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Categorical Aspects of the Theory of Quasi-uniform Spaces

Dedicated to the memory of Doitchin Doitchinov

G. C. L. Brümmer (*)

Summary. - *This is a survey for the working topologist of several categorical aspects of the bicompletion of functorial quasi–uniformities. We consider functors* $F : \mathbf{Top_o} \longrightarrow \mathbf{QU_o}$ *from the* T_o*–topological spaces to the* T_o*–quasi–uniform spaces which endow the* T_o*–spaces with compatible quasi–unformities. Regarding the bicompletion as a functor* $K\colon \mathbf{QU_o} \longrightarrow \mathbf{QU_o}$ *, we ask when the composite* $R = TKF$ *is an epireflection in* $\mathbf{Top_o}$ *and when the equality* $KF = FR$ *holds. Thereby we obtain analogues of important classical results from the theory of uniform spaces. We also present some new results concerning weaker versions of the above questions, e.g. when the pointed endofunctor given by* TKF *can be augmented to a monad. We prove that every epireflective subcategory of* $\mathbf{Top_o}$ *between the subcategory of sober spaces and the subcategory of topologically bicomplete spaces can be obtained from a reflection of the type* TKF*. We give full proofs of all new results and of some less known results whose proofs in the literature are in some way inaccessible. The exposition is intended for readers with little knowledge of category theory.*

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1. Introduction

The rapid growth of the subject of quasi-uniformity — or nonsymmetric topology — over the last twenty years is amply indicated by the monograph of Fletcher and Lindgren [22] and several recent surveys e.g. by Deák [18] and Künzi [34], [35], [32]. There is nevertheless a need also for a fairly detailed survey of the categorical aspects of this theory. Categorical methods are by their nature (naturality) ideally suited for the exploration — or even creation — of the links between the area of quasi-uniformity and those parts of the theory of topology and order that are being applied in theoretical computer science, as well as other parts of topology and analysis. The various completeness notions for quasi-uniform spaces already have ramifications which can only be disentangled by categorical analysis. Consequently a proper categorical survey of the field would be a vast undertaking, and the author has chosen to limit this paper to just one line of development, the one in which he was most involved, that started with his paper [3]. Of the various completeness notions available, only one will be used: bicompleteness, which has the simplest categorical features. Nearly all we do, concerns the action of the bicompletion on functorial quasi-uniformities. Thus in particular we are dealing with natural extensions of T_o-topological spaces, and thereby studying the structure of the category $\mathbf{Top_o}$ of T_o-spaces and continuous maps.

We now explain the terminology and sketch the prerequisites needed for reading this paper.

1.1 For category theory our basic reference is [1], but for many purposes the necessary explanations can also be found in other texts such as [40]. We assume that the reader is comfortable with the notions of category and functor. We shall use quite a lot of natural

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transformations. Given functors $F, G : \mathbf{X} \longrightarrow \mathbf{Y}$ we think of a *natural transformation* $n : F \longrightarrow G$ as a family $(n_X : FX \longrightarrow GX \mid X \in \mathbf{X})$ of morphisms satisfying the *naturality* condition: for any $f : X \longrightarrow X'$ in $\mathbf{X}$, $n_Y.Ff = Gf.n_X$. Natural transformations can be composed with functors on the left and on the right. To see this, consider a functor $L : \mathbf{Y} \longrightarrow \mathbf{B}$. Clearly we get a family $(Ln_X : LFX \longrightarrow LGX \mid X \in \mathbf{X})$ which is written as a natural transformation $Ln : LF \longrightarrow LG$. Similarly, given a functor $R : \mathbf{A} \longrightarrow \mathbf{X}$ we have a family $(n_{RA} : FRA \longrightarrow GRA \mid A \in \mathbf{A})$ which is written as the natural transformation $nR : FR \longrightarrow GR$. We shall also use one kind of composition of natural transformations: If $n : F \longrightarrow G$ and $m : G \longrightarrow H$ are natural transformations of functors $F, G, H : \mathbf{X} \longrightarrow \mathbf{Y}$, the composite $mn : F \longrightarrow H$ is the natural transformation given by $(m_X n_X : FX \longrightarrow HX \mid X \in \mathbf{X})$. (In general we use no composition symbol, but where necessary we insert a dot as a separating device to avoid ambiguity.) The functors L and R, acting now on the composite mn, produce the result $L(mn)R = LmR.\,LnR$.

1.2 The notation $\mathbf{X}(A, B)$ will denote the set of morphisms in the category $\mathbf{X}$ between the objects A and B. Identity morphisms will be denoted by the symbol 1 and identity functors as well as identity natural transformations by the bold $\mathbf{1}$, ambiguity being prevented by the context.

1.3 Subcategories considered will always be full and isomorphism-closed. Such a subcategory $\mathbf{A}$ of a category $\mathbf{X}$ will be called *reflective* if there is an endofunctor $R : \mathbf{X} \longrightarrow \mathbf{X}$ and a natural transformation $r : \mathbf{1} \longrightarrow R$ — in other words $(r_X : X \longrightarrow RX \mid X \in \mathbf{X})$ — such that for each object $X \in \mathbf{X}$:

1. $RX \in \mathbf{A}$;

2. For each $A \in \mathbf{A}$ and each $\mathbf{X}$-morphism $f : X \longrightarrow A$ there exists a unique $\mathbf{X}$-morphism $f^* : RX \longrightarrow A$ with $f^* r_X = f$.

(More basic equivalent definitions of this concept are available — see [1] — but this one best suits our purposes.) We note that the object class of $\mathbf{A}$ is $\{X \in \mathbf{X} \mid r_X \text{ is iso}\}$. One calls the pair (R, r) the *reflection determined* (up to isomorphism) *by* $\mathbf{A}$. If additionally each r_X is an epimorphism/monomorphism/bimorphism/embedding, one says that $\mathbf{A}$ is an *epireflective/monoreflective/bireflective/embedding-reflective* subcategory of $\mathbf{A}$. Every monoreflection is necessarily a bireflection ("bi-" means "mono- and epi-") [1].

1.4 A *pointed endofunctor* in a category $\mathbf{X}$ is a pair (R, r) consisting of an endofunctor $R : \mathbf{X} \longrightarrow \mathbf{X}$ and a natural transformation $r : \mathbf{1} \longrightarrow R$. It is called *well-pointed* if $Rr = rR$; it is called a *prereflection* if for every $\mathbf{X}$-morphism $f : X \longrightarrow Y$ and every $h : RX \longrightarrow RY$ with $hr_X = r_Y f$ it is true that $h = Rf$. Further, the pair (R, r) is called *idempotent* if rR is a natural isomorphism, i.e. at each object X the morphism $r_{RX} : RX \longrightarrow R^2X$ is an isomorphism. Clearly, for any pointed endofunctor (R, r) the following are equivalent:

1. (R, r) is a reflection;
2. (R, r) is an idempotent prereflection [47];
3. (R, r) is idempotent and well-pointed [13].

A basic and rich reference on these matters is [47]. Concerning *monads* we shall only use the most basic notions, which may be found in [1] or [40].

1.5 The smallest epireflective subcategory of $\mathbf{Top_o}$ consists of the singletons and the empty space. The next larger epireflective subcategory of $\mathbf{Top_o}$ is the epireflective hull $\mathbf{Sob}$ of $\{\mathbf{D}_u\}$, where $\mathbf{D}_u$ is the Sierpiński space with open sets $\emptyset, \{0\}, \{0, 1\}$. It is well known (see e.g. [26]) that $\mathbf{Sob}$ consists precisely of the sober spaces. We shall denote its reflection, the sobrification, by (Σ, σ). Every epireflective subcategory of $\mathbf{Top_o}$ which contains $\mathbf{Sob}$ is embedding-reflective. If two epireflective subcategories $\mathbf{A}$ and $\mathbf{A}'$ of $\mathbf{Top_o}$ correspond to reflections (R, r) and (R', r'), and if $\mathbf{Sob} \subseteq \mathbf{A} \subseteq \mathbf{A}'$, then there is a natural embedding $e : R' \longrightarrow R$ with $r = er'$.

1.6 Among the special spaces used are $\mathbb{R}$ with its usual topology and its subspace $\mathbf{I} = [0, 1]$. When equipped with the upper topology, with basic open sets $(\leftarrow, x)$, one has the T_o-spaces $\mathbb{R}_u$, $\mathbf{I}_u$ and in fact also the Sierpiński space $\mathbf{D}_u$. Bispaces which we shall denote by $\mathbb{R}_b$, $\mathbf{I}_b$, $\mathbf{D}_b$ are obtained by taking the upper topology as first and the lower topology as second topology.

