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A Non Quasi-metric Completion for Quasi-metric Spaces

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Summary. - *The authors have previously presented a completion theory for those approach spaces which have an underlying T_0 topology – these include all quasi-metric spaces. This theory extends the existing completion theory for uniform approach spaces, which in turn generalizes that for metric spaces. This new completion theory, moreover, has an interesting relationship with the completion theory for nearness spaces. The theory allows every quasi-metric space to be completed, and remarkably such completions need not again be quasi-metric; this situation contrasts with all other previously introduced completion theories for quasi-metric spaces (e.g. [12, 3, 9]). In this paper we present an example of a non-quasi-metric completion, and we give some conditions which ensure that the completion is again quasi-metric. This investigation leads us to favour one particular form of Cauchy sequence in quasi-metric spaces.*

1. Metric spaces

Since quasi-metric spaces are a generalization of metric spaces, any sound completion theory for quasi-metric spaces should strictly generalize the usual completion theory for metric spaces. Traditionally this is done by generalizing the concept of Cauchy sequence and/or

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that of the convergence of a sequence. But the completion of metric spaces can equally well be described in the terms of minimal Cauchy filters, and indeed this view of completion is nicer in the sense that every point of the completion has a canonical representative, and so equivalence classes are not required in its construction.

It is convenient at this point to recall the constructions used in the completion of metric spaces by means Cauchy filters. A filter $\mathcal{F}$ in a metric space (X, d) is said to be *Cauchy* when

$$\forall \varepsilon > 0,\, \exists x \in X\colon\; B_\varepsilon(x) \in \mathcal{F},$$

where $B_\varepsilon(x) := \{y \in X \mid d(x,y) < \varepsilon\}$. A *minimal Cauchy filter* is a Cauchy filter $\mathcal{F}$ such that if $\mathcal{G}$ is a Cauchy filter with $\mathcal{G} \subseteq \mathcal{F}$ then $\mathcal{G} = \mathcal{F}$. A particular form of Cauchy filter is the so-called *round Cauchy filter*, which is a Cauchy filter $\mathcal{F}$ satisfying

$$\forall F \in \mathcal{F},\, \exists \varepsilon > 0,\, \forall x \in X\colon\;\; B_\varepsilon(x) \in \mathcal{F} \;\Rightarrow\; B_\varepsilon(x) \subseteq F.$$

In metric spaces, the round Cauchy filters and the minimal Cauchy filters coincide. A metric space (X, d) is said to be *complete* when every Cauchy filter $\mathcal{F}$ has a convergence point, i.e.

$$\exists x \in X,\, \forall \varepsilon > 0\colon\; B_\varepsilon(x) \in \mathcal{F}.$$

PROPOSITION 1.1. *In a metric space, the following are equivalent:*

(1) every Cauchy filter has a convergence point;
(2) every minimal Cauchy filter has a convergence point;
(3) every round Cauchy filter has a convergence point.

The set $\hat{X}$ of points in the completion is then the set of minimal Cauchy filters (= the set of round Cauchy filters), and the metric $\hat{d}$ on the completion can be defined by:

$$\hat{d}(\mathcal{F}, \mathcal{G}) := \sup_{F \in \mathcal{F},\, G \in \mathcal{G}} \; \inf_{f \in F,\, g \in G} d(f, g).$$

2. Quasi-metric spaces

In quasi-metric spaces we can define Cauchy filters, minimal Cauchy filters, and round Cauchy filters in the same way as for metric spaces:

DEFINITION 2.1. *A Cauchy filter in a quasi-metric space* (X, d) *is a filter* $\mathcal{F}$ *such that*

$$\forall \varepsilon > 0,\ \exists x \in X\colon\ B_\varepsilon(x) \in \mathcal{F},$$

where $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$. *A minimal Cauchy filter is a Cauchy filter* $\mathcal{F}$ *such that if* $\mathcal{G}$ *is a Cauchy filter with* $\mathcal{G} \subseteq \mathcal{F}$ *then* $\mathcal{G} = \mathcal{F}$. *A round Cauchy filter is a Cauchy filter* $\mathcal{F}$ *satisfying*

$$\forall F \in \mathcal{F},\ \exists \varepsilon > 0,\ \forall x \in X\colon\ \ B_\varepsilon(x) \in \mathcal{F} \ \Rightarrow\ B_\varepsilon(x) \subseteq F.$$

While the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) of Proposition 1.1 still hold for quasi-metric spaces, their converses do not; we illustrate this with two examples. To prove that (2) $\Rightarrow$ (1) does not hold for quasi-metric spaces we present a rather odd example. The following space has no minimal Cauchy filter:

EXAMPLE 2.2. *The underlying set is* $X := \mathbb{N}$, *and the quasi-metric is:*

$$d(a, b) := \begin{cases} 10 & \text{if } a < b, \\ 0 & \text{if } a = b, \\ 1/a & \text{if } a > b. \end{cases}$$

First we show that every Cauchy filter in (X, d) *contains a set of the form* $\{0, 1, \ldots, n\}$. *Let* $\mathcal{F}$ *be a Cauchy filter. It is possible that* $\mathcal{F}$ *has a convergence point* n, *in which case* $\mathcal{F} = \dot{n} = \{A \subseteq X \mid n \in A\}$. *Otherwise there exist arbitrarily large* $n \in \mathbb{N}$ *such that each* $B_{2/n}(n) \in \mathcal{F}$. *Choosing any of these, we find that* $B_{2/n}(n) = \{0, 1, \ldots, n\}$.

