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Transfinite Order Dimension

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Summary. - *We give two different transfinite extensions of the covering dimension based on the Borst's order of certain families of boundaries of basic open sets. We compare them and we study their main properties.*

1. The first order dimension.

In this section we will use the most known of the characterizations of the covering dimension by means of bases of open sets. We propose a transfinite extension using Borst's order (see [3]) and we develop its main properties.

We first give the definition of the first order dimension of a metrizable space.

Definition 1.1. *Let X be a metrizable space and let α be an ordinal number or* 0. *We define the first order dimension of X,* $\mathrm{Or\text{–}dim}_1(X)$, *as follows:*

1. $\mathrm{Or\text{–}dim}_1(X) = -1$ *if and only if* $X = \emptyset$;
2. $\mathrm{Or\text{–}dim}_1(X) \leq \alpha$ *if and only if there is an open base* $\mathcal{B}$ *of* X

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that is σ-locally finite and such that $\operatorname{Ord} O_1(\mathcal{B}) \leq \alpha$, *where*

$$O_1(\mathcal{B}) = \{\sigma \in \operatorname{Fin}(\mathcal{B}) : \bigcap_{B \in \sigma} \operatorname{Fr}(B) \neq \emptyset\}.$$

3. $\operatorname{Or–dim}_1(X) = \alpha$ *if and only if* $\operatorname{Or–dim}_1(X) \leq \alpha$ *and is false that* $\operatorname{Or–dim}_1(X) \leq \beta$ *for any ordinal number* $\beta < \alpha$ *or* $\beta = 0$.
4. $\operatorname{Or–dim}_1(X) = \Delta$ *if and only if for every ordinal* α *we have* $\operatorname{Or–dim}_1(X) > \alpha$, *and then we say that* $\operatorname{Or–dim}_1(X)$ *does not exist.*

This definition gives an extension of the covering dimension to the transfinite case in metrizable separable spaces, since we have the equality between ind and dim in such spaces.

PROPOSITION 1.2. *Let X be a metrizable separable space. Then the covering dimension of X* $\dim(X)$ *exists if and only if* $\operatorname{Or–dim}_1(X)$ *is finite, and in that case*

$$\dim(X) = \operatorname{Or–dim}_1(X).$$

Proof. See [1], corollary 6.12. □

In metrizable spaces we can obtain the following relation between the dimensions trind and $\operatorname{Or–dim}_1$.

PROPOSITION 1.3. *Let X be a metrizable space. Then*

$$\operatorname{trind}(X) \leq \operatorname{Or–dim}_1(X).$$

Proof. We proceed by induction over $\operatorname{Or–dim}_1(X)$.

If $\operatorname{Or–dim}_1(X) = -1$ or $\operatorname{Or–dim}_1(X)$ does not exist, the result is clear. Suppose that the result is true whenever $\operatorname{Or–dim}_1(X) < \alpha$, where α is an ordinal number or 0. Let $\operatorname{Or–dim}_1(X) = \alpha$, then X has an open σ-locally finite base $\mathcal{B}$ such that $\operatorname{Ord} O_1(\mathcal{B}) \leq \alpha$. Now we see that for each $B \in \mathcal{B}$, we have $\operatorname{Or–dim}_1(\operatorname{Fr} B) < \alpha$. Let $B_0 \in \mathcal{B}$, then we have that $\{\mathrm{Fr}\,(B_0)\cap B : B\in\mathcal{B}'\}$ is an open σ-locally finite base of $\mathrm{Fr}\,(B_0)$, where $\mathcal{B}'\subseteq\mathcal{B}$ is such that $B_0\notin\mathcal{B}'$ (since B_0 is open, $\mathrm{Fr}\,(B_0)\cap B_0=\emptyset$) and given $B\in\mathcal{B}$ there exists only one $B'\in\mathcal{B}'$ with $B\cap\mathrm{Fr}\,(B_0)=B'\cap\mathrm{Fr}\,(B_0)$.

