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# The Representation of Weighted Quasi-Metric Spaces

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SUMMARY. - *We show that every weighted quasi-metric space can be identified with a subspace of a space of some canonical type, which is constructed from a metric space.*

*We also present a very simple method to construct a weighted quasi-metric space, as the graph of a function defined on a metric space, and show that every weighted quasi-metric space arises in this way.*

*Similar results may be obtained if we drop the requirement that the weight function have nonnegative values.*

In this note we continue the investigation carried out in [3]. In particular we fix our attention on a special class of quasi-metric spaces: the so-called *weighted* quasi-metric spaces (see [1]). The study of these spaces is motivated mainly because they are a useful tool in programming language semantics (see e.g. [2]).

We define a *weighted quasi-metric space* as a triple $(X, q, w)$, where $X$ is a set, $q$ and $w$ are nonnegative real-valued functions defined on $X \times X$ and $X$ respectively, and the following conditions hold:

1. $q(x, x) = 0$ for every $x \in X$;
2. $q(x, z) \leq q(x, y) + q(y, z)$ for every $x, y, z \in X$;
3. if $q(x, y) = q(y, x) = 0$ then $x = y$;

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4. $q(x,y) + w(x) = q(y,x) + w(y)$ for every $x, y \in X$.

The function $q$ is called *quasi-metric*, and $w$ is the *weight function* (or simply the *weight*).

Every metric space can be viewed as a weighted quasi-metric space, where the weight function is constant (usually zero). On the other hand, if $(X, q, w)$ is a weighted quasi-metric space, then (the restriction of) $q$ is a metric on each fiber of $w$.

We say that the weighted quasi-metric space $(X_0, q_0, w_0)$ is a *subspace* of $(X, q, w)$ if $X_0 \subset X$, $q_0$ is the restriction of $q$ to $X_0 \times X_0$ and $w_0$ is the restriction of $w$ to $X_0$.

If $(X, q, w)$ and $(E, r, v)$ are weighted quasi-metric spaces, a *morphism* of $(X, q, w)$ into $(E, r, v)$ is a mapping $\varphi: X \to E$ such that

$$\forall x, y \in X \qquad r(\varphi(x), \varphi(y)) \leq q(x,y), \tag{1}$$

$$\forall x \in X \qquad v(\varphi(x)) \leq w(x). \tag{2}$$

We say that the morphism $\varphi$ is *isometric* if in (1) we have equality; in this case $w$ and $v \circ \varphi$ differ by a constant.

An *isomorphism* of a weighted quasi-metric space onto another one is a bijection between the underlying sets which preserves both the quasi-metric and the weight function. An *embedding* of $(X', q', w')$ into $(X'', q'', w'')$ is an isomorphism of $(X', q', w')$ onto a subspace of $(X'', q'', w'')$.

A *generalized weighted quasi-metric space* is like a weighted quasi-metric space, except that neither the quasi-metric nor the weight are required to be nonnegative; in this case they are called *generalized quasi-metric* and *generalized weight*, respectively. All the above considerations apply to generalized weighted quasi-metric spaces as well.

Given a weighted quasi-metric space $(X, q, w)$, define $\hat{q}: X \times X \to [0, +\infty[$ as

$$(x,y) \mapsto \frac{q(x,y) + q(y,x)}{2} = q(x,y) + \frac{1}{2}w(x) - \frac{1}{2}w(y);$$

then $(X, \hat{q})$ is a metric space, which we call the *symmetrization* of $(X, q, w)$. Let us observe that

$$\forall x, y \in X \qquad \left|\frac{1}{2}w(x) - \frac{1}{2}w(y)\right| \leq \hat{q}(x,y). \tag{3}$$

Hence, in any weighted quasi-metric space $(X, q, w)$, the weight is (uniformly) continuous as a function on the symmetrization $(X, \hat{q})$ to $[0, +\infty[$ with the euclidean metric.

In the sequel, we will regard $[0, +\infty[$ as a generalized weighted quasi-metric space by endowing it with the generalized quasi-metric $\delta\colon (\xi, \eta) \mapsto \eta - \xi$ and with the (generalized) weight $\mu\colon \xi \mapsto 2\xi$.

PROPOSITION 0.1. *Let $(S, d)$ be a metric space. Consider the cartesian product $S \times [0, +\infty[$ and denote by $p$ and $\pi$ its projections onto $S$ and $[0, +\infty[$ respectively.*

*We can construct a generalized weighted quasi-metric space, having $S \times [0, +\infty[$ as underlying set, by defining the generalized quasi-metric as*

$$Q\colon ((x, \xi),\ (y, \eta)) \mapsto d(x, y) + \eta - \xi \tag{4}$$

*and taking $2 \cdot \pi$ as (generalized) weight function. The fibers of $\pi$ are isomorphic to $S$, and the fibers of $p$ are isomorphic to $([0, +\infty[, \delta, \mu)$.*

*Proof.* It is clear that $Q((x, \xi), (x, \xi)) = 0$ and that $Q((x, \xi), (z, \zeta)) \leq Q((x, \xi), (y, \eta)) + Q((y, \eta), (z, \zeta))$, for every $(x, \xi), (y, \eta), (z, \zeta) \in S \times [0, +\infty[$. Now suppose that both $Q((x, \xi), (y, \eta)) = 0$ and $Q((y, \eta), (x, \xi)) = 0$; by adding these two equalities we get $2d(x, y) = 0$, whence $x = y$, so that $\xi = \eta$, too. Finally, letting $W(x, \xi) = 2 \cdot \pi(x, \xi) = 2\xi$, we have $Q((x, \xi), (y, \eta)) + W(x, \xi) = d(x, y) + \xi + \eta = Q((y, \eta), (x, \xi)) + W(y, \eta)$ for each $(x, \xi), (y, \eta) \in S \times [0, +\infty[$. Hence $(S \times [0, +\infty[, Q, W)$ is a generalized weighted quasi-metric space. The statement about fibers is obvious. $\square$

We call the generalized weighted quasi-metric space constructed in the above proposition *the bundle over* $(S, d)$.