Thus we have filters, such as the one generated by the one set $\{0, 1\}$, *which have no convergence point.*

Now we show that there are no minimal Cauchy filters in this space. Let $\mathcal{F}$ be a Cauchy filter. Then there exists a set of the form $\{0,1,\ldots,n\}$ in $\mathcal{F}$. Thus $\{0,1,\ldots,n+1\} \in \mathcal{F}$, and so, if we define $\mathcal{G}$ to be the filter generated by $\{0,1,\ldots,n+1\}$, then $\mathcal{G}$ is a Cauchy filter which is strictly coarser than $\mathcal{F}$.

So in the above example, every minimal Cauchy filter has a convergence point, while the Cauchy filter generated by $\{0,1\}$ has no convergence point.

EXAMPLE 2.3. *The quasi-metric space (X,d) has a minimal Cauchy filter which is not round, where $X := \{0,1\} \times \mathbb{N}_0$ and d is defined by:*

$$d((i,m),(j,n)) := \begin{cases} 0 & \text{if } (i,m) = (j,n), \\ 1/m + 1/n & \text{if } i = 0 \text{ and } j = 1 \text{ and } m \leq n, \\ 10 & \text{otherwise.} \end{cases}$$

For each $m \in \mathbb{N}_0$ we define $C_m := \{(1,n) \mid n \geq m\}$. Note that each $B_{1/m}((0,m)) = \{(0,m)\} \cup C_m$. Let $\mathcal{F}$ be the filter with base $\{C_m \mid m \in \mathbb{N}_0\}$. $\mathcal{F}$ is Cauchy, since each $B_{1/m}((0,m)) \in \mathcal{F}$. $\mathcal{F}$ is not round, since $\{1\} \times \mathbb{N}_0 \in \mathcal{F}$ and, for each $m \in \mathbb{N}_0$, we have both $B_{1/m}((0,m)) \in \mathcal{F}$ and $B_{1/m}((0,m)) \not\subseteq \{1\} \times \mathbb{N}_0$.

Now we show that $\mathcal{F}$ is a minimal Cauchy filter. Let $\mathcal{G} \subseteq \mathcal{F}$ also be Cauchy filter. Consider any $\varepsilon > 0$ and any basic $F \in \mathcal{F}$, i.e. $F = C_m$ for some $m \in \mathbb{N}_0$. Then there exists $p \geq 1/\varepsilon$ such that $B_{1/p}((0,p)) \in \mathcal{G}$ and there exists $q > p$ such that $B_{1/q}((0,q)) \in \mathcal{G}$. Now the intersection of these sets, namely C_q, is a member of $\mathcal{G}$, and hence $F \in \mathcal{G}$. Thus $\mathcal{G} = \mathcal{F}$, i.e. $\mathcal{F}$ is minimal Cauchy.

In Example 2.3 every round Cauchy filter has a convergence point, since this space has no round Cauchy filters, while the minimal Cauchy filter $\mathcal{F}$ has no convergence point.

Thus, if we are to use Cauchy filters to describe the completeness of quasi-metric spaces and their completions, we must decide whether to use minimal Cauchy filters or round Cauchy filters. We choose round Cauchy filters; our reasons relate to completion theory in a categorical context, and they require us to take an alternative view of Cauchy filters in §3, and then an alternative view of quasi-metric spaces themselves in §4.

3. Grills, clusters, and nearness

Every filter $\mathcal{F}$ on a set X is a *stack*, i.e.

$$\forall A, B \subseteq X\colon\ (\, A \in \mathcal{F} \text{ and } A \subseteq B\,) \ \Rightarrow\ B \in \mathcal{F}.$$

When applied to stacks, the operator $\sec : \mathcal{P}(\mathcal{P}(X)) \Leftrightarrow\!\!\!\!\to \mathcal{P}(\mathcal{P}(X))$ is involutive, i.e. $\sec \circ \sec = 1$, and has two equivalent definitions:

$$\begin{aligned}\sec(\mathcal{A}) &:= \{B \subseteq X \ \mid\ \forall A \in \mathcal{A}\colon\ A \cap B \neq \emptyset\}\\ &= \{B \subseteq X \ \mid\ X \setminus B \notin \mathcal{A}\}.\end{aligned}$$

Thus we have a one-to-one correspondence between the filters on a given set and their duals via the 'sec' operator; the dual of a filter is called a 'grill'.

DEFINITION 3.1. *A grill on a set X is a non-empty collection $\mathcal{G} \subseteq \mathcal{P}(X)$ of non-empty sets satisfying*

$$\forall G, H \subseteq X\colon\ \ G \cup H \in \mathcal{G} \ \Leftrightarrow\ (\, G \in \mathcal{G} \text{ or } H \in \mathcal{G}\,).$$

The advantage of working with grills in a quasi-metric space is that we can consider the 'distance' of a point to a grill, namely

$$\alpha(x, \mathcal{G}) := \sup_{G \in \mathcal{G}} d(x, G),$$

where, as is conventional, $d(x, G) := \inf_{g \in G} d(x, g)$. In particular, a filter $\mathcal{F}$ has convergence point x precisely when $\alpha(x, \sec(\mathcal{F})) = 0$, and is Cauchy precisely when $\inf_{x \in X} \alpha(x, \sec(\mathcal{F})) = 0$.

