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Twistor Bundles of Almost Symplectic Manifolds

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Summary. - *In this paper we introduce the twistor bundle of a $2n$-dimensional almost symplectic manifold M as the quotient bundle $\frac{P(M,Sp(2n))}{U(n)}$. Given a symplectic connection on M we introduce a natural almost Hermitian structure on the twistor bundle and we prove that this structure is Kähler if and only if M is symplectic and the chosen connection has vanishing curvature and (0,2)-part of the torsion. Moreover we prove that in the case of $\mathbb{R}^{2n}$ with standard symplectic structure the twistor bundle turns out to be Kähler with constant scalar curvature for a certain class of symplectic connections.*

Contents

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1. Introduction

Let (M, ω) be an almost symplectic manifold of real dimension $2n$, let us denote $P_\omega := P(M, Sp(2n))$ the principal bundle of symplectic frames on (M, ω) and let $Z_\omega := \frac{P_\omega}{U(n)}$ be the associated bundle with respect to the standard action of $Sp(2n)$ on $P_\omega \times \frac{Sp(2n)}{U(n)}$. Z_ω is a bundle on M with structure group $Sp(2n)$ and standard fibre $\frac{Sp(2n)}{U(n)}$; the fibre at a point $x \in M$ parameterizes complex structures J on T_xM which are ω-calibrated, that is such that $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$ and $\omega(\cdot, J\cdot)$ is positive definite, then we call Z_ω *twistor bundle* of the almost symplectic manifold (M, ω).

Given a symplectic connection θ on M we can define a connection on Z_ω and, as it is a bundle of complex structures, with complex fibre, this allows us to define an almost complex structure on Z_ω, $\mathbb{J}_\theta$, in a tautologically way, also we can define an almost Hermitian metric, $\mathbb{G}_\theta$, in a natural way. Such constructions are similar to those in the twistor theory of an oriented even dimensional Riemannian manifold ([2], [11], [4]), however in this case there are some relevant differences: first the fibre is not compact, second there is not a canonical symplectic connection, like Levi-Civita in the Riemannian case, so things here depend on the chosen connection.

In this paper we study the differential geometry of $(Z_\omega, \mathbb{J}_\theta, \mathbb{G}_\theta)$, in particular we prove that it is Kähler if and only if (M, ω) is symplectic and the curvature of the given connection and the $(0, 2)$-part of the torsion vanish (section 5), this is the case for example of the twistor bundle of $\mathbb{R}^{2n}$ with standard symplectic structure, with respect to a certain class of connections (section 6). Also we study local sections defined by local almost complex structures on M, ω-calibrated, giving an interplay between symplectic connections and calibrated almost complex structures (section 4).

The paper is organized as follows. In section 2 we describe the geometry of the standard fibre $\frac{Sp(2n)}{U(n)}$ as Hermitian symmetric space of non compact type. In section 3 we define the concept of twistor bundle of an almost symplectic manifold and, given a symplectic connection θ, we introduce the almost complex structure $\mathbb{J}_\theta$ computing integrability conditions. Section 4 is devoted to the study of local sections of Z_ω. In section 5 we study the almost Hermitian metric $\mathbb{G}_\theta$ and we state Kähler and semi-Kähler condition. Finally in section 6 we describe in details the twistor bundle of $\mathbb{R}^{2n}$ with the standard symplectic structure ω_n, proving that for a certain class of symplectic connections is Kähler with constant scalar curvature.

For an introduction to almost symplectic manifolds and symplectic geometry we refer to [5], [8].

2. The geometry of Sp(2n)/U(n)

Let $J_n := \begin{pmatrix} 0 & \Leftrightarrow I_n \\ I_n & 0 \end{pmatrix} \in \mathbb{R}(2n)$ and let

$$Sp(2n) := \left\{ A \in \mathbb{R}(2n) \,\middle|\, {}^t A J_n A = J_n \right\}$$

be the real *symplectic group* of order $2n$; let

$$sp(2n) = \left\{ A \in \mathbb{R}(2n) \,\middle|\, {}^t A J_n = \Leftrightarrow J_n A \right\}$$

be the Lie algebra of $Sp(2n)$ and let $U(n) := \{A \in \mathbb{R}(2n) \,|\, AJ_n = J_n A\}$ be the *unitary group* of order n. It is well known that the quotient space $Sp(2n)/U(n)$ is a $\frac{n(n+1)}{2}$-dimensional Hermitian symmetric space of non compact type and then it is an Einstein-Kähler manifold of negative scalar curvature, [6]. We will describe in details this structure on $Sp(2n)/U(n)$ because it will be useful later.

Let us denote $\Im(n) := \left\{ P \in Sp(2n) \,\middle|\, P^2 = \Leftrightarrow I_{2n},\ \Leftrightarrow J_n P > 0 \right\}$, we have the following:

Lemma 2.1. $\Im(n) = \left\{ P \in \mathbb{R}(2n) \,\middle|\, P = AJ_nA^{-1},\ A \in Sp(2n) \right\}$.

Proof. The inclusion $\left\{ P \in \mathbb{R}(2n) \,\middle|\, P = AJ_nA^{-1},\ A \in Sp(2n) \right\} \subseteq \Im(n)$ is obvious. Now let $P \in \Im(n)$ and let $B = \Leftrightarrow J_n P$, we have that $B \in Sp(2n)$, $B = {}^t B$, $B > 0$, then $B^{\frac{1}{2}} \in Sp(2n)$, [8] (Lemma 2.19), and $P = ((\Leftrightarrow J_n P)^{\frac{1}{2}})^{-1} J_n (\Leftrightarrow J_n P)^{\frac{1}{2}}$. □

Moreover we get:

Lemma 2.2. $Sp(2n)/U(n) \cong \Im(n)$.