1.7 The bireflective hull in $\mathbf{Top_o}$ of the space $\mathbf{I}$ (or just as well of $\mathbb{R}$) is $\mathbf{CregTop}$, the category of completely regular topological spaces. By **Tych** = $\mathbf{CregTop_o}$ we denote the category of completely regular T_o-spaces. The epireflective hull of { **I**} in **Tych** is $\mathbf{CptT_2}$, the sub-category of compact Hausdorff spaces.

1.8 By **2Top** we denote the category of bitopological spaces ("bispaces") and bicontinuous maps. The bireflective hull of $\{\mathbf{I}_b\}$ (or of $\{\mathbb{R}_b\}$) in **2Top** is **Creg2Top**, the category of completely regular bispaces. Imposing T_o-separation on the join of the two topologies of such a bispace makes both topologies T_o, and gives the category $\mathbf{Creg2Top_o}$. The epireflective hull of $\{\mathbf{I}_b\}$ in $\mathbf{Creg2Top_o}$ consists of the "sup"-compact regular T_o-bispaces. We shall denote the corresponding reflection functor by β_b. See [42] and [43]. In passing we shall also refer to **CregPOTop**, the category of completely regular partially ordered spaces ([44], [45], [22]).

1.9 Our blanket reference for quasi-uniform spaces is [22]. These spaces, with quasi-uniformly continuous maps, form the category **QU**. Imposing T_o-separation gives the subcategory $\mathbf{QU_o}$. Taking the usual induced topology, which we call the *first topology*, gives the forgetful functor $T : \mathbf{QU} \longrightarrow \mathbf{Top}$ and its like-named restriction $T : \mathbf{QU_o} \longrightarrow \mathbf{Top_o}$. The literature abounds with quasi-uniformities which are canonically imposed on all the topological spaces in such a way that continuous maps become uniformly continuous. Such a construction amounts to a functor $F : \mathbf{Top} \longrightarrow \mathbf{QU}$ such that $TF = \mathbf{1}$, in other words F is a right inverse or *section* of T, briefly a T*-section*, also called a *functorial quasi-uniformity*. We shall only need the restricted T-sections $F : \mathbf{Top_o} \longrightarrow \mathbf{QU_o}$. The best known examples are:

The (Császár-) Pervin quasi-uniformity functor $=: \mathcal{C}_1^*$
The semicontinuous quasi-uniformity functor $=: \mathcal{C}_1$
The well-monotone covering quasi-uniformity functor $=: W$
The fine transitive quasi-uniformity functor $=: \Phi_t$
The fine quasi-uniformity functor $=: \Phi_1$,

as well as others induced via the Fletcher construction [22, Theorem 2.6] by certain kinds of interior-preserving open covers such as the point-finite or the locally finite open covers. The functorial features of the Fletcher construction are analysed in [9]; it accounts for precisely the class of all transitive T-sections. (Detailed information on the well-monotone functor W is to be found in [37].) Of the above examples only the fine functor Φ_1 fails to be transitive.

1.10 Any section of T : $\mathbf{QU} \longrightarrow \mathbf{Top_o}$ can be obtained by a construction called *spanning*, first used in this context in [3], and further discussed in [4], [5], [7], [8] and [9].

Let $\mathbf{A}$ be any class of quasi-uniform spaces. For any $X \in \mathbf{Top}$ a quasi-uniform space FX is obtained by putting on X the *initial* (i.e. coarsest) quasi-uniformity which renders all continuous maps $X \longrightarrow TA$ $(A \in \mathbf{A})$ uniformly continuous into the corresponding A. For any continuous $f : X \longrightarrow Y$ we let $Ff : FX \longrightarrow FY$ be the same function as f. Hereby a functor $F : \mathbf{Top} \longrightarrow \mathbf{QU}$ has been set up. We call F the *functor spanned by* $\mathbf{A}$, writing $F = \langle \mathbf{A} \rangle$. For F to be a T-section, i.e. for the constructed quasi-uniformity on every $X \in \mathbf{Top}$ to be compatible with the topology, it is necessary and sufficient that X have the initial topology for all the continuous maps $X \longrightarrow TA$ $(A \in \mathbf{A})$, i.e. that the class $T[\mathbf{A}] := \{TA \mid A \in \mathbf{A}\}$ be initially dense ([4], [1]) in $\mathbf{Top}$. More about initial sources can be found in [1]. We have a partial ordering of T-sections, written $F \leq G$ iff FX is coarser than GX (also written $FX \leq GX$) for each $X \in \mathbf{Top}$. The coarsest T-section is $\mathcal{C}_1^*$, the finest Φ_1. There are at least as many T-sections as there are infinite cardinal numbers, and the class of all T-sections has the structure of a complete lattice [3].

1.11 We have a forgetful functor T_b : $\mathbf{QU_o} \longrightarrow \mathbf{Creg2Top_o}$ which takes the first and second topologies. We shall need the special quasi-uniform space $\mathbf{I}_q$ which is [0, 1] with the upper quasi-uniformity, satisfying $T_b\mathbf{I}_q = I_b$. In context, the forgetful functors $\mathbf{Unif_o} \longrightarrow \mathbf{Tych}$ and $\mathbf{QU_o} \longrightarrow \mathbf{CregPOTop}$ will both be denoted by T. Furthermore, there is a useful functor S : $\mathbf{Creg2Top} \longrightarrow \mathbf{CregTop}$, called the symmetriser or supremum functor, which takes the supremum of the two topologies of a bispace. There is a corresponding symmetriser $s : \mathbf{QU} \longrightarrow \mathbf{Unif}$, in fact the uniform coreflection, which takes the supremum of a quasi-uniformity and its inverse. One has $Ts = ST_b$.

1.12 It is well known that the epimorphisms in $\mathbf{CregTop_o}$ are the dense maps (i.e. continuous maps with dense image). Salbany's crucial discovery, which led to his categorically motivated theory of bispaces in [41], was that a map $f : X \longrightarrow Y$ in $\mathbf{Creg2Top_o}$ is epimorphic if and only if $f[X]$ is dense in SY (one can say : f is

S-dense). From this he deduced, among other things, the known fact that the epimorphisms in $\mathbf{Top_o}$ are the b-dense maps ([41], see also [5]). It has been known for some time that a map $f : X \longrightarrow Y$ in $\mathbf{QU_o}$ is epi if and only if $f[X]$ is dense in TsY. The first published proof is in [20]. See also [19].

1.13 A quasi-uniform space X is *bicomplete* if the uniform space sX is complete. The bicomplete T_o-quasi-uniform spaces form an embedding-reflective (hence epireflective) subcategory of $\mathbf{QU_o}$. We shall denote the reflection by (K, k). Two particular features of the notion of bicompleteness, which distinguish it as the simplest of the several completeness notions for quasi-uniform spaces, are:

1. For any $X \in \mathbf{QU_o}$ one has : X is bicomplete if and only if X is injective with respect to the class of all epimorphic embeddings in $\mathbf{QU_o}$ [12].
2. For any map $f : X \longrightarrow Y$ in $\mathbf{QU_o}$, the map $Kf : KX \longrightarrow KY$ is an isomorphism if and only if f is an epimorphic quasi-uniform embedding [11].

In these respects, bicompleteness in $\mathbf{QU_o}$ is strictly analogous to classical completeness in $\mathbf{Unif_o}$. The property (2) implies the uniqueness of bicompletions, and plays a role in the directness of the reflection (K, k) – see the discussion following Definition 3.10, or [13].

2. Completion of functorial uniformities

A well-known fact was stated thus in [23, Theorem 15.13]:

Let X be a completely regular space [i.e. Tychonoff space].

(a) The completion of X in the uniform structure $\mathcal{C}(X)$ is $[\upsilon X; \mathcal{C}(\upsilon X)]$.

(a) The completion of X in the uniform structure $\mathcal{C}^*(X)$ is $[\beta X; \mathcal{C}^*(\beta X)]$.*

Let $T : \mathbf{Unif_o} \longrightarrow \mathbf{Tych}$ be the forgetful functor. The uniform structures mentioned above are given by two T-sections, which we shall also denote by $\mathcal{C}$ and $\mathcal{C}^*$, spanned respectively by $\mathbb{R}$ and [0, 1], each with the usual uniformity. If we denote by (K, k) the completion in $\mathbf{Unif_o}$, the above results read as follows:

$$KCX = CvX;\quad KC^*X = C^*\beta X$$

with v and β the realcompact and compact reflection functors, respectively. Since the equations hold for all $X \in$ **Tych**, we have equivalently:

$$KC = Cv;\quad KC^* = C^*\beta.$$

Applying the functor T and using $TC = TC^* = \mathbf{1}$ we get

$$TKC = v;\quad TKC^* = \beta$$

and substituting these back into the previous equations, we have

$$KC = CTKC;\quad KC^* = C^*TKC^*.$$

With Φ denoting the finest T-section one similarly has:

$$K\Phi = \Phi TK\Phi.$$

Here $TK\Phi$ is the Dieudonné completion functor, i.e. the reflector in **Tych** onto the subcategory of topologically complete spaces.