In order to calculate $\mathrm{Ord}\,O_1(\mathcal{B}')$, since $\mathrm{Ord}\,O_1(\mathcal{B})\le\alpha$, we have $\mathrm{Ord}\,(O_1(\mathcal{B}))^{B_0}<\alpha$ (see the definition of Borst's order in [3]). The inclusion map $\Phi:\mathcal{B}'\to\mathcal{B}$ verifies:

1. $|\tau|=|\Phi(\tau)|$, for each $\tau\in O_1(\mathcal{B}')$.
2. If $\tau\in O_1(\mathcal{B}')$, then $\Phi(\tau)\in(O_1(\mathcal{B}))^{B_0}$, since $B_0\notin\Phi(\tau)$ and

$$\mathrm{Fr}_{\,\mathrm{Fr}\,(B_0)}\Big(B\cap\mathrm{Fr}\,(B_0)\Big)\subseteq\mathrm{Fr}\,(B)\cap\mathrm{Fr}\,(B_0)$$

hence

$$\emptyset\neq\bigcap_{B\in\tau}\mathrm{Fr}_{\,\mathrm{Fr}\,(B_0)}\Big(B\cap\mathrm{Fr}\,(B_0)\Big)\subseteq\bigcap_{B\in\Phi(\tau)}\mathrm{Fr}\,(B)\cap\mathrm{Fr}\,(B_0).$$

Now lemma 2.1.6 of [3] give us $\mathrm{Ord}\,O_1(\mathcal{B}')\le\mathrm{Ord}\,(O_1(\mathcal{B}))^{B_0}<\alpha$. Hence $\mathrm{Or\text{--}dim}_1\,\mathrm{Fr}\,(B_0)<\alpha$ and from the inductive hypothesis we have $\mathrm{trind}\,\mathrm{Fr}\,(B_0)<\alpha$. Since X is regular, $\mathrm{trind}\,(X)\le\alpha$. □

Now we study some properties of this dimension. First we have the subspace theorem.

PROPOSITION 1.4. *Let X be a metrizable space and $A\subseteq X$. Then:*

$$\mathrm{Or\text{--}dim}_1\,(A)\le\mathrm{Or\text{--}dim}_1\,(X).$$

Proof. If $\mathrm{Or\text{--}dim}_1\,(X)=-1$ the result is clear; let α be an ordinal and $\mathrm{Or\text{--}dim}_1\,(X)=\alpha$. Suppose that $\mathcal{B}$ is a σ-locally finite open base of X with $\mathrm{Ord}\,O_1(\mathcal{B})\le\alpha$. Now we get the following base of open sets in A: $\mathcal{B}_A=\{B\cap A : B\in\mathcal{B}'\}$, where $\mathcal{B}'\subseteq\mathcal{B}$ is such that for every $B\in\mathcal{B}$ there exists only one $B'\in\mathcal{B}'$ with $B\cap A=B'\cap A$. This base is σ-locally finite since $\mathcal{B}$ is.

The inclusion map $\Phi:\mathcal{B}'\to\mathcal{B}$ from $\mathcal{B}'$ to $\mathcal{B}$, verifies:

1. $|\tau| = |\Phi(\tau)|$, for each $\tau \in O_1(\mathcal{B}')$.
2. If $\tau \in \mathcal{B}'$, then $\Phi(\tau) \in O_1(\mathcal{B})$, since $\mathrm{Fr}_A(B \cap A) \subseteq \mathrm{Fr}\,(B)$.

Now lemma 2.1.6 of [3] gives us, $\mathrm{Ord}\,(\mathcal{B}') \leq \mathrm{Ord}\,O_1(\mathcal{B}) \leq \alpha$ and hence $\text{Or–dim}_1\,(A) \leq \alpha$. □

This dimension also verifies the topological sum theorem.

