Rend. Istit. Mat. Univ. Trieste
Suppl. Vol. XXIX, 215–265 (1998)

Measures in Convex Geometry

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Summary. - *By convex geometry we understand here the geometry of convex bodies in Euclidean space. In this field, measure theory enters naturally and is useful under several different aspects. First, like in many other fields, measures are employed to quantify the smallness of certain exceptional sets. In our first chapter, we give examples showing how Hausdorff measures of different dimensions are appropriate tools for describing sets of singular points or directions related to the boundary structure of convex bodies. In the second chapter we treat measures that are designed to reflect the local behaviour of convex bodies in a similar way as curvatures are used in differential geometry. The third connection between convex geometry and measure theory that we want to explain is of an entirely different nature. Here we treat a special class of convex bodies, the zonoids, which can be defined in terms of measures, and we show by an example from stochastic geometry how they are related to other fields. The second of these topics will be treated in greater detail than the other two.*

Naturally, some facts from the geometry of convex bodies will have to be used without proof. The fundamental notions will be explained and are easy to understand, due to their intuitive character. As a reference where proofs can be found, we mention the book [42].

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First presented at "Workshop di Teoria della Misura e Analisi Reale", Grado (Italy), September 18–29, 1995.

1. Hausdorff measures of singular sets

We work in n-dimensional Euclidean vector space $\mathbf{E}^n$ $(n \geq 2)$ with scalar product $\langle \cdot\,,\cdot \rangle$ and norm $\|\cdot\|$. By $B^n := \{x \in \mathbf{E}^n : \|x\| \leq 1\}$ we denote its unit ball and by $S^{n-1} := \{x \in \mathbf{E}^n : \|x\| = 1\}$ its unit sphere. Lebesgue measure in $\mathbf{E}^n$ is denoted by λ_n, and κ_n is the volume of B^n. For $p \geq 0$, $\mathcal{H}^p$ is the p-dimensional Hausdorff measure. A *convex body* in $\mathbf{E}^n$ is a non-empty compact convex subset. The set of these convex bodies is denoted by $\mathcal{K}^n$ and the subset of bodies with interior points by $\mathcal{K}^n_0$. For $K \in \mathcal{K}^n$ and a vector $u \in \mathbf{E}^n$ one defines

$$h(K,u) := \max\{\langle x,u \rangle : x \in K\}$$

and, for $u \neq 0$,

$$H(K,u) := \{x \in \mathbf{E}^n : \langle x,u \rangle = h(K,u)\}.$$

The function $h(K,\cdot)$ is known as the *support function* of K, and $H(K,u)$ is the *supporting hyperplane*, or briefly the *support plane*, of K with outer normal vector u.

Each point x in ∂K, the boundary of K, lies in at least one support plane $H(K,u)$ with suitable u, und we say that $H(K,u)$ is a *support plane at* x. If there are different support planes at x, then x is called a *singular point* of K. The example of a polytope shows that the set of singular points can be of positive $(n-2)$-dimensional Hausdorff measure.

Exercise 1.0.1: Construct an example in $\mathbf{E}^2$ showing that the set of singular points of a convex body K can be dense in the boundary ∂K.

The following useful theorem restricts the possible size of the set of singular points.

Theorem 1.0.2 (Reidemeister). *The set of singular points of a convex body $K \in \mathcal{K}^n$ is of $(n-1)$-dimensional Hausdorff measure zero.*

A considerably sharper result can be proved without much effort. We classify the singular points in the following way. The point $x \in \partial K$ is called *r-singular* if it lies in $n-r$ support planes with linearly independent normal vectors. It is also convenient to define the *normal cone* $N(K,x)$ of K at x as the set of all vectors $u \in \mathbf{E}^n$ such that either $u = 0$ or $H(K,u)$ is a support plane of K at x with outer normal vector u. Then $N(K,x)$ is a closed convex cone, and the point x is r-singular if and only if $\dim N(K,x) \geq n - r$.