In order to distinguish the distance $d : X \times X \Leftrightarrow\!\!\!\!\to [0, \infty)$ from the distance $d : X \times \mathcal{P}(X) \Leftrightarrow\!\!\!\!\to [0, \infty)$, we shall always use δ_d to denote the latter:

DEFINITION 3.2. *If (X, d) is a quasi-metric space, then we define $\delta_d : X \times \mathcal{P}(X) \longrightarrow [0, \infty]$ by:*

$$\delta_d(x, A) := \inf_{a \in A} d(x, a).$$

For convenience we also define $\alpha_d : X \times \mathcal{P}(\mathcal{P}(X)) \longrightarrow [0, \infty]$ by:

$$\alpha_d(x, \mathcal{A}) := \sup_{A \in \mathcal{A}} \delta_d(x, A).$$

A point $x \in X$ is called an adherence point of a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ when $\alpha_d(x, \mathcal{A}) = 0$; a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is said to be near when $\inf_{x \in X} \alpha_d(x, \mathcal{A}) = 0$. A maximal near collection is called a cluster.

We find (see [15]) that every cluster is a grill, and is therefore a maximal near grill. Since the 'sec' operator is order-reversing, we find that the maximal near grills correspond to the minimal Cauchy filters. In fact the clusters correspond to the round Cauchy filters. Thus we know that a metric space is complete whenever every cluster has an adherence point, or equivalently whenever every maximal near grill has an adherence point.

4. Approach spaces

The convergence concepts in the previous section required only the distance δ_d derived from d. Thus we should be able to consider completeness, and indeed construct completions, using only δ_d. Fortunately, such functions derived from quasi-metrics have already been extensively studied:

DEFINITION 4.1. *If X is a set, then a function $\delta : X \times \mathcal{P}(X) \longrightarrow [0, \infty]$ is called an approach distance if it satisfies:*

- *(D1)* $\forall x \in X: \ \delta(x, \{x\}) = 0,$
- *(D2)* $\forall x \in X: \ \delta(x, \emptyset) = \infty,$
- *(D3)* $\forall x \in X, \ \forall A, B \subseteq X: \ \delta(x, A \cup B) = \min\{\delta(x, A), \ \delta(x, B)\},$
- *(D4)* $\forall x \in X, \ \forall A, B \subseteq X: \ \delta(x, A) \leq \delta(x, B) + \sup_{b \in B} \delta(b, A).$

A pair (X,δ)*, where* δ *is a distance on* X*, is called an* approach space*. A function* $f : X \longrightarrow Y$ *is called a* contraction $f : (X,\delta) \longrightarrow (Y,\delta')$ *if*

$$\forall x \in X,\ \forall A \subseteq X:\ \delta'(f(x), f(A)) \leq \delta(x,A).$$

The resulting topological category (see [13, 14]) is denoted by AP*.*

Lemma 4.2. *If* $\delta : X \times \mathcal{P}(X) \longrightarrow [0,\infty]$ *satisfies (D3), then each of the following is equivalent to (D4):*

(D4′) $\quad \forall x \in X,\ \forall A \subseteq X,\ \forall \beta \in [0,\infty]:\ \delta(x,A) \leq \delta(x, A^{(\beta)}) + \beta,$

(D4″) $\quad \forall x \in X,\ \forall A \subseteq X,\ \forall \beta,\gamma \in [0,\infty]:\ \left(A^{(\beta)}\right)^{(\gamma)} \subseteq A^{(\beta+\gamma)},$

where $A^{(\beta)} := \{x \in X \mid \delta(x,A) \leq \beta\}$. □

Thus every quasi-metric space can be represented uniquely as an approach space. This allows the completion theory from [15] to be applied to quasi-metric spaces. Those approach spaces (X,δ) in which $\delta(x,A) = \inf_{a\in A} \delta(x,\{a\})$ always holds are almost the quasi-metric spaces: they are the extended pseudo-quasi-metric spaces:

Definition 4.3. *An extended pseudo-quasi-metric (or* ∞*pq-metric) on a set* X *is a function* $d : X \times X \longrightarrow [0,\infty]$ *such that*

(M1) $\quad \forall x \in X:\ d(x,x) = 0,$

(M2) $\quad \forall x,y,z \in X:\ d(x,z) \leq d(x,y) + d(y,z).$

Clearly every quasi-metric space is an ∞pq-metric space. The latter spaces are in a sense more natural than the former: they form a topological category (indeed the MacNeille completion [16] of the pseudo-quasi-metric spaces) when contractions are used as the morphisms; contractions here are the functions $f : (X,d) \longrightarrow (Y,e)$ such that $(e \times e) \circ f \leq d$.

5. The completion

Definition 5.1. *If* (X,δ) *is an approach space, then we define* $\alpha_\delta : X \times \mathcal{P}(\mathcal{P}(X)) \longrightarrow [0,\infty]$ *by*

$$\alpha_\delta(x,\mathcal{A}) := \sup_{A\in\mathcal{A}} \delta(x,A).$$

We say that a collection $\mathcal{A}\subseteq\mathcal{P}(X)$ *is near when* $\inf_{x\in X}\alpha_\delta(x,\mathcal{A})=0$. *A maximal near collection is called a cluster (i.e.* $\mathcal{C}$ *is a cluster if, whenever* $\mathcal{C}\subseteq\mathcal{D}$ *and* $\mathcal{D}$ *is near we have* $\mathcal{C}=\mathcal{D}$*); we denote the set of clusters in an approach space by* $\mathcal{K}(X)$. *Note that every cluster is a maximal near grill ([15]). If* $\delta=\delta_d$ *for some* ∞pq*-metric* d*, then we use* α_d *instead of* α_{δ_d}.

Definition 5.2. *An approach space* (X,δ) *is said to be* complete *if every cluster has an adherence point, i.e. if*

$$\forall\mathcal{A}\in\mathcal{K}(X),\ \exists x\in X:\ \alpha_\delta(x,\mathcal{A})=0.$$

To embed an approach space nicely in its completion, we need a minimal degree of separation:

Definition 5.3. *An approach space* (X,δ) *is said to be* T_0 *when its topological coreflection is* T_0*, i.e. if and only if*

$$\forall x,y\in X:\qquad x\neq y\ \Rightarrow\ \delta(x,\{y\})\vee\delta(y,\{x\})>0.$$

In T_0 spaces, an adherence point of a cluster (even of a maximal near grill) is necessarily unique.

