Rend. Istit. Mat. Univ. Trieste
Vol. XXXI, 203–234 (1999)

Everywhere Regularity for a Class of Elliptic Systems with p, q Growth Conditions

Anna Paola Migliorini [(*)]

Summary. - *We shall prove everywhere regularity for weak solutions of elliptic systems of the form*

$$\sum \frac{\partial}{\partial x_i} a\left(x, |Du|\right) u^\alpha_{x_i} = 0$$

under general p, q growth conditions and in particular for minimizers of a class of variational integrals, both degenerate and non degenerate ones, whose models are

$$\begin{aligned} I_1\left(u\right) &= \int_\Omega a\left(x\right) |Du|^{b(x)}\, dx, \\ I_2\left(u\right) &= \int_\Omega a\left(x\right) \left(1 + |Du|^2\right)^{\frac{b(x)}{2}} dx. \end{aligned}$$

1. Introduction

In this paper we study everywhere regularity for weak solutions of elliptic systems of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^\alpha_i\left(x, Du\right) = 0 \tag{1}$$

(*) Author's address: Dipartimento di Matematica "U. Dini", Universita' di Firenze, viale Morgagni 67/A, I-50134 Firenze, Italy email: anna.migliorini@math.unifi.it

AMS classification: primary 35J50, 49N60; secondary 35B45, 35D10, 49J45.
Keywords: Elliptic Systems, Regularity, Calculus of Variations, General Growth Conditions.
I wish to thank Paolo Marcellini for many hints and encouragements.

for $\alpha = 1, 2, ..., N$ and $x \in \Omega$, where Ω is an open bounded subset of $\mathbf{R}^n$ $(n \geqslant 2)$ and Du is the gradient of a vector-valued function $u : \Omega \to \mathbf{R}^N$ $(N \geqslant 1)$. We assume that the functions $a_i^\alpha (x, \xi)$ depend only on the modulus of the gradient $|Du|$ in the following way

$$a_i^\alpha (x, Du) = a (x, |Du|) u_{x_i}^\alpha$$

for a positive function $a (x, t)$ increasing with respect to t. Therefore the vector field $\{a_i^\alpha\}$ is the gradient with respect to the ξ-variable of a real function $f = f (x, \xi)$, $(x, \xi) \in \Omega \times \mathbf{R}^{Nn}$ and a weak solution of the system (1) is a minimizer of the integral of the Calculus of Variations

$$I (u) = \int_\Omega f (x, Du) \, dx, \qquad \text{with} \qquad f (x, Du) = g (x, |Du|) \tag{2}$$

where, since

$$a_i^\alpha (x, \xi) = f_{\xi_i^\alpha} (x, \xi) = \frac{g_t (x, |\xi|)}{|\xi|} \xi_i^\alpha,$$

$a (x, t)$ is related to $g (x, t)$ by

$$a (x, t) = \frac{g_t (x, t)}{t}.$$

We assume *general p, q growth conditions*, with $2 \leqslant p \leqslant q$, for the integrand f and we extend the classical regularity results known for the so-called *natural growth conditions* when $p = q$ (we refer to the books of M. Giaquinta [6] and E. Giusti [8]).

In the context of vector-valued problems $(N > 1)$, the only kind of regularity we can expect in general is *partial regularity*, introduced by Morrey [14] in the late 60's. Nevertheless K. Uhlenbeck [16], in a fundamental paper of 1977, proved everywhere $C^{1,\alpha}$ regularity for local minimizers $u \in W_{loc}^{1,p} \left(\Omega, \mathbf{R}^N\right)$ of the integral

$$\int_\Omega |Du|^p \, dx,$$

where $p \geqslant 2$ and, in general, for local minimizers of the integral

$$\int_\Omega g (|Du|) \, dx,$$

where $g(t)$ behaves like t^p. This result has been generalized in different ways. Dependence of the integrand on (x,u) is allowed by Giaquinta-Modica in [7], where the authors consider integrands of the type

$$f(x,u,\xi) = g(x,u,|\xi|)\,.$$

They proved everywhere regularity in the scalar case ($N = 1$) and partial regularity in the vectorial one. These results have been extended to $1 < p < 2$ by Acerbi-Fusco in [5]. In both works, only natural growth conditions are allowed for the integrand.

Non standard growth conditions have been introduced in the scalar case by Marcellini in [10], [11], [12], where everywhere regularity has been proved. Specific studies of regularity in the vector-valued case can be found in the papers by Acerbi-Fusco [1], Choe [3] and Lieberman [9]. General growth conditions have been considered in the vectorial case by Marcellini in [13], where everywhere regularity has been proved in the case independent of (x,u).

Recently Chiadò Piat-Coscia in [15] obtained Hölder continuity of local minimizers of integral functionals with variable growth exponent, whose model is

$$\int_\Omega |Du|^{b(x)}\,dx$$

and this result have been extended to the vectorial case by Coscia-Mingione in [4].

In this paper, more generally, we obtain regularity of minimizers, for example, of the model problem

$$\int_\Omega a(x)\left(\mu + |Du|^2\right)^{\frac{b(x)}{2}}\,dx$$

with $a(x)$, $b(x) \in W^{1,\infty}(\Omega)$, $a(x) \geqslant a_0 > 0$, $\mu = 0$ or $\mu = 1$ and $2 \leqslant p \leqslant b(x) \leqslant q$ with a bound on the ratio $\frac{q}{p}$.

More precisely, we give some *a priori* estimates when p and q satisfy

$$2 \leqslant p \leqslant q < \frac{n}{n-2}p \tag{3}$$

(simply $2 \leqslant p \leqslant q$ if $n = 2$), while we prove local Lipschitz continuity if

$$2 \leqslant p \leqslant q < \frac{n+2}{n} p. \tag{4}$$

We assume the following p, q *growth conditions* on the integrand f:

$$m \left(\mu + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \leqslant \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} (x, \xi) \lambda_i^\alpha \lambda_j^\beta \leqslant M \left(\mu + |\xi|^2\right)^{\frac{q-2}{2}} |\lambda|^2 \tag{5}$$

$$\left| f_{\xi_i^\alpha x_s} (x, \xi) \right| \leqslant M \left(\mu + |\xi|^2\right)^{\frac{p+q-2}{4}} \tag{6}$$

and we consider exponents p and q related by (3) or by (4). In order to prove the *a priori* estimates we have to use different methods for the case $\mu = 0$ or $\mu = 1$.

In order to state one of the main results of this paper, let us denote by B_ρ and B_R balls compactly contained in Ω of radii ρ and R respectively and with the same center. We prove the following theorem.

THEOREM 1.1. *Under the assumptions* (3), (4), (5) *and* (6), *every weak solution u of the system* (1) *and every minimizer of the integral* (2) *is of class $W^{1,\infty}_{loc}\left(\Omega, \mathbf{R}^N\right)$ and, for every ρ, R, with $0 < \rho \leqslant R < 1$, there exists a constant $c = c\left(\rho, R, n, N, p, q, m, M\right)$ and an exponent $\alpha = \alpha\left(p, q, n\right)$ such that*

$$\|Du\|_{L^\infty(B_\rho, \mathbf{R}^{Nn})} \leqslant c \left\{ \int_{B_R} \left[1 + f\left(x, |Du|\right)\right] dx \right\}^{\frac{\alpha}{p}}.$$

The exponent α can be estimated explicitly by

$$\alpha = \frac{2p}{(n+2)\, p - nq}$$

if $n > 2$; otherwise, if $n = 2$ and $\frac{q}{p} > 1$, then

$$\alpha = \frac{\theta \frac{p}{q}}{1 - \theta \left(1 - \frac{p}{q}\right)}$$

where θ is any number such that $\frac{q}{p} < \theta < \frac{q}{q-p}$; finally, if $n = 2$ and $p = q$, then $\alpha = 1$.

We make use of the two methods introduced by Marcellini in [11] and [13], combining them in order to handle the technical problems due to the x-dependence. We obtain an explicit estimate of the L^∞-norm of the gradient Du in term of its L^q-norm and, by an interpolation technique, an estimate of the L^∞-norm of the gradient Du in term of its L^p-norm. Hence, by using these a priori estimates and by an approximation of the original problem with regular integrals, we prove the local boundedness of the gradient of minimizers.

2. Regularity

We consider the integral of the Calculus of Variation

$$I(u) = \int_\Omega f(x, Du)\, dx, \qquad \text{with} \qquad f(x, Du) = g(x, |Du|) \tag{7}$$

where Ω is an open bounded subset of $\mathbf{R}^n$ $(n \geqslant 2)$, Du is the gradient of a vector-valued function $u : \Omega \to \mathbf{R}^N$ $(N \geqslant 1)$ and $f : \Omega \times \mathbf{R}^{Nn} \to \mathbf{R}$ has the form $f(x, \xi) = g(x, |\xi|)$ for $x \in \Omega$ and $\xi \in \mathbf{R}^{Nn}$ $(\xi = (\xi_i^\alpha)$, $i = 1, 2, ..., n,$ $\alpha = 1, 2, ..., N)$. We assume that the function

$$g = g(x, t) : \Omega \times [0, +\infty] \to [0, +\infty]$$

is of class C^2, with $g_t(x, t) = \frac{\partial g(x,t)}{\partial t}$ positive and increasing with respect to t for *a.e.* $x \in \Omega$.

