

Cofinal Bicompleteness and Quasi-Metrizability

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SUMMARY. - *We introduce the notions of a cofinally bicomplete quasi-uniformity and of a cofinally bicomplete quasi-pseudometric. The Sorgenfrey quasi-metric and the Kofner quasi-metric are interesting examples of cofinally bicomplete quasi-metrics. We observe that the finest quasi-uniformity of any quasi-pseudometrizable bitopological space is cofinally bicomplete and characterize those quasi-pseudometrizable bitopological spaces which admit a cofinally bicomplete quasi-pseudometric. A necessary and sufficient condition for cofinal bicompleteness of quasi-pseudometrizable topological spaces is derived. Finally, quasi-metrizable bitopological spaces whose supremum topology is locally compact are characterized.*

1. Introduction

Our basic references for quasi-uniform and quasi-metric spaces are [6] and [10].

Given a quasi-uniform space (X, \mathcal{U}) we shall denote by \mathcal{U}^s the coarsest uniformity finer than \mathcal{U} and its conjugate \mathcal{U}^{-1} (i.e. $\mathcal{U}^s =$

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$\mathcal{U} \vee \mathcal{U}^{-1}$). If $U \in \mathcal{U}$ we denote by U^s the entourage of \mathcal{U}^s , $U \cap U^{-1}$.

Every quasi-uniformity \mathcal{U} on a set X induces a topology $T(\mathcal{U}) = \{A \subseteq X : \text{for each } x \in A \text{ there is } U \in \mathcal{U} \text{ such that } U(x) \subseteq A\}$, where $U(x) = \{y \in X : (x, y) \in U\}$.

According to [6], a quasi-uniform space (X, \mathcal{U}) is called bicomplete if (X, \mathcal{U}^s) is a complete uniform space.

A quasi-pseudometric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$ and (ii) $d(x, y) \leq d(x, z) + d(z, y)$. In the context of this paper, a quasi-metric on X is a quasi-pseudometric d on X which satisfies the additional condition (iii) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$.

Every quasi-pseudometric d on X generates a topology $T(d)$ on X which has as a base the family of sets of the form $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$. The conjugate d^{-1} of the quasi-pseudometric d is given by $d^{-1}(x, y) = d(y, x)$. Then we shall denote by d^s the pseudometric defined on X by $d^s = d \vee d^{-1}$. Note that d^s is a metric if and only if d is a quasi-metric on X . In this case, d^{-1} is also a quasi-metric on X .

Every quasi-pseudometric d on a set X generates a quasi-uniformity \mathcal{U}_d on X which has as a base the family of sets of the form $U_n = \{(x, y) \in X \times X : d(x, y) < 2^{-n}\}$, for $n = 0, 1, 2, \dots$ (see [6, p. 3]). The uniformity $(\mathcal{U}_d)^s$ will be denoted simply by \mathcal{U}_d^s .

Bitopological spaces appear in a natural way when one consider a quasi-uniformity (or a quasi-pseudometric) and its conjugate. A bitopological space is an ordered triple (X, S, T) such that X is a (nonempty) set and S and T are topologies on X [9]. A bitopological space (X, S, T) is called quasi-uniformizable if there is a quasi-uniformity \mathcal{U} on X compatible with (X, S, T) , where \mathcal{U} is compatible with (X, S, T) provided that $T(\mathcal{U}) = S$ and $T(\mathcal{U}^{-1}) = T$. Let us recall that (X, S, T) is quasi-uniformizable if and only if it is pairwise completely regular [12]. (X, S, T) is said to be quasi-(pseudo)metrizable if there is a quasi-(pseudo)metric d on X compatible with (X, S, T) , where d is compatible with (X, S, T) provided that $T(d) = S$ and $T(d^{-1}) = T$.

A topological space (X, S) is said to be quasi-(pseudo)metrizable if there is a quasi-(pseudo)metric d on X such that $T(d) = S$.

