

Quasi-Metric Spaces, Quasi-Metric Hyperspaces and Uniform Local Compactness

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SUMMARY. - We show that every locally compact quasi-metrizable Moore space admits a uniformly locally compact quasi-metric. We also observe that every equinormal quasi-metric is cofinally complete. Finally we prove that for any small-set symmetric quasi-uniform space, uniform local compactness is preserved by the Hausdorff-Bourbaki quasi-uniformity on compact sets. Several illustrative examples are given.

1. Introduction and preliminaries

Throughout this paper the letters \mathbb{N} and \mathbb{R} denote the sets of positive integer numbers and real numbers, respectively. Locally compact spaces are assumed to be Tychonoff. Terms and undefined concepts are used as in [10] and in [7].

A quasi-pseudometric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$, and

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- (ii) $d(x, y) \leq d(x, z) + d(z, y)$. If in addition d satisfies the condition
 (iii) $d(x, y) = 0 \Leftrightarrow x = y$, then d is called a quasi-metric on X .

If d is a quasi-pseudometric on a set X and $x \in X$, the set $\{y \in X : d(x, y) < r\}$ is called the open r -sphere around x and is denoted by $B_d(x, r)$. We shall denote by d^s the pseudometric $d \vee d^{-1}$, i.e. $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$. Note that if d is a quasi-metric, then d^s is a metric. A quasi-(pseudo)metric space is a pair (X, d) such that X is a non-empty set and d is a quasi-(pseudo)metric on X .

Each quasi-pseudometric d on X generates a quasi-uniformity \mathcal{U}_d on X which has as a base the family of sets of the form $\{(x, y) \in X \times X : d(x, y) < 2^{-n}\}$, $n \in \mathbb{N}$. Then the topology $T(\mathcal{U}_d)$, induced by \mathcal{U}_d , will be denoted simply by $T(d)$. Note that the family $\{B_d(x, r) : x \in X, r > 0\}$ is a base for $T(d)$.

A space (X, T) is said to be quasi-metrizable if there is a quasi-metric d on X such that $T = T(d)$. In this case we say that d is compatible with T .

In Section 2 of this paper we study conditions under which a space admits a uniformly locally compact quasi-metric. Since every locally compact cofinally complete quasi-uniform space is uniformly locally compact, we also study conditions under which a space admits a cofinally complete quasi-metric. Section 3 is devoted to the study of uniform local compactness and cofinal completeness in hyperspaces. In particular, it is shown that for any small-set symmetric quasi-uniform space, uniform local compactness is preserved by the Hausdorff-Bourbaki quasi-uniformity on compact sets.

2. Uniformly locally compact quasi-metric spaces

Recall that a quasi-uniform space (X, \mathcal{U}) is uniformly locally compact [9], [10], provided that there is a $V \in \mathcal{U}$ such that for each $x \in X$, $\overline{V(x)}$ is compact. A quasi-pseudometric space (X, d) is called uniformly locally compact if (X, \mathcal{U}_d) is a uniformly locally compact quasi-uniform space.

A filter \mathcal{F} on a quasi-uniform space (X, \mathcal{U}) is said to be weakly Cauchy [9], [10], [5], if for each $U \in \mathcal{U}$, $\bigcap_{F \in \mathcal{F}} U^{-1}(F) \neq \emptyset$. (X, \mathcal{U}) is cofinally complete provided that every weakly Cauchy filter has a

cluster point. The notion of a cofinally complete quasi-pseudometric space is defined in the obvious manner.

In [18] it is introduced the notion of a cofinally Čech complete space in order to characterize those spaces that admit a cofinally complete metric. A Tychonoff space (X, T) is called cofinally Čech complete if there is a countable collection $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of X satisfying the property that whenever \mathcal{F} is a filter on X such that for each $n \in \mathbb{N}$ there is some $G_n \in \mathcal{G}_n$ which meets all the members of \mathcal{F} , then \mathcal{F} has a cluster point. In this case we say that $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a cofinally Čech complete collection for (X, T) .

