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Multi Valued Analytic Functionals on Compact Riemann Surfaces of Genus $g \geq 1$

Irene Sabadini (*)

Summary. - *In this paper we study analytic functionals on compact Riemann surfaces of genus $g \geq 1$, from the modern point of view of hyperfunctions. We will give some topological duality theorems and an integral representation for these functionals.*

1. Introduction

The purpose of this paper is the study of analytic functionals defined on closed sets of a compact Riemann surface of genus $g \geq 1$.
This problem, in a primitive form, was proposed by L. Fantappié, who treated the case $g = 0$, while some of his students dealt with the general case (see [10], [13], [14]). In all these papers no topology is considered on the spaces used (for obvious historical reasons). Moreover, the modern theory of hyperfunctions of Sato, which seems the natural tool for this kind of duality, did not appear until a few years later, and so it seems interesting to link this approach to more classical theories.
In [2], Fantappié defined the space of "ultraregular functions", denoted with $S^{(1)}$, on an open set U of $\mathbf{CP^1}$, as the space of holomorphic functions in U, vanishing at infinity if this point belongs to U. He then proceeded to define a linear region R as a subset of $S^{(1)}$, closed with respect to the $\mathbf{C}$–linear combinations of its elements. Let K be the intersection of all regions of $\mathbf{CP^1}$ where the functions of R are defined. We then have that the set of holomorphic functions

(*) Author address: Irene Sabadini, Dipartimento di Matematica del Politecnico di Milano, via Bonardi 9, 20133 Milano, Italy, e-mail sabadini@mate.polimi.it

on K, denoted with (K), coincides with R and so we can define an analytic functional as an element of the dual of (K). We have the following:

THEOREM 1.1. *([2]) Let F be an analytic functional defined on (K). The value it assumes for any function $y(t) \in (K)$ is given by:*

$$F[y(t)] = \frac{1}{2\pi i} \int_{\mathcal{C}} y(t)u(t)dt \tag{1}$$

where $\mathcal{C}$ is a closed smooth curve on the Riemann sphere, encircling all points in which $y(t)$ is not regular and not containing points in which $u(t)$ is not defined.
The function $u(t)$, called indicatrix, is a holomorphic function defined, on the complement of K, by

$$u_F(t) := F_x \left[\frac{1}{x-t} \right]$$

REMARK 1.2. *This theorem shows that analytic functionals carried by a compact K are in bijective correspondence with their indicatrices i.e. holomorphic functions defined on the complement of K.*

If the indicatrix $u(t)$ is a multi–valued function, then there are difficulties in the interpretation of (1) which gives different values according to the chosen determination of $u(t)$. Fantappié suggested to replace the curve $\mathcal{C}$ on the Riemann sphere with a closed curve on the Riemann surface associated with the indicatrix. In such a way we can explain the different values obtained from the integral (1) as integrations along cycles which are not homologous. M. Vaccaro (see [14]) and S. Martis Biddau (see [10]) have worked in this direction, and in sections 1 and 2 we will try to give a more modern flavour to their results.
Our approach will follow A. Martineau who, first, understood that Sato's hyperfunctions are a generalization of indicatrices of analytic functionals. To sketch his ideas we need the following:

DEFINITION 1.3. *The space of hyperfunctions on $\mathbf{R}^n$ is*

$$\mathcal{B}(\boldsymbol{R}^n) := H^n(\boldsymbol{C}^n, \boldsymbol{C}^n \backslash \boldsymbol{R}^n; \mathcal{O})$$

where $\mathcal{O}$ is the sheaf of germs of analytic functions.

The duality theorem proved by Martineau, see [8], is the following:

THEOREM 1.4. *Let $K \subset \boldsymbol{C}^n$ be a compact set. We have:*

$$H^p(\boldsymbol{C}^n, \boldsymbol{C}^n \backslash K; \mathcal{O}) = \begin{cases} 0 & \text{if } p \neq n \\ \mathcal{O}(K)' & \text{if } p = n \end{cases} \tag{2}$$

where

$$\mathcal{O}(K) := \operatorname{ind}\lim_{K \subset U \subset \boldsymbol{C}^n} \mathcal{O}(U)$$

and U varies in the family of open sets containing K.

