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μ-Embedded Sets in Topological Spaces

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Summary. - *We define the concept of a μ-embedded set in a completely regular topological space X and we state that every ν-embedded set in X is μ-embedded in X. Also, we give an example which proves that the converse is not true.*

1. Introduction.

All spaces considered in this paper are completely regular and Hausdorff. It is known that if βX is the Stone-Cěch compactification of a topological space X; νX, its realcompactification of Hewitt, and μX, the topological completion of X defined by Dieudonné (see [2]), then $X \subset \mu X \subset \nu X \subset \beta X$. R.L. Blair defines in [1] the concept of a ν-embedded set in a completely regular and Hausdorff space X and he characterizes such subsets $S \subset X$ by the condition $\nu S \subset \nu X$. According to this definition, a subset S of X is ν-embedded in X if the extension $t' : \nu S \to \nu X$ of the inclusion map $i : S \to X$ induces a homeomorphism from νS onto its image by t'. Following this definition, we can define the notion of a μ- embedded set S in X by the condition that $\mu S \subset \mu X$. One easily sees that every ν- embedded set in X is μ-embedded in it. However, one question naturally arises: is every μ-embedded set in X, ν-embedded in such a space?. We an-

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swer negatively this question by means of an example of a set which is μ-embedded in a completely regular and Hausdorff space but it is not ν-embedded in it.

2. μ-embedded sets in topological spaces.

Let S be a subset of a space X, and i the inclusion of S in X. We call t' the Hewitt extension $t':\nu S \to \nu X$ of the inclusion i, and t, the extension of i onto the topological completion μS of S, t: $\mu S \to \mu X$, such that $t'|_{\mu S} = t$, with $\mu S \subset \nu S$.

DEFINITION 2.1. *A subset S of a space X is μ-embedded in X if $t{:}\mu S \to \mu X$ is a homeomorphism from μS onto $t(\mu S)$.*

If S is ν-embedded in X, t': $\nu S \to t'(\nu S)$ is a homeomorphism and since μS is a subset of νS, we have

PROPOSITION 2.2. *If S is a ν-embedded subset of X, then S is μ-embedded in X.*

DEFINITION 2.3. ([1]) *A subset S of a space X is c-(resp. c∗-) embedded in X in case every function in C(S) (resp. C∗(S)) has a continuous extension over X. By [3], S is c∗-embedded in X if and only if $\beta S = Cl_{\beta X} S$. S is z- embedded in X in case every zero-set Z in S is of the form $Z' \cap S$ for some zero-set Z' in X.*

In [1] Blair gives an example of a space X and a ν- embedded subset S of X, such that S is neither realcompact nor z-embedded in X. Furthermore, S is a μ-embedded subset of X, non realcompact.

One easily states the following implications: S is c-embedded $\to S$ is c^*-embedded $\to S$ is z-embedded $\to S$ is ν-embedded $\to S$ is μ- embedded.

Since every realcompact subset of X is $c*$-embedded in X ([1]) and every cozero-set in X is z-embedded in X, we have

PROPOSITION 2.4. *Every realcompact subset of X and every cozero set in X is μ-embedded in X.*

PROPOSITION 2.5. *Let S be a subset of a space X, if $\mu S \subset \mu X$, S is μ-embedded in X. This is an immediate consequence of the fact that t: $\mu S \to \mu S \subset \mu X$, is a homeomorphism with $t|_S = i$.*

Proposition 2.6. *Every topologically complete subset of X is μ-embedded in X. This follows immediately from $\mu S = S \subset X \subset \mu X$, and then $\mu S \subset \mu X$.*

3. There are subsets which are μ-embedded but non ν-embedded in X.

In [3] R.Walker states the following theorem

Theorem 3.1. ([3]) *X is $c*$-embedded in βX iff every point of βX is limit of a unique z-ultrafilter in X. Besides if we replace βX for any other space in which X is a dense subspace, the equivalence remains valid.*

Since X is dense in μX and X is $c*$-embedded in μX, we have:

Corollary 3.2. *Every point of μX is limit of a unique z- ultrafilter in X.*

Proposition 3.3. *If A_1 and A_2 are topologically complete spaces and c-embedded in X, then $A_1 \cup A_2$ is topologically complete space.*

Proof. Let F be a real z-ultrafilter on X. From the countable intersection property of F, A_1 or A_2 meets every member of F. Let $A_1 \cap F_i$ non empty, for every $F_i \in F$. Since A_1 is c-embedded in X, $F_{|A_1}$ is a real z-ultrafilter on the topologically complete space A_1. Hence, $F_{|A_1}$ converges to $x \in \mu(A_1)$ by virtue of Corollary 1 and thus F converges to $x \in X$. Consequently, X is a topologically complete space. □

From Propositions 4 and 5 we have

Corollary 3.4. *The addition of two topologically complete spaces, c-embedded in X, is μ-embedded in X.*

Finally, the next example find out that there exist μ- embedded sets in a completely regular and Hausdorff space which are not ν-embedded.

EXAMPLE 3.5. *Let S_1 y S_2 two copies of a topologically complete non realcompact space. Let S be the topological sum of S_1 and S_2, and let X be the one-point compactification of S. We have*

1. *$S_1 \cup S_2$ is μ-embedded in X.*
2. *$S_1 \cup S_2$ is not ν-embedded in X.*

Proof. 1) S_1 and S_2 are topologically complete subsets of X and evidently, both are c-embedded in X. Then, from the last corollary, $S_1 \cup S_2$ is μ-embedded in X, $(\mu S \subset \mu X = X)$.

2) $S_1 \cup S_2$ is not realcompact, since S_1 is closed in S but S_1 is not realcompact. Hence, $\nu S \neq S$. Since S is clearly not c-embedded in X, $\nu S \neq X$. Hence $\nu S \nsubseteq \nu X = X$, and S is not ν-embedded in X. □

REFERENCES

[1] R.L. BLAIR, *On ν-embedded sets in topological spaces*, L.N.M. **378** (1972), 46–79.
[2] K. MORITA, *Topological completions and M-spaces*, Sci. Rep. Tokyo Kyoiku Daigaku **10** (1970), 271–288.
[3] R.C. WALKER, *The Stone-Čech compactification*, Springer–Verlag, 1974.

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