{v}) = 1 \quad \text{and} \quad E_1(|\bar{v}|) = E_1(\bar{v}) = M_0 \, ,$$

therefore it is not restrictive to consider $\bar{v} \geq 0$. $\square$

6.1.3. Step 3

Let $\bar{v} \geq 0$ be the maximizer of (6.6) found in Lemma 6.1.3, and set

$$\bar{u}(x) := \bar{t} \cdot \bar{v}(x)$$

with $\bar{t} = t(\bar{v}) > 0$ defined by (6.4), i.e.

$$\bar{t} = +\Big[E_1(\bar{v})\Big]^{-\frac{1}{p-2}}.$$

By applying the fibering method it follows that $\bar{u}$ is a nonnegative solution of the BVP (6.1). Clearly, $\bar{u} \neq 0$ because $\bar{v} \neq 0$ and $E_1(\bar{v}) = M_0 > 0$. The proof of Theorem 6.1.1 is complete. $\square$

6.1.4. The case $1 < p < 2$

Here we consider again the BVP (6.1), but with $1 < p < 2$. In this case we have the same representation (6.2) for the Euler functional $E(u)$. Let us consider for a fixed v the behaviour of $\tilde{E}$ with respect to $t > 0$. If $E_1(v) < 0$, then $\tilde{E}$ is decreasing in t, anyhow. If $E_1(v) < 0$, then $\tilde{E}$ is eventually increasing or eventually decreasing in t for $2 < p < 2^*$ or $1 < p < 2$, respectively.

We get the following result.

THEOREM 6.1.4. *Assume* $(A1)$–$(A3)$ *and* $1 < p < 2$. *Then there exists a nonnegative (nontrivial) solution of* (6.1).

Proof. The proof is completely coincident with that of Theorem 6.1.1. $\blacksquare$

$\square$

6.2. The case of the p-Laplacian

In this section we generalize the previous results to the existence of positive solutions for the BVP

$$\begin{cases} \Delta_p u + (\nabla\psi, \nabla u)|\nabla u|^{p-2} + a(x)|u|^{q-2}u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \partial\Omega\,. \end{cases} \tag{6.10}$$

Here $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ with $p > 1$, and $\psi \in C^1(\overline{\Omega})$. As before, we consider this problem under assumptions $(A1)$–$(A3)$:

$(A1)$ $\quad a \in L^\infty(\Omega)$;

$(A2)$ $\quad a^+(x)$ is not identically zero;

$(A3)$ $\quad \displaystyle\int_\Omega a(x)\rho(x)\,dx < 0$, where $\rho(x) := e^{\psi(x)}$.

Concerning q we suppose

$$1 < q < p^*\ , \quad q \neq p\ , \quad p^* = \begin{cases} \frac{Np}{N-p} & \text{for } p < N \\ +\infty & \text{for } p \geq N \end{cases} \tag{6.11}$$

(notice that the role of q was played by p in the previous section). We have then the following result.

THEOREM 6.2.1. *Let the assumptions* $(A1)$–$(A3)$ *and* (6.11) *be satisfied. Then the BVP* (6.10) *admits a nonnegative (nontrivial) solution*

$$u \in W_p^1(\Omega) \cap L^\infty(\Omega) \cap C^{1,\alpha}_{\mathrm{loc}}(\Omega)\,.$$

The proof is divided into some steps.

6.2.1. Step 1

Similarly as in the previous section,

$$E(u) := -\frac{1}{p}\int_\Omega |\nabla u|^p \rho + \frac{1}{q}\int_\Omega a|u|^q \rho$$

is the Euler functional associated with (6.10). From our assumptions it follows that $E(u)$ is well-defined on the Sobolev space

$$W := {}_\rho W_p^1(\Omega)\ , \quad \|u\|_W = \left(\int_\Omega |\nabla u|^p \rho\right)^{1/p} .$$

Following the fibering method we set $u(x) = t \cdot v(x)$ with $t \neq 0$ and $v \in W$. Under the (spherical) fibering constraint

$$H(v) := \int_\Omega |\nabla u|^p \rho = 1 \tag{6.12}$$

the functional E reduces to

$$\tilde{E}(t,v) = -\frac{|t|^p}{p} + \frac{|t|^q}{q} E_1(v) \tag{6.13}$$

where

$$E_1(v) := \int_\Omega a(x)|v(x)|^q \rho(x)\,dx\,.$$

From the bifurcation equation

$$\tilde{E}_t' = -|t|^{p-2}t + |t|^{q-2}t\,E_1(v) = 0$$

we find for $t \neq 0$

$$|t(v)| = \Big[E_1(v)\Big]^{\frac{1}{p-q}} \tag{6.14}$$

with the necessary condition $E_1(v) > 0$. Now, by substituting (6.14) in (6.13) we obtain

$$\hat{E}(v) = \left(\frac{1}{q} - \frac{1}{p}\right)\Big[E_1(v)\Big]^{+\frac{p}{p-q}} . \tag{6.15}$$

6.2.2. Step 2

Now we search for an extremal point of E_1, i.e. of $\hat{E}$, under the fibering constraint $H(v) = 1$.