It is known that cofinal completeness constitutes a very interesting strong notion of uniform and metric completeness (see [2], [8], [3], [5], [13], [7], [14], [11], etc.). We then think that an appropriate nonsymmetric version of this concept may be interesting in the development of the theory of bitopological completeness. In this direction, we here introduce the notions of a cofinally bicomplete quasi-uniform space and of a cofinally bicomplete quasi-metric space, and show that they provide several satisfactory results in the context of bitopological spaces. Thus, in Section 2, we observe that the finest quasi-uniformity of any quasi-pseudometrizable bitopological space is cofinally bicomplete and characterize the bitopological spaces which admit a cofinally bicomplete quasi-metric. From our bitopological results we deduce a characterization of the topological spaces which admit a cofinally bicomplete quasi-metric. We also characterize the quasi-metrizable bitopological spaces whose supremum topology is locally compact. In Section 3, some illustrative examples are given. In particular, the Sorgenfrey quasi-metric and the Kofner quasi-metric are cofinally bicomplete. Finally, the hedgehog metric $J(\alpha)$ space with α spines (an interesting example of a non locally compact cofinally complete metric space) is splitted in two non comparable topologies S and T such that the bitopology (S, T) admits a cofinally bicomplete quasi-metric.

For the sake of brevity we shall use in the following the term *bispace* instead of bitopological space.

2. Cofinal bicompleteness in quasi-metrizable spaces

According to Howes [7], a uniformity \mathcal{U} on a set X is cofinally complete provided that each weakly Cauchy filter on (X, \mathcal{U}) has a cluster point, where a filter \mathcal{F} on X is said to be weakly Cauchy if for each $U \in \mathcal{U}$, $\bigcap_{F \in \mathcal{F}} U(F) \neq \emptyset$ [2]. A uniform space (X, \mathcal{U}) is called cofinally complete if \mathcal{U} is a cofinally complete uniformity on X . A metric d on X is called cofinally complete if the uniformity \mathcal{U}_d generated by d is cofinally complete. Cofinally complete uniform spaces were called ultracomplete by Császár [3], who showed that the Euclidean metric on \mathbb{R} is cofinally complete.

DEFINITION 2.1. A quasi-uniform space (X, \mathcal{U}) is called cofinally bicomplete if \mathcal{U}^s is a cofinally complete uniformity on X . In this case we say that \mathcal{U} is a cofinally bicomplete quasi-uniformity on X . A quasi-pseudometric d on X is cofinally bicomplete provided that \mathcal{U}_d is a cofinally bicomplete quasi-uniformity on X .

Corson proved in [2] that a completely regular topological space is paracompact if and only if its fine uniformity is cofinally complete. Therefore the fine uniformity of any pseudometrizable topological space is cofinally complete. For bispaces we have the following result.

PROPOSITION 2.2. The finest quasi-uniformity of any quasi-pseudometrizable bispaces is cofinally bicomplete.

Proof. Let (X, S, T) be a quasi-pseudometrizable bispaces. Denote by \mathcal{FN} the finest quasi-uniformity of (X, S, T) (see [17]). It is shown in [15, proof of Theorem 1] that the uniformity $(\mathcal{FN})^s$ coincides with the fine uniformity of the pseudometrizable topological space $(X, S \vee T)$. Therefore $(\mathcal{FN})^s$ is cofinally complete by an observation made above. Hence \mathcal{FN} is cofinally bicomplete. \square

It is shown in [16, Theorem 2.1] that a quasi-pseudometrizable bispaces (X, S, T) admits a bicomplete quasi-pseudometric if and only if the topological space $(X, S \vee T)$ admits a complete pseudometric. Our next result provides an analogue of this theorem to cofinal bicompleteness.

THEOREM 2.3. A quasi-pseudometrizable bispaces (X, S, T) admits a cofinally bicomplete quasi-pseudometric if and only if the topological space $(X, S \vee T)$ admits a cofinally complete pseudometric.