Proof. Let us define $\rho : Sp(2n) \to \Im(n)$ by: $\rho(A) = AJ_nA^{-1}$, we have immediately that ρ defines a map $\widehat{\rho} : Sp(2n)/U(n) \to \Im(n)$ by $\widehat{\rho}([A]) = \rho(A)$. $\widehat{\rho}$ is injective, in fact is $\rho(A) = \rho(B)$ if and only if $B^{-1}A \in U(n)$; $\widehat{\rho}$ is surjective, in fact, from Lemma 2.1, given $P \in \Im(n)$ is $P = \widehat{\rho}([(\Leftrightarrow J_nP)^{-\frac{1}{2}}])$. Thus $Sp(2n)/U(n)$ and $\Im(n)$ are identified by $\widehat{\rho}$. ☐

From now on we will refer to $Sp(2n)/U(n)$ as to the set:
$\{P \in \mathbb{R}(2n) \,|\, P = AJ_nA^{-1},\ A \in Sp(2n)\}$, and we will denote it $\Im(n)$.

Following Gromov [9], we recall the following definition:

Definition 2.1. *Let (V, ω) be a symplectic vector space and let J be a complex structure on V, J is called $\omega\Leftrightarrow$**calibrated** if the following conditions are satisfied:*
a) $\omega(v, Jv) > 0$ for any $v \in V \setminus \{0\}$,
b) $\omega(Jv, Jw) = \omega(v, w)$ for any $v, w \in V$.

Let ω_n be the standard symplectic form on $\mathbb{R}^{2n}$ defined by: $\omega_n(X, Y) = \Leftrightarrow^t XJ_nY$, we have immediately that $\Im(n)$ represents the set of complex structures on $\mathbb{R}^{2n}$, ω_n-calibrated and inducing positive orientation.

Let $P \in \Im(n)$ and let $T_P\Im(n)$ be the tangent space to $\Im(n)$ at the point P, denoted by $[.,.]$ the Lie bracket of matrices, we have the following:

Lemma 2.3. $T_P\Im(n) = \{[Y, P] \,|\, Y \in sp(2n)\}$.

Proof. Let $P = \rho(A)$, where $\rho : Sp(2n) \to \Im(n)$ is the map defined by $\rho(A) = AJ_nA^{-1}$, then the tangent map $\rho_{*A} : T_ASp(2n) \to T_P\Im(n)$ is given by:

$$\begin{aligned}\rho_{*A}(X) = XJ_nA^{-1} \Leftrightarrow AJ_nA^{-1}XA^{-1} = XA^{-1}\rho(A) \Leftrightarrow \rho(A)XA^{-1} = \\ = \left[XA^{-1}, P\right]\end{aligned}$$

but is $T_ASp(2n) = \{AZ \in \mathbb{R}(2n) \,|\, Z \in sp(2n)\}$, then $Y = XA^{-1} = AZA^{-1} \in sp(2n)$ and $T_P\Im(n) = \rho_{*A}T_ASp(2n) = \{[Y, P] \,|\, Y \in sp(2n)\}$. ☐

In the following we will use the notation: $\widehat{Y}(P) := [Y,P]$, thus we get $T_P\Im(n) = \left\{\widehat{Y}(P)\,|Y \in sp(2n)\right\}$.

Using the Killing form on $sp(2n)$ we define the Riemannian metric $\overset{\vee}{g}$ on $\Im(n)$ by: $\overset{\vee}{g}\,(X,Y) := \frac{1}{2}TrXY$, where $X,Y \in T_P\Im(n)$ and Tr means trace, in particular if $X = \widehat{A}(P)$, $Y = \widehat{B}(P)$, we have:

$\overset{\vee}{g}\,(\widehat{A}(P), \widehat{B}(P)) = \frac{1}{2}Tr\widehat{A}(P)\widehat{B}(P) = Tr(APBP + AB)$.

Also we define, in a natural way, the almost complex structure $\overset{\vee}{J}$ on $\Im(n)$ by: $\overset{\vee}{J}\,(P)(X) := PX$, for $P \in \Im(n)$ and $X \in T_P\Im(n)$.

Direct computations, similar to those in [4], give that $(\Im(n), \overset{\vee}{J}, \overset{\vee}{g})$ is an Einstein-Kähler manifold of negative scalar curvature.

3. Twistor bundles

Let (M,ω) be an almost symplectic manifold of real dimension $2n$, let us denote by $P_\omega := P(M, Sp(2n))$ the S$p(2n)$-principal bundle of symplectic frames on M and let $\pi : P_\omega \to M$ be the canonical projection. $Sp(2n)$ acts on $P_\omega \times \Im(n)$ in the following natural way:

$$\begin{array}{c} Sp(2n) \times P_\omega \times \Im(n) \to P_\omega \times \Im(n) \\ (A,a,P) \to (aA, A^{-1}PA) \end{array} \quad . \tag{1}$$

We pose the following:

Definition 3.1. *The associated bundle to P_ω, $P_\omega \times_{Sp(2n)} \Im(n) \cong \frac{P_\omega}{U(n)}$, defined as the quotient of $P_\omega \times \Im(n)$ with respect to the action* (1)*, is called* ***twistor bundle*** *of (M,ω), it will be denoted by $Z(M,\omega)$, or simply by Z_ω.*

Z_ω is a bundle over M with standard fibre $\frac{Sp(2n)}{U(n)} \cong \Im(n)$ and structure group $Sp(2n)$. Denote by $p : Z_\omega \to M$ and by $r : P_\omega \to Z_\omega$ the bundle projections. Z_ω is a manifold of real dimension $n(n+3)$ and the fibre of Z_ω at the point $x \in M$, $p^{-1}(x)$, parameterizes all complex structures on T_xM which are ω_x-calibrated.