THEOREM 1.5. *Let* $X = \bigoplus_{i \in I} X_i$ *where* X_i *are metric spaces. Then*

$$\text{Or–dim}_1\,(X) = \mathrm{Sup}\,\{\text{Or–dim}_1\,(X_i) : i \in I\}.$$

Proof. Suppose that $\text{Or–dim}_1\,(X_i) = \alpha_i$, with α_i an ordinal or 0 for each $i \in I$ and let $\mathcal{B}_i$ be an open σ-locally finite base for each X_i such that $\mathrm{Ord}\,O_1(\mathcal{B}_i) \leq \alpha_i$. Then $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ is an open σ-locally finite base for X. Moreover, $O_1(\mathcal{B}) = \bigcup_{i \in I} O_1(\mathcal{B}_i)$ (disjoint union) and using lemma 3.3 from [2] we obtain $\mathrm{Ord}\,O_1(\mathcal{B}) = \mathrm{Sup}\,\{\mathrm{Ord}\,O_1(\mathcal{B}_i) : i \in I\} \leq \mathrm{Sup}\,\{\text{Or–dim}_1\,(X_i) : i \in I\}$, hence $\text{Or–dim}_1\,(X) \leq \mathrm{Sup}\,\{\text{Or–dim}_1\,(X_i) : i \in I\}$. The other inequality comes from the subspace theorem. □

We also have obtained the following weight theorem.

THEOREM 1.6. *Let* X *be a metrizable space such that* $\text{Or–dim}_1\,(X)$ *exists. Then*

$$|\text{Or–dim}_1\,(X)| \leq W(X).$$

Proof. Suppose that $\text{Or–dim}_1\,(X) = \alpha$ and let $\mathcal{B}'$ be a σ-locally finite base of X such that $\mathrm{Ord}\,O_1(\mathcal{B}') = \alpha$. Then there exists a σ-locally finite base $\mathcal{B} \subseteq \mathcal{B}'$ with $|\mathcal{B}| = W(X)$. So $O_1(\mathcal{B}) \subseteq O_1(\mathcal{B}')$ and $\mathrm{Ord}\,O_1(\mathcal{B}) \leq \mathrm{Ord}\,O_1(\mathcal{B}') = \alpha$ (see [3], lemma 2.1.1 (3)) in fact $\mathrm{Ord}\,O_1(\mathcal{B}) = \alpha$ because $\text{Or–dim}_1\,(X) = \alpha$. The map $\varphi : O_1(\mathcal{B}) \to \alpha$, defined as $\varphi(\sigma) = \mathrm{Ord}\,O_1(\mathcal{B})^\sigma$, satisfies:

1. if $\beta < \alpha$ there exists $B \in \mathcal{B}$ such that $\beta \leq \mathrm{Ord}\,(O_1(\mathcal{B}))^B = \varphi(\{B\}) < \alpha$.

2. Let $\beta = \varphi(\sigma) = \mathrm{Ord}\,(O_1(\mathcal{B}))^{\sigma}$ with $\sigma \in O_1(B)$ and let $\gamma < \beta$, then there exists $B \in (O_1(\mathcal{B}))^{\sigma}$ such that $\gamma \leq \mathrm{Ord}\,((O_1(\mathcal{B}))^{\sigma})^{B} = \mathrm{Ord}\,(O_1(\mathcal{B}))^{\sigma \cup \{B\}} < \beta$. So $\sigma \cup \{B\} \in O_1(\mathcal{B})$ and $\gamma \leq \varphi(\sigma \cup \{B\}) \in \varphi(O_1(\mathcal{B})) \cap \beta$.

And using lemma 2.8 of [6], $|\alpha| = |\varphi(O_1(\mathcal{B}))| \leq |O_1(\mathcal{B})| \leq |\mathrm{Fin}\,(\mathcal{B})| = |\mathcal{B}| = W(X)$. □

2. The second order dimension.

Now we consider another point of view in the following definition that appears in the finite case in [1].