Note that every locally compact space is cofinally Čech complete and that every cofinally Čech complete space is Čech complete. It is known that the converse implications do not hold.

PROPOSITION 2.1. *Every Tychonoff cofinally complete quasi-metric space is cofinally Čech complete.*

Proof. Let (X, d) be a Tychonoff cofinally complete quasi-metric space. For each $n \in \mathbb{N}$ let $\mathcal{G}_n = \{B_d(x, 2^{-n}) : x \in X\}$. Clearly $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a cofinally Čech complete collection for (X, d) . \square

COROLLARY 2.2. *Every metrizable space which admits a cofinally complete quasi-metric admits a cofinally complete metric.*

Proof. Let (X, T) be a metrizable space which admits a cofinally complete quasi-metric. By Proposition 2.1, it is cofinally Čech complete and by [18, Theorem 1], it admits a cofinally complete metric. \square

It is well-known that a Tychonoff space is fully normal if and only if its fine uniformity is a Lebesgue uniformity. This result suggests the following definition. A topological space is called *quasi-fully normal* if its fine quasi-uniformity is a Lebesgue quasi-uniformity (see [10, p. 97] for the notion of a Lebesgue quasi-uniformity). It is clear that every orthocompact space is quasi-fully normal (see [10, p. 100]).

PROPOSITION 2.3. *Every quasi-fully normal cofinally Čech complete space with a G_δ -diagonal admits a cofinally complete quasi-metric.*

Proof. Let (X, T) be a quasi-fully normal cofinally Čech complete space with a G_δ -diagonal. Then there is a cofinally Čech complete collection $\{\mathcal{G}_n : n \in \mathbb{N}\}$ for (X, T) and a sequence $(\mathcal{H}_n)_{n \in \mathbb{N}}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap_{n \in \mathbb{N}} St(x, \mathcal{H}_n)$, where as usual $St(x, \mathcal{H}_n) = \bigcup \{H \in \mathcal{H}_n : x \in H\}$. For each $n \in \mathbb{N}$ let $\mathcal{C}_n = \{C \subseteq X : C \text{ is open and } \overline{C} \subseteq G \cap H \text{ for some } G \in \mathcal{G}_n \text{ and } H \in \mathcal{H}_n\}$. Then \mathcal{C}_n is an open cover of X . Since (X, T) is quasi-fully normal there is a sequence $(U_n)_{n \in \mathbb{N}}$ of members of the fine quasi-uniformity of (X, T) such that $U_{n+1}^3 \subseteq U_n$ and $\{U_n(x) : x \in X\}$ refines \mathcal{C}_n for all $n \in \mathbb{N}$. By Kelley's metrization lemma there is a quasi-pseudometric d on X such that for each $n \in \mathbb{N}$,

$$U_{n+1} \subseteq \{(x, y) \in X \times X : d(x, y) < 2^{-n}\} \subseteq U_n.$$

We first show that d is compatible with T . Clearly $T(d) \subseteq T$. Now let $x \in X$ and let V be a neighborhood of x in (X, T) . Suppose that for each $n \in \mathbb{N}$, $B_d(x, 2^{-n}) \setminus V \neq \emptyset$. Put $\mathcal{F} = \text{fil}\{B_d(x, 2^{-n}) \setminus V : n \in \mathbb{N}\}$. Since for each $n \in \mathbb{N}$ there is $C_n \in \mathcal{C}_n$ such that $B_d(x, 2^{-n}) \subseteq U_n(x) \subseteq C_n$ it follows that $B_d(x, 2^{-n}) \setminus V \subseteq \overline{C_n} \subseteq G_n \cap H_n$ for some $G_n \in \mathcal{G}_n$ and $H_n \in \mathcal{H}_n$. Since $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a cofinally Čech complete collection, \mathcal{F} has a cluster point $y \in X \setminus V$. However $y = x$ since $y \in \overline{B_d(x, 2^{-n})} \subseteq St(x, \mathcal{H}_n)$ for all $n \in \mathbb{N}$. We have obtained a contradiction. Hence $T = T(d)$ and thus d is actually a quasi-metric on X .