A connection θ on P_ω defines a connection on Z_ω, that is a splitting of the tangent bundle in horizontal and vertical subbundles:

$TZ_\omega = H \oplus V$, and, as Z_ω is a bundle of complex structures, with complex fibre, this allow us to define an almost complex structure, $\mathbb{J}_\theta$, on Z_ω in a tautologically way. We will describe $\mathbb{J}_\theta$ in terms of local coordinates.

Let $\{U_i\}_{i\in I}$ be an open covering of M where local symplectic frames are defined, let $\left\{E_1^{(i)}, ..., E_{2n}^{(i)}\right\}$ be a symplectic frame on TU_i and let $\varphi_i : \pi^{-1}(U_i) \to U_i \times Sp(2n)$ be the induced trivialisation of the principal bundle P_ω, let $a \in \pi^{-1}(U_i)$, $a = \{X_1, ..., X_{2n}\}$, let $X_k = X_k^l E_l^{(i)}$ and $X^{(i)} = (X_k^l) \in Sp(2n)$, then $\varphi_i(a) = (\pi(a), X^{(i)})$, where we used, as we will do in the following, Einstein's convention on repeated indices. Let $\mathcal{A}_{(i)} \in \Lambda^1(U_i) \otimes sp(2n)$ be the local form of the connection, or *gauge potential*, [10], denote $\mathcal{A}_{(i)}(E_k^{(i)}) = ,_k^{(i)} \in sp(2n)$, then the horizontal lifting of $E_k^{(i)}$ on P_ω is given by: $\widetilde{E_k^{(i)}} = E_k^{(i)} \Leftrightarrow ,_k^{(i)} X^{(i)}$; thus we have $T_a P_\omega = \widetilde{H_a} \oplus \widetilde{V_a}$ where the horizontal subbundle $\widetilde{H_a}$ is spanned by $\left\{\widetilde{E_1^{(i)}}, ..., \widetilde{E_{2n}^{(i)}}\right\}$. From now on, for sake of simplicity, we omit the index (i) in notations.

Let $\xi \in T_a P_\omega$, we denote $\xi = \xi^h + \xi^v$ the decomposition in horizontal and vertical component; for the Lie bracket of horizontal vector fields we have:

$$\left[\widetilde{E_k}, \widetilde{E_l}\right] = \widetilde{[E_k, E_l]} + \left[\widetilde{E_k}, \widetilde{E_l}\right]^v$$

where

$$\left[\widetilde{E_k}, \widetilde{E_l}\right]^v = (\Leftrightarrow E_k(,_l) + E_l(,_k) + [,_l, ,_k])\, X.$$

In particular, denoted by $\mathcal{F} = (F_{kl}) \in \Lambda^2(U_i) \otimes sp(2n)$ the local form of the curvature of the connection θ, or *strength field*, [10], we have: $\left[\widetilde{E_k}, \widetilde{E_l}\right]^v = \Leftrightarrow F_{kl} X$.

Let $J = r(a)$ and let $H_J = r_{*a}(\widetilde{H_a})$, then $T_J Z_\omega = H_J \oplus V_J$, H_J is the horizontal tangent space to Z_ω at the point J defined by the connection θ, and V_J is the vertical tangent space.

We get: $H_J = \left\{\widehat{E_1}, ..., \widehat{E_{2n}}\right\}$, where $\widehat{E_i} = E_i \Leftrightarrow [,_i, J] \frac{\partial}{\partial J} = E_i \Leftrightarrow \widehat{,_i}$; and $V_J = \left\{\widehat{Y}(J)\,|Y \in sp(2n)\right\}$.

The almost complex structure $\mathbb{J}_\theta$ on Z_ω is then defined by:

$$\begin{cases} \mathbb{J}_\theta(J)(\widehat{E_i}) = J_i^k \widehat{E_k} \\ \mathbb{J}_\theta(J)(\widehat{Y}(J)) = \widehat{JY}(J) \end{cases}.$$

A direct computation gives conditions on the connection θ under which $\mathbb{J}_\theta$ is integrable.

Precisely we have:

$$\left[\widehat{E_i}, \widehat{E_j}\right] = \left(,\,{}^k_{ij} \Leftrightarrow ,\,{}^k_{ji} \Leftrightarrow T^k_{ij} \right) \widehat{E_k} \Leftrightarrow \widehat{F_{ij}}$$

$$\left[\widehat{E_i}, \widehat{A}\right] = \widehat{E_i(A)} + \widehat{[,\,_i, A]}$$

$$\left[\widehat{A}, \widehat{B}\right] = \Leftrightarrow\widehat{[A, B]}$$

where T is the torsion of the connection and $\widehat{A}, \widehat{B}$ are vertical vector fields defined by $A = A(x)$, $B = B(x) \in sp(2n)$ for any $x \in U$; then, denoted by $N_{\mathbb{J}_\theta}$ the Nijenhuis tensor of $\mathbb{J}_\theta$, we have:

$$\begin{aligned} N_{\mathbb{J}_\theta}(J)(\widehat{E_i}, \widehat{E_j}) &= \left[\mathbb{J}_\theta(\widehat{E_i}), \mathbb{J}_\theta(\widehat{E_j})\right] \Leftrightarrow \mathbb{J}_\theta \left[\mathbb{J}_\theta(\widehat{E_i}), \widehat{E_j}\right] + \\ &\Leftrightarrow \mathbb{J}_\theta \left[\widehat{E_i}, \mathbb{J}_\theta(\widehat{E_j})\right] \Leftrightarrow \left[\widehat{E_i}, \widehat{E_j}\right] \\ &= \left(\Leftrightarrow J^k_i J^l_j T^r_{kl} + J^k_i T^s_{kj} J^r_s + J^l_j T^s_{il} J^r_s + T^r_{ij}\right) \widehat{E_r} + \\ &+ \left(\Leftrightarrow J^k_i J^l_j \widehat{F_{kl}} + J^k_i \widehat{JF_{kj}} + J^l_j \widehat{JF_{il}} + \widehat{F_{ij}}\right) \end{aligned}$$