In term of systems, we deal with

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i^\alpha (x, Du) = 0 \qquad \forall \alpha = 1, 2, ..., N, \tag{8}$$

where

$$a_i^\alpha (x, \xi) = f_{\xi_i^\alpha} (x, \xi) = \frac{g_t(x, |\xi|)}{|\xi|} \xi_i^\alpha \qquad \forall \alpha = 1, 2, ..., N, \ \forall i = 1, 2, ..., n.$$

We consider exponents p and q such that

$$2 \leqslant p \leqslant q < \frac{n}{n-2} p \tag{9}$$

(simply $2 \leqslant p \leqslant q < +\infty$, if $n = 2$). About the function $f(x, \xi)$ and its derivatives with respect to x and ξ, we assume that there are two positive constants m and M such that for every λ and $\xi \in \mathbf{R}^{Nn}$ and for $a.e.x \in \Omega$ we have

$$m\left(\mu + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \leqslant \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, \xi)\, \lambda_i^\alpha \lambda_j^\beta \leqslant M\left(\mu + |\xi|^2\right)^{\frac{q-2}{2}} |\lambda|^2 \tag{10}$$

$$\left|f_{\xi_i^\alpha x_s}(x, \xi)\right| \leqslant M\left(\mu + |\xi|^2\right)^{\frac{p+q-2}{4}} \tag{11}$$

for $\mu = 0$ or $\mu = 1$, $\forall \alpha = 1, 2, ..., N, \qquad \forall i,\ s = 1, 2, ..., n.$

A minimizer of the integral (7) is a function $u \in W^{1,p}\left(\Omega, \mathbf{R}^N\right)$ such that $f(x, Du) \in L^1_{loc}(\Omega)$ with the property that $I(u) \leqslant I(u + \varphi)$ for every $\varphi \in C_0^1\left(\Omega, \mathbf{R}^N\right)$. A weak solution of (8) is a function $u \in W^{1,q}_{loc}\left(\Omega, \mathbf{R}^N\right)$ such that for every $\Omega' \subset\subset \Omega$ and for every test function $\varphi \in W_0^{1,q}\left(\Omega', \mathbf{R}^N\right)$, u satisfies

$$\int_\Omega \sum_{i=1}^n a_i^\alpha(x, Du)\, \varphi_{x_i}^\alpha(x)\, dx = 0, \quad \forall \alpha = 1, 2, ..., N. \tag{12}$$

By assumption (10), every minimizer u of the integral (7) of class $W^{1,q}_{loc}\left(\Omega, \mathbf{R}^N\right)$ satisfies the Euler's first variation

$$\int_\Omega \sum_{i=1}^n f_{\xi_i^\alpha}(x, Du)\, \varphi_{x_i}^\alpha(x)\, dx = 0, \quad \forall \alpha = 1, 2, ..., N, \qquad \forall \varphi \in W_0^{1,q}\left(\Omega, \mathbf{R}^N\right) \tag{13}$$

and thus u is a weak solution of (8).

Let B_ρ and B_R balls compactly contained in Ω of radii ρ and R respectively and with the same center, and such that $0 < \rho \leqslant R < 1$. The main result of this section is the following *a priori* estimate.

Theorem 2.1. *Let* (9) *to* (11) *hold. Then every minimizer* u *of the integral* (7), *of class* $W^{1,q}_{loc}\left(\Omega, \mathbf{R}^N\right)$, *is of class* $W^{1,\infty}_{loc}\left(\Omega, \mathbf{R}^N\right)$.

Moreover there are positive numbers C, C', β, θ such that, for $\mu = 1$ we have

$$\sup_{x \in B_\rho} \left(1 + |Du|^2\right)^{\frac{1}{2}} \leqslant \frac{C}{(R-\rho)^{2\beta\theta}} \left\| \left(1 + |Du|^2\right)^{\frac{1}{2}} \right\|_{L^q(B_R)}^{\theta}$$

and for $\mu = 0$

$$\sup_{x \in B_\rho} (1 + |Du|) \leqslant \frac{C'}{(R-\rho)^{2\beta\theta}} \|(1 + |Du|)\|_{L^q(B_R)}^{\theta} .$$

Let us start with some lemmas from linear algebra. They can be proved using the Cauchy-Schwarz inequality (as in [11], lemmas 2.4 and 2.5).

Lemma 2.2. *Under the assumption* (10), *there is a constant c_1 such that for every λ, ξ, $\eta \in \mathbf{R}^{Nn}$ and for a.e.$x \in \Omega$ we have*

$$\left| \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} (x, \xi)\, \eta_i^\alpha \lambda_j^\beta \right| \leqslant c_1 \left(\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} (x, \xi)\, \lambda_i^\alpha \lambda_j^\beta \right)^{\frac{1}{2}} \left(\mu + |\xi|^2\right)^{\frac{q-2}{4}} |\eta| .$$

Lemma 2.3. *Under the assumptions* (10) *and* (11) *there is a constant c_2 such that for every λ, $\xi \in \mathbf{R}^{Nn}$ and for a.e.$x \in \Omega$ we have*

$$\left| \sum_{i,\alpha} f_{\xi_i^\alpha x_s} (x, \xi)\, \lambda_i^\alpha \right| \leqslant c_2 \left(\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} (x, \xi)\, \lambda_i^\alpha \lambda_j^\beta \right)^{\frac{1}{2}} \left(\mu + |\xi|^2\right)^{\frac{q}{4}}$$

$$\forall s = 1, 2, ..., n.$$

By using (10), with the technique of the different quotient (see, for example Theorem 1.1 of Chapter II of [6]; in this context, see [11]), we obtain that u admits second derivatives, precisely that $u \in W^{2,2}_{loc}\left(\Omega, \mathbf{R}^N\right)$ and satisfies the second variation

$$\int_\Omega \left\{ \sum_{i,\alpha} f_{\xi_i^\alpha x_s} (x, Du)\, \varphi_{x_i}^\alpha (x) + \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta} (x, Du)\, \varphi_{x_i}^\alpha u_{x_s x_j}^\beta \right\} dx = 0 \tag{14}$$

$$\forall s = 1, 2, ..., n, \qquad \forall \varphi = (\varphi^\alpha) \in W_0^{1,q}\left(\Omega, \mathbf{R}^N\right).$$

Formally, we derive this equation from (13), taking as test function $\varphi = \psi_{x_s}$and integrating by parts (see [11] for details).

Fixed $1 \leqslant s \leqslant n$, let η be a positive function of class $C_0^1(\Omega)$ and we choose $\varphi^\alpha = \eta^2 u^\alpha_{x_s}\Phi(|Du|)$ for every $\alpha = 1, 2, ..., N$, where Φ is a positive, increasing, bounded, Lipschitz continuous function defined in $[0, +\infty)$, (in particular Φ and Φ' are bounded, so that $\varphi \in W_0^{1,q}\left(\Omega, \mathbf{R}^N\right)$). Then

$$\varphi^\alpha_{x_i} = 2\eta\eta_{x_i}u^\alpha_{x_s}\Phi(|Du|) + \eta^2 u^\alpha_{x_sx_i}\Phi(|Du|) + \eta^2 u^\alpha_{x_s}\Phi'(|Du|)(|Du|)_{x_i}$$

and from (14) we obtain

$$\begin{aligned}
0 \quad = \quad & \int_\Omega 2\eta\Phi \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x, Du)\,\eta_{x_i}u^\alpha_{x_s}dx && (15)\\
& + \int_\Omega \eta^2\Phi \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x, Du)\,u^\alpha_{x_sx_i}dx && (16)\\
& + \int_\Omega \eta^2\Phi' \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x, Du)\,u^\alpha_{x_s}(|Du|)_{x_i}dx && (17)\\
& + \int_\Omega 2\eta\Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x, Du)\,\eta_{x_i}u^\alpha_{x_s}u^\beta_{x_sx_j}dx && (18)\\
& + \int_\Omega \eta^2\Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x, Du)\,u^\alpha_{x_sx_i}u^\beta_{x_sx_j}dx && (19)\\
& + \int_\Omega \eta^2\Phi' \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x, Du)\,u^\alpha_{x_s}u^\beta_{x_sx_j}(|Du|)_{x_i}dx. && (20)
\end{aligned}$$

Let us start with the integral in (15). By the assumption (11), we have

$$
\begin{aligned}
&\left|\int_{\Omega} 2\eta\Phi \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x, Du)\, \eta_{x_i} u_{x_s}^\alpha\, dx\right| \qquad (21)\\
\leqslant\ & M \int_{\Omega} 2\eta\Phi \left(\mu + |Du|^2\right)^{\frac{p+q-2}{4}} \sum_{i,\alpha} \left|\eta_{x_i} u_{x_s}^\alpha\right| dx \\
\leqslant\ & c_0 \int_{\Omega} 2\eta\, |D\eta|\, \Phi \left(\mu + |Du|^2\right)^{\frac{p+q}{4}} dx.
\end{aligned}
$$

About the integral in (16), from lemma (2.3) and by using the inequality $|ab| \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, we obtain

$$
\begin{aligned}
&\left|\int_{\Omega} \eta^2\Phi \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x, Du)\, u_{x_s x_i}^\alpha dx\right| \qquad (22)\\
\leqslant\ & c_2 \int_{\Omega} \eta^2\Phi \left(\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u_{x_s x_i}^\alpha u_{x_s x_j}^\beta\right)^{\frac{1}{2}} \left(\mu + |Du|^2\right)^{\frac{q}{4}} dx \\
\leqslant\ & c_2\varepsilon_0 \int_{\Omega} \eta^2\Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u_{x_s x_i}^\alpha u_{x_s x_j}^\beta dx \\
&\qquad + \frac{c_2}{4\varepsilon_0} \int_{\Omega} \eta^2\Phi \left(\mu + |Du|^2\right)^{\frac{q}{2}} dx.
\end{aligned}
$$

Similarly, by lemma (2.2), from the integral (18) we have

$$\left|\int_\Omega 2\eta\Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du)\,\eta_{x_i} u_{x_s}^\alpha u_{x_s x_j}^\beta\,dx\right| \tag{23}$$

$$\leqslant c_1 \int_\Omega 2\eta\Phi \left(\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du)\,u_{x_i x_s}^\alpha u_{x_s x_j}^\beta\right)^{\frac{1}{2}}$$

$$\cdot\left(\mu+|Du|^2\right)^{\frac{q-2}{4}}\left(\sum_{\alpha,i}\left|\eta_{x_i} u_{x_s}^\alpha\right|^2\right)^{\frac{1}{2}} dx$$

$$\leqslant c_3\varepsilon_1 \int_\Omega \eta^2\Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du)\,u_{x_i x_s}^\alpha u_{x_s x_j}^\beta\,dx$$

$$+\frac{c_3}{4\varepsilon_1}\int_\Omega |D\eta|^2\,\Phi\left(\mu+|Du|^2\right)^{\frac{q}{2}} dx.$$