Finally, let \mathcal{F} be a weakly Cauchy filter on (X, \mathcal{U}_d) . For each $n \in \mathbb{N}$ there is $x_n \in X$ such that $B_d(x_n, 2^{-n}) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Since for each $n \in \mathbb{N}$ there is $G_n \in \mathcal{G}_n$ such that $B_d(x_n, 2^{-n}) \subseteq U_n(x_n) \subseteq G_n$ we deduce that \mathcal{F} has a cluster point. We conclude that d is a cofinally complete quasi-metric on X . \square

According to [20] and [12], a quasi-metric d on a set X is said to be strong (point symmetric in [10]) if $T(d) \subseteq T(d^{-1})$. Let us recall [10], [12], that a regular quasi-metrizable space admits a strong quasi-metric if and only if it is a Moore space.

COROLLARY 2.4. *Every cofinally Čech complete quasi-metrizable Moore space admits a cofinally complete strong quasi-metric.*

Proof. Let (X, T) be a cofinally Čech complete quasi-metrizable Moore space. By [10, Theorem 7.24] and Proposition 2.3 above,

(X, T) admits a cofinally complete quasi-metric d . Let p be a compatible strong quasi-metric for (X, T) . Then $d + p$ is a compatible cofinally complete strong quasi-metric for (X, T) . \square

In [9, Example 3.2] it is presented an example of a locally compact separable Moore space which does not admit a cofinally complete quasi-uniformity. Our next result shows that the situation is quite different when the space is quasi-metrizable.

PROPOSITION 2.5. *Every locally compact quasi-metrizable Moore space admits a uniformly locally compact strong quasi-metric.*

Proof. Let (X, T) be a locally compact quasi-metrizable Moore space. By Corollary 2.4, (X, T) admits a cofinally complete strong quasi-metric. The result follows from the well-known fact [9, Theorem 2.4] (or [10, Proposition 5.32 and Theorem 5.33]) that every locally compact cofinally complete quasi-uniform space is uniformly locally compact. \square

EXAMPLE 2.6. *Let $X = \{(m/n^2, 1/n) : m \text{ is an integer and } n \in \mathbb{N}\} \cup \{(x, 0) : x \in \mathbb{R}\}$. Define a topology T on X in terms of the neighborhood system of a point as follows. If $(x, y) \in X$ and $y \neq 0$, then for each $n \in \mathbb{N}$ let $U_n(x, y) = \{(x, y)\}$. If $y = 0$, let $U_n(x, 0) = \{(u, v) : v \leq 1/2^n \text{ and } |u \ominus x| \leq v\}$.*

Dieudonné showed in [6] that (X, T) is a locally compact nonnormal topological space and Stoltenberg showed in [20] that it admits a compatible strong quasi-metric. By Proposition 2.5 this space admits a uniformly locally compact strong quasi-metric.

EXAMPLE 2.7. *The rational sequence topology [19, Example 65]: Enumerate the rationals as $\{q_j : j \in \mathbb{N}\}$. For each irrational x choose a sequence $(x_k)_{k \in \mathbb{N}}$ of rationals converging to x in the Euclidean topology. For each $x \in \mathbb{R}$ let $d(x, x) = 0$. For each irrational x and each $k \in \mathbb{N}$ let $d(x, x_k) = 2^{-j}$, where $x_k = q_j$, and let $d(x, y) = 1$ otherwise. Then d is a uniformly locally compact quasi-metric on \mathbb{R} such that $T(d)$ is the rational sequence topology as it is observed in [9, Example 3.3]. Moreover d is strong since $T(d^{-1})$ is the discrete topology on \mathbb{R} .*

EXAMPLE 2.8. *The Niemytzki plane is an interesting example of an orthocompact Čech complete quasi-metrizable Moore space. We shall prove that it does not admit a cofinally complete quasi-metric.*

Let X be the union of the upper half plane and the x -axis. The upper half plane has the Euclidean subspace topology E . For each $x \in \mathbb{R}$ and $n \in \mathbb{N}$ set $G(n, x) = \{(x, 0)\} \cup G_n$, where G_n is the open disk of radius $1/n$ in the upper half plane, tangent to the x -axis at $(x, 0)$. Then $\{G(n, x) : n \in \mathbb{N}, x \in \mathbb{R}\} \cup E$ is a base for the topology T of the Niemytzki plane and it follows from [13, Proposition 7] that it admits a convergence complete (strong) quasi-metric.