Definition 2.1. *[7] A T-section F is called* completion-true*, or more briefly* K-true*, if $KF = FTKF$.*

The following facts are known about this concept.

Theorem 2.2. *Let F be a section of T:* $\mathbf{Unif_o} \longrightarrow$ **Tych**.

1. *F is K-true if and only if F is spanned by a class of complete uniform spaces [7].*
2. *Always $KF \geq FTKF$, i.e. KF is finer than $FTKF$ [33].*
3. *The subcategory $\mathbf{E}(F)$ of* **Tych** *with object class $\{X \in$ **Tych** $\mid FX$ is complete$\}$ is epireflective in* **Tych** *[14].*
4. *If F is K-true, then $(TKF,\ TkF)$ is the reflection onto $\mathbf{E}(F)$ [7], [14].*
5. *If F is spanned by a class $\mathbf{A}$ of complete uniform spaces (see (1) above), then $\mathbf{E}(F)$ is the epireflective hull of $T[\mathbf{A}]$ in* **Tych** *[7].*
6. *There exists a T-section F for which $(TKF,\ TkF)$ is not a reflection [14].*

7. $\mathbf{CptT_2} \subseteq \mathbf{E}(F) \subseteq \mathbf{Topcpl}$, *where* $\mathbf{CptT_2}$ *denotes the category of compact Hausdorff spaces and* **Topcpl** *that of topologically complete Tychonoff spaces [14].*

8. *If* $\mathbf{B}$ *is any epireflective subcategory of* **Tych** *with* $\mathbf{CptT_2} \subseteq \mathbf{B} \subseteq \mathbf{Topcpl}$, *then there exists a K-true T-section F with* $\mathbf{B} = \mathbf{E}(F)$, *e.g.* $F = \langle \Phi[\mathbf{B}] \rangle$ *[14].*

We note further that for two T-sections F and G, $F \leq G \Longrightarrow \mathbf{E}(F) \subseteq \mathbf{E}(G)$. Also, $\mathbf{E}(\mathcal{C}^*) = \mathbf{CptT_2}$ and $\mathbf{E}(\Phi) = \mathbf{Topcpl}$. A given epireflective subcategory $\mathbf{B}$ of **Tych** can be of the form $\mathbf{E}(F)$ for many different K-true T-sections F.

3. Bicompletion of functorial quasi-uniformities

Throughout this section we consider the forgetful functor T: $\mathbf{QU_o} \longrightarrow \mathbf{Top_o}$, and (K, k) will denote the bicompletion in $\mathbf{QU_o}$. We aim at obtaining analogues of the uniform results mentioned in Section 2. The most striking difference is the failure of the result 2.2(2) to carry over: in the quasi-uniform setting, KF need not be finer than $FTKF$. We shall see that those T-sections which do satisfy this inequality have a behaviour quite analogous to the uniform paradigm, while those which do not satisfy it, give rise to a range of new phenomena.

Definition 3.1. *[10]. Let F be a section of* $T : \mathbf{QU_o} \longrightarrow \mathbf{Top_o}$.

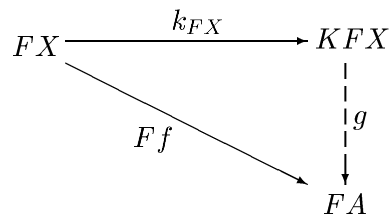
1. *F is* bicompletion-true, *or just K*-true, *if $KF = FTKF$.*
2. *F is* upper *K-true if $KF \geq FTKF$.*
3. *F is* lower *K-true if $KF \leq FTKF$.*
4. $\mathbf{E}(F)$ *is the subcategory of* $\mathbf{Top_o}$ *with object class* $\{X \in \mathbf{Top_o} \mid FX$ *is bicomplete*$\}$.

Trivially, F is K-true iff it is both upper and lower K-true.

Proposition 3.2. *If a T-section F is K-true, then* $(R, r) := (TKF,\ TkF)$ *is an epireflection of* $\mathbf{Top_o}$ *to the subcategory* $\mathbf{E}(F)$.

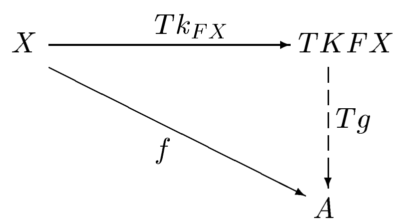
Proof. For $X \in \mathbf{Top_o}$, $TKFX \in \mathbf{E}(F)$ since $F(TKFX) = KFX$ which is bicomplete. For $A \in \mathbf{E}(F)$, FA is bicomplete. Consider any continuous $f : X \longrightarrow A$.

(A)



Then $Ff : FX \longrightarrow FA$ is uniformly continuous, and there exists unique $g : KFX \longrightarrow FA$ making diagram (A) commute. The image of diagram (A) under T is the commutative diagram (B). To prove reflectiveness we only have to show that Tg is the unique map which makes diagram (B) commute.

(B)



Consider any continuous map $h : TKFX \longrightarrow A$ such that $h.Tk_{FX} = f$. Applying the functor F gives the equation $Fh.FTk_{FX} = Ff$. The map FTk_{FX} has the same domain (FX) and the same codomain $(FTKFX = KFX)$ as the map k_{FX}; moreover $T(FTk_{FX}) = T(k_{FX})$, and therefore FTk_{FX} is the same map as k_{FX}. Thus $Fh.k_{FX} = Ff$, and by the uniqueness of the map g in diagram (A) we have $Fh = g$, and therefore $h = T(Fh) = Tg$, which proves the uniqueness. Lastly, the epireflectivity now follows from the categorical result in 1.3 since each reflection map Tk_{FX} is an embedding, hence mono. $\square$

Proposition 3.3. *[10, Theorem 6.1 and Example 6.4 (4)]. A section F of $T : \mathbf{QU_o} \longrightarrow \mathbf{Top_o}$ is lower K-true if and only if F is* *spanned by a class of bicomplete quasi-uniform spaces.*

EXAMPLE 3.4. *The Császár-Pervin T-section $\mathcal{C}_1^*$ is lower K-true since it is spanned by the bicomplete Sierpiński quasi-uniform space. We shall see that $\mathcal{C}_1^*$ is far from being upper K-true. It follows from [26, Corollary 3.2] that $\mathbf{E}(\mathcal{C}_1^*)$ is the class of all sober hereditarily compact spaces.*

EXAMPLE 3.5. *In [37, Proposition 4 and Corollary 4] it is shown that the well-monotone covering quasi-uniformity functor W is bicompletion-true, with the sobrification reflection (Σ, σ) in $\mathbf{Top_o}$ as induced reflection:*

$$(TKW,\ TkW) = (\Sigma, \sigma).$$

Thus a T_o-space X is sober if and only if WX is bicomplete. Writing **Sob** *for the full subcategory of sober spaces of $\mathbf{Top_o}$, we have $\mathbf{E}(W)$ = **Sob**.*

EXAMPLE 3.6. *In [37, Corollary 5] it is proved that the fine quasi-uniformity functor Φ_1 is bicompletion-true. The epireflective subcategory $\mathbf{E}(\Phi_1)$ of $\mathbf{Top_o}$ consists of those T_o-spaces whose fine quasi-uniformity is bicomplete, or equivalently (by [37, Corollary 1]) which admit a bicomplete quasi-uniformity. We shall call these spaces* topologically bicomplete *and shall denote the category by*

$$\mathbf{TopBiCpl} := \mathbf{E}(\Phi_1).$$

Künzi and Ferrario give useful information about this category: e.g. an uncountable cofinite space does not belong to it. However, a practical characterisation of the topologically bicomplete spaces still does not seem to be known.