If we sum with respect to s from 1 to n these estimates, they remain the same except for the constants. We continue to use c_0, c_2 and c_3 even if changed. Let us consider the integral (17) summed with respect to s. By the assumption (11), we have

$$\left|\int_\Omega \eta^2\Phi' \sum_{i,\alpha,s} f_{\xi_i^\alpha x_s}(x,Du)\,u_{x_s}^\alpha\left(|Du|\right)_{x_i} dx\right| \tag{24}$$

$$\leqslant M\int_\Omega \eta^2\Phi' \sum_{i,\alpha,s}\left(\mu+|Du|^2\right)^{\frac{p+q-2}{4}}\left|u_{x_s}^\alpha\left(|Du|\right)_{x_i}\right| dx$$

$$\leqslant M\int_\Omega \eta^2\Phi' \sum_{i,\alpha,s}\left(\mu+|Du|^2\right)^{\frac{p-2}{4}}\left|\left(|Du|\right)_{x_i}\right|$$

$$\cdot\left(\mu+|Du|^2\right)^{\frac{1}{2}}\left(\mu+|Du|^2\right)^{\frac{q}{4}} dx$$

$$\leqslant c_4\varepsilon_2\int_\Omega \eta^2\Phi'\left(\mu+|Du|^2\right)^{\frac{1}{2}}\left(\mu+|Du|^2\right)^{\frac{p-2}{2}}\sum_i\left|\left(|Du|\right)_{x_i}\right|^2 dx$$

$$+\frac{c_4}{4\varepsilon_2}\int_\Omega \eta^2\Phi'\left(\mu+|Du|^2\right)^{\frac{1}{2}}\left(\mu+|Du|^2\right)^{\frac{q}{2}} dx.$$

In order to estimate the integral in (20), summed with respect to s, we remember $f(x,\xi) = g(x,|\xi|)$ and we calculate

$$\begin{aligned} f_{\xi_i^\alpha}(x,\xi) &= \frac{g_t(x,|\xi|)}{|\xi|}\xi_i^\alpha \\ f_{\xi_i^\alpha \xi_j^\beta}(x,\xi) &= \left(\frac{g_{tt}(x,|\xi|)}{|\xi|^2} - \frac{g_t(x,|\xi|)}{|\xi|^3}\right)\xi_j^\beta \xi_i^\alpha + \frac{g_t(x,|\xi|)}{|\xi|}\delta_{\xi_i^\alpha \xi_j^\beta}. \end{aligned}$$

Moreover we have

$$(|Du|)_{x_i} = \frac{1}{|Du|}\sum_{s,\alpha} u_{x_s}^\alpha u_{x_i x_s}^\alpha. \tag{25}$$

Since $\frac{g_t(x,t)}{t}$ is increasing with respect to t, it follows that

$$0 \leqslant \frac{\partial}{\partial t}\frac{g_t(x,t)}{t} = \frac{g_{tt}(x,t)\,t - g_t(x,t)}{t^2}$$

and, using also the fact that $g_t(x,t)$ is positive, we can prove that

$$\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du)\, u_{x_s}^\alpha u_{x_s x_j}^\beta (|Du|)_{x_i} \geqslant 0. \tag{26}$$

In fact

$$\begin{aligned} &\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du)\, u_{x_s}^\alpha u_{x_s x_j}^\beta (|Du|)_{x_i} \\ = &\left(\frac{g_{tt}(x,|\xi|)}{|\xi|^2} - \frac{g_t(x,|\xi|)}{|\xi|^3}\right)\sum_{i,j,s,\alpha,\beta} u_{x_i}^\alpha u_{x_j}^\beta u_{x_s x_j}^\beta u_{x_s}^\alpha (|Du|)_{x_i} \\ &\qquad + \frac{g_t(x,|\xi|)}{|\xi|}\sum_{i,s,\alpha} u_{x_s}^\alpha u_{x_i x_s}^\alpha (|Du|)_{x_i} \\ = &\left(\frac{g_{tt}(x,|\xi|)}{|\xi|} - \frac{g_t(x,|\xi|)}{|\xi|^2}\right)\sum_{i,s,\alpha} u_{x_i}^\alpha (|Du|)_{x_s} u_{x_s}^\alpha (|Du|)_{x_i} \\ &\qquad + g_t(x,|\xi|)\sum_i \left((|Du|)_{x_i}\right)^2 \\ = &\left(\frac{g_{tt}(x,|\xi|)}{|\xi|} - \frac{g_t(x,|\xi|)}{|\xi|^2}\right)\sum_{i,s,\alpha} \left(u_{x_i}^\alpha (|Du|)_{x_s}\right)^2 \\ &\qquad + g_t(x,|\xi|)\sum_i \left((|Du|)_{x_i}\right)^2 \geqslant 0 \end{aligned}$$

and thus(26).

From the second variation equation (14) and the previous estimates (21), (22), (23), (24), (26), we obtain that

$$
\begin{aligned}
&\int_\Omega \eta^2 \Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u^\alpha_{x_s x_i} u^\beta_{x_s x_j}\, dx \\
\leqslant\; & c_0 \int_\Omega 2\eta\, |D\eta|\, \Phi \left(\mu + |Du|^2\right)^{\frac{p+q}{4}} dx \\
& + c_2 \varepsilon_0 \int_\Omega \eta^2 \Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u^\alpha_{x_s x_i} u^\beta_{x_s x_j}\, dx \\
& + \frac{c_2}{4\varepsilon_0} \int_\Omega \eta^2 \Phi \left(\mu + |Du|^2\right)^{\frac{q}{2}} dx \\
& + c_3 \varepsilon_1 \int_\Omega \eta^2 \Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u^\alpha_{x_i x_s} u^\beta_{x_s x_j}\, dx \\
& + \frac{c_3}{4\varepsilon_1} \int_\Omega |D\eta|^2\, \Phi \left(\mu + |Du|^2\right)^{\frac{q}{2}} dx \\
& + c_4 \varepsilon_2 \int_\Omega \eta^2 \Phi' \left(\mu + |Du|^2\right)^{\frac{1}{2}} \left(\mu + |Du|^2\right)^{\frac{p-2}{2}} \sum_i \left|(|Du|)_{x_i}\right|^2 dx \\
& + \frac{c_4}{4\varepsilon_2} \int_\Omega \eta^2 \Phi' \left(\mu + |Du|^2\right)^{\frac{1}{2}} \left(\mu + |Du|^2\right)^{\frac{q}{2}} dx.
\end{aligned}
$$

Now we can choose ε_0, ε_1 both in the second and the fourth integral in order to have the same integral as in the first member. From now on we relabel the constants in a generic c, whose value may change from line to line. Thus the inequality above reduces to

$$\begin{aligned} & c\int_{\Omega}\eta^{2}\Phi\sum_{i,j,\alpha,\beta}f_{\xi_{i}^{\alpha}\xi_{j}^{\beta}}\left(x,Du\right)u_{x_{s}x_{i}}^{\alpha}u_{x_{s}x_{j}}^{\beta}dx \\ \leqslant\ & \int_{\Omega}2\eta\left|D\eta\right|\Phi\left(\mu+\left|Du\right|^{2}\right)^{\frac{p+q}{4}}dx \\ & +\int_{\Omega}\eta^{2}\Phi\left(\mu+\left|Du\right|^{2}\right)^{\frac{q}{2}}dx \\ & +\int_{\Omega}\left|D\eta\right|^{2}\Phi\left(\mu+\left|Du\right|^{2}\right)^{\frac{q}{2}}dx \\ & +c_{4}\varepsilon_{2}\int_{\Omega}\eta^{2}\Phi'\left(\mu+\left|Du\right|^{2}\right)^{\frac{1}{2}}\left(\mu+\left|Du\right|^{2}\right)^{\frac{p-2}{2}}\sum_{i}\left|\left(\left|Du\right|\right)_{x_{i}}\right|^{2}dx \\ & +\frac{c_{4}}{4\varepsilon_{2}}\int_{\Omega}\eta^{2}\Phi'\left(\mu+\left|Du\right|^{2}\right)^{\frac{1}{2}}\left(\mu+\left|Du\right|^{2}\right)^{\frac{q}{2}}dx. \end{aligned} \tag{27}$$

From (25), by using the Cauchy-Schwartz inequality, we see that

$$\left|D\left(\left|Du\right|\right)\right|^{2}=\sum_{i}\left|\left(\left|Du\right|\right)_{x_{i}}\right|^{2}\leqslant\sum_{i,s,\alpha}\left|u_{x_{s}x_{i}}^{\alpha}\right|^{2}=\left|D^{2}u\right|^{2}$$

and therefore we infer from assumption (10) that

$$\begin{aligned} & \int_{\Omega}\eta^{2}\Phi\sum_{i,j,\alpha,\beta}f_{\xi_{i}^{\alpha}\xi_{j}^{\beta}}\left(x,Du\right)u_{x_{s}x_{i}}^{\alpha}u_{x_{s}x_{j}}^{\beta}dx \\ \geqslant\ & m\int_{\Omega}\eta^{2}\Phi\left(\mu+\left|Du\right|^{2}\right)^{\frac{p-2}{2}}\left|D^{2}u\right|^{2}dx \\ \geqslant\ & m\int_{\Omega}\eta^{2}\Phi\left(\mu+\left|Du\right|^{2}\right)^{\frac{p-2}{2}}\left|D\left(\left|Du\right|\right)\right|^{2}dx. \end{aligned} \tag{28}$$