Suppose that (X, T) admits a cofinally complete quasi-metric d . For each $x \in \mathbb{R}$ and each $r > 0$ denote by $C(x, r)$ the circle of radius r in the upper half plane tangent to the x -axis at $(x, 0)$. Since d is compatible with T , for each $n \in \mathbb{N}$ there is $\delta_n > 0$, with $\delta_n \leq 2^{-n}$, such that $C((n, 0), \delta_n) \subseteq B_d((n, 0), 2^{-n})$. For each $n \in \mathbb{N}$ choose a sequence $(x_m(n))_{m \in \mathbb{N}}$ of distinct points in $C((n, 0), \delta_n)$, convergent to $(n, 0)$ in the Euclidean topology. Now for each $n \in \mathbb{N}$ set $F_n = \{x_m(k) : m \geq n, k \in \mathbb{N}\}$. Then $\{F_n : n \in \mathbb{N}\}$ is a base for a filter \mathcal{F} on X and $B_d((n, 0), 2^{-n}) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. However \mathcal{F} has no cluster point in (X, T) .

A quasi-metric d on a set X is called equinormal [10] if $d(A, B) > 0$ whenever A and B are two disjoint nonempty closed subsets of X .

PROPOSITION 2.9. *Every equinormal quasi-metric is cofinally complete.*

Proof. Let d be an equinormal quasi-metric on a set X . Let \mathcal{F} be a weakly Cauchy filter on (X, d) . Then there is a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that $B_d(x_n, 2^{-n}) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.

If the sequence $(x_n)_{n \in \mathbb{N}}$ has a cluster point $y \in X$, then y is a cluster point for \mathcal{F} . Hence we will suppose in the following that $(x_n)_{n \in \mathbb{N}}$ is a sequence of distinct points which has no cluster point. Let us assume that the filter \mathcal{F} has no cluster point. Then there exist a decreasing sequence $(F_n)_{n \in \mathbb{N}}$ of members of \mathcal{F} and a sequence $(r(x_n))_{n \in \mathbb{N}}$ of positive real numbers such that $r(x_n) < 2^{-n}$ and $B_d(x_n, r(x_n)) \cap F_n = \emptyset$ for all $n \in \mathbb{N}$.

Suppose that for each $n \in \mathbb{N}$ there is an $m(n) > n$ such that $x_{m(n)} \in B_d(x_n, 2^{-n})$. Then we can construct two subsequences $(a_n)_{n \in \mathbb{N}}$

and $(b_n)_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that $\{a_n : n \in \mathbb{N}\} \cap \{b_n : n \in \mathbb{N}\} = \emptyset$ and $d(a_n, b_n) \rightarrow 0$, which is not possible since d is equinormal. Therefore, we can assume, without loss of generality, that for each $n \in \mathbb{N}$, $B_d(x_n, 2^{-n}) \cap \{x_m : m > n\} = \emptyset$. Consequently, for each $n \in \mathbb{N}$ there is $y_n \in [B_d(x_n, 2^{-n}) \setminus \{x_m : m \in \mathbb{N}\}] \cap F_n$. (In fact, otherwise there is a $k \in \mathbb{N}$ such that $B_d(x_k, 2^{-k}) \cap F_k$ is a nonempty subset of $\{x_m : m \leq k\}$. Since $B_d(x_k, 2^{-k}) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, there is $m \leq k$ for which $x_m \in \cap_{F \in \mathcal{F}} F$, a contradiction.)