Examples 2.2 and 2.3 demonstrate that taking the set of clusters, or even maximal near grills, as the set underlying the completion could result in an empty completion! So we form a 'simple completion' of any approach space by adjoining all clusters to the space: these extra points in the completion will serve as adherence points for those same clusters. However, some clusters are indistinguishable from points in X; indeed in well-behaved spaces we can associate a cluster (a 'point-cluster') with each point via the function $\iota : X \Leftrightarrow\!\!\!\rightarrow \mathcal{P}(\mathcal{P}(X))$ defined by:

$$\iota x := \{A\subseteq X\ \mid\ \delta(x,A)=0\}.$$

This well-behavedness can be expressed, not surprisingly, as a form of symmetry, which we call 'weak symmetry'. But even for an arbitrary quasi-metric space, we can construct a completion, and embed the original space using ι. The following theorems are proved in [15].

Definition 5.4. $\mathcal{K}^*(X):=\mathcal{K}(X)\setminus\iota X$.

Definition 5.5. *If* $\mathcal{A} \subseteq \mathcal{P}(X)$ *and* $\beta \in [0, \infty]$ *then* $\mathcal{A}^{[\beta]} := \{A^{(\beta)} \mid A \in \mathcal{A}\}$.

Theorem 5.6. *If* (X, δ) *is a* T_0 *approach space then* $(\hat{X}, \hat{\delta})$ *is a complete* T_0 *approach space, where*

$$\begin{aligned} &\hat{X} := \iota X \cup \mathcal{K}^*(X), \\ &\hat{\delta} : \hat{X} \times \mathcal{P}(\hat{X}) \Leftrightarrow\!\!\rightarrow [0, \infty], \\ &\hat{\delta}(\underline{x}, \underline{A}) := \inf\{\beta > 0 \mid (\cap \underline{A})^{[\beta]} \subseteq \underline{x}\}, \end{aligned}$$

and $\iota : (X, \delta) \Leftrightarrow\!\!\rightarrow (\hat{X}, \hat{\delta})$ *is a dense embedding.*

Since each $\underline{x} \in \hat{X}$ is a stack, it is straightforward to verify that

$$\hat{\delta}(\underline{x}, \underline{A}) = \sup_{A \in \cap \underline{A}} \inf_{B \in \underline{x}} \sup_{b \in B} \delta(b, A).$$

Theorem 5.7. *The completion defined in Theorem 5.6, when applied to metric spaces, yields the usual (metric) completion.*

In [15] the authors exhibited a quasi-metric space whose completion is not a quasi-metric space, or even an ∞pq-metric:

Example 5.8. *The underlying set is* $X := \{0, 1\} \times \mathbb{N}_0$, *and the quasi-metric is:*

$$d((i, m), (j, n)) := \begin{cases} 1/m \Leftrightarrow 1/n & \text{if } i = j = 0 \text{ and } m \leq n, \\ 1/m & \text{if } i = 0 \text{ and } j = 1 \text{ and } m \leq n, \\ 10 & \text{otherwise.} \end{cases}$$

Now if a set $A \subseteq X$ *is finite, let us say* $A \subseteq \{0, 1\} \times \{1, \ldots, n\}$ *for some* $n \in \mathbb{N}_0$, *then* $\inf_{x \in X \setminus A} \delta_d(x, A) \geq 1/(n^2 \Leftrightarrow n)$. *Thus if* $\mathcal{A} \subseteq \mathcal{P}(X)$ *is near, then either* $\cap \mathcal{A} \neq \emptyset$ *or every* $A \in \mathcal{A}$ *is infinite.*

So for each $x \in X$, *the collection*

$$\iota x = \{A \subseteq X \mid \delta_d(x, A) = 0\} = \{A \subseteq X \mid x \in A\}$$

is a cluster. But note that $\mathcal{C} := \{A \subseteq X \mid A \text{ is infinite}\}$ *is also a cluster since, for each* $n \in \mathbb{N}_0$*, we have* $\alpha_{\widehat{d}}((0,n),\mathcal{C}) = 1/n$*. Now let* $D := \{1\} \times \mathbb{N}_0$*. Then* $D \in \mathcal{C}$*, and hence* $\widehat{\delta_d}(\mathcal{C}, \iota D) = 0$*. But for each* $n \in \mathbb{N}_0$ *we have*

$$\widehat{\delta_d}(\,\mathcal{C}, \{\iota(1,n)\}\,) \;=\; \inf\{\beta > 0 \mid \{(1,n)\}^{(\beta)} \text{ is infinite}\} \;=\; 10,$$

and so there can exist no ∞*pq-metric space* $(\hat{X}, e)$ *such that* $(\hat{X}, \hat{\delta}) = (\hat{X}, \delta_e)$*.* □

6. Nearness and regularity

The theory outlined in §5 also holds when maximal near grills are used for $\mathcal{K}(X)$ instead of clusters. But there are two reasons for using clusters: they give a good correspondence with completion theory in nearness spaces and they allow us to describe completions using a form of Cauchy sequence. The correspondence with nearness spaces requires the following condition, which of course can be applied to quasi-metric spaces:

Definition 6.1. *An approach space* (X, δ) *is said to be* weakly symmetric *if*

$$\forall x \in X,\ \forall A \subseteq X:\ \inf_{a \in A} \delta(a, \{x\}) = 0 \;\Rightarrow\; \delta(x, A) = 0.$$

One should note that this condition still admits many quasi-metric spaces, including all the examples given in this paper except Example 2.2, and including, for instance, any subspace of any quasi-metric space $(\mathbb{R}, d_\beta)$, where $\beta > 0$, $d_\beta(x,y) := y \Leftrightarrow x$ when $x \leq y$, and $d_\beta(x,y) := \beta(x \Leftrightarrow y)$ when $y < x$.