THEOREM 3.7. *For a T-section $F : \mathbf{Top_o} \longrightarrow \mathbf{QU_o}$ the following are equivalent.*

1. *$KF \geq FTKF$, i.e. F is upper K-true.*
2. *$KF \geq GTKF$ for some (every) T-section $G \leq F$.*
3. *TkF is objectwise epi (i.e. Tk_{FX} is a $\mathbf{Top_o}$-epimorphism for each X in $\mathbf{Top_o}$).*
4. *(TKF, TkF) is a prereflection.*

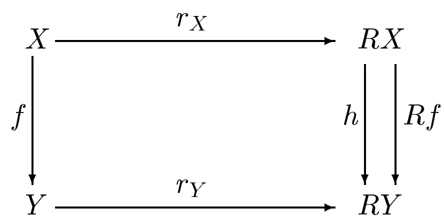
5. *(TKF, TkF) is well-pointed.*

6. *F is finer than the well-monotone covering quasi-uniformity functor W.*

7. *For every $X \in \mathbf{Top_o}$, the extension $Tk_{FX} : X \longrightarrow TKFX$ factors into the sobrification $\sigma_X : X \longrightarrow \Sigma X$ via a natural embedding $TKFX \longrightarrow \Sigma X$.*

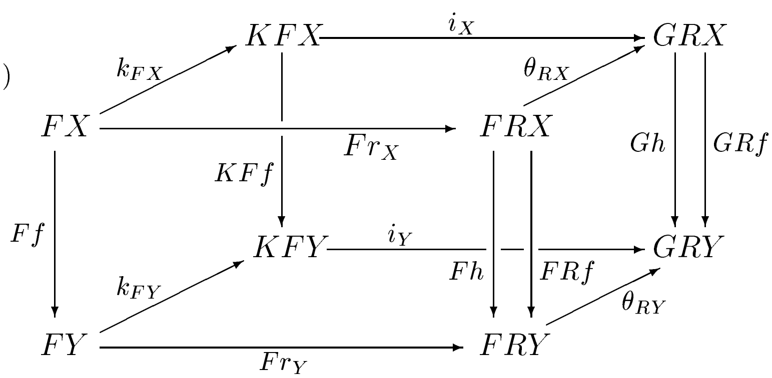
Proof. We shall as always abbreviate $(TKF, TkF) =: (R, r)$.
(1) $\Longrightarrow$ (2): This is trivial for every $G \leq F$.
(2) $\Longrightarrow$ (4): We have $i : KF \longrightarrow GR$ with $Ti = \mathbf{1}$ and $\theta : F \longrightarrow G$ with $T\theta = \mathbf{1}$. Consider $f : X \longrightarrow Y$ in $\mathbf{Top_o}$. Assume that the inner rectangle of diagram (C) commutes.

(C)



We only have to show that $h = Rf$. We let F map (C) to the front face of the cubic diagram (D).

(D)



In the front face the inner and outer squares commute. The bottom square commutes since T is faithful and $T\theta = \mathbf{1}$ and $Ti = \mathbf{1}$.

The top square commutes for the same reasons. The left hand face commutes by naturality of k. In the right hand face, the inner and outer squares commute since $T\theta = \mathbf{1}$ and T is faithful.

Back face, outer square: $GRf.i_X = i_Y.KFf$ since $Ti = \mathbf{1}$ and T is faithful. Furthermore,
$Gh.i_X.k_{FX} = Gh.\theta_{RX}.Fr_X = \theta_{RY}.Fh.Fr_X = \theta_{RY}.Fr_Y.Ff = i_Y.k_{FY}.Ff = i_Y.KFf.k_{FX}$. Since k_{FX} is epi, $Gh.i_X = i_Y.KFf$. Apply $T : h = TKFf = Rf$, as required.

(4) $\Longrightarrow$ (5): This is a well-known and simple categorical fact [47].

(5) $\Longrightarrow$ (1): [10, Theorem 1.4].

Thus far we have (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5).

(1) $\Longrightarrow$ (6): [34, Proposition 10].

(6) $\Longrightarrow$ (7): [37, Proposition 5] proves the embedding, and naturality is clear from the proof given.

(7) $\Longrightarrow$ (3): The sobrification $\sigma_X : X \longrightarrow \Sigma X$ is well known to be a $\mathbf{Top_o}$-epimorphism, i.e. dense with respect to the b-topology on ΣX. By (7) $TKFX$ lies between X and ΣX, so X is b-dense in $TKFX$, i.e. Tk_{FX} is a $\mathbf{Top_o}$-epimorphism.

(3) $\Longrightarrow$ (4): Trivial categorial fact.

Herewith all seven properties are equivalent. □

EXAMPLE 3.8. *It is known [22] that the bicompletion of a transitive quasi-uniform space is again transitive. Therefore the fine transitive quasi-uniformity functor* Φ_t *satisfies the inequality* $K\Phi_t \leq \Phi_t TK\Phi_t$, *i.e. is lower* K*-true. Trivially* $\Phi_t \geq W$. *Thus by Theorem 3.7* Φ_t *is* K*-true. The epireflective subcategory* $\mathbf{E}(\Phi_t)$ *of* $\mathbf{Top_o}$ *is characterised in [37, Corollary 2; see also Corollary 3]. Characterisations are also given in [31, Remark 2.4.12], e.g.:* $\mathbf{E}(\Phi_t)$ *is the epireflective hull in* $\mathbf{Top_o}$ *of the class of Alexandroff-discrete* T_o*-spaces.*

EXAMPLE 3.9. *The Császár-Pervin quasi-uniformity functor* $\mathcal{C}_1^*$ *and the semicontinuous quasi-uniformity functor* $\mathcal{C}_1$ *both fail to be finer than* W, *i.e. fail to be upper* K*-true.*

The implication (6) $\Longrightarrow$ (7) in Theorem 3.6 is a special case of the following result, of which various cases have occurred in the literature, e.g. in [6, Proposition 5.8], [37, Proposition 5] and [31, Proposition 2.3.8].

PROPOSITION 3.10. *Let F, G : $\mathbf{Top_o} \longrightarrow \mathbf{QU_o}$ be T-sections with $F \geq W$, i.e. F finer than the well-monotone covering quasi-uniformity functor.*

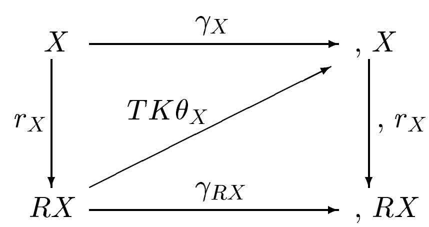
(1) If $F \geq G$, then the extension $X \longrightarrow TKFX$ is naturally embedded in the extension $X \longrightarrow TKGX$.

(2) In particular we have the following natural embeddings: $TKFX \hookrightarrow \Sigma X \hookrightarrow \beta_1 X$ where ΣX denotes the sobrification and $\beta_1 X$ the one-sided Stone-Čech compactification of X.

Proof. (1) Since $F \geq G$ we have $\theta : F \longrightarrow G$ with $T\theta = \mathbf{1}$. We abbreviate $(TKG,\ TkG) = (,\ ,\gamma)$ and as always $(TKF,\ TkF) = (R, r)$.

Applying the bicompletion (K, k) to the arrow $\theta_X : FX \longrightarrow GX$ we obtain $K\theta_X.k_{FX} = k_{GX}.\theta_X$ by the naturality of k. Using $T\theta_X = 1_X$ we deduce $TK\theta_X.Tk_{FX} = Tk_{GX}$, i.e. $TK\theta_X.r_X = \gamma_X$, i.e. the upper triangle in diagram (E) commutes.

(E)



The outer rectangle commutes by naturality of γ. We have thus

$$,\ r_X.TK\theta_X.r_X = ,\ r_X.\gamma_X = \gamma_{RX}.r_X$$

in which r_X is epi by the above Theorem ((6) $\Longrightarrow$ (3)). Cancellation of r_X gives

$$,\ r_X.TK\theta_X = \gamma_{RX}.$$

Since γ_{RX} is an embedding, so is $TK\theta_X$, which thus naturally embeds $TKFX$ into $TKGX$.