Since $(F_n)_{n \in \mathbb{N}}$ is a decreasing sequence and for each $n \in \mathbb{N}$, $B_d(x_n, r(x_n)) \cap F_n = \emptyset$ we deduce that $\{x_n : n \in \mathbb{N}\} \cap \{y_n : n \in \mathbb{N}\} = \emptyset$. However, $d(x_n, y_n) < 2^{-n}$ for all $n \in \mathbb{N}$. Thus we have obtained a contradiction. Hence d is a cofinally complete quasi-metric. \square

3. Uniformly locally compact quasi-metric hyperspaces

Let (X, \mathcal{U}) be a quasi-uniform space. If we denote by $\mathcal{CL}_o(X)$ the collection of all nonempty closed subsets of $(X, T(\mathcal{U}))$, then the Hausdorff-Bourbaki quasi-uniformity of (X, \mathcal{U}) is defined as the quasi-uniformity \mathcal{U}_* on $\mathcal{CL}_o(X)$ which has as a base the family of sets of the form

$$U_H = \{(A, B) \in \mathcal{CL}_o(X) \times \mathcal{CL}_o(X) : B \subseteq U(A) \text{ and } A \subseteq U^{-1}(B)\}$$

whenever $U \in \mathcal{U}$ ([1], [16]).

Now let (X, d) be a bounded quasi-metric space. Then the Hausdorff quasi-pseudometric d_* on $\mathcal{CL}_o(X)$ is defined by

$$d_*(A, B) = \max\{\sup_{b \in B} d(A, b), \sup_{a \in A} d(a, B)\}$$

whenever $A, B \in \mathcal{CL}_o(X)$ ([1], [16]). This quasi-pseudometric generates on $\mathcal{CL}_o(X)$ the Hausdorff-Bourbaki quasi-uniformity of the quasi-uniform space (X, \mathcal{U}_d) .

Given a quasi-uniform space (X, \mathcal{U}) we denote by $\mathcal{K}_o(X)$ the collection of all nonempty compact subsets of $(X, T(\mathcal{U}))$. Then $\mathcal{K}_o(X) \subseteq \mathcal{CL}_o(X)$ whenever (X, \mathcal{U}) is a Hausdorff quasi-uniform space. In this case the quasi-uniform (hyper)space $(\mathcal{K}_o(X), \mathcal{U}_*|_{\mathcal{K}_o(X)})$ will be

denoted simply by $(\mathcal{K}_o(X), \mathcal{U}_*)$. Similarly, if (X, d) is a Hausdorff bounded quasi-metric space, the quasi-metric (hyper)space $(\mathcal{K}_o(X), d_*|_{\mathcal{K}_o(X)})$ will be denoted simply by $(\mathcal{K}_o(X), d_*)$.

In [2] Burdick characterized uniform local compactness of $(\mathcal{CL}_o(X), \mathcal{U}_*)$ in the case that \mathcal{U} is a uniformity on X , and showed that the Hausdorff uniformity of the Euclidean uniformity on \mathbb{R} is not locally compact on $\mathcal{CL}_o(\mathbb{R})$. However, its restriction to $\mathcal{K}_o(\mathbb{R})$ is uniformly locally compact as our next results show. In this context it seems appropriate to recall that Coban proved in [4] that a uniform space (X, \mathcal{U}) is locally compact if and only if $(\mathcal{K}_o(X), \mathcal{U}_*)$ is locally compact.

In a first attempt to obtain conditions under which the Hausdorff-Bourbaki quasi-uniformity on $\mathcal{K}_o(X)$ preserves uniform local compactness, it seems interesting to recall Example 3.8 in [3], which provides an example of a compact locally symmetric quasi-metric space (X, d) such that $(\mathcal{K}_o(X), d_*)$ is not cofinally complete (so, not uniformly locally compact). However, we shall prove that the problem has an affirmative solution in the setting of small-set symmetric quasi-uniform spaces. The notion of a small-set symmetric quasi-uniform space was introduced in [8]. Let us recall that a quasi-uniform space (X, \mathcal{U}) is small-set symmetric if and only if $T(\mathcal{U}^{-1}) \subseteq T(\mathcal{U})$ [15].