We refer the reader to [6, 7] for details of completion in nearness spaces. Here it is sufficient to state that every R_0 topological space is a nearness space, and that the (strict) completion of nearness spaces describes all strict extensions of topological spaces, including the Wallman and Čech–Stone compactifications and the Hewitt realcompactification, and also describes the Weil completion and Samuel compactification of uniform spaces.

If (X, δ) is an approach space, then the following are equivalent:

1. (X, δ) is weakly symmetric,
2. $\forall x \in X:\ \iota x$ is a cluster,
3. $\hat{X} = \mathcal{K}(X)$,
4. $\{\mathcal{A} \subseteq \mathcal{P}(X) \mid \mathcal{A} \text{ is near }\}$ is a nearness whose topological coreflection is the same as (X, δ)'s topological coreflection.

Moreover, if (X, δ) is a weakly symmetric approach space, then the functor from *AP* to *NEAR* described by 4. above commutes with completion (in the appropriate category), i.e. if ξ_δ denotes the nearness associated with the weakly symmetric approach distance δ and if * denotes the strict completion of nearness spaces, then $(X^*, {\xi_\delta}^*) = (\hat{X}, \xi_{\hat{\delta}})$.

The uniqueness of a completion is always desirable. Unfortunately, given our defintion of completeness, there is no unique completion amongst the quasi-metric spaces, even for metric spaces:

EXAMPLE 6.2. *A metric space which is a dense subspace of two distinct complete quasi-metric spaces. The underlying set is* $X = \{2, 3, 4, \dots\}$, *and the metric on* X *is:*

$$d(a, b) := |1/a \Leftrightarrow 1/b\,|.$$

Of course the usual (metric) completion of (X, d), *which coincides with the completion used in this paper, is*

$$\begin{aligned} &\hat{X} = X \cup \{\omega\}, \\ &\hat{d}(\underline{a}, \underline{b}) = |1/\underline{a} \Leftrightarrow 1/\underline{b}\,|, \qquad\qquad \textit{where } 1/\omega := 0. \end{aligned}$$

But consider the following quasi-metric on $\hat{X}$*:*

$$e(\underline{a}, \underline{b}) := \begin{cases} |1/\underline{a} \Leftrightarrow 1/\underline{b}\,| & \textit{if } \underline{a} \neq \omega, \\ 1/\underline{b}^2 & \textit{if } \underline{a} = \omega, \end{cases}$$

where $1/\omega^2 := 0$. *Then* $(\hat{X}, e)$ *is a complete quasi-metric space, and* (X, d) *is a dense subspace of* $(\hat{X}, e)$.

However, amongst 'limit-regular' approach spaces, our completion is the unique 'smallest' completion, i.e. a completion which defines an epireflection. Thus if (X,δ) is a limit-regular approach space which is embedded (not necessarily densely) in a complete limit-regular approach space (Y,ρ), then there is a unique morphism $f:(\hat{X},\hat{\delta}) \Leftrightarrow\!\!\!\rightarrow (Y,\rho)$ leaving X unchanged. In limit-regular approach spaces, the three properties listed in Proposition 1.1 are equivalent. Surprisingly, limit regularity is a Hausdorff condition. More specifically, in topological spaces (which are also approach spaces), limit regularity coincides with the H_0 property of [2] and the R_1 property of [5], i.e. the non-T_0 part of the Hausdorff property. Limit regularity is stronger than weak symmetry.

DEFINITION 6.3. *An approach space (X,δ) is said to be limit-regular if, whenever $A \subseteq X$ and $\mathcal{G}$ is a grill on X, we have*

$$\inf_{a\in A} \alpha_\delta(a,\mathcal{G}) = 0 \quad\Rightarrow\quad \forall x \in X\colon\ \delta(x,A) \le \alpha_\delta(x,\mathcal{G}).$$

7. Cauchy sequences

In this section we define a 'Cauchy sequence' and show its correspondence with clusters. In electing the following definition for a Cauchy sequence, we are motivated by Theorem 7.10. We also note that this type of sequence arises naturally from a categorical view of convergence [11] and facilitates a Baire category theorem for quasi-metric spaces [4] (which also appears in [10]); indeed it was considered by Kelley in [8].

DEFINITION 7.1. *If (X,d) is a quasi-metric space, then a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be a* Cauchy sequence *iff*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall m,n \ge N\colon\ \ m \le n \ \Rightarrow\ d(x_m,x_n) \le \varepsilon.$$

Sequences of this type are called 'left K-Cauchy' in [12].

REMARK 7.2. *A collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is near in an approach space (X,δ) if and only if there exists a sequence (x_n) in X such that $(x_n) \to \mathcal{A}$, where*

$$(x_n) \to \mathcal{A} \quad\Leftrightarrow\quad \forall\varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \ge N\colon\ \alpha_\delta(x_n,\mathcal{A}) \le \varepsilon.$$

In particular when $\mathcal{A}$ is a cluster, we obtain the following proposition.