Proof. Suppose that (X, S, T) is a quasi-pseudometrizable bispaces such that $(X, S \vee T)$ admits a cofinally complete pseudometric p . Let d be a quasi-pseudometric on X compatible with (X, S, T) . Then for each $x \in X$ there is a sequence $(r_n(x))_{n \in \mathbb{N}}$ of positive real numbers such that $B_{d^s}(x, r_n(x)) \subseteq B_p(x, 2^{-n})$ and $5r_{n+1}(x) < r_n(x) < 2^{-n}$ for all $n \in \mathbb{N}$. Now set for each $n \in \mathbb{N}$,

$$U_n = \bigcup_{x \in X} (B_{d^{-1}}(x, r_n(x)) \times B_d(x, r_n(x)))$$

Then, as in [16, Theorem 2.1], $U_{n+1}^3 \subseteq U_n$ for all $n \in \mathbb{N}$, and thus, by Kelley's metrization lemma, there is a quasi-pseudometric q on X such that

$$U_{n+1} \subseteq \{(x, y) \in X \times X : q(x, y) < 2^{-n}\} \subseteq U_n$$

for all $n \in \mathbb{N}$. Moreover, by a similar argument to the one given in the proof of [16, Theorem 2.1], q is compatible with (X, S, T) . Finally we show that q is cofinally bicomplete. Let \mathcal{F} be a weakly Cauchy filter on the pseudometric space (X, q^s) . Then, for each $n \in \mathbb{N}$ there exists an $x_n \in X$ such that $B_{q^s}(x_n, 2^{-n}) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Let $y_{n,F} \in B_{q^s}(x_n, 2^{-n}) \cap F$ where $F \in \mathcal{F}$. Then there exist $a_{n,F}$ and $b_{n,F}$ in X such that $d(x_n, a_{n,F}) < r_n(a_{n,F})$, $d(a_{n,F}, y_{n,F}) < r_n(a_{n,F})$, $d(y_{n,F}, b_{n,F}) < r_n(b_{n,F})$ and $d(b_{n,F}, x_n) < r_n(b_{n,F})$. Assume, without loss of generality, that $r_n(b_{n,F}) \leq r_n(a_{n,F})$. Then $d^s(a_{n,F}, x_n) < 3r_n(a_{n,F})$ and $d^s(a_{n,F}, y_{n,F}) < 3r_n(a_{n,F})$, so $p(a_{n,F}, x_n) < 3 \cdot 2^{-n}$ and $p(a_{n,F}, y_{n,F}) < 3 \cdot 2^{-n}$, so $p(x_n, y_{n,F}) < 3 \cdot 2^{-(n-1)}$. We have shown that, for each $n \in \mathbb{N}$, $B_p(x_{n+3}, 2^{-n}) \cap F \neq \emptyset$. Hence, \mathcal{F} is a weakly Cauchy filter on the cofinally complete pseudometric space (X, p) , so it has a cluster point in $(X, S \vee T)$. The converse is clear. \square

COROLLARY 2.4. *A quasi-metrizable bispace (X, S, T) admits a cofinally bicomplete quasi-metric if and only if the set of points of X that admit no $S \vee T$ -compact neighborhood in $(X, S \vee T)$ is $S \vee T$ -compact.*

Proof. It follows immediately from Theorem 2.3 and the fact that a metrizable topological space admits a cofinally complete metric if and only if the set of points that admit no compact neighborhood is compact [14, Theorem 2]. \square

COROLLARY 2.5. *A quasi-pseudometrizable topological space (X, S) admits a cofinally bicomplete quasi-pseudometric if and only if it admits a quasi-pseudometric d such that the topological space $(X, T(d^s))$ admits a cofinally complete pseudometric.*

A uniformity \mathcal{U} on a set X is called uniformly locally compact if there is a $V \in \mathcal{U}$ such that for each $x \in X$, $\overline{V(x)}$ is compact ([18], [5]).

DEFINITION 2.6. A quasi-uniformity \mathcal{U} on a set X is called 2-uniformly locally compact if \mathcal{U}^s is a uniformly locally compact uniformity.

A quasi-(pseudo)metric d on a set X is called 2-uniformly locally compact if \mathcal{U}_d is a 2-uniformly locally compact quasi-uniformity on X .

THEOREM 2.7. A quasi-metrizable bispaces (X, S, T) admits a 2-uniformly locally compact quasi-metric if and only if $(X, S \vee T)$ is a locally compact topological space.