PROPOSITION 3.1. *Let (X, \mathcal{U}) be a small-set symmetric quasi-uniform space. Then $(\mathcal{K}_o(X), \mathcal{U}_*)$ is uniformly locally compact if and only if (X, \mathcal{U}) is uniformly locally compact.*

Proof. Suppose that (X, \mathcal{U}) is a small-set symmetric uniformly locally compact quasi-uniform space. Then there is $W \in \mathcal{U}$ such that $\overline{W(x)}$ is compact for all $x \in X$. Choose $Z \in \mathcal{U}$ such that $Z^6 \subseteq W$. Denote by U the closure of Z with respect to the topology $T(\mathcal{U} \times \mathcal{U}^{-1})$. Then $U \in \mathcal{U}$ and $U \subseteq Z^3$. First we show that for each $K \in \mathcal{K}_o(X)$, $U(K)$ is compact in (X, \mathcal{U}) . Let $K \in \mathcal{K}_o(X)$ and let $(x_\alpha)_{\alpha \in I}$ be a net in $U(K)$. Then there is a net $(a_\alpha)_{\alpha \in I}$ in K such that $(a_\alpha, x_\alpha) \in U$ for all $\alpha \in I$. Furthermore, $(a_\alpha)_{\alpha \in I}$ has a subnet $(b_\beta)_{\beta \in J}$ which converges to a point $b \in K$. So $(x_\alpha)_{\alpha \in I}$ has a subnet $(y_\beta)_{\beta \in J}$ such that $(b_\beta, y_\beta) \in U$ for all $\beta \in J$. Hence, $(y_\beta)_{\beta \in J}$ is eventually in $U^2(b)$, so is eventually in $W(b)$. Therefore $(y_\beta)_{\beta \in J}$ has a cluster point $y \in \overline{W(b)}$. By the small-set symmetry of (X, \mathcal{U}) ,

y is a cluster point of $(y_\beta)_{\beta \in J}$ with respect to $T(\mathcal{U}^{-1})$. Suppose that $(b, y) \notin U$. Then there is $H \in \mathcal{U}$ such that $(H(b) \times H^{-1}(y)) \cap U = \emptyset$, which contradicts that $(a_\alpha, x_\alpha) \in U$ for all $\alpha \in I$. Hence $y \in U(b)$ and thus $U(K)$ is compact in (X, \mathcal{U}) .

Since every compact small-set symmetric quasi-uniform space is uniform [14, proof of Proposition 1], $(U(K), \mathcal{U})$ is a compact uniform space, so $(\mathcal{K}_o(U(K)), \mathcal{U}_*)$ is a compact uniform (hyper)space.

Choose a $V \in \mathcal{U}$ such that $V^2 \subseteq U$. We shall prove that for each $K \in \mathcal{K}_o(X)$, $\overline{V_H(K)}$ is compact in $(\mathcal{K}_o(X), \mathcal{U}_*)$. Let $K \in \mathcal{K}_o(X)$ and let $(A_\alpha)_{\alpha \in I}$ be a net in $\overline{V_H(K)}$. Since (X, \mathcal{U}) is small-set symmetric it follows from [3, Theorem 2.10] that $(\mathcal{K}_o(X), \mathcal{U}_*)$ is small-set symmetric, so $\overline{V_H(K)} \subseteq U_H(K)$. Thus $A_\alpha \in U_H(K)$ for all $\alpha \in I$. Therefore $A_\alpha \subseteq U(K)$, so that $A_\alpha \in \mathcal{K}_o(U(K))$ for all $\alpha \in I$. By the compactness of $(\mathcal{K}_o(U(K)), \mathcal{U}_*)$ it follows that $(A_\alpha)_{\alpha \in I}$ has a cluster point $C \in \mathcal{K}_o(U(K))$. Thus $C \in \mathcal{K}_o(X)$. Furthermore $C \in \overline{V_H(K)}$ because each A_α is in $\overline{V_H(K)}$. We conclude that $\overline{V_H(K)}$ is compact in $(\mathcal{K}_o(X), \mathcal{U}_*)$. Hence $(\mathcal{K}_o(X), \mathcal{U}_*)$ is uniformly locally compact.