(2) This follows from (1) since $\Sigma X = TKWX$, $\beta_1 X = TK\mathcal{C}_1^* X$ and $F \geq \mathcal{C}_1^*$. □

REMARK 3.11. *(1) Let for the moment T be any one of the forgetful functors* $\mathbf{Unif_o} \longrightarrow \mathbf{CregTop_o}$, $\mathbf{QU_o} \longrightarrow \mathbf{Creg2Top_o}$, $\mathbf{QU_o} \longrightarrow \mathbf{CregPOTop_o}$, *with (K,k) the completion in* $\mathbf{Unif_o}$ *respectively the bicompletion in* $\mathbf{QU_o}$. *In these three settings T preserves epimorphisms, so that in the proof of Proposition 3.10 part (1) the morphism r_X is again epi and can be cancelled. This gives the following result without any additional assumptions:*

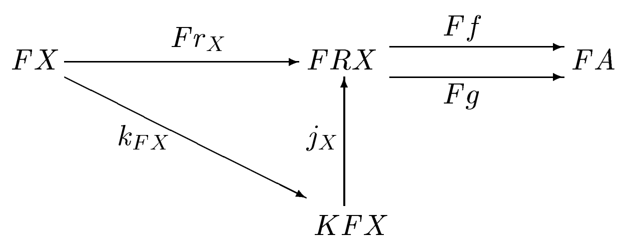
> *If F and G are T-sections with $F \geq G$, then the extension $X \longrightarrow TKFX$ is naturally embedded into the extension $X \longrightarrow TKGX$.*

(2) In Theorem 3.7 the conditions (1), (2), (3), (4), (5) are of a categorical form which makes sense for any forgetful functor $T : \mathbf{Y} \longrightarrow \mathbf{X}$, T-section $F : \mathbf{X} \longrightarrow \mathbf{Y}$ and pointed endofunctor (K,k) in $\mathbf{Y}$. In [10, Theorem 1.4] it was shown that (1) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) holds in any abstract categorical setting that satisfies the following assumptions:

> *(*) Let T be faithful and let k_{FX} be a T-initial epimorphism in $\mathbf{Y}$ for every X in $\mathbf{X}$.*

Our proof of Theorem 3.6 above already establishes (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) for any categorical setting satisfying (*). We also have (3) $\Longrightarrow$ (4) trivially, and it remains to show (1) $\Longrightarrow$ (3). With our usual abbreviation $(TKF, TkF) = (R, r)$ we assume $KF \geq FR$ and we have to show r_X epi in $\mathbf{X}$ for each X in $\mathbf{X}$. We have $j : KF \longrightarrow FR$ with $Tj = \mathbf{1}$. Let $fr_X = gr_X$ for $f, g : RX \longrightarrow A$. It remains to be shown that $f = g$.

(F)



One sees that the diagram (F) commutes by applying the faithful functor T to it. Thus

$$Ff.(j_X.k_{FX}) = Fg.(j_X.k_{FX}).$$

But j_X is epi since $Tj_X = 1_X$, and k_{FX} is epi by (*). Thus $Ff = Fg$, and $f = T(Ff) = T(Fg) = g$, so that r_X is epi.

We have established the following result:

> *in any categorical setting satisfying the assumptions (*), the conditions (1), (2), (3), (4), (5) of Theorem 3.7 are equivalent.*

(3) In particular, for the settings $\mathbf{Unif_o} \longrightarrow \mathbf{CregTop_o}$, $\mathbf{QU_o} \longrightarrow \mathbf{Creg2Top_o}$, $\mathbf{QU_o} \longrightarrow \mathbf{CregPOTop_o}$ the conditions (1), (2), (3), (4), (5) of Theorem 3.7 are not only equivalent, but they hold unconditionally for every T-section F. (This follows again because T preserves epis in each of these settings, so that Tk_{FX} is always epi.)

(4) The embedding $\Sigma X \hookrightarrow \beta_1 X$ (see Proposition 3.10 (2) above) reduces to an isomorphism if and only if X is hereditarily compact [36, Theorem 3].

Our next major objective is to present necessary and sufficient conditions on a section F of T: $\mathbf{QU_o} \longrightarrow \mathbf{Top_o}$ so that (TKF, TkF) will be a reflection in $\mathbf{Top_o}$. For this we need some preparations.

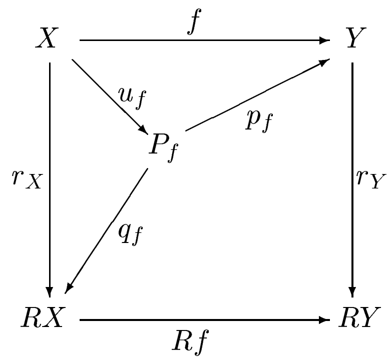
DEFINITION 3.12. *A pointed endofunctor (R, r) in a category $\mathbf{X}$ is called* direct *if for every $f : X \longrightarrow Y$ in $\mathbf{X}$ the following hold:*

1. *The pullback (P_f, p_f, q_f) of Rf against r_Y exists (see commutative diagram below);*
2. *The unique morphism $u_f : X \longrightarrow P_f$ satisfying $p_f u_f = f$ and $q_f u_f = r_X$ is also such that Ru_f is an isomorphism.*

The above definition is due to Brümmer and Giuli (1993) and is explored in [29], [30], [13] and [28]. In case (R, r) is a reflection and the category $\mathbf{X}$ has pullbacks, then (R, r) is direct if and only if it is a simple reflection in the sense of [17].

Many of the well-known reflections occurring in general topology are direct. Examples are given in [13]. In particular the compact and the realcompact reflections in $\mathbf{Tych}$ are direct, the T_0-reflection in $\mathbf{Top}$ is direct, the sobrification in $\mathbf{Top_o}$ is direct, the totally bounded reflections in $\mathbf{Unif_o}$ and in $\mathbf{QU_o}$ are direct, and so are the completion

(G)



in $\mathbf{Unif_o}$ and the bicompletion in $\mathbf{QU_o}$, as well as the Samuel compactification in $\mathbf{Unif_o}$ and the Samuel "bi"-compactification in $\mathbf{QU_o}$. Directness is closely linked to the theory of perfect morphisms relative to a pointed endofunctor (R, r) in a category $\mathbf{X}$. In fact a morphism is called (R, r)-*perfect* ([29], [30], [28], [13]) if f is a pullback of Rf (i.e. the diagram $r_Y f = Rf.r_X$ is a pullback). This idea goes back to Herrlich [24], who showed that a map f in **Tych** is perfect in the usual sense if and only if f is a pullback of its Stone-Čech extension βf. Herrlich also observed that in **Tych** one has the morphism factorisation structure ({compact-extendible dense}, {perfect}) [25], and he duly generalised this observation. This phenomenon in fact characterises directness: If a pointed endofunctor (R, r) is idempotent, then (R, r) is direct $\Longleftrightarrow$ ($L(R)$,$\{(R, r)$-perfect$\}$) is a factorisation structure for morphisms in $\mathbf{X}$ [30]. Here $L(R)$ denotes the class $\{g \in \text{Mor } \mathbf{X} \mid Rg \text{ is an isomorphism}\}$. If (R, r) is a reflection, and if we let $\mathbf{R}\colon = \text{Fix}(R, r)$, then $L(R)$ coincides with the class of all $\mathbf{R}$-dense $\mathbf{R}$-extendible morphisms [11]. (A morphism $g : X \longrightarrow Y$ is $\mathbf{R}$-*dense* if for all $s, t : Y \longrightarrow A$, with $A \in \mathbf{R}$, $sg = tg$ implies $s = t$. It is $\mathbf{R}$-*extendible* if for all $h : X \longrightarrow A$, with $A \in \mathbf{R}$, there exists $h^* : Y \longrightarrow A$ with $h^* g = h$.)

The following result adds three equivalent conditions to Example 5.10 of [13].

PROPOSITION 3.13. *Let F be any section of $T : \mathbf{QU_o} \longrightarrow \mathbf{Top_o}$ and let $(R, r) := (TKF, TkF)$. The following conditions are equivalent.*

1. *(R, r) is a reflection in $\mathbf{Top_o}$.*
2. *(R, r) is direct and $F \geq W$.*

3. (R, r) *is idempotent and* $F \geq W$.

4. Rr *is a natural isomorphism and* $F \geq W$.

5. (R, r) *can be augmented to a monad and* $F \geq W$.