If K is a compact set in $\mathbf{R}$, we can define the space of real analytic functions as

$$\mathcal{A}(K) = \operatorname{ind}\lim_{K \subset U \subset \mathbf{C}} \mathcal{O}(U)$$

where U is an open as above. The link with hyperfunctions is now clear: the dual of the space $\mathcal{A}(K)$, according to formula (2), is:

$$\mathcal{A}(K)' \cong \mathcal{B}_K(\mathbf{R}),$$

i.e. the space of hyperfunctions supported by K (unlike analytic functionals, hyperfunctions form a sheaf and we can therefore use the notion of support).
It is known that $\mathcal{B}_K(\mathbf{R})$ is an FS–space, while $\mathcal{A}(K)$ is a DFS–space. $\mathcal{B}_K(\mathbf{R})$ is even a Montel space, so it is reflexive, and we have:

$$\mathcal{B}_K(\mathbf{R})' \cong \mathcal{A}(K).$$

In this paper we will prove some similar results on Riemann surfaces. In section 1 we will give a duality theorem and we will explain a fact first mentioned in a paper by S. Martis Biddau. It is in fact possible to show that in the case of multi–valued analytic functionals the indicatrix is nothing but a hyperfunction. We will then give, in section 2, an integral representation for these analytic functionals.

2. Hyperfunctions on Riemann Surfaces

Sato, in his fundamental paper [11] introduced hyperfunctions on a m–dimensional real analytic manifold M which can be complexified (or "analytically prolonged") to a paracompact m–dimensional

complex analytic manifold X as a relative m–cohomology class of $X \text{mod}(X\backslash M)$ with coefficients in the sheaf of analytic functions. Harvey and Wells have completed (see [5]) the study of this subject in a quite general setting.
Even though it is possible to adapt the results of [5] to the case of Riemann surfaces, for our purposes we prefer to introduce the main results we need. Keeping in mind that hyperfunctions on an open set $\Omega \subset \mathbf{R}$ are defined by:

$$\mathcal{B}(\Omega) = H^1(V\backslash\Omega, V; \mathcal{O}) \cong \frac{\mathcal{O}(V\backslash\Omega)}{\mathcal{O}(V)}$$

we will restate some preliminaries on hyperfunctions on a Riemann surface.

Definition 2.1. *Let Σ be a Riemann surface. A subset Γ of Σ is called a 1–dimensional submanifold if every point $P \in \Sigma$ has an open neighbourhood U with local homomorphism φ, such that φ maps homeomorphically U onto the open unit disc and in such a way that the intersection $U \cap \Gamma$ corresponds to the real diameter. Γ is also said to be an analytic submanifold.*

We now define the sheaf of hyperfunctions on Γ as follows:
let Ω be an open set of Γ; an open set $U \subset \Sigma$ is called a neighbourhoodof Ω if Ω is a relatively closed set of U. We associate to each open set Ω of Γ the following vector space:

$$\frac{\mathcal{O}(U\backslash\Omega)}{\mathcal{O}(U)} = \mathcal{B}(\Omega), \tag{3}$$

which will be called called the space of hyperfunctions.

The following is a quite expected fact in the theory of hyperfunctions:

Theorem 2.2. *The correspondence $\Omega \longrightarrow \mathcal{B}(\Omega)$, Ω open set in Γ, defines a flabby sheaf of vector spaces on Γ.*

If we define the support of a hyperfunction $f \in \mathcal{B}(\Omega)$ as the complement in Ω of the largest open set on which f is 0 and if $\mathcal{B}[K]$ is the space of hyperfunctions with support in a compact K, it is possible to show the following result:

Proposition 2.3. $\mathcal{B}[K] \cong \frac{\mathcal{O}(U\backslash K)}{\mathcal{O}(U)}$.