Now we allow only test function Φ satisfying

$$\Phi'\left(t\right)\left(\mu+t^{2}\right)^{\frac{1}{2}}\leqslant c_{\Phi}\Phi\left(t\right) \tag{29}$$

for a certain constant $c_{\Phi}\geqslant 1$ depending on the test function. From

(27) and (28), we obtain

$$\begin{aligned}
& c\int_\Omega \eta^2\Phi\left(|Du|\right)\left(\mu+|Du|^2\right)^{\frac{p-2}{2}}|D\left(|Du|\right)|^2\,dx \\
\leqslant\ & \int_\Omega 2\eta\,|D\eta|\,\Phi\left(|Du|\right)\left(\mu+|Du|^2\right)^{\frac{p+q}{4}}dx \\
& +\int_\Omega\left(\eta^2+|D\eta|^2\right)\Phi\left(|Du|\right)\left(\mu+|Du|^2\right)^{\frac{q}{2}}dx \\
& +c_4\varepsilon_2 c_\Phi\int_\Omega \eta^2\Phi\left(|Du|\right)\left(\mu+|Du|^2\right)^{\frac{p-2}{2}}|D\left(|Du|\right)|^2\,dx \\
& +\frac{c_4}{4\varepsilon_2}c_\Phi\int_\Omega \eta^2\Phi\left(|Du|\right)\left(\mu+|Du|^2\right)^{\frac{q}{2}}dx.
\end{aligned}$$

By choosing ε_2 in the second integral above, we can have the same integral as in the first member. Hence

$$\begin{aligned}
& c\int_\Omega \eta^2\Phi\left(|Du|\right)\left(\mu+|Du|^2\right)^{\frac{p-2}{2}}|D\left(|Du|\right)|^2\,dx \\
\leqslant\ & \int_\Omega 2\eta\,|D\eta|\,\Phi\left(|Du|\right)\left(\mu+|Du|^2\right)^{\frac{p+q}{4}}dx \\
& +\left(c_\Phi\right)^2\int_\Omega\left(\eta^2+|D\eta|^2\right)\Phi\left(|Du|\right)\left(\mu+|Du|^2\right)^{\frac{q}{2}}dx.
\end{aligned} \tag{30}$$

If we consider a general function Φ not bounded, with derivative Φ' not bounded too, for which (29) is true, then we can approximate Φ by a sequence of Lipschitz functions Φ_r bounded with Φ_r' bounded, in the following way:

$$\Phi_r\left(t\right)=\begin{cases}\Phi\left(t\right) & \text{for } t\in[0,r]\\ \Phi\left(r\right) & \text{for } t\in(r,+\infty)\end{cases}\quad r\in\mathbf{N}.$$

Since

$$\Phi_r'\left(t\right)\left(\mu+t^2\right)^{\frac{1}{2}}=\begin{cases}\Phi'\left(t\right)\left(\mu+t^2\right)^{\frac{1}{2}}\leqslant c_\Phi\Phi\left(t\right) & \text{for } t\in[0,r)\\ 0\leqslant c_\Phi\Phi\left(t\right) & \text{for } t\in(r,+\infty)\end{cases}$$

(while $\Phi_r'\left(r^+\right)$ and $\Phi_r'\left(r^-\right)$ are uniformly bounded), the condition (29) holds for Φ_r with the same constant of Φ. Thus (30) holds for Φ_r. By monotone convergence theorem, letting r tend to $+\infty$, we infer that (30) holds for every Φ positive, increasing, Lipschitz continuous function defined in $[0,+\infty)$ which satisfies (29).

Now we choose

$$\Phi(t) = \left(\mu + t^2\right)^{\frac{\gamma-1}{2}} \qquad \text{with } \gamma \geqslant 1$$

and since

$$\Phi'(t)\left(\mu + t^2\right)^{\frac{1}{2}} \leqslant (\gamma - 1)\left(\mu + t^2\right)^{\frac{\gamma-1}{2}} \leqslant \gamma\Phi(t)$$

the condition (29) is satisfied with $c_\Phi = \gamma$. With this choice of Φ, (30) reduces to

$$\begin{aligned} & c\int_\Omega \eta^2\left(\mu + |Du|^2\right)^{\frac{\gamma+p-3}{2}} |D(|Du|)|^2\,dx \\ \leqslant\ & \int_\Omega 2\eta\,|D\eta|\left(\mu + |Du|^2\right)^{\frac{\gamma-1}{2}+\frac{p+q}{4}} dx \\ & +\gamma^2\int_\Omega \left(\eta^2 + |D\eta|^2\right)\left(\mu + |Du|^2\right)^{\frac{\gamma+q-1}{2}} dx. \end{aligned} \tag{31}$$

Now we have to consider the two cases $\mu = 0$ and $\mu = 1$ separately.

Case $\mu = 1$.

Since $2 \leqslant p \leqslant q$ and $\gamma \geqslant 1$, the inequality in (31) can be written in the form

$$\begin{aligned} & c\int_\Omega \eta^2\left(1 + |Du|^2\right)^{\frac{\gamma+p-3}{2}} |D(|Du|)|^2\,dx \\ \leqslant\ & \gamma^2\int_\Omega \left(\eta^2 + |D\eta|^2\right)\left(1 + |Du|^2\right)^{\frac{\gamma+q-1}{2}} dx. \end{aligned} \tag{32}$$

Let us compute

$$\begin{aligned} & \left| D\left[\eta\left(1 + |Du|^2\right)^{\frac{\gamma+p-1}{4}}\right]\right|^2 \\ \leqslant\ & 2\,|D\eta|^2\left(1 + |Du|^2\right)^{\frac{\gamma+p-1}{2}} + \\ & \eta^2\frac{(\gamma+p-1)^2}{2}\left(1 + |Du|^2\right)^{\frac{\gamma+p-3}{2}} |D(|Du|)|^2 \\ \leqslant\ & 2\,|D\eta|^2\left(1 + |Du|^2\right)^{\frac{\gamma+q-1}{2}} + c\gamma^2\eta^2\left(1 + |Du|^2\right)^{\frac{\gamma+p-3}{2}} |D(|Du|)|^2 \end{aligned}$$

where, from now on we assume that c depends also on p. Therefore by (32) we infer that

$$\begin{aligned}
&\int_\Omega \left| D\left[\eta\left(1+|Du|^2\right)^{\frac{\gamma+p-1}{4}}\right]\right|^2 dx \\
\leqslant\; & 2\int_\Omega \left(\eta^2+|D\eta|^2\right)\left(1+|Du|^2\right)^{\frac{\gamma+q-1}{2}} dx \\
& +c\gamma^4 \int_\Omega \left(\eta^2+|D\eta|^2\right)\left(1+|Du|^2\right)^{\frac{\gamma+q-1}{2}} dx \\
\leqslant\; & c\gamma^4 \int_\Omega \left(\eta^2+|D\eta|^2\right)\left(1+|Du|^2\right)^{\frac{\gamma+q-1}{2}} dx.
\end{aligned}$$

By Sobolev's inequality, (remember the Sobolev's exponent $2^* = \frac{2n}{n-2}$ if $n \geqslant 3$, while is 2^* any fixed real number greater than 2 if $n=2$) we deduce

$$\begin{aligned}
&\left\{\int_\Omega \eta^{2^*}\left(1+|Du|^2\right)^{\frac{\gamma+p-1}{2}\frac{2^*}{2}} dx\right\}^{\frac{2}{2^*}} \\
\leqslant\; & c\gamma^4 \int_\Omega \left(\eta^2+|D\eta|^2\right)\left(1+|Du|^2\right)^{\frac{\gamma+q-1}{2}} dx.
\end{aligned}$$

Fixed $0<\rho\leqslant R<1$, let us denote by B_ρ and B_R balls compactly contained in Ω of radii ρ and R respectively and with the same center. Let η be a positive test function equal to 1 in B_ρ, whose support is contained in B_R, such that $|D\eta| \leqslant \frac{2}{R-\rho}$. Hence we obtain

$$\begin{aligned}
&\left\{\int_{B_\rho} \left(1+|Du|^2\right)^{\frac{\gamma+p-1}{2}\frac{2^*}{2}} dx\right\}^{\frac{2}{2^*}} \\
\leqslant\; & c\frac{\gamma^4}{(R-\rho)^2} \int_{B_R} \left(1+|Du|^2\right)^{\frac{\gamma+q-1}{2}} dx.
\end{aligned} \tag{33}$$

Since $\frac{\gamma+p-1}{2}\frac{2^*}{2} > \frac{\gamma+q-1}{2}$, this inequality gives an higher integrability of the gradient.