Proof. (1) $\Longleftrightarrow$ (2): This is proved in [13, Example 5.10]. (We recall from Theorem 3.7 that $F \geq W$ iff F is upper K-true.)
(1) $\Longleftrightarrow$ (3): This follows at once from [13, Proposition 1.2] since (R, r) is well-pointed iff $F \geq W$ (Theorem 3.7).
(3) $\Longleftrightarrow$ (4): Immediate, since $F \geq W \Longrightarrow Rr = rR$.
(1) $\Longrightarrow$ (5): It is well known that a reflection (R, r) gives rise to a monad $(R, r, (rR)^{-1})$.
(5) $\Longrightarrow$ (3): This follows at once from [47, Corollary 1] again since $F \geq W$ implies that (R, r) is well-pointed. □

REMARK 3.14. *In the above Proposition there is one ingredient that is easily manageable and well understood, namely the condition* $F \geq W$ *with its various equivalents given by Theorem 3.7. All the other ingredients lack usable characterisations. Since every* T*-section* F *can be defined by a spanning class (in many ways, see e.g. [15]), one may require conditions in terms of some spanning class for* F*, in order that* (R, r) *be a reflection, or that* (R, r) *be direct, or idempotent, or that* Rr *be iso, or that* (R, r) *can be augmented to a monad. The author does not know such conditions. Moreover, one would like to know what implications exist among the five mentioned conditions. (The analogous problems for* $\mathbf{Unif_o}$ *are as open as they are for* $\mathbf{QU_o}$*.)*

REMARK 3.15. *One of the desired implications is: If* (R, r) *is direct, then* Rr *is a natural isomorphism. To see this, take the special case of Diagram (G) in the definition of directness where* Y *is a singleton. Then* r_Y *is iso and its pullback* q_f *is iso. Directness says that* Ru_f *is iso; since* $r_X = q_f u_f$, Rr_X *is iso.*

REMARK 3.16. *The above Proposition has an immediate analogue for the sections of the forgetful functors* $\mathbf{Unif_o} \longrightarrow \mathbf{CregTop_o}$, $\mathbf{QU_o} \longrightarrow \mathbf{Creg2Top_o}$, $\mathbf{QU_o} \longrightarrow \mathbf{CregPOTop_o}$.
In fact one only has to delete the condition $F \geq W$*, because in these settings every* T*-section is upper* K*-true [10].*

REMARK 3.17. *Kimmie [31, pp. 71–77] has constructed a functorial quasi-uniformity F which fails to be lower K-true, but for which $(TKF,\ TkF)$ coincides with the sobrification reflection (Σ, σ), so that F is in fact upper K-true. In his construction Kimmie used a superrigid space due to Van Douwen.*

REMARK 3.18. *In [6] an example due to Salbany was given which showed that the pointed endofunctor $(\beta_1, \eta) := (TK\mathcal{C}_1^*,\ Tk\mathcal{C}_1^*)$ induced by the Császár-Pervin quasi-uniformity is not a reflection, in fact not idempotent. Salbany discusses it further in [43, p. 491]. He considers the space $\mathbf{N}_u$ of the natural numbers with the upper topology, i.e. having basic open sets of the form $[1, n]$, and obtains $\beta_1\mathbf{N}_u$ by adjoining a point at infinity to the right of $\mathbf{N}_u$. Then he produces the iteration $\beta_1(\beta_1\mathbf{N}_u) = \beta_1^2\mathbf{N}_u$ by adding a second point at infinity to the right of $\beta_1\mathbf{N}_u$. One sees immediately from Salbany's construction that $\beta_1^2\mathbf{N}_u$ is not homeomorphic to $\beta_1\mathbf{N}_u$. Therefore in particular the natural transformation $Rr := \beta_1\eta$ fails to be an isomorphism at the object $\mathbf{N}_u$, and therefore from Remark 3.15(2) above we have:*

The one-sided Stone-Čech compactification (β_1, η) in $\mathbf{Top_o}$ fails to be direct.

This tells us that if $(R, r) := (TKF, TkF)$ is induced by a lower K-true F, neither Rr nor rR need be an isomorphism, nor need (R, r) be direct.

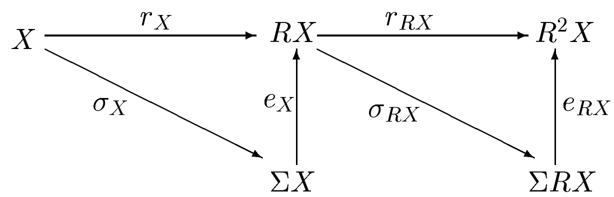
The following result extends Salbany's example.

PROPOSITION 3.19. *Let $(R, r)\ :\ =\ (TK\mathcal{C}_1^*,\ Tk\mathcal{C}_1^*)$ and let $X \in \mathbf{Top_o}$. Then, r_{RX} is an iso (i.e. homeomorphism) if and only if X is hereditarily compact.*

Proof. By Proposition 3.10(2) the sobrification (Σ, σ) admits a natural embedding e into (R, r).

By [36, Theorem3], for any $Y \in \mathbf{Top_o}$, ΣY coincides with RY, i.e. e_Y is iso, if and only if Y is hereditarily compact. Assume that r_{RX} is iso. Since $r_{RX} = e_{RX}\sigma_{RX}$, the embedding e_{RX} is then surjective, hence iso. This means that RX is hereditarily compact. But X is embedded into RX, and thus X is hereditarily compact. Conversely, if X is hereditarily compact, so is ΣX (this well known fact

(H)



follows merely from the b-density) and we also know that e_X is iso. Thus RX is sober, and σ_{RX} is iso. Again since e_X is iso, the hereditary compactness of ΣX implies that of RX and hence e_{RX} is iso. Thus $r_{RX} = e_{RX}\sigma_{RX}$ is iso. □

Remark 3.20. *In view of condition (5) of Proposition 3.13 it is of interest to note that every lower K-true functorial quasi-uniformity F induces a monad (TKF, TkF, μ) in* $\mathbf{Top_o}$. *This was in effect proved in [6], modulo terminology. That the converse result does not hold follows from Kimmie's example cited in 3.17(4) above. We shall now present a streamlined version of the construction in [6]. A filter-theoretic construction of the monad is given in [31] for the special case of transitive lower K-true F.*

Proposition 3.21. *If F is a lower K-true section of $T : \mathbf{QU_o} \longrightarrow \mathbf{Top_o}$, then $(TKF,\ TkF)$ can be augmented to a monad.*

Proof. The construction is achieved via an extension of $\mathbf{Top_o}$ to the category $\mathbf{Creg2Top_o}$ of completely regular T_0 bitopological spaces. The procedure is successful because the obvious forgetful functor $T_b : \mathbf{QU_o} \longrightarrow \mathbf{Creg2Top_o}$ (assigning first and second topologies) preserves epimorphisms (since sup-dense maps go to sup-dense maps). We consider the first topology functor $E : \mathbf{Creg2Top_o} \longrightarrow \mathbf{Top_o}$ which forgets the second topology. Salbany ([41], [42]) discovered that E has precisely one right inverse, $Q : \mathbf{Top_o} \longrightarrow \mathbf{Creg2Top_o}$, given by

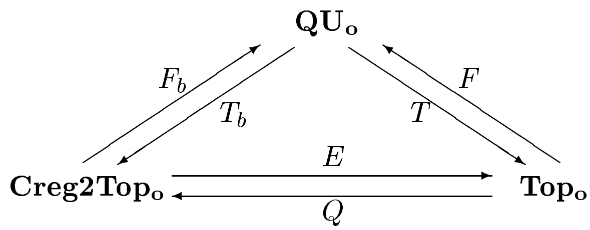
$$Q(X, \mathcal{T}) = (X, \mathcal{T}, \mathcal{T}^*)$$

where $\mathcal{T}^*$ is the topology for which the closed sets of $\mathcal{T}$ form a base. (A more accessible proof of the uniqueness of Q is given in [5].) Moreover, Q is left adjoint to E, so that we have an adjunction

$(Q, E, 1, i)$ where the co-unit $i : QE \to \mathbf{1}$ is given by QE being finer than the identity. Note that $ET_b = T$.

We consider any lower K-true T-section $F : \mathbf{Top_o} \longrightarrow \mathbf{QU_o}$.