DEFINITION 2.1. *Let X be a metrizable space and α an ordinal number or* 0. *We define the second order dimension of X,* $\mathrm{Or\text{-}dim}_2\,(X)$ *as follows:*

1. $\mathrm{Or\text{-}dim}_2\,(X) = -1$ *if and only if* $X = \emptyset$;
2. $\mathrm{Or\text{-}dim}_2\,(X) \leq \alpha$ *if and only if there is an open base $\mathcal{B}$ of X that is σ-locally finite and such that* $\mathrm{Ord}\,O_2(\mathcal{B}) \leq \alpha$, *where*

$$O_2(\mathcal{B}) = \{\sigma \in \mathrm{Fin}\,(\mathrm{Fr}\,\mathcal{B}) : \bigcap_{\mathrm{Fr}\,(B) \in \sigma} \mathrm{Fr}\,(B) \neq \emptyset\}$$

$$\text{and } \mathrm{Fr}\,(\mathcal{B}) = \{\mathrm{Fr}\,(B) : B \in \mathcal{B}\}.$$

3. $\mathrm{Or\text{-}dim}_2\,(X) = \alpha$ *if and only if* $\mathrm{Or\text{-}dim}_2\,(X) \leq \alpha$ *and is not true that* $\mathrm{Or\text{-}dim}_2\,(X) \leq \beta$ *for any* $\beta < \alpha$.
4. $\mathrm{Or\text{-}dim}_2\,(X) = \Delta$, *if and only if for every ordinal α we have* $\mathrm{Or\text{-}dim}_2\,(X) > \alpha$, *and we say that* $\mathrm{Or\text{-}dim}_2\,(X)$ *does not exist.*

Note that the difference between both dimensions is the set used to calculate the Borst's order: in the first case we take *finite families of basic open sets* whose boundaries verify certain property, in the second case we take *finite families of boundaries of basic open sets* that verify the same property as in the first case. To realize that the sets are different in general, note that *two different open sets may share the same boundary.* However there is always a relation between $\operatorname{Ord} O_2(\mathcal{B})$ and $\operatorname{Ord} O_1(\mathcal{B})$ for a fixed base $\mathcal{B}$, and we use that relation to prove the following.

PROPOSITION 2.2. *Let X be a metrizable space. Then:*

$$\text{Or–dim}_2(X) \leq \text{Or–dim}_1(X).$$

Proof. If $\text{Or–dim}_1(X) = -1$ or $\text{Or–dim}_1(X)$ does not exist, it is clear.

Suppose that $\text{Or–dim}_1(X) = \alpha$ where α is an ordinal number or 0 and let $\mathcal{B}$ a σ-locally finite open base of X with $\operatorname{Ord} O_1(\mathcal{B}) \leq \alpha$. We will see that $\operatorname{Ord} O_2(\mathcal{B}) \leq \alpha$. To this end it suffices to consider the map $\Phi : \operatorname{Fr} \mathcal{B} \to \mathcal{B}$ defined as $\Phi\Big(\operatorname{Fr}(B)\Big) = B$ for each $\operatorname{Fr}(B) \in \operatorname{Fr}(\mathcal{B})$. Clearly that map fulfills the hypothesis of lemma 2.1.6 of [3], hence $\operatorname{Ord} O_2(\mathcal{B}) \leq \operatorname{Ord} O_1(\mathcal{B}) \leq \alpha$, so $\text{Or–dim}_2(X) \leq \text{Or–dim}_1(X)$. □

This dimension Or–dim$_2$ is a transfinite extension of the covering dimension in metrizable spaces, as we can see in the following result, since Ind and dim are equal in such spaces.

PROPOSITION 2.3. *Let X be a metrizable space. Then* $\dim(X)$ *exists if and only if* $\text{Or–dim}_2(X)$ *is finite, and in that case*

$$\dim(X) = \text{Or–dim}_2(X).$$

Proof. See [4], theorem 4.2.2. □

In the finite-dimensional case of metrizable separable spaces Or–dim$_1$ and Or–dim$_2$ are equal, since both coincide with the covering dimension. However, if one consider the modification made replacing the empty set with another class of topological spaces, the results are different. See the book [1] for a comprehensive study of such kind of invariants and its relations with the answer given by Kimura in 1988 to the long-standing problem raised by De Groot in 1942.