Case $\mu = 0$.
The inequality in (31) reduces to

$$\begin{aligned} & c \int_{\Omega} \eta^2 |Du|^{\gamma+p-3} |D(|Du|)|^2 \, dx \qquad (34) \\ \leqslant & \int_{\Omega} 2\eta |D\eta| |Du|^{\gamma-1+\frac{p+q}{2}} \, dx \\ & +\gamma^2 \int_{\Omega} \left(\eta^2 + |D\eta|^2\right) |Du|^{\gamma+q-1} \, dx. \end{aligned}$$

Let us define the function $G(t)$ for $t \in [0, +\infty)$ in the following way

$$G(t) = 1 + \int_0^t \sqrt{s^{p+\gamma-3}} ds;$$

since the function $t^{p+\gamma-3}$ is increasing and $p \leqslant q$, we have

$$[G(t)]^2 \leqslant \left[1 + t\sqrt{t^{\gamma+p-3}}\right]^2 \leqslant 2\left(1 + t^{\gamma+p-1}\right) \leqslant 4\left(1 + t^{\gamma+q-1}\right).$$

Let us compute

$$\begin{aligned} & |D[\eta G(|Du|)]|^2 \\ \leqslant & 2|D\eta|^2 [G(|Du|)]^2 + 2\eta^2 \left[G'(|Du|)\right]^2 |D(|Du|)|^2 \\ \leqslant & 8|D\eta|^2 \left(1 + |Du|^{\gamma+q-1}\right) + 2\eta^2 |Du|^{\gamma+p-3} |D(|Du|)|^2. \end{aligned}$$

Therefore by (34) we infer that

$$\begin{aligned} & \int_{\Omega} |D[\eta G(|Du|)]|^2 \, dx \\ \leqslant & 8 \int_{\Omega} |D\eta|^2 \left(1 + |Du|^{\gamma+q-1}\right) dx \\ & +c \int_{\Omega} 2\eta |D\eta| |Du|^{\gamma-1+\frac{p+q}{2}} \, dx \\ & +c\gamma^2 \int_{\Omega} \left(\eta^2 + |D\eta|^2\right) |Du|^{\gamma+q-1} \, dx. \end{aligned}$$

Finally, since

$$|Du|^{\gamma-1+\frac{p+q}{2}}, \quad |Du|^{\gamma+q-1} \leqslant \left(1+|Du|^{\gamma+q-1}\right)$$

and $\gamma \geqslant 1$, we obtain

$$\begin{aligned} & \int_\Omega |D[\eta G(|Du|)]|^2 \, dx \\ \leqslant \; & c\gamma^2 \int_\Omega \left(\eta^2+|D\eta|^2\right)\left(1+|Du|^{\gamma+q-1}\right) dx. \end{aligned}$$

By Sobolev's inequality, we deduce

$$\begin{aligned} & \left\{\int_\Omega \eta^{2^*}\left[G(|Du|)\right]^{2^*} dx\right\}^{\frac{2}{2^*}} \\ \leqslant \; & c\gamma^2 \int_\Omega \left(\eta^2+|D\eta|^2\right)\left(1+|Du|^{\gamma+q-1}\right) dx. \end{aligned} \tag{35}$$

Let us compute

$$\begin{aligned} [G(t)]^{2^*} &= \left[1+\int_0^t \sqrt{s^{\gamma+p-3}}ds\right]^{2^*} = \left(1+\frac{2}{\gamma+p-1}t^{\frac{\gamma+p-1}{2}}\right)^{2^*} \\ &= \frac{1}{(\gamma+p-1)^{2^*}}\left(p+\gamma-1+2t^{\frac{\gamma+p-1}{2}}\right)^{2^*} \\ &\geqslant \frac{2^{2^*}}{(\gamma+p-1)^{2^*}}\left(1+t^{\frac{\gamma+p-1}{2}}\right)^{2^*} \\ &\geqslant \frac{2^{2^*}}{(\gamma+p-1)^{2^*}}\left(1+t^{\frac{2^*}{2}(\gamma+p-1)}\right). \end{aligned}$$

Thus from (35) we have

$$\begin{aligned} & \left\{\int_\Omega \eta^{2^*}\left(1+|Du|^{\frac{2^*}{2}(\gamma+p-1)}\right) dx\right\}^{\frac{2}{2^*}} \\ \leqslant \; & c\frac{(\gamma+p-1)^2}{4}\gamma^2 \int_\Omega \left(\eta^2+|D\eta|^2\right)\left(1+|Du|^{\gamma+q-1}\right) dx \\ \leqslant \; & c\gamma^4 \int_\Omega \left(\eta^2+|D\eta|^2\right)\left(1+|Du|^{\gamma+q-1}\right) dx. \end{aligned}$$

Fixed $0 < \rho \leqslant R < 1$, let us denote by B_ρ and B_R balls compactly contained in Ω of radii ρ and R respectively and with the same center. Let η be a positive test function equal to 1 in B_ρ, whose support is contained in B_R, such that $|D\eta| \leqslant \frac{2}{R-\rho}$. Hence we obtain

$$\left\{\int_{B_\rho}\left(1+|Du|^{\frac{2^*}{2}(\gamma+p-1)}\right)dx\right\}^{\frac{2}{2^*}} \leqslant c\frac{\gamma^4}{(R-\rho)^2}\int_{B_R}\left(1+|Du|^{\gamma+q-1}\right)dx. \tag{36}$$

In both cases $\mu = 1$ and $\mu = 0$, we define a sequence of exponents γ_i in the following way

$$\gamma_1 = 1$$

$$\gamma_{i+1} = \frac{2^*}{2}(\gamma_i + p - 1) - (q-1), \quad \forall i = 1, 2, ... \tag{37}$$

As in [11] (lemmas 2.11 and 2.12), we can prove the following lemmas.

LEMMA 2.4. *Let γ_i the sequence defined in* (37). *Then the following representation formulas hold*

$$\gamma_i = 1 + \left(\frac{2^*}{2}p - q\right)\sum_{k=0}^{i-2}\left(\frac{2^*}{2}\right)^k, \quad \forall i \geqslant 2$$

$$\gamma_i = 1 + \frac{\frac{2^*}{2}p - q}{\frac{2^*}{2} - 1}\left[\left(\frac{2^*}{2}\right)^{i-1} - 1\right], \quad \forall i \geqslant 1.$$

In particular, since γ_i is a polynomial expression in $\frac{2^*}{2} > 1$, $\lim_{i\to+\infty}\gamma_i = +\infty$.

LEMMA 2.5. *Let θ be defined by*

$$\theta = \prod_{k=1}^{+\infty}\frac{\gamma_k + q - 1}{\gamma_k + p - 1}$$

then θ is finite and it is given by

$$\theta = \frac{q}{p}\frac{\frac{2^*}{2} - 1}{\frac{2^*}{2} - \frac{q}{p}}. \tag{38}$$

Proof. It follows easily from the second formula in lemma (2.4) (see [11] lemma 2.12). □

REMARK 2.6. *Note that $\theta \geqslant 1$ and $\theta = 1$ if and only if $\frac{q}{p} = 1$. Explicitly we have*

$$\theta = \frac{2q}{np - (n-2)\,q} \qquad \text{if } n > 2 \tag{39}$$

and if $n = 2$ and $p < q$, then we can choose $\frac{2^*}{2}$ so large that θ in (38) is as close to $\frac{q}{p}$ as we like.

LEMMA 2.7. *The product*

$$\prod_{k=1}^{+\infty} \left[c \frac{4^{k+1}\gamma_k^4}{(R_0 - \rho_0)^2} \right]^{\frac{1}{\gamma_k + p - 1} \prod_t \frac{\gamma_t + q - 1}{\gamma_t + p - 1}}$$

is finite, and defining

$$\beta = \sum_{k=1}^{+\infty} \frac{1}{\gamma_k + p - 1} \quad \text{and} \quad C = \exp \theta \sum_{k=1}^{+\infty} \frac{\lg\left[c 4^{k+1} \gamma_k^4\right]}{\gamma_k + p - 1},$$

we have

$$\prod_{k=1}^{+\infty} \left[c \frac{4^{k+1}\gamma_k^4}{(R_0 - \rho_0)^2} \right]^{\frac{1}{\gamma_k + p - 1} \prod_t \frac{\gamma_t + q - 1}{\gamma_t + p - 1}} \leqslant C \left(R_0 - \rho_0\right)^{-2\theta\beta}.$$

Proof. Since γ_k grows exponentially, the series $\sum_{k=1}^{+\infty} \frac{1}{\gamma_k + p - 1}$ and

$\sum_{k=1}^{+\infty} \frac{\lg[c4^{k+1}\gamma_k^4]}{\gamma_k+p-1}$ converge. Therefore we have

$$\prod_{k=1}^{i}\left[c\frac{4^{k+1}\gamma_k^4}{(R_0-\rho_0)^2}\right]^{\frac{1}{\gamma_k+p-1}\prod_t \frac{\gamma_t+q-1}{\gamma_t+p-1}} \leqslant \prod_{k=1}^{i}\left[c\frac{4^{k+1}\gamma_k^4}{(R_0-\rho_0)^2}\right]^{\frac{1}{\gamma_k+p-1}\theta}$$
$$\leqslant (R_0-\rho_0)^{-2\theta\sum_{k=1}^{i}\frac{1}{\gamma_k+p-1}} \prod_{k=1}^{i}\left[c4^{k+1}\gamma_k^4\right]^{\frac{1}{\gamma_k+p-1}\theta}$$
$$\leqslant (R_0-\rho_0)^{-2\theta\beta}\exp\lg\left[\prod_{k=1}^{i}\left[c4^{k+1}\gamma_k^4\right]^{\frac{1}{\gamma_k+p-1}}\right]^{\theta}$$
$$\leqslant (R_0-\rho_0)^{-2\theta\beta}\exp\theta\sum_{k=1}^{i}\frac{\lg\left[c4^{k+1}\gamma_k^4\right]}{\gamma_k+p-1}$$
$$\leqslant (R_0-\rho_0)^{-2\theta\beta}\exp\theta\sum_{k=1}^{+\infty}\frac{\lg\left[c4^{k+1}\gamma_k^4\right]}{\gamma_k+p-1}.$$

□

Case $\mu = 1$.