Proof. Suppose that (X, S, T) is a quasi-metrizable bispaces such that $(X, S \vee T)$ is locally compact. By [14, Corollary 1], $(X, S \vee T)$ admits a cofinally complete metric, so (X, S, T) admits a cofinally bicomplete quasi-metric d , by Theorem 1. Therefore d^s is a cofinally complete metric compatible with $(X, S \vee T)$, and by [5, Theorem 2.4] it is uniformly locally compact. We conclude that d is a 2-uniformly locally compact quasi-metric compatible with (X, S, T) . The converse is clear. \square

3. Examples

EXAMPLE 3.1. Let d be the quasi-metric defined on \mathbb{R} by $d(x, y) = \max\{y \Leftrightarrow x, 0\}$. Then, $T(d)$ and $T(d^{-1})$ are the upper topology and the lower topology on \mathbb{R} , respectively. Since d^s is the Euclidean metric on \mathbb{R} , d is a cofinally bicomplete quasi-metric. Note that, actually, d is 2-uniformly locally compact.

EXAMPLE 3.2. Let d be the quasi-metric defined on \mathbb{R} by $d(x, y) = y \Leftrightarrow x$ if $x \leq y$ and $d(x, y) = 1$, otherwise. d is the so-called Sorgenfrey quasi-metric. Since for each $x, y \in \mathbb{R}$ such that $x \neq y$, $d^s(x, y) \geq 1$, it immediately follows that d is 2-uniformly locally compact (in fact, for each $x \in \mathbb{R}$, $B_{d^s}(x, 1)$ is a compact neighborhood of x in (\mathbb{R}, d^s)).

EXAMPLE 3.3. Let d be the so-called Kofner quasi-metric (see [6, Example 7.7]). Since d^s is the discrete metric, it follows that d is a 2-uniformly locally compact quasi-metric.

We conclude the paper with an example of a cofinally bicomplete quasi-metric q such that $T(q)$ and $T(q^{-1})$ are not comparable and such that the topology $T(q^s)$ is not locally compact.

EXAMPLE 3.4. *Let S be a set of cardinality $\alpha \geq \aleph_0$. Let $I_s = [0, 1] \times \{s\}$ for all $s \in S$. By letting,*

$$(x, s_1)E(y, s_2) \Leftrightarrow x = 0 = y \text{ or, } x = y \text{ and } s_1 = s_2,$$

we define an equivalence relation E on the set $\bigcup_{s \in S} I_s$. Then, we can define a metric d on the set $J(\alpha)$ of equivalence classes of E , as follows: $d((x, s_1), (y, s_2)) = |x \Leftrightarrow y|$ if $s_1 = s_2$, and $d((x, s_1), (y, s_2)) = x + y$, otherwise. $(J(\alpha), d)$ is called the hedgehog metric space with α spines (see [4, Example 4.1.5]). It is known (see [7, p. 96–97]) that d is a cofinally complete metric on $J(\alpha)$, but $J(\alpha)$ is not locally compact.

We split the topology $T(d)$ as follows:

Let q be the quasi-metric defined on $J(\alpha)$ by

$$q((x, s_1), (y, s_2)) = x \text{ if } s_1 \neq s_2,$$

$$q((x, s_1), (y, s_2)) = \max\{x \Leftrightarrow y, 0\}, \text{ otherwise.}$$

Clearly, $T(q)$ and $T(q^{-1})$ are not comparable topologies. On the other hand, we have that $q^s((x, s_1), (y, s_2)) = \max\{x, y\}$ if $s_1 \neq s_2$, and $q^s((x, s_1), (y, s_2)) = |x \Leftrightarrow y|$, otherwise. It is easily seen that q^s generates the hedgehog topology $T(d)$ on $J(\alpha)$. Hence, by Theorem 2.3, the bispaces $(J(\alpha), T(q), T(q^{-1}))$ admits a cofinally bicomplete quasi-metric, but it does not admit a 2-uniformly locally compact quasi-metric by Theorem 2.7.

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