LEMMA 6.2.2. *The variational problem*

$$M_1 := \sup_{v \in W} \left\{ E_1(v) \,\Big|\, H(v) \le 1 \right\} \tag{6.16}$$

admits a nonnegative maximizer $\bar{v} \in W$ *with* $H(\bar{v}) = 1$ *and* $E_1(\bar{v}) = M_1 > 0$.

Proof. The proof follows exactly the same lines as that of Lemma 6.1.2 together with the proof of Lemma 6.1.3. ☐

6.2.3. Step 3

By virtue of the fibering method we derive that $\bar{u}(x) = \bar{t} \cdot \bar{v}(x)$, with

$$\bar{t} = t(\bar{v}) = +\Big[E_1(\bar{v})\Big]^{-\frac{1}{p-q}}$$

(see equality (6.14)), is a nonnegative solution of (6.10). Clearly, $\bar{u}$ is nontrivial since $\bar{v} \neq 0$. By the bootstrap argument (used e.g. in [9]) we can prove that $u \in L^\infty(\Omega)$; then from the result of Tolksdorf [24] it follows that $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$. Theorem 6.2.1 is proved. ☐

Conclusion

So, we showed applications of the fibering method to:

1. the existence problem;
2. the existence of multiple solutions;
3. the existence of infinitely many solutions;
4. the problem of non-existence of solutions.

The above mentioned results have been obtained by using the one-parametric fibering method. If we consider the imbedding of the original space X into $\tilde{X} := \mathbb{R}^k \times X$ we get the k-parametric fibering method. As an application of the 2-parametric fibering method, we consider the following nonlinear problem.

The authors of [7] considered a **quasilinear elliptic problem with a resonance**, that is, a linear problem with spectral value of the parameter and with a nonlinear perturbation "small" at infinity. We consider the problem when the perturbation is superlinear and anticoercive. It is natural to base the investigation of such problems no longer on a one-parametric fibration, but on a multiparametric one; the simplest variant of a two-parametric fibration is used in the following problem. Let Ω be a bounded region in $\mathbb{R}^N$, $N \leq 3$, with locally Lipschitz boundary $\partial\Omega$. We consider in $\overset{\circ}{W}{}^1_2(\Omega)$ the boundary value problem

$$\begin{cases} \Delta u + \lambda_1 u + u^3 = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6.17}$$

where λ_1 is the first eigenvalue of $-\Delta$ under Dirichlet boundary conditions, with eigenfunction $e_1 \in \overset{\circ}{W}{}_2^1(\Omega)$ such that

$$\int_\Omega |\nabla e_1|^2 = \lambda_1 .$$

We represent $u = te_1 + sv$, where $\langle e_1, v\rangle = 0$ and $t, s \in \mathbb{R}$, $s > 0$; then from the Euler functional f associated to problem (6.17) we obtain the functional $\tilde{f}$ defined by

$$\tilde{f}(t,s,v) := f(te_1+sv) = \frac{s^2}{2}\left(\int_\Omega |\nabla v|^2 - \lambda_1 \int_\Omega v^2\right) - \frac{1}{4}\int_\Omega (te_1+sv)^4.$$

By the minimax realization of the fibering method, the corresponding functional $\hat{f}$ takes the form

$$\hat{f}(v) = \max_{s>0,\, t\in\mathbb{R}} \tilde{f}(t,s,v);$$

for $v \neq 0$ and $\langle e_1, v\rangle = 0$, we have that $\hat{f}$ belongs to the class C^2. The fibering constraint is

$$H(v) = -1 \quad \text{with} \quad H(v) := \tilde{f}''_{ss}(t(v), s(v), v)\,, \tag{6.18}$$

where $t(v)$ and $s(v) > 0$ are uniquely determined by the relation

$$\tilde{f}(t(v), s(v), v) = \hat{f}(v)$$

with respect to $v \neq 0$. We apply the Lyusternik-Shnirel'man theory to the even functional $\hat{f}$, considered on the manifold (6.18). Then we get that the boundary value problem (6.17) has a countable set of nontrivial solutions.

Remark 6.2.3: The application of the fibering method to systems of nonlinear ordinary differential equations can be find in [20]. Some other applications can be found in [4] and [17].

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