THEOREM 1.0.3. *Let $K \in \mathcal{K}^n$ and $r \in \{0,\dots,n-1\}$. The set of r-singular points of K can be covered by countably many compact sets of finite r-dimensional Hausdorff measure.*

This implies, in particular, that the set of r-singular points of K has σ-finite $\mathcal{H}^r$-measure and, hence, $\mathcal{H}^p$-measure zero for any $p > r$.

The proof of Theorem 1.0.3 is surprisingly simple. It makes use of the *nearest-point map* (or *metric projection*) of K,

$$p(K,\cdot) : \mathbf{E}^n \to K,$$

which is defined by letting $p(K,x)$ be the unique point in K nearest to x.

EXERCISE 1.0.4: Show the existence of the map $p(K,\cdot)$ and show that it is contractive, that is, $\|p(K,x) - p(K,y)\| \leq \|x - y\|$ for all $x, y \in \mathbf{E}^n$.

Let $x \in \partial K$. It is not difficult to see that

$$p(K,y) = x \Leftrightarrow y - x \in N(K,x) \qquad \text{for } y \in \mathbf{E}^n \setminus K.$$

To prove Theorem 1.0.3, we choose a closed ball B containing K in its interior. By an *r-flat* we understand an r-dimensional affine subspace of $\mathbf{E}^n$, and we call it *rational* if it can be spanned by points with rational coordinates (with respect to a given basis of $\mathbf{E}^n$). Now let x be an r-singular boundary point of K. The normal cone $N(K,x)$ is of dimension at least $n-r$, hence the translated cone $N(K,x)+x$ must meet some rational r-flat F inside B. A point $y \in F \cap (N(K,x) + x)$ satisfies $p(K,y) = x$; thus $x \in p(K, F \cap B)$. The compact set $F \cap B$ has finite r-dimensional Hausdorff measure. The same is true for its image $p(K, F \cap B)$, since the nearest-point map is contracting. Since there are only countably many rational r-flats, we see that the set of r-singular points of K can be covered by countably many compact sets of finite $\mathcal{H}^r$-measure. This completes the proof of Theorem 1.0.3.

Our second example for the use of Hausdorff measures in describing the boundary structure of convex bodies concerns segments in the boundary. Can the boundary of a convex body contain segments in all directions? The answer is “no”, but the proof is not easy. It gives a stronger result; a simpler proof of the weaker result is not known.

For a convex body $K \in \mathcal{K}^n$, we denote by $U(K) \subset S^{n-1}$ the set of all unit vectors that are parallel to some segment in the boundary of K. The example of a polytope shows that $U(K)$ can contain great subspheres of S^{n-1}. Obviously, there are convex bodies K for which $U(K)$ contains infinitely many great subspheres, which are sets of positive $(n-2)$-dimensional measure. The following theorem is, therefore, best possible.

THEOREM 1.0.5 (EWALD-LARMAN-ROGERS). *For every convex body $K \in \mathcal{K}^n$, the set $U(K)$ is of σ-finite $(n-2)$-dimensional Hausdorff measure.*

We cannot give here the full proof, but want to sketch the principal ideas. These become already clear if we restrict ourselves to the three-dimensional case. Let $K \in \mathcal{K}_0^3$ be a convex body. We consider two parallel planes H_0, H_1 intersecting K. It suffices to prove that the set U^* of all unit vectors parallel to a segment in ∂K that cuts both planes H_0, H_1 has finite one-dimensional Hausdorff measure. This is sufficient, since every segment in ∂K cuts some pair of rational planes, and there are only countably many pairs of this kind.