$$\begin{aligned} N_{\mathbb{J}_\theta}(J)(\widehat{E_i}, \widehat{A}) &= \left[\mathbb{J}_\theta(\widehat{E_i}), \mathbb{J}_\theta(\widehat{A})\right] \Leftrightarrow \mathbb{J}_\theta \left[\mathbb{J}_\theta(\widehat{E_i}), \widehat{A}\right] + \\ &\Leftrightarrow \mathbb{J}_\theta \left[\widehat{E_i}, \mathbb{J}_\theta(\widehat{A})\right] \Leftrightarrow \left[\widehat{E_i}, \widehat{A}\right] \\ &= J^k_i \left(\mathcal{L}_{\widehat{E_k}} \mathbb{J}_\theta\right) \left(\widehat{A}\right) \Leftrightarrow \mathbb{J}_\theta \left(\mathcal{L}_{\widehat{E_i}} \mathbb{J}_\theta\right) (\widehat{A}) \\ &= 0 \end{aligned}$$

where $\mathcal{L}$ means Lie derivative and $\left(\mathcal{L}_{\widehat{E_i}} \mathbb{J}_\theta\right) \left(\widehat{A}\right) = 0$ for all $i = 1, ..., 2n$, for all $\widehat{A} \in V$ because of the definition of $\mathbb{J}_\theta$ on vertical vector fields (the Lie derivative of $\mathbb{J}_{\theta_{|V}}$ along horizontal liftings vanishes); moreover:

$$N_{\mathbb{J}_\theta}(J)(\widehat{A}, \widehat{B}) = 0$$

because $\mathbb{J}_\theta$ restricted to the fibre is the standard complex structure $\overset{\vee}{J}$.

Thus the **integrability condition** can be written as:

$$\begin{cases} T(JX,JY) \Leftrightarrow JT(JX,Y) \Leftrightarrow JT(X,JY) \Leftrightarrow T(X,Y) = 0 \\ F(\widehat{JX,JY}) \Leftrightarrow JF(\widehat{JX,Y}) \Leftrightarrow JF(\widehat{X,JY}) \Leftrightarrow F(\widehat{X,Y}) = 0 \end{cases} \tag{2}$$

for all $X, Y \in T_xM$, for all $x \in M$, for all $J \in p^{-1}(x)$.

We remark that the problem of integrability condition also has been considered in [3], and [1].

4. Local sections

Let (M, ω) be an almost symplectic manifold and let Z_ω be its twistor bundle as before, let $U \subset M$ be an open set and let J be an almost complex structure on U, ω-calibrated, that is ω_x-calibrated for any $x \in U$. J defines a local section of Z_ω over U by $J(x) := J_x \in p^{-1}(x)$.

Let $\mathbb{J} = \mathbb{J}_\theta$ be the almost complex structure on Z_ω defined by a given connection θ on P_ω, we want to exploit conditions on J under which $(J(U), J)$ is an almost complex local submanifold of $(Z_\omega, \mathbb{J})$. We have the following results:

Lemma 4.1. *$(J(U), J)$ is an almost complex local submanifold of $(Z_\omega, \mathbb{J})$ if and only if for any $x \in U$, $X, Y \in T_xM$ we have:*

$$A(X,Y) := (\nabla_{JX} J)Y \Leftrightarrow J(\nabla_X J)Y = 0$$

where ∇ is the covariant derivative on M defined by the given connection on P_ω, using previous notations: $\nabla_{E_i} E_j = ,\,^k_{ij} E_k$.

Proof. We have: $\mathbb{J} \circ J_* = J_* \circ J$ if and only if $\mathbb{J}(J_*(E_l)) = J_*(J(E_l))$ for all $l = 1, ..., 2n$, or: $\mathbb{J}(\widehat{E_l} + \widehat{,\,_l} + \frac{1}{2}\widehat{E_l(J)}) = J^k_l(\widehat{E_k} + \widehat{,\,_k} + \frac{1}{2}\widehat{E_k(J)})$, that is: $\widehat{J,\,_l} = J^k_l \widehat{,\,_k}$, on the other hand is: $A(E_i, E_j) = (J^k_i \widehat{,\,_k} \Leftrightarrow \widehat{J,\,_i})^r_j E_r$. □

Lemma 4.2. *$A(X,Y) \Leftrightarrow A(Y,X) = N_J(X,Y) + T^{0,2}(J)(X,Y)$, where N_J denotes the Nijenhuis tensor of J and $T^{0,2}(J)(X,Y) := T(JX,JY) \Leftrightarrow JT(JX,Y) \Leftrightarrow JT(X,JY) \Leftrightarrow T(X,Y)$.*

Proof. The proof is just a computation:

$$
\begin{aligned}
&A(X,Y) \Leftrightarrow A(Y,X) = \nabla_{JX}JY \Leftrightarrow \nabla_{JY}JX + \nabla_Y X \Leftrightarrow \nabla_X Y + \\
&\Leftrightarrow J(\nabla_{JX}Y \Leftrightarrow \nabla_Y JX) + \Leftrightarrow J(\nabla_X JY \Leftrightarrow \nabla_{JY}X) \\
&= [JX,JY] + T(JX,JY) \Leftrightarrow [X,Y] \Leftrightarrow T(X,Y) \Leftrightarrow J\,[JX,Y] + \\
&\Leftrightarrow JT(JX,Y) \Leftrightarrow J\,[X,JY] + \Leftrightarrow JT(X,JY) \\
&= N_J(X,Y) + T^{0,2}(J)(X,Y).
\end{aligned}
$$

□

COROLLARY 4.3. *Let us suppose that the given connection satisfies* $T^{0,2}(J) \equiv 0$*, if* J *defines an almost complex local submanifold of* $(Z_\omega, \mathbb{J})$ *then* J *is integrable.*