Now we are going to study the properties of this new dimension. First we obtain an open subspace theorem, whose proof is similar to that of the subspace theorem for Or–dim$_1$.

PROPOSITION 2.4. *Let X be a metrizable space and A an open subspace of X. Then* $\mathrm{Or\text{--}dim}_2\,(A) \leq \mathrm{Or\text{--}dim}_2\,(X)$.

Proof. If $\mathrm{Or\text{--}dim}_2\,(X) = -1$ or $\mathrm{Or\text{--}dim}_2\,(X)$ does not exist, is clear.

Let α be an ordinal number or 0 with $\mathrm{Or\text{--}dim}_2\,(X) = \alpha$. Suppose that $\mathcal{B}$ is a σ-locally finite open base of X with $\mathrm{Ord}\, O_2(\mathcal{B}) \leq \alpha$. Let consider the following open base of A, $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}'\}$, with $\mathcal{B}' \subseteq \mathcal{B}$ such that for each $B \in \mathcal{B}$ there exists only one $B' \in \mathcal{B}'$ with $B \cap A = B' \cap A$. This is a σ-locally finite base since $\mathcal{B}$ is.

To calculate $\mathrm{Ord}\, O_2(\mathcal{B}_A)$, let consider the map $\Phi : \mathrm{Fr}\, \mathcal{B}_A \to \mathrm{Fr}\, \mathcal{B}$ defined as $\Phi\Big(\mathrm{Fr}_A(B \cap A)\Big) = \mathrm{Fr}\,(B)$ for each $\mathrm{Fr}_A(B \cap A) \in \mathrm{Fr}\, \mathcal{B}_A$. We have:

1. If $\tau \in \mathcal{B}'$, then $\Phi(\tau) \in O(\mathcal{B})$, since $\mathrm{Fr}_A(B \cap A) \subseteq \mathrm{Fr}\,(B)$.
2. $|\tau| = |\Phi(\tau)|$, for every $\tau \in O_2(\mathcal{B}')$, since A open in X means $\mathrm{Fr}_A(B \cap A) = A \cap \mathrm{Fr}\,(B)$, hence $\mathrm{Fr}_A(B \cap A) \neq \mathrm{Fr}_A(B' \cap A)$ implies $\mathrm{Fr}\,(B) \neq \mathrm{Fr}\,(B')$.

So $\mathrm{Ord}\, O_2(\mathcal{B}_A) \leq \mathrm{Ord}\, O_2(\mathcal{B}) \leq \alpha$, (see lemma 2.1.6 of [3]) then $\mathrm{Or\text{--}dim}_2\,(A) \leq \alpha$. □

Note that we need A open in X to ensure the second property of Φ; in the proof for $\mathrm{Or\text{--}dim}_1$ however, we do not need any condition over A. Moreover, in [1], example 6.13, we see that we need A open if we want to use here the same argument as in the proof for $\mathrm{Or\text{--}dim}_1$.

We also have a topological sum theorem.

THEOREM 2.5. *Let $X = \bigoplus_{i \in I} X_i$, where X_i are metrizable. Then:*

$$\mathrm{Or\text{--}dim}_2\,(X) = \mathrm{Sup}\,\{\mathrm{Or\text{--}dim}_2\,(X_i) : i \in I\}.$$