Definition 7.3. *If (x_n) is a sequence in a set X, then we say that a set $T \subseteq X$* spans *(x_n) whenever*

$$\forall N \in \mathbb{N},\ \exists n \geq N \colon\ x_n \in T.$$

Proposition 7.4. *If $\mathcal{C}$ is a cluster in an ∞pq-metric space (X, d) then every sequence (x_n) in X such that $(x_n) \to \mathcal{C}$ contains a subsequence which is Cauchy.*

Proof. Let $\mathcal{C}$ be a cluster and let $(x_n) \to \mathcal{C}$. Now every T which spans (x_n) defines a subsequence (z_n) of (x_n), and hence $(z_n) \to \mathcal{C}$. But $(z_n) \to \mathcal{C} \cup \{T\}$, making $\mathcal{C} \cup \{T\}$ near, and therefore $T \in \mathcal{C}$. Thus $\mathcal{C}$ contains all sets which span (x_n). Now

$$\exists N_0 \in \mathbb{N},\ \forall n \geq N_0 \colon\ \alpha_d(x_n, \mathcal{C}) \leq 1/2.$$

For each $m > N_0$, $T_m := \{x_n \mid n \geq m\}$ spans (x_n). Thus for each $m > N_0$, we have $\delta_d(x_{N_0}, T_m) \leq 1/2$, and therefore there exists $n \geq m$ such that $d(x_{N_0}, x_n) \leq 1$. So we obtain a subsequence (x_n^0) of (x_n) such that

(a) $x_0^0 = x_{N_0}$,

(b) $\forall n \in \mathbb{N} \colon\ \alpha_d(x_n^0, \mathcal{C}) \leq 1/2$, and

(c) $\forall n \in \mathbb{N} \colon\ d(x_0^0, x_n^0) \leq 1$.

Now we construct a subsequence (x_n^1) of (x_n^0). Again, $(x_n^0) \to \mathcal{C}$, and so

$$\exists N_1 \in \mathbb{N}_0,\ \forall n \geq N_1 \colon\ \alpha_d(x_n^0, \mathcal{C}) \leq 1/4.$$

And again for each $m > N_1$, $T_m := \{x_n^0 \mid n \geq m\}$ spans (x_n), and this gives rise to a subsequence (x_n^1) of (x_n^0) satisfying

(a) $x_0^1 = x_{N_1}$,

(b) $\forall n \in \mathbb{N} \colon\ \alpha_d(x_n^1, \mathcal{C}) \leq 1/4$, and

(c) $\forall n \in \mathbb{N}\colon\ d(x_0^1, x_n^1) \leq 1/2$.

Continuing in this way, we obtain a subsequence (x_0^n) of (x_n) such that

$$\forall m \in \mathbb{N},\ \forall n \geq m\colon\ d(x_0^m, x_0^n) \leq 2^{-m}. \qquad \square$$

LEMMA 7.5. *If (x_n) is a Cauchy sequence in a quasi-metric space (X,d) then $\{T \subseteq X \mid T$ spans $(x_n)\}$ is a near grill.* □

PROPOSITION 7.6. *If (x_n) is a Cauchy sequence in a limit-regular quasi-metric space (X,d), then $\mathcal{C}(x_n)$ is a cluster, where*

$$\mathcal{C}(x_n) := \{A \subseteq X \mid \forall n \in \mathbb{N}\colon\ \delta_d(x_n, A) \leq \limsup_{m\to\infty} d(x_n, x_m)\}.$$

Proof. Clearly $\mathcal{C}(x_n)$ is a near grill. Hence by limit regularity and by Lemma 6.8 of [15], $\mathcal{C}(x_n)$ is contained in a unique cluster $\mathcal{D}$. By Lemma 6.15 of [15], each $\alpha_d(x_n, \mathcal{D}) = \alpha_d(x_n, \mathcal{C}(x_n)) \leq \limsup_{m\to\infty} d(x_n, x_m)$, and therefore $\mathcal{D} \subseteq \mathcal{C}(x_n)$. Hence $\mathcal{C}(x_n)$ is a cluster. □

PROPOSITION 7.7. *If $\mathcal{C}$ is a cluster in a limit-regular quasi-metric space and if (x_n) is a Cauchy sequence satisfying $(x_n) \to \mathcal{C}$, then $\mathcal{C}(x_n) = \mathcal{C}$.*

Proof. We have $(x_n) \to \mathcal{C} \cup \mathcal{C}(x_n)$, and therefore $\mathcal{C}(x_n) \subseteq \mathcal{C}$. But by Proposition 7.6, $\mathcal{C}(x_n)$ is a cluster, and hence $\mathcal{C}(x_n) = \mathcal{C}$. □

REMARK 7.8. *Thus we have a many-to-one correspondence between the Cauchy sequences and the clusters in a limit-regular quasi-metric space. This of course induces an equivalence relation on the Cauchy sequences; it also allows us to describe completeness in terms of the convergence of Cauchy sequences:*

DEFINITION 7.9. *If (X,d) is a ∞pq-metric space, $y \in X$, and (x_n) is a sequence in X, then we say that (x_n) converges to y when $\lim_{n\to\infty} d(y, x_n) = 0$.*

THEOREM 7.10. *A limit-regular quasi-metric space is complete if and only if every Cauchy sequence converges.*

Proof. We need only show that a Cauchy sequence (x_n) converges to a point y if and only if y is an adherence point of $\mathcal{C}(x_n)$. This follows by noting that (x_n) converges to y iff y is an adherence point of the grill of spanning sets of (x_n) iff y is an adherence point of $\mathcal{C}(x_n)$. $\square$

PROPOSITION 7.11. *In a limit-regular quasi-metric space (X, d), the Cauchy sequences form equivalence classes generated by the relation:*

$$(x_n) \sim (y_n) \quad \Leftrightarrow \quad \limsup_{m\to\infty} \limsup_{n\to\infty} d(x_m, y_n) = 0.$$

Proof. Clearly we have an equivalence relation $\sim$ on the Cauchy sequences, namely that two Cauchy sequences (x_n) and (y_n) are equivalent iff $\mathcal{C}(x_n) = \mathcal{C}(y_n)$.