Since directions of boundary segments do not change under homotheties and under the addition of a ball to the convex body, we may assume that $K = K' + B^3$ with a convex body K' and that $H_0 = \mathbf{E}^2 - e$, $H_1 = \mathbf{E}^2 + e$ with $\mathbf{E}^2 \subset \mathbf{E}^3$ and a unit vector e orthogonal to $\mathbf{E}^2$. Let $K_0 := K \cap H_0$, $K_1 := K \cap H_1$ and

$$\Lambda := \{x_1 - x_0 : x_1 \in K_1, x_0 \in K_0,\ \exists u \in S^{n-1} : x_0, x_1 \in H(K,u)\}.$$

If $S \subset \partial K$ is a segment meeting H_0 and H_1, then it lies in a support plane of K, hence the points x_0, x_1 defined by $S \cap H_0 = \{x_0\}$, $S \cap H_1 = \{x_1\}$ satisfy $x_1 - x_0 \in \Lambda$. This observation has a converse, and it follows that

$$U^* = \left\{ \frac{y}{\|y\|} : y \in \Lambda \cup (-\Lambda) \right\}.$$

Since U^* is the image of $\Lambda \cup (-\Lambda)$ under a Lipschitz map, it is sufficient to prove that Λ has finite one-dimensional Hausdorff measure.

Let $x_1 - x_0 \in \Lambda$, with $x_0 \in K_0$ and $x_1 \in K_1$ lying in a support plane of K. Then the points $x_1 - e$ and $x_0 + e$ lie in parallel support lines of the two-dimensional convex bodies $K_1' := K_1 - e$ and $K_0' := K_0 + e$, respectively. Observe that K_0', K_1' are convex bodies in $\mathbf{E}^2$. Let Λ' be the set of all differences $y_1 - y_0$ of boundary points y_1 of K_1' and $y_0 \in K_0'$ lying in parallel support lines. Essentially, we have to show that Λ' is a small set. This is achieved by using certain cap coverings. By a *cap* of a convex body in $\mathbf{E}^2$ one understands the part of the body cut off by a line. The following lemma is elementary.

LEMMA 1.0.6. *If $C_1, \dots, C_m$ are caps of $K_0' + K_1'$ covering the boundary of $K_0' + K_1'$, then*

$$\Lambda' \subset \bigcup_{i=1}^{m} (\mathrm{D}C_i + a_i),$$

where $\mathrm{D}C_i := \{x - y : x, y \in C_i\}$ and $a_1, \dots, a_m$ are suitable translation vectors.

Let $C_1, \dots, C_m$ be as in the lemma. If now x_0, x_1 are as above, then $(x_1 - e) - (x_0 + e) \in \Lambda'$, hence there is an index i such that $(x_1 - e) - (x_0 + e) \in \mathrm{D}C_i + a_i$ and thus $x_1 - x_0 \in \mathrm{D}C_i + a_i + 2e$. Thus

$$\Lambda \subset \bigcup_{i=1}^{m} (\mathrm{D}C_i + t_i)$$

with suitable translation vectors t_i.

The bulk of the work now consists in showing that the boundary of the two-dimensional body $K_0' + K_1'$ can be covered by caps $C_1, \dots, C_m$ in a very economical way, that is, without too much overlapping. More precisely, for every sufficiently small $\epsilon > 0$ one needs a covering of the boundary by caps whose widths are essentially of order ϵ and such that the sum of their volumes is also of order ϵ. This can be done, but requires a series of technical arguments from convex geometry. The result is as follows. There exists a constant c such that, for every sufficiently small $\epsilon > 0$, the set Λ can be covered by less than $c\epsilon^{-1}$ cubes of diameter ϵ. It follows that Λ has finite one-dimensional Hausdorff measure.

This was only a brief sketch of a proof that works in all dimensions. Theorem 1.0.5 implies, in particular, that the set $U(K)$ of directions of boundary segments is of $(n-1)$-dimensional Hausdorff measure zero. It is this weaker result that is needed in applications to the integral geometry of convex bodies.

PROBLEM 1.0.7: Is there a short proof of this weaker result?