(J)



The functor F has the canonical spanning class $KF[\mathbf{Top_o}]$ with respect to T [10, Theorem 6.1]. Then $KF[\mathbf{Top_o}] \cup \{\mathbf{I}_q\}$ is a class of bicomplete quasi-uniform spaces which spans a T_b-section F_b. Thus F_b is lower K-true (with respect to T_b) by [10, Theorem 6.1]. Since T_b preserves epis, F_b is in fact K-true (see Remark 3.11(3) above). Moreover the T_b-section F_b is an extension of the T-section F in the only possible sense (see [5, Proposition 3.1]), namely that $F_bQ = F$ (we note here that Q is a full embedding of $\mathbf{Top_o}$ into $\mathbf{Creg2\,Top_o}$). From the fact that F_b is K-true now follows, analogously to Proposition 3.2 above, that the pair $(M, m) := (T_bKF_b\,, T_bkF_b)$ is a reflection and can be augmented to a monad

$$\mathbf{M} := (M,\, m,\, \mu) = (M,\, m,\, (mM)^{-1})$$

in $\mathbf{Creg2Top_o}$. Our next task is to transport the monad $\mathbf{M}$ to a monad in $\mathbf{Top_o}$ along the adjunction $(Q, K, 1, i)$. In [6] this was done by extensive computation. The author is indebted to K.A. Hardie for the hint to achieve the same purpose more efficiently via the Eilenberg-Moore category $\mathbf{X}^{\mathrm{M}}$ of $\mathbf{X} := \mathbf{Creg2Top_o}$ as follows. There is the well-known adjunction $(U^{\mathrm{M}}, V^{M}, \eta^{M}, \varepsilon^{M})$ given by $\mathbf{X}^{\mathrm{M}}$ (see e.g. [40, p. 136] or [1, p. 304]) which induces the monad $\mathbf{M}$. In particular $V^{\mathrm{M}}U^{\mathrm{M}} = M$, $\eta^{\mathrm{M}} = m$ and $V^{\mathrm{M}}\varepsilon^{\mathrm{M}}U^{\mathrm{M}} = \mu$. The composition of this adjunction with the adjunction $(Q, E, 1, i)$ is the following adjunction (see e.g. [40, p. 101] or [1, Proposition 19.13]):

$$(U^{\mathrm{M}}Q,\, EV^{\mathrm{M}},\, E\eta^{\mathrm{M}}Q.1,\, \varepsilon^{\mathrm{M}}.U^{\mathrm{M}}iV^{\mathrm{M}}).$$

This adjunction induces the following monad in $\mathbf{Top_o}$:

$$\begin{aligned} & (EV^{\mathrm{M}}U^{\mathrm{M}}Q,\ E\eta^{\mathrm{M}}Q,\ EV^{\mathrm{M}}(\varepsilon^{\mathrm{M}}.U^{\mathrm{M}}iV^{\mathrm{M}})U^{\mathrm{M}}Q) \\ = \ & (EMQ,\ EmQ,\ EV^{\mathrm{M}}\varepsilon^{\mathrm{M}}U^{\mathrm{M}}Q.EV^{\mathrm{M}}U^{\mathrm{M}}iV^{\mathrm{M}}U^{\mathrm{M}}Q) \\ = \ & (EMQ,\ EmQ,\ E\mu Q.EMiMQ) \end{aligned}$$

Now $EMQ = ET_bKF_bQ = TKF$ and $EmQ = ET_bkF_bQ = TkF$ so that the induced monad in $\mathbf{Top_o}$ simplifies to $(TKF,\ TkF,\ E\mu Q.EMiMQ)$, as desired. □

REMARK 3.22. *For a special class of transitive, lower K-true T-sections F, Kimmie identified the corresponding categories of Eilenberg-Moore algebras of the monads described above [31]. For the case $F = \mathcal{C}_1^*$ the monad has been studied in several guises (e.g. as the prime open filter space monad, [46]) and a number of authors have given various realisations of its Eilenberg-Moore category, e.g.: (1) The category of compact regular T_o-bispaces as full subcategory of* $\mathbf{Creg2Top_o}$ *[43] — this is evident from the construction given in the proof of 3.21 above, since in this case $F_b = \mathcal{C}_b^*$, the coarsest T_b-section, which gives $T_bKF_b = T_bK\mathcal{C}_b^*$, the bitopological Stone-Čech compactification; (2) the category of compact partially ordered spaces and continuous isotone maps [27], [48]; (3) the category of stably compact topological spaces and perfect continuous maps [46], [2].*

4. Induced epireflections in $\mathbf{Top_o}$

Throughout this section we shall deal with the forgetful functor $T : \mathbf{QU_o} \longrightarrow \mathbf{Top_o}$. For any T-section F, we recall from Definition 3.1 that $\mathbf{E}(F)$ is the full subcategory of $\mathbf{Top_o}$ consisting of spaces on which F is bicomplete. One of the results below, Proposition 4.2, was proved in [10] in a more general abstract setting. For the reader's convenience, and because it is more instructive, we shall give a more concrete proof. The other major results of this section, Propositions 4.6, 4.7 and 4.8, are new.

It is well known that if a uniform space A is complete and coarser than a uniform space B which has the same topology as A, then B is complete. The quasi-uniform analogue of this fact requires an additional proviso, and plays a crucial role in the results below. Strangely, we have in Lemma 4.1 two quasi-uniform analogues of this principle, and they seem to be located at opposite ends of a scale.

Moreover, in the proof of Proposition 4.2 the one principle works and the other does not, whereas in the proof of Proposition 4.4 the situation is reversed.

LEMMA 4.1. *1. [37, p. 180]. Let* $X \in \mathbf{Top_o}$, $A, B \in \mathbf{QU_o}$ *and* $\mathcal{C}_1^* X \leq A \leq B$ *with* $X = TA = TB$. *If* A *is bicomplete, then* B *is bicomplete.*

2. [10, item 9]. Let F *be an upper* K*-true section of* $T : \mathbf{QU_o} \longrightarrow \mathbf{Top_o}$. *Then for any bicomplete quasi-uniform space* A, $A \leq FTA \Longrightarrow FTA$ *is bicomplete.*

Proof. For (1), see [37, p. 180]. Since (2) is not fully proved in [10], we prove it here: We have a map $i : FTA \longrightarrow A$ with $Ti = 1$. Naturality of $k : \mathbf{1} \longrightarrow K$ applied to i gives $k_A.i = Ki.k_{FTA}$. Applying T to this equation we have

$$Tk_A = TKi.Tk_{FTA}\,.$$

Since A is bicomplete, Tk_A is an iso, and thus the map Tk_{FTA} is a section. Since F is upper K-true, by Theorem 3.7(3) Tk_{FTA} is epi. Each epi section is iso [1, 7.43]. Being iso, Tk_{FTA} is then surjective, so that also the embedding k_{FTA} is surjective, hence iso. Thus FTA is bicomplete. □

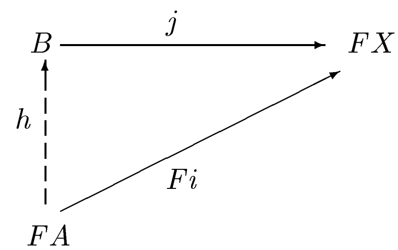
PROPOSITION 4.2. *If* F *is an upper* K*-true section of* $T : \mathbf{QU_o} \longrightarrow \mathbf{Top_o}$, *then* $\mathbf{E}(F)$ *is epireflective in* $\mathbf{Top_o}$.

Proof. We shall show that $\mathbf{E}(F)$ is closed for the taking of products and b-closed subspaces (i.e. extremal subobjects). Epireflectivity will then follow from [1, Corollary 16.9] since $\mathbf{Top_o}$ satisfies the conditions given there (of being co-wellpowered and strongly complete).

First, let $X_i \in \mathbf{E}(F)$ $(i \in I)$ and consider the product $X = \prod X_i$ with projections $\pi_i : X \longrightarrow X_i$ in $\mathbf{Top_o}$. In $\mathbf{QU_o}$ the product $Y := \prod FX_i$ is bicomplete; let its projections be $p_i : Y \longrightarrow FX_i$. By the universal property of this product there exists a unique map $j : FX \longrightarrow Y$ with $p_i j = F\pi_i$ and hence $Tp_i.Tj = TF\pi_i = \pi_i$ (all $i \in I$). Since $Tp_i = \pi_i$ it follows that $Tj = 1$, i.e. $Y \leq FX = FTY$. By Lemma 4.1(2), FX is bicomplete, i.e. $X \in \mathbf{E}(F)$.

Secondly, for any $X \in \mathbf{E}(F)$, consider a b-closed topological subspace A of X, calling the inclusion map $i : A \longrightarrow X$. Let B denote the quasi-uniform subspace of FX with the same underlying set as A, and denote the inclusion map by $j : B \longrightarrow FX$.