Proof. Suppose that $\mathrm{Or\text{--}dim}_2\,(X_i) = \alpha_i$, with α_i an ordinal number or 0, for each $i \in I$ and let $\mathcal{B}_i$ be a σ-locally finite open base for each X_i such that $\mathrm{Ord}\, O_2(\mathcal{B}_i) \leq \alpha_i$. Then $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$ is a σ-locally finite open base of X. Moreover is clear that $O_2(\mathcal{B}) = \bigcup_{i \in I} O_2(\mathcal{B}_i)$ (disjoint union) and using lemma 3.3 of [2] we get $\mathrm{Ord}\, O_2(\mathcal{B}) =$

$\mathrm{Sup}\,\{\mathrm{Ord}\,O_2(\mathcal{B}_i) : i \in I\} \leq \mathrm{Sup}\,\{\mathrm{Or\text{–}dim}_2\,(X_i) : i \in I\}$ hence $\mathrm{Or\text{–}dim}_2\,(X) \leq \mathrm{Sup}\,\{\mathrm{Or\text{–}dim}_2\,(X_i) : i \in I\}$.

The other inequality comes from the open subspace theorem since each X_i is open in X. □

Note that using 2.2, if $\mathrm{Or\text{–}dim}_1\,(X)$ exists then $\mathrm{Or\text{–}dim}_2\,(X)$ exists and $|\mathrm{Or\text{–}dim}_2\,(X)| \leq |\mathrm{Or\text{–}dim}_1\,(X)| \leq W(X)$. But this result can be obtained without the hypothesis about the existence of $\mathrm{Or\text{–}dim}_1\,(X)$.

THEOREM 2.6. *Let X be a metrizable space such that* $\mathrm{Or\text{–}dim}_2\,(X)$ *exists. Then*

$$|\mathrm{Or\text{–}dim}_2\,(X)| \leq W(X).$$

Proof. Suppose that $\mathrm{Or\text{–}dim}_2\,(X) = \alpha$ and let $\mathcal{B}'$ be a σ-locally finite base of X such that $\mathrm{Ord}\,O_2(\mathcal{B}') = \alpha$. As in theorem 1.6, we can find an open σ-locally finite base $\mathcal{B}$ of X such that $|\mathcal{B}| = W(X)$ and $\mathrm{Ord}\,O_2(\mathcal{B}) = \alpha$.

The map $\varphi : O_2(\mathcal{B}) \to \alpha$, defined as $\varphi(\sigma) = \mathrm{Ord}\,O_2(\mathcal{B})^\sigma$, satisfies, with the same arguments that theorem 1.6, the hypothesis of lemma 2.8 in [6], so $|\alpha| = |\varphi(O_2(\mathcal{B}))| \leq |O_2(\mathcal{B})| \leq |\mathrm{Fin}\,(\mathcal{B})| = |\mathcal{B}| = W(X)$. □

Finally we study under what conditions we can ensure the existence of the dimension $\mathrm{Or\text{–}dim}_2$ in separable metrizable spaces.

PROPOSITION 2.7. *Let X be a metrizable separable space. Then:*

$$\mathrm{trind}\,(X) \textit{ exists if and only if } \mathrm{Or\text{–}dim}_2\,(X) \textit{ exists.}$$

Proof. From theorem 2 of [5], for a metrizable separable space X, $\mathrm{trind}\,(X)$ exists if and only if X has a countable base $\mathcal{B}$ such that $\mathrm{Fr}\,\mathcal{B}$ is an strongly point-finite family (see section 2 of [5]), that is, no infinite subfamily of $\mathrm{Fr}\,\mathcal{B}$ has the finite intersection property. In other words, for every sequence $\{\mathrm{Fr}\,(B_i)\}_{i\in\mathbb{N}}$ of distinct elements of $\mathrm{Fr}\,(\mathcal{B})$, there exists $n \in \mathbb{N}$ such that $\bigcap_{i=1}^n \mathrm{Fr}\,(B_i) = \emptyset$.

So, from lemma 2.1.3 of [3], trind (X) exists if and only if Or–dim_2 (X) exists. $\square$

QUESTION 2.8: Is $\text{Or–dim}_2 X = \text{Or–dim}_1 X$ for every separable metrizable space X?

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