REMARK 4.4. *In the case of the twistor space of an oriented even dimensional Riemannian manifold we have a canonical choice for the connection, namely the Levi Civita connection, and the analogous study of local sections gives that* J *defines an almost local submanifold if and only if* J *is integrable, [4], in this case, instead, we can construct examples of integrable local complex structures* J *such that* $A \neq 0$*.*

LEMMA 4.5. *Let* J *be an almost complex structure* ω*-calibrated on an open set* $U \subset M$*, then* $J(U)$ *is a horizontal section if and only if* $\nabla J = 0$*.*

Proof. We have that $J_*(E_i) \in H$ if and only if $\widehat{,}_i + \frac{1}{2}\widehat{JE_i(J)} = 0$ for any $i = 1, ..., 2n$, on the other hand is:

$$(\nabla_{E_i}J)(E_j) = E_i(J^l_j)E_l + J^l_j ,^k_{il} E_k \Leftrightarrow ,^k_{ij} J^l_k E_l = (E_i(J) + [,_i, J])^l_j E_l$$
$$= (\tfrac{1}{2}\widehat{JE_i(J)} + \widehat{,}_i)^l_j E_l.$$ □

COROLLARY 4.6. *If* $T^{0,2}(J) = 0$ *then* $\nabla J = 0$ *implies that* J *is integrable.*

COROLLARY 4.7. *If* $\nabla J = 0$ *and* J *is integrable then* $T^{0,2}(J) = 0$*.*

Proof. $\nabla J = 0$ implies that J is a horizontal section, in particular it defines an almost complex local submanifold and then $A = 0$, as $0 = N_J + T^{0,2}(J)$, $N_J = 0$ implies immediately that $T^{0,2}(J) = 0$. □

5. Almost Hermitian structure

Let Z_ω be the twistor bundle of the almost symplectic manifold (M,ω), let θ be a connection on P_ω and let $\mathbb{J} = \mathbb{J}_\theta$ be the corresponding almost complex structure on Z_ω, we can define, in a natural way, a Riemannian metric, $\mathbb{G} = \mathbb{G}_\theta$, on Z_ω: let $J \in Z_\omega$ and let $X, Y \in T_J Z_\omega$, decompose X and Y in horizontal and vertical components, $X = X^h + X^v$, $Y = Y^h + Y^v$, let $\overset{\vee}{g}$ be the canonical metric of the fibre $p^{-1}(p(J)) \cong \frac{Sp(2n)}{U(n)}$ defined by the Killing form on $sp(2n)$, then we pose $\mathbb{G}(X,Y) := \mathbb{G}(X^h,Y^h) + \mathbb{G}(X^v,Y^v)$, where: $\mathbb{G}(X^h,Y^h) := \omega(p_*(X^h), Jp_*(Y^h))$ and $\mathbb{G}(X^v,Y^v) := \overset{\vee}{g}(X^v,Y^v)$.

Using previous notations, in local coordinates, we have the following expression:

$$\begin{cases} \mathbb{G}(\widehat{E_i},\widehat{E_j}) & = & \omega(E_i, JE_j) = \Leftrightarrow {}^tE_i J_n J E_j = (\Leftrightarrow J_n J)^i_j \\ \mathbb{G}(\widehat{E_i},\widehat{A}) & = & 0 \\ \mathbb{G}(\widehat{A},\widehat{B}) & = & \frac{1}{2}Tr\widehat{A}\widehat{B} = Tr(AB + AJBJ) \end{cases} .$$

We get immediately that $(Z_\omega, \mathbb{J}, \mathbb{G})$ is an almost Hermitian manifold.

We want to exploit conditions under which $(Z_\omega, \mathbb{J}, \mathbb{G})$ is a Kähler manifold. Let $\eta(X,Y) = \mathbb{G}(X, \mathbb{J}Y)$ be the Kähler form, we have:

$$\begin{cases} \eta(\widehat{E_i},\widehat{E_j}) = (J_n)^i_j \\ \eta(\widehat{E_i},\widehat{A}) = 0 \\ \eta(\widehat{A},\widehat{B}) = \frac{1}{2}Tr(\widehat{AJB}) \end{cases}$$

thus, a direct computation of the differential of η gives:

$$\begin{cases} d\eta(\widehat{E_i},\widehat{E_j},\widehat{E_k}) & = & \Leftrightarrow d\omega(E_i,E_j,E_k) \\ d\eta(\widehat{E_i},\widehat{E_j},\widehat{A}) & = & \eta(\widehat{F_{ij}},\widehat{A}) = \frac{1}{2}Tr(\widehat{F_{ij}}\widehat{JA}) \\ d\eta(\widehat{E_i},\widehat{A},\widehat{B}) & = & 0 \\ d\eta(\widehat{A},\widehat{B},\widehat{C}) & = & 0 \end{cases} . \tag{3}$$

We have the following:

PROPOSITION 5.1. $(Z_\omega, \mathbb{J}, \mathbb{G})$ *is an almost Kähler manifold if and only if the following conditions hold: a)* (M,ω) *is a symplectic manifold; b) the curvature of the chosen connection vanishes.*