Fixed $0 < \rho_0 \leqslant R_0 < 1$, let us define $R_i = \rho_0 + \frac{R_0-\rho_0}{2^i}$ for $i \geqslant 1$ and insert in (33) $R = R_i$, $\rho = R_{i+1}$ and $\gamma = \gamma_i$. Since $R-\rho = \frac{R_0-\rho_0}{2^{i+1}}$, we obtain

$$\left\{\int_{B_{R_{i+1}}}\left(1+|Du|^2\right)^{\frac{\gamma_{i+1}+q-1}{2}}dx\right\}^{\frac{2}{2*}} \tag{40}$$
$$\leqslant \quad c\frac{4^{i+1}\gamma_i^4}{(R_0-\rho_0)^2}\left\{\int_{B_{R_i}}\left(1+|Du|^2\right)^{\frac{\gamma_i+q-1}{2}}dx\right\}.$$

For every $i = 1, 2, ...$ we define

$$A_i = \left\{\int_{B_{R_i}}\left(1+|Du|^2\right)^{\frac{\gamma_i+q-1}{2}}dx\right\}^{\frac{1}{\gamma_i+q-1}} \tag{41}$$

thus, from the definition (37), the inequality (40) can be written in the form

$$A_{i+1} \leqslant \left[c\frac{4^{i+1}\gamma_i^4}{(R_0-\rho_0)^2}\right]^{\frac{1}{\gamma_i+p-1}} A_i^{\frac{\gamma_i+q-1}{\gamma_i+p-1}}. \tag{42}$$

LEMMA 2.8. *For the positive constant β and C previously defined, we have*

$$A_{i+1} \leqslant C\left[(R_0-\rho_0)^{-2\beta} A_1\right]^{\theta}, \quad \textit{for } i=1,2,\dots.$$

Proof. By iterating (42), we obtain

$$\begin{aligned} A_{i+1} &\leqslant \prod_{k=0}^{i-1}\left[c\frac{4^{i+1-k}\gamma_{i-k}^4}{(R_0-\rho_0)^2}\right]^{\frac{1}{\gamma_{i-k}+p-1}\prod_t \frac{\gamma_t+q-1}{\gamma_t+p-1}} A_1^{\prod_{k=0}^{i-1}\frac{\gamma_{i-k}+q-1}{\gamma_{i-k}+p-1}} \\ &= \prod_{k=1}^{i}\left[c\frac{4^{k+1}\gamma_k^4}{(R_0-\rho_0)^2}\right]^{\frac{1}{\gamma_k+p-1}\prod_t \frac{\gamma_t+q-1}{\gamma_t+p-1}} A_1^{\prod_{k=1}^{i}\frac{\gamma_k+q-1}{\gamma_k+p-1}}. \end{aligned}$$

Thus the result follows immediately from lemma (2.7) and from the definition of C, β and θ. □

Recall the definition of A_i in (41). Since $\rho_0 \leqslant R_i \leqslant R_0$ for every $i=1,2,\dots$, from lemma (2.8) we have

$$\begin{aligned} &\left\{\int_{B_{\rho_0}}\left(1+|Du|^2\right)^{\frac{\gamma_{i+1}+q-1}{2}}dx\right\}^{\frac{1}{\gamma_{i+1}+q-1}} \\ &\leqslant \frac{C}{(R_0-\rho_0)^{2\beta\theta}}\left\{\int_{B_{R_0}}\left(1+|Du|^2\right)^{\frac{q}{2}}dx\right\}^{\frac{\theta}{q}}. \end{aligned}$$

Since $\lim_{i\to+\infty}\gamma_{i+1}+q-1=+\infty$, the left hand side converges to the essential supremum of $\left(1+|Du|^2\right)^{\frac{1}{2}}$ in B_{ρ_0} and thus the theorem (2.1) is proved in the case $\mu=1$.

Case $\mu=0$.

Fixed $0<\rho_0\leqslant R_0<1$, let us define $R_i=\rho_0+\frac{R_0-\rho_0}{2^i}$ for $i\geqslant 1$ and insert in (36) $R=R_i$, $\rho=R_{i+1}$ and $\gamma=\gamma_i$. Since $R-\rho=\frac{R_0-\rho_0}{2^{i+1}}$, we obtain

$$\begin{aligned} &\left\{\int_{B_{R_{i+1}}}\left(1+|Du|^{\gamma_{i+1}+q-1}\right)dx\right\}^{\frac{2}{2*}} \qquad (43)\\ &\leqslant c'\frac{4^{i+1}\gamma_i^4}{(R_0-\rho_0)^2}\left\{\int_{B_{R_i}}\left(1+|Du|^{\gamma_i+q-1}\right)dx\right\}. \end{aligned}$$

For every $i = 1, 2, \ldots$ we define

$$A_i = \left\{ \int_{B_{R_i}} \left(1 + |Du|^{\gamma_i + q - 1} \right) dx \right\}^{\frac{1}{\gamma_i + q - 1}} \tag{44}$$

thus, from the definition (37), the inequality (43) can be written in the form (42). Therefore lemma (2.8) holds also in this case. Since $\lim_{i \to +\infty} \gamma_{i+1} + q - 1 = +\infty$, we have

$$\sup_{x \in B_{\rho_0}} |Du| = \lim_{i \to +\infty} \left\{ \int_{B_{\rho_0}} |Du|^{\gamma_{i+1} + q - 1} \, dx \right\}^{\frac{1}{\gamma_{i+1} + q - 1}} .$$

From the definition of A_i in (44), since $\rho_0 \leqslant R_i \leqslant R_0$ for every $i = 1, 2, \ldots$, we have

$$\begin{aligned} A_i &= \left\{ \int_{B_{R_i}} \left(1 + |Du|^{\gamma_i + q - 1} \right) dx \right\}^{\frac{1}{\gamma_i + q - 1}} \\ &\geqslant \left\{ \int_{B_{\rho_0}} \left(1 + |Du|^{\gamma_i + q - 1} \right) dx \right\}^{\frac{1}{\gamma_i + q - 1}} \\ &\geqslant \left\{ |B_{\rho_0}| + \int_{B_{\rho_0}} |Du|^{\gamma_i + q - 1} \, dx \right\}^{\frac{1}{\gamma_i + q - 1}} \\ &\geqslant |B_{\rho_0}|^{\frac{1}{\gamma_i + q - 1}} + \left\{ \int_{B_{\rho_0}} |Du|^{\gamma_i + q - 1} \, dx \right\}^{\frac{1}{\gamma_i + q - 1}} . \end{aligned}$$

From lemma (2.8), we deduce

$$\begin{aligned} & |B_{\rho_0}|^{\frac{1}{\gamma_i + q - 1}} + \left\{ \int_{B_{\rho_0}} |Du|^{\gamma_i + q - 1} \, dx \right\}^{\frac{1}{\gamma_i + q - 1}} \\ \leqslant \; & \frac{C'}{(R_0 - \rho_0)^{2\beta\theta}} \left\{ \int_{B_{R_0}} (1 + |Du|^q) \, dx \right\}^{\frac{\theta}{q}} \\ \leqslant \; & \frac{C'}{(R_0 - \rho_0)^{2\beta\theta}} \left\{ \int_{B_{R_0}} (1 + |Du|)^q \, dx \right\}^{\frac{\theta}{q}} . \end{aligned}$$

For $i \to +\infty$, we obtain

$$\begin{aligned}\|(1+|Du|)\|_{L^\infty(B_{\rho_0})} &= \sup_{x\in B_{\rho_0}} (1+|Du|) = 1 + \sup_{x\in B_{\rho_0}} |Du| \\ &= \lim_{i\to+\infty} \left\{ |B_{\rho_0}|^{\frac{1}{\gamma_i+q-1}} + \left\{ \int_{B_{\rho_0}} |Du|^{\gamma_i+q-1}\, dx \right\}^{\frac{1}{\gamma_i+q-1}} \right\} \\ &\leqslant \frac{C'}{(R_0-\rho_0)^{2\beta\theta}} \|(1+|Du|)\|^{\theta}_{L^q(B_{R_0})}\end{aligned}$$

and thus the theorem (2.1) is proved also in the case $\mu = 0$.

3. Interpolation

We recall the well-known interpolation inequality

$$\|v\|_{L^q} \leqslant \|v\|_{L^p}^{\frac{p}{q}} \cdot \|v\|_{L^\infty}^{1-\frac{p}{q}}$$

(for the proof, see for example Brezis [2]) and let us consider it for $v_1 = \left(1+|Du|^2\right)^{\frac{1}{2}}$ when $\mu = 1$ and for $v_0 = (1+|Du|)$ when $\mu = 0$; in both cases, by theorem (2.1) we have

$$\|v_i\|_{L^\infty(B_\rho)} \leqslant c\,\|v_i\|^{\theta}_{L^q(B_R)} \quad i = 0,\,1.$$

Formally (up to the different radii R and ρ), we infer that

$$\|v_i\|_{L^\infty} \leqslant c\,\|v_i\|^{\theta}_{L^q} \leqslant c\,\|v_i\|_{L^p}^{\theta\left(\frac{p}{q}\right)} \cdot \|v_i\|_{L^\infty}^{\theta\left(1-\frac{p}{q}\right)}$$

and if $\theta\left(1-\frac{p}{q}\right) < 1$, we obtain the inequality

$$\|v_i\|_{L^\infty}^{1-\theta\left(1-\frac{p}{q}\right)} \leqslant c\,\|v_i\|_{L^p}^{\theta\left(\frac{p}{q}\right)}$$

which gives the local boundedness of the gradient Du in terms of its L^p norm.

From the explicit formula of θ in (39), the condition $\theta\left(1-\frac{p}{q}\right) < 1$ holds if and only if

$$\frac{q}{p} < \frac{n+2}{n}.$$

Therefore let us consider exponents p and q related by

$$2 \leqslant p \leqslant q < \frac{n+2}{n} p \tag{45}$$

Fixed $0 < \rho \leqslant R < 1$, let us denote again by B_ρ and B_R balls compactly contained in Ω of radii ρ and R respectively and with the same center. Let α and θ be defined by

$$\alpha = \frac{p}{q} \frac{\theta}{1-\theta\left(1-\frac{p}{q}\right)} \quad \text{and} \quad \theta = \frac{2q}{np-(n-2)\,q} \tag{46}$$

if $n > 2$; otherwise, if $n = 2$ and $\frac{q}{p} > 1$, then let θ be any number such that $\frac{q}{p} < \theta < \frac{q}{q-p}$ and let $\alpha = \frac{\theta \frac{p}{q}}{1-\theta\left(1-\frac{p}{q}\right)}$; finally, if $n = 2$ and $p = q$, then let $\alpha = \theta = 1$.