Let now Σ be a compact Riemann surface of genus $g \geq 1$, let $K \subset \Sigma$ be a closed subset of Σ and let $V \subset \Sigma$ be a Stein neighbourhood of K. We recall that all proper, open and connected sets V of Σ are Stein, because they are open and connected Riemann surfaces. Since the space $H^q(K, \mathcal{O})$, $q \geq 1$, is a direct limit of $H^q(V, \mathcal{O})$, V Stein open containing K, and since $H^q(V, \mathcal{O}) = 0$, $q \geq 1$, by Cartan's Theorem B, (see [1]), then $H^1(K, \mathcal{O}) = 0$. This fact will now be used to prove a duality theorem.
Let us write the exact sequence of cohomology with compact support (see [4]):

$$\begin{aligned} 0 \longrightarrow H_c^0(V\backslash K, \mathcal{O}) \longrightarrow H_c^0(V, \mathcal{O}) \longrightarrow H^0(K, \mathcal{O}) \longrightarrow \\ \longrightarrow H_c^1(V\backslash K, \mathcal{O}) \longrightarrow H_c^1(V, \mathcal{O}) \longrightarrow 0. \end{aligned}$$

It is known that, if V is Stein, then $H_c^n(V, \mathcal{O}) = 0$, for any $n \neq \dim V = 1$, we then obtain:

$$0 \longrightarrow H^0(K, \mathcal{O}) \longrightarrow H_c^1(V\backslash K, \mathcal{O}) \longrightarrow H_c^1(V, \mathcal{O}) \longrightarrow 0. \tag{4}$$

The well known Serre duality theorem (see [12]) assures that the spaces $H^q(V, \Omega^p)$ and $H_c^{1-q}(V, \Omega^{1-p})$ are in duality, so $H_c^1(V\backslash K, \mathcal{O}) \cong H^0(V, \Omega^1)$ and $H_c^1(V, \mathcal{O}) \cong H^0(V, \Omega^1)$. The dual sequence of (4) is:

$$0 \longleftarrow \mathcal{O}(K)' \longleftarrow H^0(V\backslash K, \Omega^1) \longleftarrow H^0(V, \Omega^1) \longleftarrow 0$$

from which we obtain:

$$\mathcal{O}(K)' \cong \frac{H^0(V\backslash K, \Omega^1)}{H^0(V, \Omega^1)} \cong \frac{\Omega^1(V\backslash K)}{\Omega^1(V)}. \tag{5}$$

On an open set V, and in general on an open, connected Riemann surface, we have $\mathcal{O} \cong \Omega^1$, so that

$$\mathcal{O}(K)' \cong \frac{\mathcal{O}(V\backslash K)}{\mathcal{O}(V)}$$

which formally coincides with result (2) if we recall that $\frac{\mathcal{O}(V\backslash K)}{\mathcal{O}(V)} \cong H^1(V, V\backslash K; \mathcal{O})$. The main difference with the complex case is that

it is not possible to replace V with the compact surface Σ which is not Stein.
Under the hypotheses of this section, that is K is a compact of a real analytic submanifold Γ of Σ, we have

$$\mathcal{O}(K)' \cong \mathcal{B}_K \tag{6}$$

We recall that $\mathcal{O}(K)$ is a Montel space (see [9]), so it is a reflexive space. We can conclude that

$$\mathcal{B}'_K \cong \mathcal{O}(K).$$

The isomorphism (6) allows us to explain the difficulty discussed in a paper of S. Martis Biddau (see [10]), concerning the integral representation of the functionals associated to multi–valued functions. Let us restate the isomorphism (6) in a more suitable form:

THEOREM 2.4. *Let $u = u(x)$ an algebraic multi–valued function defined by $f(u, x) = 0$, with f polynomial of degree m in the variable u. Let us cut the Riemann sphere by the cuts $\lambda_1 \dots \lambda_h$, in such a way that we can separate one of the determinations $u_r = u_r(x)$ of the function $u(x)$. Let $K = \cup_{s=1}^{h} \lambda_s$, we have:*

$$\mathcal{O}(K)' = \mathcal{B}_K.$$

Proof. It is a consequence of the isomorphism (6). □

APPLICATION 2.5: Theorem 2.4 allows us to give a modern flavour to the following argument (see section 1 of [10]).
Without loss of generality we can suppose that the cuts $\lambda_1, \dots \lambda_h$ have an extreme in a regular point and the other extreme in a branch point of u_r; moreover cuts must not intersect. The function u_r may also have isolated singularities which are however located on branch points. This fact involves some difficulties that are beyond our purposes and that do not affect the validity of our statements. So we will suppose that u_r is regular on the Riemann sphere except on cuts $\lambda_1 \dots \lambda_h$ and consider the functional

$$F_r[y(x)] = \frac{1}{2\pi i} \int_{\mathcal{C}} u_r(x) y(x) dx$$

where $y(x)$ is a function regular on cuts, $\mathcal{C}$ is a smooth curve containing cuts and leaving out points on which $y(t)$ is not regular. We can also suppose that the curve $\mathcal{C}$ is composed by h curves, one for each cut, leaving the regular extreme t_s of the cut λ_s, wrapping it and returning in t_s. So we have:

$$F_r[y(x)] = \sum_{s=1}^{h} \frac{1}{2\pi i} \int_{\mathcal{C}_s} u_r(x) y(x) dx = \sum_{s=1}^{h} F_{rs}[y(x)]$$

If we call "abelian linear functional" a functional like:

$$F[y(x)] = \frac{1}{2\pi i} \int_{\mathcal{C}} u(x) y(x) dx$$

where $u(x)$ is an algebraic function, $\mathcal{C}$ is a closed or open curve on the relative Riemann surface and $y(x)$ is a regular function on $\mathcal{C}$, we have that every functional F_{rs} is an abelian functional. In effect, it suffices to think at the curve $\mathcal{C}$ as drawn on the Riemann surface of $u(x)$ and joining two different points having the same abscissa t_s. Varying the curve $\mathcal{C}$ with continuity until it coincides with edges of the cut λ_s and denoting with $u_r^+(x)$ and $u_r^-(x)$ the values of $u_r(x)$ on the edges of the cut, we obtain:

$$F_r[y(x)] = \sum_{s=1}^{h} F_{rs}[y(x)] = \sum_{s=1}^{h} \frac{1}{2\pi i} \int_{\lambda_s} (u_r^+(x) - u_r^-(x)) y(x) dx =$$

$$= \frac{1}{2\pi i} \int_{K} (u_r^+(x) - u_r^-(x)) y(x) dx$$

Now we observe that, under our assumptions, $K = \cup_{s=1}^{h} \lambda_s$ can be seen as a compact on a 1–dimensional submanifold. Then we notice that the difference $u_r^+(x) - u_r^-(x)$ is the boundary value representation of $[u_r(x)]$ on the compact K i.e. the class of $u_r(x)$ in $\mathcal{B}_K$, so we can write

$$F_r[y(x)] = \frac{1}{2\pi i} \int_{K} [u_r(x)] y(x) dx.$$

This equality shows that in the case of multi–valued functions the indicatrix is a hyperfunction as prescribed by the isomorphism $\mathcal{O}(K)' \cong \mathcal{B}_K$.

3. Integral Representation of Functionals

In this paragraph we introduce an indicatrix, which will allow us to calculate the value of a functional by an integral, adapting an idea originally given in [14].

LEMMA 3.1. *Let F_{t_0} be the functional in $\mathcal{O}(K)'$ defined by:*

$$F_{t_0}[y(x)] := y(t_0)$$

where K is a compact set on a compact Riemann surface Σ contained in an open set U_x with chart x. Then there exists a 1–differential form $v(t_0, x)dx \in \Omega^1(V \backslash K)$, with V open set containing K, such that for all $y(x) \in \mathcal{O}(K)$ we have:

$$F_{t_0}[y(x)] = \frac{1}{2\pi i} \int_{\mathcal{C}} y(x) v(t_0, x) dx$$

where $\mathcal{C}$ is a closed smooth curve encircling all points in which $y(x)$ is not regular and containing points in which $v(t_0, x)dx$ is not defined.