THEOREM 0.2. *Every weighted quasi-metric space $(X, q, w)$ is embeddable in the bundle over a suitable metric space $(S, d)$.*

*Proof.* Take as $(S, d)$ the symmetrization $(X, \hat{q})$ of $(X, q, w)$, and denote by $(T, Q, W)$ the corresponding bundle, so that, in particular, the set $T$ is $X \times [0, +\infty[$. Define a mapping $\varphi\colon X \to T$ as follows:

$$x \mapsto \Big(x,\ \frac{1}{2} w(x)\Big).$$

Then, for every $x, y \in X$, we have, according to (4):

$$Q(\varphi(x), \varphi(y)) = \hat{q}(x, y) + \frac{1}{2}w(y) - \frac{1}{2}w(x) = q(x, y).$$

Clearly we also have $W(\varphi(x)) = w(x)$ for each $x \in X$. Thus $\varphi$ is an embedding. □

Now we present a simple construction that give rise to any weighted quasi-metric space starting from a metric space and a suitable nonnegative function defined on it.

THEOREM 0.3. *Given a metric space* $(S, d)$ *and a function* $f: S \to [0, +\infty[$ *such that*

$$\forall s, t \in S \qquad |f(s) - f(t)| \leq d(s, t), \tag{5}$$

*let* $G = \{ (s, f(s)) \mid s \in S \}$ *be the graph of* $f$*; if* $\rho: G \times G \to [0, +\infty[$ *is defined by*

$$((s, f(s)),\ (t, f(t))) \mapsto d(s, t) + f(t) - f(s),$$

*then* $(G, \rho, 2 \cdot f)$ *is a weighted quasi-metric space. Moreover, every weighted quasi-metric space can be constructed in this way.*

*Proof.* Indeed $(G, \rho, 2 \cdot f)$ turns out to be a subspace of the bundle over $(S, d)$, and $\rho$ is nonnegative on $G$ by (5).

That every weighted quasi-metric space $(X, q, w)$ arises in this way follows from the proof of Theorem 0.2, taking into account inequality (3). □

In order to construct a $T_1$-separated weighted quasi-metric space (i.e. one in which the distance between different points is always positive), it is necessary and sufficient that in the above theorem the function $f$ satisfy a suitable strengthening of condition (5), namely the following:

$$\forall s \neq t \in S \qquad |f(s) - f(t)| < d(s, t).$$

Recall that a *quasi-metric space* is a pair $(X, q)$ where $X$ is a set and $q$ is a quasi-metric. The *dual* of $(X, q)$ is $(X, \bar{q})$, where $\bar{q}: (x, y) \mapsto q(y, x)$.

We say that the quasi-metric space $(X, q)$ *admits* the (generalized) weight $w$ if $(X, q, w)$ is a (generalized) weighted quasi-metric space; if $(X, q)$ admits a weight we also say that it is *weightable*.

Let $(X, q, \gamma)$ be a generalized weighted quasi-metric space: then $\gamma'$ is another generalized weight for $(X, q)$ if and only if $\gamma' - \gamma$ is constant; similary $\tilde{\gamma}$ is a generalized weight for $(X, \bar{q})$ if and only if $\tilde{\gamma} + \gamma$ is constant. Hence, given a quasi-metric space $(X, q)$ which admits a weight $\gamma$, the dual space $(X, \bar{q})$ is weightable if and only if $\gamma$ is bounded.

By replacing $[0, +\infty[$ with $\mathbb{R}$ throughout, Theorems 0.2 and 0.3 readily extend to the case in which the function $w$ is a generalized weight.

We conclude with a characterization of those quasi-metric spaces which admit a (generalized) weight.

PROPOSITION 0.4. *A quasi-metric space $(X, q)$ admits a generalized weight if and only if*

$$\forall x, y, z \in X \qquad q(x, y) + q(y, z) + q(z, x) = q(x, z) + q(z, y) + q(y, x). \tag{6}$$

*Proof.* Necessity is a straightforward consequence of condition 4. Let us prove sufficiency. Fix an element $a \in X$, and for each $x \in X$ put $\gamma_a(x) = q(a, x) - q(x, a)$: then, from (6), it follows that

$$\forall x, y \in X \qquad q(x, y) + \gamma_a(x) = q(y, x) + \gamma_a(y),$$

so that $\gamma_a$ is a generalized weight for $(X, q)$. □

COROLLARY 0.5. *A quasi-metric space $(X, q)$ is weightable if and only if it satisfies (6) and, for some (equivalently, for each) $a \in X$, the set*

$$T_a = \{\, q(a, x) - q(x, a) \mid x \in X \,\}$$

*is bounded below.*

*Proof.* Indeed, if $w$ is a weight for $(X, q)$ then $-w(a)$ is a lower bound for $T_a$. Conversely, let $\gamma_a$ be as in the proof of the previous proposition: a weight for $(X, q)$ can be defined by $x \mapsto \gamma_a(x) - \ell_a$, where $\ell_a$ is a lower bound for $T_a$. □

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