Let $(x_n) \not\sim (y_n)$. Then

$$\begin{aligned} &\exists \varepsilon > 0,\ \forall M \in \mathbb{N},\ \exists m \geq M,\ \forall N \in \mathbb{N},\ \exists n \geq N\colon\ d(x_m, y_n) \geq \varepsilon \\ \Rightarrow\ &\exists \varepsilon > 0,\ \forall M \in \mathbb{N},\ \exists m \geq M,\ \exists T \text{ spanning } (y_n)\colon\ \delta_d(x_m, T) \geq \varepsilon. \end{aligned}$$

But there exists $M \in \mathbb{N}$ such that for all $m \geq M$ we have $\limsup_{p\to\infty} d(x_m, x_p) \leq \varepsilon/2$. So $T \notin \mathcal{C}(x_n)$, and hence $\mathcal{C}(y_n) \not\subseteq \mathcal{C}(x_m)$.

Conversely let $(x_n) \sim (y_n)$, and let T span (y_n). We shall show that

$$\limsup_{m\to\infty} \delta_d(x_m, T) = 0. \tag{1}$$

Assume (1) to be false. Then

$$\begin{aligned} &\exists \varepsilon > 0,\ \forall M \in \mathbb{N},\ \exists m \geq M\colon\ \delta_d(x_m, T) \geq \varepsilon \\ \Rightarrow\ &\exists \varepsilon > 0,\ \forall M \in \mathbb{N},\ \exists m \geq M,\ \forall N \in \mathbb{N},\ \exists n \geq N\colon\ d(x_m, y_n) \geq \varepsilon, \end{aligned}$$

contradicting $(x_n) \sim (y_n)$. Therefore (1) holds, and hence $(x_n) \to (\mathcal{C}(x_n) \cup \{T\})$. Applying Axiom (D4) we find that $(x_n) \to (\mathcal{C}(x_n) \cup \mathcal{C}(y_n))$. Now applying Proposition 7.6 we obtain $\mathcal{C}(x_n) = \mathcal{C}(y_n)$. $\square$

8. Completion by Cauchy sequences

Now we can consider the underlying set of the completion of a quasi-metric space to be the equivalence classes of the Cauchy sequences.

It will be more convenient to use the slightly broader class of ∞pq-metric spaces (see Definition 4.3). But Example 5.8 shows that the completion is not necessarily an ∞pq-metric space. In this section we investigate when the completion of an ∞pq-metric space is again an ∞pq-metric.

DEFINITION 8.1. *An ∞pq-metric space (X, d) is said to be* insular *iff*

$$\forall A \subseteq X, \forall \text{ Cauchy sequence } (x_n): \\ \lim_{n\to\infty} \delta_d(x_n, A) = 0 \;\Rightarrow\; \forall \varepsilon > 0,\ \exists a \in A\colon\ \limsup_{n\to\infty} d(x_n, a) \leq \varepsilon.$$

PROPOSITION 8.2. *Every complete limit-regular ∞pq-metric space is insular.*

Proof. Let (x_n) be a Cauchy sequence in a complete limit-regular ∞pq-metric space (X, d), and let $A \subseteq X$ be such that $\lim_{n\to\infty} \delta_d(x_n, A) = 0$. Then $\mathcal{C}(x_n) \cup \{A\}$ is near, and therefore $A \in \mathcal{C}(x_n)$. Now $\mathcal{C}(x_n)$ has an adherence point y, and so $\delta_d(y, A) = 0$. Consider any $\varepsilon > 0$. Then $\exists a \in A$ such that $d(y, a) \leq \varepsilon$. But $\alpha_d(y, \mathcal{C}(x_n)) = 0$, and hence by limit regularity we have

$$\begin{aligned} & \forall n \in \mathbb{N}\colon\ d(x_n, y) \leq \alpha_d(x_n, \mathcal{C}(x_n)) \\ \Rightarrow\ & \forall n \in \mathbb{N}\colon\ d(x_n, y) \leq \limsup_{p\to\infty} d(x_n, x_p) \\ \Rightarrow\ & \forall n \in \mathbb{N}\colon\ d(x_n, a) \leq \limsup_{p\to\infty} d(x_n, x_p) + \varepsilon \\ \Rightarrow\ & \limsup_{n\to\infty} d(x_n, a) \leq \varepsilon. \end{aligned}$$

□

PROPOSITION 8.3. *If (X, d) is an insular T_0 ∞pq-metric space, then $(\hat{X}, \widehat{\delta_d})$ is an ∞pq-metric space.*

Proof. We must show, for $\underline{x} \in \hat{X}$ and $\underline{A} \subseteq \hat{X}$, that

$$\widehat{\delta_d}(\underline{x}, \underline{A}) = \bigwedge_{\underline{a} \in \underline{A}} \widehat{\delta_d}(\underline{x}, \{\underline{a}\}).$$

The $\leq$ part follows from the fact that $\underline{A} \subseteq \underline{B} \;\Rightarrow\; \hat{\delta}(\underline{x}, \underline{B}) \leq \hat{\delta}(\underline{x}, \underline{A})$.