(K)



We have a (unique) map $h : FA \longrightarrow B$ with $Th = 1$, i.e. $B \leq FA$. Now FX is bicomplete, and applying the symmetrising or "sup" functor s (i.e. the uniform coreflector) to $j : B \longrightarrow FX$ we get $s(j) : sB \longrightarrow sFX$ which is a uniform subspace inclusion into the complete uniform space sFX. Now $TsFX = bX$ is the Skula modification of X (this is well known — see [37, Lemma 2] or [41] — and can be seen as follows: Considering the functors E and Q in the proof of Proposition 3.21, since Q is the unique section of E, $T_bF = Q$ and thus $TsFX = ST_bFX = SQX = bX$). Given that A is b-closed in X we now have sB a closed subspace of the complete uniform space sFX, so that sB is complete and B bicomplete. Having shown that $B \leq FA = FTB$ we see from Lemma 4.1(2) that FA is bicomplete, i.e. $A \in \mathbf{E}(F)$, as required. □

REMARK 4.3. *The more general version of a proof of the epireflectivity of* $\mathbf{E}(F)$ *in [10, Proposition 7 and 9] has the advantage of avoiding the appeal to co-wellpoweredness that we have used here.*

REMARK 4.4. *There is an important difference between the outcomes of Propositions 3.2 and 4.2. In both cases* $\mathbf{E}(F)$ *is epireflective in* $\mathbf{Top_o}$. *When* F *is* K*-true, the reflection is* (TKF, TkF), *but when* F *fails to be* K*-true,* (TKF, TkF) *need not be a reflection.*
It is well known (see e.g. [47]) that transfinite iteration of a prereflection sometimes converges to a reflection. This is the case with the prereflection (TKF, TkF) *induced in* $\mathbf{Top_o}$ *by an upper* K*-true* T*-section* F. *It should be interesting to know how the length of the iteration depends on the given* F.

PROPOSITION 4.5. *Let* F *and* G *be sections of* T: $\mathbf{QU_o} \longrightarrow \mathbf{Top_o}$.

1. $F \leq G \implies \mathbf{E}(F) \subseteq \mathbf{E}(G)$.

2. If F is upper K-true, then $\mathbf{Sob} \subseteq \mathbf{E}(F) \subseteq \mathbf{TopBiCpl}$.

Proof. Immediate from Lemma 4.1(1).

(2) This follows from (1) since $\mathbf{Sob} = \mathbf{E}(W)$ and $\mathbf{TopBiCpl} = \mathbf{E}(\Phi_1)$ by Examples 3.5 and 3.6. □

PROPOSITION 4.6. *Let F be an upper K-true T-section spanned by a class $\mathbf{A}$ of bicomplete T_0 quasi-uniform spaces (so that F is in fact K-true). Then $\mathbf{E}(F)$ is the epireflective hull of the class $T[\mathbf{A}]$ in* $\mathbf{Top_o}$.

Proof. Since $\mathbf{A}$ spans F, each $A \in \mathbf{A}$ satisfies $A \leq FTA$ ([4] or [5]) so that by 4.1(2) FTA is bicomplete, i.e. $TA \in \mathbf{E}(F)$. Thus $T[\mathbf{A}] \subseteq \mathbf{E}(F)$ and by 3.2 the epireflective hull of $T[\mathbf{A}]$ is contained in $\mathbf{E}(F)$. It remains to prove that if $\mathbf{B}$ is any epireflective subcategory of $\mathbf{Top_o}$ that contains $T[\mathbf{A}]$, then $\mathbf{E}(F) \subseteq \mathbf{B}$. Let $X \in \mathbf{E}(F)$, so that FX is bicomplete. By the spanning construction we have an initial source

$$(f' : FX \longrightarrow A \mid f \ \in \mathbf{Top_o}(X, TA),\, A \in \mathbf{A}).$$

For the given X there is a set-indexed subfamily $(f'_j : FX \longrightarrow A_j \mid j \in J)$ of this source which is still an initial source. Forming the product $B := \prod(A_j \mid j \in J)$ in $\mathbf{QU_o}$ with projection maps p_j, we have a quasi-uniform embedding $e' : FX \longrightarrow B$ given by $p_j e' = f'_j$ for all $j \in J$ (e' is an embedding since it is an initial map on a T_0 domain).

Since each $A_j \in \mathbf{A}$, B is bicomplete. Applying the symmetrizer s we have the uniform embedding $s(e') : sFX \longrightarrow sB$ in which both sFX and sB are complete T_0 uniform spaces. Thus $Ts(e') : TsFX \longrightarrow TsB$ is a closed topological embedding. We consider the map $Te' = e : X \longrightarrow TB$ and claim that this is a b-closed topological embedding. Indeed, $B \leq \Phi_1 TB$ and therefore $TsB \leq Ts\Phi_1 TB = bTB$ (repeating an argument from the proof of 4.2 above); and since $e[X]$ is closed in TsB, it is also closed in the finer space bTB. It was given that $T[\mathbf{A}] \subseteq \mathbf{B}$, $\mathbf{B}$ being epireflective. Since each $TA_j \in \mathbf{B}$, we have $TB = \prod TA_j \in \mathbf{B}$. Moreover X admits a b-closed embedding into TB (i.e. an extremal mono) and so $X \in \mathbf{B}$ by [1, Corollary 16.9]. The result follows. □

Lemma 4.7. *If G is any K-true T-section, then G is spanned by the class $G[\mathbf{E}(G)]$.*

Proof. Suppose that G is K-true and let H be the functor spanned by $G[\mathbf{E}(G)]$. Since the Sierpiński quasi-uniform space clearly belongs to this class, H is a T-section. Since G is (trivially) spanned by its range, which contains $G[\mathbf{E}(G)]$, it is clear that $G \geq H$. To prove that $G \leq H$, consider the quasi-uniform embedding (hence initial map) $k_{GX} : GX \longrightarrow KGX$ with $KGX = GTKGX$. By Proposition 3.2 $GTKGX$ belongs to the given spanning class of H. The spanning construction therefore lifts the map $Tk_{GX} : X \longrightarrow TKGX$ to a map, say $m_X : HX \longrightarrow GTKGX$, with $Tm_X = Tk_{GX}$ and $T(HX) = T(GX) = X$.

$$\text{(L)} \qquad \begin{array}{ccc} GX & \xrightarrow{\quad k_{GX} \quad} & KGX \\ {\scriptstyle i_X} \uparrow & & \| \\ HX & \xrightarrow{\quad m_X \quad} & GTKGX \\ & & \\ X & \xrightarrow{\quad Tk_{GX} \quad} & TKGX \end{array}$$

Since k_{GX} is initial, there is a map $i_X : HX \longrightarrow GX$ with $Ti_X = 1_X$. Thus $GX \leq HX$. □

Proposition 4.8. *Let $\mathbf{B}$ be any epireflective subcategory of $\mathbf{Top_o}$ with $\mathbf{Sob} \subseteq \mathbf{B} \subseteq \mathbf{TopBiCpl}$. Then there exists a K-true T-section F such that $\mathbf{E}(F) = \mathbf{B}$. The finest such T-section is spanned by the class $\Phi_1[\mathbf{B}]$.*

Proof. Let $F := \langle \Phi_1[\mathbf{B}] \rangle$. Since $\mathbf{B} \subseteq \mathbf{TopBiCpl}$, $\Phi_1[\mathbf{B}]$ is a class of bicomplete spaces. The class also contains the Sierpiński quasi-uniform space. Thus F is a lower K-true T-section. Using Lemma 4.7 we see that

$$W = \langle W[\mathbf{Sob}] \rangle \leq \langle W[\mathbf{B}] \rangle \leq \langle \Phi_1[\mathbf{B}] \rangle = F$$

and consequently F is K-true. By Proposition refpro:4.6 $\mathbf{E}(F)$ is the epireflective hull of

$T[\Phi_1[\mathbf{B}]] = \mathbf{B}$, and since $\mathbf{B}$ is itself epireflective, $\mathbf{E}(F) = \mathbf{B}$. Finally, if G is any K-true T-section with $\mathbf{E}(G) = \mathbf{B}$, then by Lemma 4.7

$$G = \langle G[\mathbf{E}(G)]\rangle = \langle G[\mathbf{B}]\rangle \leq \langle \Phi_1[\mathbf{B}]\rangle = F.$$

□

REMARK 4.9. *When F is an upper K-true T-section of sufficient interest, it is desirable to obtain manageable characterisations of the epireflective subcategory $\mathbf{E}(F)$. The few such characterisations known to the author occur in the paper [37] by Künzi and Ferrario and in the thesis [31] by Kimmie. A more transparent description of the topologically bicomplete spaces is still needed.*

REMARK 4.10. *Even when F fails to be upper K-true, it may be of interest to characterise $\mathbf{E}(F)$, as was the case with $\mathbf{E}(\mathcal{C}_1^*)$ — see Example 3.4.*

REMARK 4.11. *The author does not know whether epireflectivity of $\mathbf{E}(F)$ by itself implies that the T-section F is upper K-true.*

REMARK 4.12. *For the completeness notions due to Sieber and Pervin [39] and Doitchinov [21] it is known that analogues of certain of the results of this section exist [10, items 4.3, 6.5(4), 10], [38].*

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