Proof. First remark that from (3) it follows that (M,ω) is a symplectic manifold if and only if $d\eta(\widehat{E_i},\widehat{E_j},\widehat{E_k}) = 0$ for all $i,j,k = 1,...,2n$;

second we have that $d\eta(\widehat{E_i}, \widehat{E_j}, \widehat{A}) = 0$ if and only if $\widehat{F_{ij}} = 0$ for all $i, j = 1, ..., 2n$, from arguments of linear algebra we get the statement in fact: $\widehat{F_{ij}} = 0 \Leftrightarrow [F_{ij}, P] = 0 \ \forall P \in \Im(n)$, in particular for $P = J_n$ we get $F_{ij} = \Leftrightarrow^t F_{ij}$, moreover, for generic $P = XJ_nX^{-1}$ is $\widehat{F_{ij}} = 0 \Leftrightarrow F_{ij}XJ_nX^{-1} \Leftrightarrow XJ_nX^{-1}F_{ij} = 0 \Leftrightarrow F_{ij}X^tXJ_n \Leftrightarrow X^tXJ_nF_{ij} = 0 \Leftrightarrow F_{ij}X^tXJ_n + X^tX^tF_{ij}J_n = 0 \Leftrightarrow F_{ij}X^tX \Leftrightarrow X^tXF_{ij} = 0 \Leftrightarrow [F_{ij}, X^tX] = 0 \ \forall X \in Sp(2n)$; if we take X diagonal matrix with all the elements distinct we get immediately that F_{ij} must be diagonal, and, as is antisymmetric, we have that $\widehat{F_{ij}} = 0$ if and only if $F_{ij} = 0$. From the expression of $d\eta$, (3), we have that the proof is complete. □

REMARK 5.2. *It is well known that (M, ω) is symplectic if and only if P_ω admits a torsion free connection, [7], then, chosen such a connection θ, or more generally with only (0,2)-part of the torsion zero, if $(Z_\omega, \mathbb{J}_\theta, \mathbb{G}_\theta)$ is almost Kähler then it is automatically Kähler because the integrability conditions* (2) *are satisfied.*

Now we will investigate conditions under which the manifold $(Z_\omega, \mathbb{J}, \mathbb{G})$ is semi-Kähler. Remember that an almost Hermitian manifold is called semi-Kähler if its Kähler form is co-closed, or equivalently if its *Lee form* is zero. We recall that given a $2k$-dimensional almost symplectic manifold (N, α) the **Lee form** of (N, α) is the 1-form Ψ defined by:

$$\Psi(X) := \frac{1}{2(k \Leftrightarrow 1)} d\alpha(X, X_i, \mu^{-1}(X_i^*))$$

where X is a vector field on N, $\{X_1, ..., X_{2k}\}$ is a local basis of vector fields, $\{X_1^*, ..., X_{2k}^*\}$ is the dual basis of 1-forms and μ is the musical isomorphism defined by α as $\mu(X)(Y) := \Leftrightarrow \alpha(X, Y)$, [5].

Using previous notations we compute the Lee form of (Z_ω, η), we have:
$\Psi(X) = \frac{1}{n(n+3)-2} d\eta(X, X_i, \mu^{-1}(X_i^*))$, where $\mu(X_i)(X_l) = \Leftrightarrow \eta(X_i, X_l)$; in particular $\mu^{-1}(\widehat{E_i}^*) = (J_n)^i_l \widehat{E_l}$, and, using (3), we get:

$$\begin{aligned} \Psi(\widehat{E_k}) &= \tfrac{1}{n(n+3)-2} d\eta(\widehat{E_k}, \widehat{E_i}, (J_n)^i_l \widehat{E_l}) \\ &= \tfrac{-1}{n(n+3)-2} (J_n)^i_l d\omega(E_k, E_i, E_l) \\ &= \tfrac{-2(n-1)}{n(n+3)-2} \widetilde{\Psi}(E_k) \end{aligned} \qquad (4)$$

where $\widetilde{\Psi}$ is the Lee form of (M,ω),
moreover:

$$\begin{aligned}\Psi(\widehat{A}) &= \tfrac{1}{n(n+3)-2} d\eta(\widehat{A},\widehat{E_i},(J_n)^i_l\widehat{E_l}) \\ &= \tfrac{1}{n(n+3)-2}(J_n)^i_l\eta(\left[\widehat{E_i},\widehat{E_l}\right],\widehat{A}) \\ &= \tfrac{-1}{n(n+3)-2}\mathbb{G}((J_n)^i_l\widehat{F_{il}},\widehat{JA})\end{aligned} \quad . \tag{5}$$

We can now state the following:

Proposition 5.3. $(Z_\omega,\mathbb{J},\mathbb{G})$ *is a semi-Kähler manifold if and only if the following conditions hold: a)* (M,ω) *is semi-Kähler; b)* ω *is in the* ker *of the curvature operator of the given connection.*

Proof. $(Z_\omega,\mathbb{J},\mathbb{G})$ is semi-Kähler if and only if $\Psi\equiv 0$, in particular condition a) follows immediately from (4). Moreover from condition (5) we have that $\Psi(\widehat{A})=0$ for any $\widehat{A}$ if and only if $\widehat{(J_n)^i_l F_{il}}=0$; the argument used in the proof of Proposition 5.1. gives that this condition is equivalent to $(J_n)^i_l F_{il}=0$, this is exactly condition b), after considering the curvature operator acting on 2-forms identifying vectors and 1-forms by ω. □

6. The twistor bundle $Z(\mathbb{R}^{2n},\omega_n)$

In this section we describe in details some interesting properties of the twistor bundle of $\mathbb{R}^{2n}$ with the standard symplectic form ω_n.