If $\underline{x} \in \mathcal{K}(X)$ then we know by Proposition 7.4 that there is a Cauchy sequence (x_n) in X such that $\lim_{n\to\infty} \alpha_d(x_n, \underline{x}) = 0$. For each $N \in \mathbb{N}$ we shall define $B_N := \{x_n \mid n \geq N\}$. Note that, by the maximality of $\underline{x}$, we have each $B_N \in \underline{x}$. If $\underline{x} \notin \mathcal{K}(X)$ then $\underline{x} = \iota(x)$ for some $x \in X$, and then we let (x_n) be the constant sequence at x; we define every B_N to be $\{x\}$, and again we have each $B_N \in \underline{x}$.

Now let $\beta := \widehat{\delta_d}(\underline{x}, \underline{A})$; consider any $\varepsilon > 0$ and, for each $\underline{a} \in \underline{A}$, any $A_{\underline{a}} \in \underline{a}$. Let $C := \cup_{\underline{a}\in\underline{A}} A_{\underline{a}}$. Then, since each $\underline{a}$ is a stack, we have $C \in \cap\underline{A}$. Therefore $C^{(\beta+\varepsilon)} \in \underline{x}$. Thus $\lim_{n\to\infty} \delta_d(x_n, C^{(\beta+\varepsilon)}) = 0$. So, by the insularity of (X, d), there exists a $c \in C^{(\beta+\varepsilon)}$ such that $\limsup_{n\to\infty} d(x_n, c) \leq \varepsilon$. But since (X, d) in an ∞pq-metric, $C^{(\beta+\varepsilon)} = \cup_{\underline{a}\in\underline{A}}({A_{\underline{a}}}^{(\beta+\varepsilon)})$. Therefore

$$\begin{aligned} &\exists \underline{a} \in \underline{A},\ \exists a \in {A_{\underline{a}}}^{(\beta+\varepsilon)}\colon\ \limsup_{n\to\infty} d(x_n, a) \leq \varepsilon \\ \Rightarrow\ &\exists \underline{a} \in \underline{A}\colon\ \limsup_{n\to\infty} \delta_d(x_n, {A_{\underline{a}}}^{(\beta+\varepsilon)}) \leq \varepsilon \\ \Rightarrow\ &\exists \underline{a} \in \underline{A}\colon\ \limsup_{n\to\infty} \delta_d(x_n, A_{\underline{a}}) \leq \beta + 2\varepsilon. \end{aligned}$$

To summarise,

$$\begin{aligned} &\forall \varepsilon > 0,\ \forall \{A_{\underline{a}}\}_{\underline{a}\in\underline{A}} \in \prod_{\underline{a}\in\underline{A}} \underline{a},\ \exists \underline{a} \in \underline{A}, \\ &\qquad\qquad \exists N \in \mathbb{N},\ \forall n \geq N\colon\ \delta_d(x_n, A_{\underline{a}}) \leq \beta + 3\varepsilon \\ \Rightarrow\ &\forall \varepsilon > 0,\ \exists \underline{a} \in \underline{A},\ \forall A \in \underline{a}, \\ &\qquad\qquad \exists B_N \in \underline{x},\ \forall b \in B_N\colon\ \delta_d(b, A) \leq \beta + 3\varepsilon \\ \Rightarrow\ &\inf_{\underline{a}\in\underline{A}} \hat{\delta}(\underline{x}, \{\underline{a}\}) \leq \beta. \end{aligned}$$

□

Using Proposition 8.2, Proposition 8.3, and the fact [15] that the completion of a limit-regular approach space is limit-regular, we obtain:

PROPOSITION 8.4. *If (X, d) is a limit-regular insular T_0 ∞pq-metric space, then its completion $(\hat{X}, \hat{d})$ is insular.*

THEOREM 8.5. *Within the T_0 limit-regular insular ∞pq-metric spaces, completion is an epireflection.*

Proof. The completion of a limit-regular insular T_0 ∞pq-metric space is again a limit-regular insular T_0 ∞pq-metric space. But completion is an epireflection for T_0 limit-regular approach spaces, and $pqMET^\infty$ is a full subcategory of AP. □

PROPOSITION 8.6. *If (X, d) is a limit-regular insular T_0 ∞pq-metric space, then its completion can be described by:*

$$\hat{X} := \{\, [(x_n)] \mid (x_n) \text{ Cauchy sequence}\},$$
$$\text{where } (x_n) \sim (y_n) \Leftrightarrow \limsup_{m\to\infty} \limsup_{n\to\infty} d(x_m, y_n) = 0,$$
$$\hat{d}(\, [(x_n)], [(y_n)]\,) := \sup_{M\in\mathbb{N}} \inf_{N\in\mathbb{N}} \sup_{n\geq N} \inf_{m\geq M} d(x_m, y_n).$$

Proof. In the following proof we implicitly show that the description of $\hat{d}$ is independent of the choice of representatives for the two equivalence classes.

$$\begin{aligned}
\widehat{\delta_d}(\, [(x_m)], \{[(y_n)]\}\,) &= \sup_{U\in\mathcal{C}(y_n)} \inf_{T\in\mathcal{C}(x_n)} \sup_{t\in T} \inf_{u\in U} d(t,u) \\
&= \sup_{U \text{ spans } (y_n)} \inf_{T \text{ spans } (x_n)} \sup_{t\in T} \inf_{u\in U} d(t,u) \\
&= \sup_{U \text{ spans } (y_n)} \sup_{M\in\mathbb{N}} \inf_{m\geq M} \inf_{u\in U} d(x_m,u) \\
&= \sup_{M\in\mathbb{N}} \sup_{U \text{ spans } (y_n)} \inf_{u\in U} \inf_{m\geq M} d(x_m,u) \\
&= \sup_{M\in\mathbb{N}} \inf_{N\in\mathbb{N}} \sup_{n\geq N} \inf_{m\geq M} d(x_m,y_n).
\end{aligned}$$

□

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