Proof. If $y(x) \in \mathcal{O}(K)$, then $y(x) \in \mathcal{O}(U)$, where U is an open set containing K. Let $V \subset \Sigma$ be an open set containing the open U where y is regular. An indicatrix on V must satisfy two requests:

i) $\forall t_0 \in V$, $v(t_0, x)dx$ is regular on V;

ii) $\forall t_0 \in V$ the residue at t_0, calculated along a smooth curve $\mathcal{C}$ encircling t_0 is 1.

Let $\mathcal{C}$ be a curve in V. Let $\mathcal{C}$ be the contour of a region $\bar{U}$ containing K. The differential form $y(x)v(t_0, x)dx$ is:

i) regular in all points of $\mathcal{C}$, $\forall t_0 \in \bar{U}$;

ii) regular in all points of $\bar{U}$ unless t_0.

Since the residue at t_0 is $y(t_0)$, $\forall t_0 \in \bar{U}$, we can write the following equality which is a generalization of the Cauchy formula:

$$y(t) = \frac{1}{2\pi i} \int_{\mathcal{C}} y(x) v(t, x) dx \qquad \forall t \in \bar{U} \tag{7}$$

Let M and N be two points on the Riemann surface Σ and let $D = M - N$ a divisor on Σ. It is known that there is on Σ an abelian differential of the third kind with poles in M and N and residues $+1$ and -1 respectively. The difference between two such differentials is an abelian differential of the first kind. It is also known that on a Riemann surface of genus g there are ∞^g differentials of the first kind. Now we consider g points $P_1, P_2, \dots, P_g$ on Σ and a divisor $D_g = \sum_{i=1}^{g} n_i P_i$, $n_i \in Z$. We can suppose that D_g is not special and that D_g does not contain M and N. The differential vanishing on the points P_i is unique, because D_g is not special, so there is a unique differential form $v(t,x)dx$ having as zeros, fixed with respect to t, the points P_i, a pole fixed at N and a pole at M of abscissa t with residue 1. The 1–differential form $v(t,x)dx$ is an indicatrix of F_{t_0} in a region V not containing D_g and N. In other words, we have:

$$F_t[y(x)] = y(t) = \frac{1}{2\pi i}\int_{\mathcal{C}} y(x)v(t,x)dx \qquad \forall t \in \bar{U}$$

where $\mathcal{C}$ belongs to the region U of regularity of $y(t)$ and it is the contour of $\bar{U}$ not containing D_g and N. □

THEOREM 3.2. *Let $F \in \mathcal{O}(K)'$, K compact on a Riemann surface Σ. Then there exists a 1–differential form $u(x)dx \in \Omega^1(V \backslash K)$, with V open set containing K, such that for all $y(x) \in \mathcal{O}(K)$ we have:*

$$F[y(x)] = \frac{1}{2\pi i}\int_{\mathcal{C}} y(x)u(x)dx$$

where $\mathcal{C}$ is a closed smooth curve encircling all points in which $y(x)$ is not regular and containing points in which $u(x)dx$ is not defined.

Proof. We will assume that K is contained in an open set U_x with coordinate chart x.
Let $F \in \mathcal{O}(K)'$, $F : \mathcal{O}(K) \longrightarrow \mathbf{C}$. We recall the following facts whose proofs are easily obtained ([2], [3] and [14]):

$$dF(y(t,x)dx) = F(\frac{\partial}{\partial x} y(t,x)dx) \tag{8}$$

$$F(\int_{\mathcal{C}} y(t,x)dx) = \int_{\mathcal{C}} F_t(y(t,x)dx) \tag{9}$$

We will use (9) to construct an indicatrix of the functional $F \in \mathcal{O}(K)'$. By lemma 3.1, we have an indicatrix for the functional F_t defined as follows:

$$F_t[y(x)] := y(t).$$

With the notation above we can say that the function in (7) is defined for $t \in \bar{U}$ while the function $y(t)$ is defined and regular in U. We will call such function $\tilde{y}(t)$ instead of $y(t)$. It is clear that $\tilde{y}(t)$ is a analytic continuation of $y(t)$, so, given a functional $F \in \mathcal{O}(K)'$, F can be applied to the function $\tilde{y}(t)$. We have:

$$F[\tilde{y}(t)] = F[y(t)] = F[\frac{1}{2\pi i}\int_{\mathcal{C}} y(x)v(t,x)dx]$$

where $y(x)v(t,x)dx$ is regular for any $x \in \mathcal{C}$, and for any $t \in K$. From property (9), we obtain

$$F[y(t)] = \frac{1}{2\pi i}\int_{\mathcal{C}} y(x)F[v(t,x)dx].$$

We observe that we can apply F to the differential form $v(t,x)dx$ thought as analytic function in t, because $x \in \mathcal{C}$, $t \in K$, so $t \neq x$. The indicatrix we get is:

$$u(x)dx = F[v(t,x)dx].$$

In the general case, we have a complex coordinate covering (U_α, x_α) on Σ with transition functions $x_\alpha = f_{\alpha\beta}(x_\beta)$. An analytic functional F is a collection $\{F_\alpha\}$ of analytic functionals on $x_\alpha(U_\alpha) \subset C$, such that, if $U_\alpha \cap U_\beta \neq \emptyset$, we have

$$F_\beta = f^*_{\alpha\beta}(F_\alpha)$$

where

$$f^*_{\alpha\beta}(F_\alpha(f) := F_\alpha[(f \cdot f_{\beta\alpha})J_{\beta\alpha}]$$

$J_{\beta\alpha}$ is the Jacobian determinant of the map $f_{\beta\alpha}$. Let $y \in \mathcal{O}(K)$: we can cover K with finitely many coordinate neighbourhoods $U_1 \dots U_N$. Since $\mathcal{B}$ is flabby we have the decomposition:

$$y = y_1 + \dots y_N \qquad K_\alpha = \mathrm{supp} y_\alpha \subset U_\alpha, \qquad \alpha = 1 \dots N.$$

Then we obtain

$$F[y] = F[\sum_{\alpha=1}^{N} y_\alpha] = \sum_{\alpha=1}^{N} F_\alpha[y_\alpha]$$

where $F_\alpha \in \mathcal{O}(K_\alpha)'$, $K_\alpha \subset U_\alpha$ with chart x_α. By virtue of the above considerations, we have an indicatrix for each F_α, and we can write:

$$\sum_{\alpha=1}^{N} F_\alpha[y_\alpha] = \sum_{\alpha=1}^{N} \frac{1}{2\pi i} \int_{\mathcal{C}_\alpha} y_\alpha(x_\alpha) u_\alpha(x_\alpha) dx_\alpha$$

this ends the proof. $\square$

Application 3.3: The point M is variable in K while points P_i and N are chosen out of K so we can find a region V containing K and not containing D_g and N. So we have as many indicatrices as choices of D_g and N out of K.
Let $v'(t,x)dx$ e $v''(t,x)dx$ two indicatrices of the functional F_{t_0}. Their difference is an indicatrix of the functional $0 \in \mathcal{O}(K)'$.
Let $\bar{v}(t,x)dx = v'(t,x)dx - v''(t,x)dx$. We have two cases:

i) $\bar{v}(t,x)dx$ is an abelian differential of the first kind;

ii) $\bar{v}(t,x)dx$ is an abelian differential of the third kind with N' and N'' (the notation is obvious) poles of the first order with residues +1 and –1 respectively.

Let $u'(x)dx$ and $u''(x)dx$ be indicatrices obtained applying the functional F to v' and v'' respectively, and let $\bar{u}(x)dx = u'(x)dx - u''(x)dx$. By linearity we have:

$$\bar{u}(x)dx = F[\bar{v}(t,x)dx].$$

So we can say that an indicatrix $u(x)dx$ belongs to $\Omega^1(V \backslash K)$ and that two indicatrices differ in a form $\bar{u}(x)dx \in \Omega^1(V)$. Then an indicatrix is an element of the quotient $\frac{\Omega^1(V\backslash K)}{\Omega^1(V)}$ as prescribed by the isomorphism in (5).

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