Theorem 3.1. *Under the assumptions* (10), (11) *and* (45), *and with* α *and* θ *defined in* (46), *there are positive numbers* C, C' *and* β *such that for* $\mu = 1$ *we have*

$$\left\| \left(1+|Du|^2\right)^{\frac{1}{2}} \right\|_{L^q(B_\rho)} \leqslant C \left\{ \frac{1}{(R-\rho)^{2\beta\left(\frac{q-p}{p}\right)}} \left\| \left(1+|Du|^2\right)^{\frac{1}{2}} \right\|_{L^p(B_R)}^{\frac{1}{\theta}} \right\}^{\alpha} \tag{47}$$

$$\left\| \left(1+|Du|^2\right)^{\frac{1}{2}} \right\|_{L^\infty(B_\rho)} \leqslant C \left\{ \frac{1}{(R-\rho)^{2\beta\frac{q}{p}}} \left\| \left(1+|Du|^2\right)^{\frac{1}{2}} \right\|_{L^p(B_R)} \right\}^{\alpha} \tag{48}$$

and for $\mu = 0$ *we have*

$$\|(1+|Du|)\|_{L^q(B_\rho)} \leqslant C' \left\{ \frac{1}{(R-\rho)^{2\beta\left(\frac{q-p}{p}\right)}} \|(1+|Du|)\|_{L^p(B_R)}^{\frac{1}{\theta}} \right\}^{\alpha} \tag{49}$$

$$\|(1+|Du|)\|_{L^\infty(B_\rho)} \leqslant C' \left\{ \frac{1}{(R-\rho)^{2\beta\frac{q}{p}}} \|(1+|Du|)\|_{L^p(B_R)} \right\}^{\alpha} \tag{50}$$

for every minimizer u *of class* $W^{1,q}_{loc}\left(\Omega, \mathbf{R}^N\right)$ *of the integral* (7).

Case $\mu = 1$.

Let us apply the interpolation inequality with $v = \left(1+|Du|^2\right)^{\frac{1}{2}}$ and use the estimate in theorem (2.1). We obtain

$$\begin{aligned}\left\|\left(1+|Du|^2\right)^{\frac{1}{2}}\right\|_{L^q(B_\rho)} &\leqslant \left\|\left(1+|Du|^2\right)^{\frac{1}{2}}\right\|_{L^p(B_\rho)}^{\frac{p}{q}} \cdot \left\|\left(1+|Du|^2\right)^{\frac{1}{2}}\right\|_{L^\infty(B_\rho)}^{1-\frac{p}{q}} \\ &\leqslant c^{1-\frac{p}{q}} \left\|\left(1+|Du|^2\right)^{\frac{1}{2}}\right\|_{L^p(B_\rho)}^{\frac{p}{q}} \\ &\quad \cdot \left(\frac{1}{(R-\rho)^{2\beta}} \left\|\left(1+|Du|^2\right)^{\frac{1}{2}}\right\|_{L^q(B_R)}\right)^{\theta\left(1-\frac{p}{q}\right)}.\end{aligned}$$

Now we define an increasing sequence of radii which converges to R: fixed $0 < \rho_0 < R_0 < 1$, for every $k \geqslant 1$ let us define $\rho_k = R_0 - (R_0 - \rho_0)\, 2^{-k}$ and insert $\rho = \rho_k$ and $R = \rho_{k+1}$ in the inequality above; we have $R - \rho = (R_0 - \rho_0)\, 2^{-(k+1)}$. We also define

$$B_k = \left\|\left(1+|Du|^2\right)^{\frac{1}{2}}\right\|_{L^q\left(B_{\rho_k}\right)} \qquad \text{for } k = 0, 1, \ldots$$

thus we have for $k = 0, 1, \ldots$

$$B_k \leqslant c^{1-\frac{p}{q}} \left\|\left(1+|Du|^2\right)^{\frac{1}{2}}\right\|_{L^p\left(B_{R_0}\right)}^{\frac{p}{q}} \left(\frac{2^{2\beta(k+1)}}{(R_0-\rho_0)^{2\beta}} B_{k+1}\right)^{\theta\left(1-\frac{p}{q}\right)}.$$

We iterate this inequality and we obtain

$$\begin{aligned}B_0 \quad &\leqslant \quad \left(c^{1-\frac{p}{q}} \left\|\left(1+|Du|^2\right)^{\frac{1}{2}}\right\|_{L^p\left(B_{R_0}\right)}^{\frac{p}{q}} \frac{1}{(R_0-\rho_0)^{2\beta\theta\left(1-\frac{p}{q}\right)}}\right)^{\sum_{i=0}^{k-1}\left[\theta\left(1-\frac{p}{q}\right)\right]^i} \\ &\qquad \cdot 2^{2\beta\sum_{i=0}^{k-1} i\left[\theta\left(1-\frac{p}{q}\right)\right]^i} B_k^{\left[\theta\left(1-\frac{p}{q}\right)\right]^k}.\end{aligned}$$

By the assumption (45), the series above are convergent and since B_k is bounded by

$$B_k \leqslant \left\|\left(1+|Du|^2\right)^{\frac{1}{2}}\right\|_{L^q\left(B_{R_0}\right)}, \quad \forall k = 1, 2, \ldots$$

we can let k tend to infinity and we infer that (for some constant C_1)

$$B_0 \leqslant C_1 \left(\frac{1}{(R_0 - \rho_0)^{2\beta\theta\left(1-\frac{p}{q}\right)}} \left\| \left(1 + |Du|^2\right)^{\frac{1}{2}} \right\|_{L^p\left(B_{R_0}\right)}^{\frac{p}{q}} \right)^{\frac{1}{1-\theta\left(1-\frac{p}{q}\right)}}$$

which proves (47). The second estimates (48) can be proved either in the same way, or by combining (47) and theorem (2.1). In fact, if $\rho' = \frac{R+\rho}{2}$, from theorem (2.1) we have

$$\left\| \left(1 + |Du|^2\right)^{\frac{1}{2}} \right\|_{L^\infty(B_\rho)} \leqslant C \left(\frac{1}{(\rho' - \rho)^{2\beta}} \left\| \left(1 + |Du|^2\right)^{\frac{1}{2}} \right\|_{L^q\left(B_{\rho'}\right)} \right)^{\theta}$$
$$\leqslant C_2 \left\{ \frac{1}{(\rho' - \rho)^{2\beta}} \frac{1}{(R - \rho')^{2\beta\left(\frac{q-p}{p}\right)\alpha}} \left\| \left(1 + |Du|^2\right)^{\frac{1}{2}} \right\|_{L^p(B_R)}^{\frac{\alpha}{\theta}} \right\}^{\theta}$$

and since $\rho' - \rho = R - \rho'$ and $1 + \left(\frac{q-p}{p}\right)\alpha = \frac{q}{p}\frac{\alpha}{\theta}$, we have the conclusion (48).

Case $\mu = 0$.

We use the same technique with $v = (1 + |Du|)$ and

$$B_k = \|(1 + |Du|)\|_{L^q\left(B_{\rho_k}\right)} \qquad \text{for } k = 0, 1, \ldots.$$

4. Approximation and passage to the limit

Let us consider for $u \in W^{1,p}\left(\Omega, \mathbf{R}^N\right)$, such that $f(x, |Du|) \in L^1_{loc}(\Omega)$, the integral

$$I(u) = \int_\Omega f(x, |Du|)\, dx \tag{51}$$

where the integrand f is a function of class C^2 of its arguments, satisfying:

$$m\left(\mu + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \leqslant \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, |\xi|)\, \lambda_i^\alpha \lambda_j^\beta \leqslant M \left(\mu + |\xi|^2\right)^{\frac{q-2}{2}} |\lambda|^2 \tag{52}$$

$$\left|f_{\xi_i^\alpha x_s}(x,|\xi|)\right| \leqslant M\left(\mu+|\xi|^2\right)^{\frac{p+q-2}{4}} \tag{53}$$

for $\mu=0$ or $\mu=1$, $\forall\alpha=1,2,...,N,\qquad \forall i,\ s=1,2,...,n.$

If we add some assumptions on $f(x,0)$ and $f_{\xi_i^\alpha}(x,0)$ of the type

$$\begin{aligned} |f(x,0)| &\leqslant c_1 \\ \left|\sum_{i,\alpha} f_{\xi_i^\alpha}(x,0)\,\xi_i^\alpha\right| &\leqslant c_2 \end{aligned}$$

it is easy to verify that f also satisfies the following growth conditions:

$$m_1|\xi|^p - m_2 \leqslant f(x,|\xi|) \leqslant M_1\left(\mu+|\xi|^2\right)^{\frac{q}{2}} \qquad a.e. x\in\Omega,\ \forall\xi\in\mathbf{R}^{Nn} \tag{54}$$

for some positive constants m_1, m_2 and M_1.