Conversely, suppose that $(\mathcal{K}_o(X), \mathcal{U}_*)$ is uniformly locally compact. Then there is a $V \in \mathcal{U}$ such that for each $K \in \mathcal{K}_o(X)$, $\overline{V_H(K)}$ is compact in $(\mathcal{K}_o(X), \mathcal{U}_*)$. We shall show that for each $x \in X$, $\overline{V(x)}$ is compact. In fact, fix $x \in X$ and let $\{x_\alpha : \alpha \in I\}$ be a net in $\overline{V(x)}$. Then $\{\{x_\alpha\} : \alpha \in I\}$ is a net in $\mathcal{K}_o(X)$ such that $\{x_\alpha\} \in \overline{V_H(\{x\})}$ for all $\alpha \in I$. Let C be a cluster point of $\{\{x_\alpha\} : \alpha \in I\}$ in $(\mathcal{K}_o(X), \mathcal{U}_*)$. Then $C \in \overline{V_H(\{x\})}$. Finally fix $c \in C$. It is clear that c is a cluster point of $\{x_\alpha : \alpha \in I\}$ and that $c \in \overline{V(x)}$. This completes the proof. \square

REMARK 3.2. Note that “small-set symmetry” is only used in the proof of the part ‘if’ of Proposition 3.1.

COROLLARY 3.3. Let (X, \mathcal{U}) be a uniform space. Then $(\mathcal{K}_o(X), \mathcal{U}_*)$ is uniformly locally compact if and only if (X, \mathcal{U}) is uniformly locally compact.

A net $\{x_\alpha : \alpha \in I\}$ in a quasi-uniform space (X, \mathcal{U}) is called cofinally Cauchy if for each $U \in \mathcal{U}$ there is an $x \in X$ such that $\{x_\alpha : \alpha \in I\}$ is frequently in $U(x)$ [11]. It is known (and easy to see) that a quasi-uniform space is cofinally complete if and only if every cofinally Cauchy net has a cluster point.

A quasi-metric space (X, d) is called small-set symmetric if (X, \mathcal{U}_d) is a small set symmetric quasi-uniform space. We remark that every small-set symmetric quasi-metric space is metrizable.

PROPOSITION 3.4. *For a small-set symmetric quasi-metric space (X, d) the following statements are equivalent:*

- (1) $(\mathcal{K}_o(X), d_*)$ is uniformly locally compact.
- (2) $(\mathcal{K}_o(X), d_*)$ is cofinally complete.
- (3) (X, d) is uniformly locally compact.

Proof. (1) \Rightarrow (2). [9, Theorem 2.4].

(2) \Rightarrow (3). By [9, Theorem 2.4] it suffices to prove that (X, d) is cofinally complete and locally compact. We first show that (X, d) is cofinally complete.

Let $\{x_\alpha : \alpha \in I\}$ be a cofinally Cauchy net in (X, d) . Then for each $n \in \mathbb{N}$ there is a $y_n \in X$ such that $\{x_\alpha : \alpha \in I\}$ is frequently in $B_d(y_n, 2^{-n})$. Therefore $\{\{x_\alpha\} : \alpha \in I\}$ is a cofinally Cauchy net in $(\mathcal{K}_o(X), d_*)$, so it has a cluster point $C \in \mathcal{K}_o(X)$. Fix $c \in C$. Since for each $n \in \mathbb{N}$ and each $\alpha \in I$ there is a $\beta \geq \alpha$ for which $\{x_\beta\} \in B_{d_*}(C, 2^{-n})$, we deduce that $x_\beta \in B_d(c, 2^{-n})$. We have shown that c is a cluster point of the cofinally Cauchy net $\{x_\alpha : \alpha \in I\}$. Therefore (X, d) is cofinally complete.

Next we show that (X, d) is locally compact.