Let $P=P(\mathbb{R}^{2n},Sp(2n))$ be the principal bundle of symplectic frames on $(\mathbb{R}^{2n},\omega_n)$, and let $Z=Z(\mathbb{R}^{2n},\omega_n)$ be the corresponding twistor bundle; let $\{x^1,...,x^{2n}\}$ be coordinates on $\mathbb{R}^{2n}$, let $E_1=\frac{\partial}{\partial x^1},...,E_{2n}=\frac{\partial}{\partial x^{2n}}$ and $e=\{E_1,...,E_{2n}\}$, e defines global trivializations $P\cong\mathbb{R}^{2n}\times Sp(2n)$ and $Z\cong\mathbb{R}^{2n}\times\Im(n)$. Consider the connection θ_0 on P defined by the gauge potential $\mathcal{A}=0$, we have immediately that θ_0 has zero torsion and curvature, in particular, from previous results we get:

Proposition 6.1. $(Z,\mathbb{J}_{\theta_0},\mathbb{G}_{\theta_0})$ *is a Kähler manifold.*

Using symplectomorphisms of $(\mathbb{R}^{2n},\omega_n)$ we can construct more general connections on P such that the induced Hermitian structure on Z is Kähler. We proceed in the following way:

let $Symp(\mathbb{R}^{2n}, \omega_n)$ denote the set of symplectic diffeomorphisms of $(\mathbb{R}^{2n}, \omega_n)$, let $\ell \in Symp(\mathbb{R}^{2n}, \omega_n)$, pose $L^i_j = \frac{\partial \ell^i}{\partial x^j}$ and $L = (L^i_j)_{1 \leq i,j \leq 2n}$, we have that $L = L(x) \in Sp(2n)$ $\forall x \in \mathbb{R}^{2n}$ and $\mathcal{A} = L^{-1}dL \in \Lambda^1(\mathbb{R}^{2n}) \otimes sp(2n)$ defines a connection θ_ℓ on P, the following holds:

Lemma 6.2. *For any $\ell \in Symp(\mathbb{R}^{2n}, \omega_n)$ the connection θ_ℓ has zero torsion and zero curvature.*

Proof. The computation of the torsion T gives:

$T^h_{kl} = \,,^h_{kl} \Leftrightarrow ,^h_{lk} = (L^{-1})^h_r \frac{\partial L^r_l}{\partial x^k} \Leftrightarrow (L^{-1})^h_r \frac{\partial L^r_k}{\partial x^l}$

$= (L^{-1})^h_r \left(\frac{\partial^2 \ell^r}{\partial x^l \partial x^k} \Leftrightarrow \frac{\partial^2 l^r}{\partial x^k \partial x^l} \right) = 0;$

for the curvature we have:

$F_{kl} = \frac{\partial}{\partial x^k} \left(L^{-1} \frac{\partial L}{\partial x^l} \right) \Leftrightarrow \frac{\partial}{\partial x^l} \left(L^{-1} \frac{\partial L}{\partial x^k} \right) + \left[L^{-1} \frac{\partial L}{\partial x^k}, L^{-1} \frac{\partial L}{\partial x^l} \right]$

$= \Leftrightarrow \frac{\partial}{\partial x^k} \left(\frac{\partial L^{-1}}{\partial x^l} L \right) + \frac{\partial}{\partial x^l} \left(\frac{\partial L^{-1}}{\partial x^k} L \right) \Leftrightarrow \frac{\partial L^{-1}}{\partial x^k} \frac{\partial L}{\partial x^l} + \frac{\partial L^{-1}}{\partial x^l} \frac{\partial L}{\partial x^k} = 0.$ □

From previous Lemma and previous results we get immediately the following:

Proposition 6.3. *For any $\ell \in Symp(\mathbb{R}^{2n}, \omega_n)$ the twistor bundle $\left(Z(\mathbb{R}^{2n}, \omega_n), \mathbb{J}_{\theta_\ell}, \mathbb{G}_{\theta_\ell} \right)$ is Kähler.*

In the following we will compute the scalar curvature of the metric $\mathbb{G} = \mathbb{G}_{\theta_\ell}$; let us compute the covariant derivative, $\widehat{\nabla}$, defined by the Levi-Civita connection of $\mathbb{G}$, using previous notations we have:

$\mathbb{G}(\widehat{\nabla}_{\widehat{E_i}} \widehat{E_j}, \widehat{E_k}) = \,,^r_{ij} (\Leftrightarrow J_n J)^r_k$

$\mathbb{G}(\widehat{\nabla}_{\widehat{E_i}} \widehat{E_j}, \widehat{A}) = \frac{1}{2} (J_n [A, J])^i_j$

$\mathbb{G}(\widehat{\nabla}_{\widehat{A}} \widehat{E_i}, \widehat{E_j}) = \frac{1}{2} (\Leftrightarrow J_n [A, J])^i_j$

$\mathbb{G}(\widehat{\nabla}_{\widehat{A}} \widehat{E_i}, \widehat{B}) = 0$

$\mathbb{G}(\widehat{\nabla}_{\widehat{A}} \widehat{B}, \widehat{E_i}) = 0$

$\mathbb{G}(\widehat{\nabla}_{\widehat{A}} \widehat{B}, \widehat{C}) = \overset{\vee}{g} (\overset{\vee}{\nabla}_{\widehat{A}} \widehat{B}, \widehat{C})$

where $\overset{\vee}{\nabla}$ is the covariant derivative of the Levi-Civita connection of

$\overset{\vee}{g}$ on the fibre. In particular we get:

$$\begin{array}{lcl}
\widehat{\nabla}_{\widehat{E_i}}\widehat{E_j} & = & ,^k_{ij}\widehat{E_k} + \frac{1}{2}(J_n[Y_\alpha, J])^i_j\widehat{Y_\alpha} \\
\widehat{\nabla}_{\widehat{A}}\widehat{E_i} & = & \Leftrightarrow\frac{1}{2}(J[A,J])^k_i\widehat{E_k} \\
\widehat{\nabla}_{\widehat{E_i}}\widehat{A} & = & \widehat{\nabla}_{\widehat{A}}\widehat{E_i} + [\widehat{E_i}, \widehat{A}] \\
\widehat{\nabla}_{\widehat{A}}\widehat{B} & = & \overset{\vee}{\nabla}_{\widehat{A}}\,\widehat{B}
\end{array} \quad . \tag{6}$$