In order to apply the regularity results of the previous sections, we have to consider an ε-approximating regular problem with minimizer in $W^{1,q}_{loc}\left(\Omega,\mathbf{R}^N\right)$. To this aim, let us define for $\varepsilon\in(0,1]$ and $v\in W^{1,q}\left(\Omega,\mathbf{R}^N\right)$ the function

$$f_\varepsilon(x,|Dv|) = f(x,|Dv|) + \varepsilon|Dv|^q \tag{55}$$

and the integral

$$I_\varepsilon(v) = \int_\Omega f_\varepsilon(x,|Dv|)\,dx. \tag{56}$$

From (54) and the definition (55), we have

$$m_1|Dv|^p - m_2 + \varepsilon|Dv|^q \leqslant f_\varepsilon(x,|Dv|) \leqslant (M_1+1)\left(\mu+|Dv|^2\right)^{\frac{q}{2}}; \tag{57}$$

moreover, from (52) and (53), we deduce that there is a constant M' independent of ε such that

$$\begin{aligned} m\left(\mu+|\xi|^2\right)^{\frac{p-2}{2}}|\lambda|^2 &\leqslant \sum_{i,j,\alpha,\beta}(f_\varepsilon)_{\xi_i^\alpha\xi_j^\beta}(x,|\xi|)\,\lambda_i^\alpha\lambda_j^\beta \\ &\leqslant M'\left(\mu+|\xi|^2\right)^{\frac{q-2}{2}}|\lambda|^2 \left|(f_\varepsilon)_{\xi_i^\alpha x_s}(x,|\xi|)\right| \leqslant M\left(\mu+|\xi|^2\right)^{\frac{p+q-2}{4}}. \end{aligned}$$

These conditions imply that for every ε, I_ε in (56) is convex, coercive and lower semicontinuous in the weak topology of $W^{1,q}\left(\Omega, \mathbf{R}^N\right)$; thus, for every fixed function $u_0 \in W^{1,q}\left(\Omega, \mathbf{R}^N\right)$, there exists a unique minimizer u_ε in the class $u_0 + W_0^{1,q}\left(\Omega, \mathbf{R}^N\right)$ and thus

$$I_\varepsilon(u_\varepsilon) \leqslant I_\varepsilon(v) \qquad \forall v \in u_0 + W_0^{1,q}\left(\Omega, \mathbf{R}^N\right). \tag{58}$$

The integrand (55) satisfies the assumptions of the theorems (2.1) and (3.1) uniformly with respect to ε and the minimizers u_ε are in $W^{1,q}\left(\Omega, \mathbf{R}^N\right)$. Therefore the estimates (47) and (48) or (49) and (50) hold for u_ε with constants c independent of ε: for every fixed $0 < \rho \leqslant R < 1$, let us denote by B_ρ and B_R balls compactly contained in Ω of radii ρ and R respectively and with the same center, by theorem (3.1) we have, for $\mu = 1$

$$\begin{aligned} \|Du_\varepsilon\|_{L^\infty(B_\rho, \mathbf{R}^{Nn})} &\leqslant \left\| \left(1 + |Du_\varepsilon|^2\right)^{\frac{1}{2}} \right\|_{L^\infty(B_\rho)} \\ &\leqslant c \left\{ \frac{1}{(R-\rho)^{2\beta\frac{q}{p}}} \left\| \left(1 + |Du_\varepsilon|^2\right)^{\frac{1}{2}} \right\|_{L^p(B_R)} \right\}^\alpha \end{aligned} \tag{59}$$

and for $\mu = 0$

$$\|Du_\varepsilon\|_{L^\infty(B_\rho, \mathbf{R}^{Nn})} \leqslant c \left\{ \frac{1}{(R-\rho)^{2\beta\frac{q}{p}}} \left\| (1 + |Du_\varepsilon|) \right\|_{L^p(B_R)} \right\}^\alpha.$$

From the definition of I_ε in (56) and by using the condition (57), we obtain

$$I_\varepsilon(v) = \int_\Omega f_\varepsilon(x, |Dv|)\, dx \geqslant m_1 \int_\Omega |Dv|^p\, dx - m_2 |\Omega| \qquad \forall v \in u_0 + W_0^{1,q}\left(\Omega, \mathbf{R}^N\right).$$

In particular for the minimizers u_ε, by choosing $v = u_0$ in (58) and using (57), we finally get

$$\begin{aligned} \||Du_\varepsilon|\|_{L^p(\Omega)}^p &= \int_\Omega |Du_\varepsilon|^p\, dx \leqslant \frac{1}{m_1} \left\{ I_\varepsilon(u_\varepsilon) + m_2 |\Omega| \right\} \\ &\leqslant \frac{1}{m_1} \left\{ I_\varepsilon(u_0) + m_2 |\Omega| \right\}, \end{aligned}$$

which gives an uniform bound of the L^p-norms of Du_ε. Up to a subsequence, we can suppose that $\{u_\varepsilon\}$ converges to a function u in the weak topology of $W^{1,p}\left(\Omega, \mathbf{R}^N\right)$. From the *a priori* estimates (59), for every B_R ball compactly contained in Ω of radius R, there exists a constant c such that

$$\|Du_\varepsilon\|_{L^\infty(B_R,\mathbf{R}^{Nn})} \leqslant c.$$

As $\varepsilon \to 0$, we obtain that u is of class $W^{1,\infty}\left(B_R, \mathbf{R}^N\right)$ for every R.

Fixed $\varepsilon_0 \in (0,1]$, from the lower semicontinuity of I_{ε_0} and by (58), we have

$$\begin{aligned}\int_\Omega f_{\varepsilon_0}(x,|Du|)\,dx &\leqslant \liminf_{\varepsilon\to 0}\int_\Omega f_{\varepsilon_0}(x,|Du_\varepsilon|)\,dx \\ &\leqslant I_{\varepsilon_0}(v) \qquad \forall v \in u_0 + W_0^{1,q}\left(\Omega,\mathbf{R}^N\right).\end{aligned}$$

As $\varepsilon_0 \to 0$, by Lebesgue's dominate convergence theorem, we infer that

$$\int_\Omega f(x,|Du|)\,dx \leqslant I(v) \qquad \forall v \in u_0 + W_0^{1,q}\left(\Omega,\mathbf{R}^N\right).$$

Thus u is a minimizer of $I(u)$ of class $W^{1,\infty}\left(B_R, \mathbf{R}^N\right)$ for every B_R compactly contained in Ω and the theorems (2.1) and (3.1) hold for

u. Moreover, from (59) when $\mu = 1$ we have

$$\|Du_\varepsilon\|_{L^\infty(B_\rho,\mathbf{R}^{Nn})} \leqslant c\left\{\frac{1}{(R-\rho)^{2\beta\frac{q}{p}}}\left\|\left(1+|Du_\varepsilon|^2\right)^{\frac{1}{2}}\right\|_{L^p(B_R)}\right\}^\alpha$$

$$\leqslant c\left\{\frac{1}{(R-\rho)^{2\beta\frac{q}{p}}}\left[1+\||Du_\varepsilon|\|_{L^p(B_R)}\right]\right\}^\alpha$$

$$\leqslant c\left\{\frac{1}{(R-\rho)^{2\beta\frac{q}{p}}}\left[1+\left(\int_{B_R}|Du_\varepsilon|^p\,dx\right)^{\frac{1}{p}}\right]\right\}^\alpha$$

$$\leqslant c\left\{\frac{1}{(R-\rho)^{2\beta\frac{q}{p}}}\left[1+\left(\frac{1}{m_1}\left\{\int_{B_R} f_\varepsilon\left(x,|Du_\varepsilon|\right)dx+m_2\,|B_R|\right\}\right)^{\frac{1}{p}}\right]\right\}^\alpha$$

$$\leqslant c\left\{\frac{1}{(R-\rho)^{2\beta\frac{q}{p}}}\left[1+\left(\frac{1}{m_1}\left\{\int_{B_R} f_\varepsilon\left(x,|Du|\right)dx+m_2\,|B_R|\right\}\right)^{\frac{1}{p}}\right]\right\}^\alpha$$

$$\leqslant c(\rho,R,n,N,p,q,m,M)\left\{\int_\Omega\left[1+f_\varepsilon\left(x,|Du|\right)\right]dx\right\}^{\frac{\alpha}{p}}$$

and, as $\varepsilon \to 0$, we finally obtain

$$\|Du\|_{L^\infty(B_\rho,\mathbf{R}^{Nn})} \leqslant c(\rho,R,n,N,p,q,m,M)\left\{\int_\Omega\left[1+f\left(x,|Du|\right)\right]dx\right\}^{\frac{\alpha}{p}}.$$

The same passages hold in the case $\mu = 0$.

References

[1] E. Acerbi and N. Fusco, *Partial regularity under anisotropic (p,q) growth conditions*, J. Differential Equations **107** (1994), 46–67.

[2] H. Brezis, *Analisi funzionale*, Liguori Editore, 1986.

[3] H.J. Choe, *Interior behaviour of minimizers for certain functionals with non standard growth*, Nonlinear Anal. Theory Methods Appl. **19** (1992), 933–945.

[4] A. Coscia and G. Mingione, *Hölder continuity of thegradient of p(x)-harmonic mappings*, Preprint Univ. Parma, 1998.

[5] N. Fusco E. Acerbi, *Regularity for minimizers of non-quadratic functionals: the case* $1 < p < 2$, Journal of Math. Anal. and Appl. **140** (1989), 115–135.

[6] M. Giaquinta and G. Modica, *Multiple integrals in the calculus of variations and non linear elliptic systems*, Annals of Math. Studies . **105** (1983).
[7] M. Giaquinta and G. Modica, *Remarks on the regularity of the minimizers of certain degenerate functionals*, Manuscripta Math. **57** (1986), 55–99.
[8] E. Giusti, *Metodi diretti nel calcolo delle variazioni*, Unione Matematica Italiana, Bologna, 1994.
[9] G.M. Lieberman, *Gradient estimates for a class of elliptic systems*, Ann. Mat. Pura Appl. **164** (1993), 103–120.
[10] P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions*, Arch. Rational Mech. Anal. **105** (1989), 267–284.
[11] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, J. Diff. Equations. **90** (1991), 1–30.
[12] P. Marcellini, *Regularity for elliptic equations with general growth conditions*, J. Diff. Equations **105** (1993), 296–333.
[13] P. Marcellini, *Everywhere regularity for a class of elliptic systems without growth conditions*, Ann. Sc. Norm. Sup. Pisa **XXIII** (1996), serie IV, Fasc. 1.
[14] C. B. Morrey, *Partial regularity results for non-linear elliptic systems*, J. Math.and Mech. **17** (1968), 649–670.
[15] V. Chiadò Piat and A. Coscia, *Hölder continuity of minimizers of functionals with variable growth exponent*, Manuscripta Math. **93** (1997), 283–299.
[16] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. **138** (1977), 219–240.

Received June 7, 1999.