Assume the contrary. Then there is a $p \in X$ such that each neighborhood of p does not have compact closure. In particular, $\overline{B_d(p, 2^{-1})}$ is not compact, so, by the small-set symmetry of (X, d) , there is a sequence $(x_n)_{n \in \mathbb{N}}$ in $B_d(p, 2^{-1})$ without cluster point in (X, d) . Now we wish to show that for each $n \in \mathbb{N}$, the point $\{p, x_n\} \in \mathcal{K}_o(X)$ has no compact neighborhood in $(\mathcal{K}_o(X), d_*)$.

Suppose that there is some $\{p, x_k\}$ having compact neighborhood. Then there exists $j \in \mathbb{N}$ such that $\overline{B_{d_*}(\{p, x_k\}, 2^{-j})}$ is compact in $(\mathcal{K}_o(X), d_*)$. But in such a case, $\overline{B_d(p, 2^{-j})}$ is compact in (X, d) as we show.

Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $\overline{B_d(p, 2^{-j})}$. Then there is a sequence $(z_n)_{n \in \mathbb{N}}$ in $B_d(p, 2^{-j})$ such that $d^s(y_n, z_n) \rightarrow 0$. Consider the sequence $(\{z_n, x_k\})_{n \in \mathbb{N}}$ of points of $\mathcal{K}_o(X)$. Since $\{z_n, x_k\} \subseteq$

$B_{d_*}(\{p, x_k\}, 2^{-j})$ for all $n \in \mathbb{N}$, the sequence $(\{z_n, x_k\})_{n \in \mathbb{N}}$ has a cluster point C in $(\mathcal{K}_o(X), d_*)$. If $C = \{x_k\}$, we obtain that x_k is a cluster point of the sequence $(z_n)_{n \in \mathbb{N}}$ in (X, d) . Hence it is a cluster point of the sequence $(y_n)_{n \in \mathbb{N}}$. Furthermore $x_k \in \overline{B_d(p, 2^{-j})}$. Otherwise, there is $c \in C \setminus \{x_k\}$, so c is a cluster point of $(z_n)_{n \in \mathbb{N}}$. Therefore c is a cluster point of $(y_n)_{n \in \mathbb{N}}$ and $c \in \overline{B_d(p, 2^{-j})}$.

We conclude that for each $n \in \mathbb{N}$, $\{p, x_n\}$ has no compact neighborhood in $(\mathcal{K}_o(X), d_*)$. Since (X, d) is small-set symmetric, $(\mathcal{K}_o(X), d_*)$ so is. Thus $(\mathcal{K}_o(X), T(d_*))$ is metrizable and, by Corollary 2.2, it admits a cofinally complete metric. Hence, by [17, Theorem 5], the sequence $(\{p, x_n\})_{n \in \mathbb{N}}$ has a cluster point C in $(\mathcal{K}_o(X), d_*)$. If $C = \{p\}$, we obtain that p is a cluster point of $(x_n)_{n \in \mathbb{N}}$, a contradiction. Otherwise, there is $c \in C \setminus \{p\}$. Thus c is a cluster point of $(x_n)_{n \in \mathbb{N}}$, which is a contradiction, again. Therefore (X, d) is locally compact.

(3) \Rightarrow (1). It follows from Proposition 3.1. \square

COROLLARY 3.5. *A metric space (X, d) is uniformly locally compact if and only if $(\mathcal{K}_o(X), d_*)$ is cofinally complete.*

REMARK 3.6. *Note that if (X, d) is a metric space such that $(\mathcal{CL}_o(X), d_*)$ is cofinally complete, then $(\mathcal{K}_o(X), d_*)$ is also cofinally complete. A. Hothi proved that the so-called hedgehog metric space with α spines provides an example of a bounded cofinally complete non locally compact metric space (\mathbb{H}, d) (see [11, p. 96-97]). It follows from Corollary 3.5 that its Hausdorff metric is not cofinally complete not even on $\mathcal{K}_o(\mathbb{H})$. Therefore, the classical result that the Hausdorff metric of any complete metric space (X, d) is complete on $\mathcal{K}_o(X)$ cannot be extended to cofinally complete metric spaces.*

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