Moreover, denoted by $R(X,Y) = \nabla_X\nabla_Y \Leftrightarrow \nabla_Y\nabla_X \Leftrightarrow \nabla_{[X,Y]}$ the Riemann curvature tensor of $\mathbb{G}$, we have:

$$\mathbb{G}(R(\widehat{E_i},\widehat{E_j})\widehat{E_k},\widehat{E_l}) = \frac{1}{4}\Big\{(J_n[Y_\alpha,J])^i_k(J_n[Y_\alpha,J])^l_j + \Leftrightarrow (J_n[Y_\alpha,J])^j_k(J_n[Y_\alpha,J])^l_i\Big\}$$

$$\mathbb{G}(R(\widehat{E_i},\widehat{Y_\alpha})\widehat{E_j},\widehat{Y_\alpha}) = \frac{1}{2}(\Leftrightarrow\frac{1}{2}(J[Y_\alpha,J])^r_j(J_n[Y_\alpha,J])^i_r + \Leftrightarrow(J_n[Y_\beta,J])^i_j\mathbb{G}(\widehat{\nabla}_{\widehat{Y_\alpha}}\widehat{Y_\beta},\widehat{Y_\alpha}) \Leftrightarrow \widehat{Y_\alpha}((J_n[Y_\alpha,J])^i_j)$$

where $1 \le \alpha, \beta \le n(n+1)$ and $\{Y_\alpha\}_{1\le\alpha\le n(n+1)}$ is an orthonormal basis of the vertical tangent bundle, then, denoted by $\mathbb{G}^{ab}$ the ab entry of the inverse matrix of $\mathbb{G}$, by $\overset{\vee}{s}$ the scalar curvature of $\overset{\vee}{g}$, the scalar curvature of the metric $\mathbb{G}$, s, is given by:

$$\begin{aligned}
s =&\mathbb{G}(R(\widehat{E_i},\widehat{E_j})\widehat{E_s},\widehat{E_k})\mathbb{G}^{sj}\mathbb{G}^{ik} + 2\mathbb{G}(R(\widehat{E_i},\widehat{Y_\alpha})\widehat{Y_\alpha},\widehat{E_k})\mathbb{G}^{ik} + \\
&+ \mathbb{G}(R(\widehat{Y_\alpha},\widehat{Y_\beta})\widehat{Y_\beta},\widehat{Y_\alpha}) = \frac{1}{4}(\Leftrightarrow(J_n[Y_\alpha,J])^j_s(J_n[Y_\alpha,J])^k_i + \\
&+ (J_n[Y_\alpha,J])^i_s(J_n[Y_\alpha,J])^k_j)(JJ_n)^s_j(JJ_n)^k_i + \\
&+ \frac{1}{2}((\Leftrightarrow J[Y_\alpha,J])^r_k(J_n[Y_\alpha,J])^i_r \Leftrightarrow 2(J_n[Y_\beta,J])^i_k\mathbb{G}(\widehat{\nabla}_{\widehat{Y_\alpha}}\widehat{Y_\beta},\widehat{Y_\alpha}) + \\
&\Leftrightarrow 2(J_n[Y_\alpha,[Y_\alpha,J]])^i_k)(JJ_n)^k_i + \overset{\vee}{s} \\
= &\Leftrightarrow \frac{1}{4}(Tr[Y_\alpha,J]J)^2 + \frac{1}{4}Tr([Y_\alpha,J]J[Y_\alpha,J]J) + \\
&+ \frac{1}{2}Tr([Y_\alpha,J]J[Y_\alpha,J]J) + (Tr(J[Y_\beta,J]))\mathbb{G}(\widehat{\nabla}_{\widehat{Y_\alpha}}\widehat{Y_\beta},\widehat{Y_\alpha}) \\
&+ Tr([Y_\alpha,[Y_\alpha,J]J) + \overset{\vee}{s} = \frac{1}{2}\mathbb{G}(\widehat{Y_\alpha},\widehat{Y_\alpha}) + \mathbb{G}(\widehat{Y_\alpha},\widehat{Y_\alpha}) + \\
&\Leftrightarrow 2\mathbb{G}(\widehat{Y_\alpha},\widehat{Y_\alpha}) + \overset{\vee}{s} = \Leftrightarrow \frac{1}{2}n(n+1) + \overset{\vee}{s}
\end{aligned}$$

so far we proved the following:

PROPOSITION 6.4. *The scalar curvature of the metric $\mathbb{G}$ is constant negative.*

REFERENCES

[1] D.V. ALEKSEEVSKY AND M.M. GRAEV, *G-structures of twistor type and their twistor spaces*, Journal of Geometry and Physics **10** (1993), 203–229.

[2] M.F. ATIYAH AND N.J. HITCHIN, *Self duality in four-dimensional riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978), 425–461.

[3] L. BÉRARD BERGERY AND T. OCHIAI, *On the generalization of the construction of twistor spaces*, Glob. Riemannian Geometry (I.J. Willmore and N.J. Hitchin, eds.), 1984, pp. 52–59.

[4] P. DE BARTOLOMEIS AND A. NANNICINI, *Introduction to differential geometry of twistor spaces*, Geometric theory of singular phenomena in P.D.E. (J.P. Bourguignon, M. Giaquinta, and P. de Bartolomeis, eds.), Symposia Mathematica Academic Press, to appear.

[5] A. Gray and M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. **123** (1980), no. IV, 35–58.

[6] S. Helgason, *Differential geometry Lie groups and symmetric spaces*, Academic Press New York, 1978.

[7] S. Kobayashi, *Transformation groups in differential geometry*, Ergebnisse der Mathematik **70** (1972).

[8] D. McDuff and D. Salamon, *Introduction to symplectic topology*, Oxford Mathematical monographs, 1995.

[9] M.Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Inventiones Mathematicæ**82** (1985), 307–347.

[10] M. Nakahara, *Topology and physics*, 1990, Graduate student series in physics.

[11] O'Brian and W.J. Rawnsley, *Twistor spaces*, Ann. Global Anal. Geom. **3** (1985), 29–58.

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