In certain investigations about integral geometry and touching probabilities of convex bodies, the following extension of Theorem 1.5 is needed. Here the Hausdorff measure on the rotation group SO_n of $\mathbf{E}^n$ is derived from a natural metric on SO_n.

THEOREM 1.0.8. *Let $K, K' \in \mathcal{K}^n$ be convex bodies, and let $U(K, K')$ be the set of all rotations $\rho \in SO_n$ for which K and $\rho K'$ contain parallel segments lying in parallel support planes. Then $U(K, K')$ has σ-finite $[\frac{1}{2}n(n-1)-1]$-dimensional Hausdorff measure.*

Observe that the Lie group SO_n has dimension $\frac{1}{2}n(n-1)$; hence it follows from Theorem 1.0.8 that $U(K, K')$ is a set of Haar measure zero in the rotation group. Theorem 1.0.5 essentially corresponds to the special case of 1.8 where K' is a segment.

It has recently turned out that for some questions in the integral geometry of convex bodies one would need a result that is in a vague sense dual to the consequence of Theorem 1.0.8 noted above. Let $K, K' \in \mathcal{K}_0^n$ be convex bodies. We say that they are *in singular position* if there is a point $x \in \partial K \cap \partial K'$ such that

$$\operatorname{lin} N(K, x) \cap \operatorname{lin} N(K', x) \neq \{0\},$$

where lin denotes the linear hull.

OPEN PROBLEM 1.0.9: Is the set of all rigid motions g such that K and gK' are in singular position a set of Haar measure zero in the motion group of $\mathbf{E}^n$?

This question came up in the work of Glasauer [21]. The answer is known to be affirmative in dimensions $n \leq 3$.

Notes. The proof of Theorem 1.0.3 given here goes back to Anderson and Klee [8]. The covering assertion of Theorem 1.0.3 can be refined considerably. Such refinements are due to Zajíček [51], Alberti, Ambrosio and Cannarsa [2], Alberti [1], Anzellotti and Serapioni [9], Fu and Ossanna [20]. From the latter paper we quote that the set of r-singular points of a convex body can be covered by a set M_0 with $\mathcal{H}^r(M_0) = 0$ and countably many r-dimensional embedded submanifolds of class C^2.

Theorem 1.0.5 was proved by Ewald, Larman and Rogers [17], and Theorem 1.0.8 by Schneider [39]. Full proofs of both Theorems are reproduced in [42], Section 2.3. The Notes for that section give references to further related results.

2. Curvature measures

As a motivation for the measures to be introduced, let us first consider the notion of curvatures in elementary differential geometry. Let F be an oriented surface in Euclidean space $\mathbf{E}^3$, and suppose that F is twice continuously differentiable and regular (in the sense of differential geometry). At every point of F there are two principal curvatures, k_1 and k_2. Their arithmetic mean is the mean curvature, $H = \frac{1}{2}(k_1 + k_2)$, and their product is the Gauss curvature, $K = k_1 k_2$. Closed surfaces with constant H or constant K are of particular interest, the first, for instance, as forms of soap bubbles in equilibrium, the second, since they locally realize the geometry of the elliptic plane. Classical theorems tell us that a closed convex surface F with $H = \text{const}$ or $K = \text{const}$ must be a sphere. Can these theorems be extended to general convex surfaces, that is, boundaries of convex bodies, without assuming any differentiability? The first step towards such a generalization has to be the extension of the mean curvature and the Gauss curvature to general convex bodies. Such an extension is possible if measures are used. A hint how this can be done is given by the familiar intuitive interpretation of the integrated Gauss curvature: If β is a measurable subset of F, then $\int_\beta K \, d\mathcal{H}^2$ is the area of the spherical image $\sigma(F, \beta)$; the latter is the subset of the unit sphere S^2 which is defined as the set of all exterior unit normal vectors to F at points of β. Clearly

$$C_0(F, \beta) := \mathcal{H}^2(\sigma(F, \beta))$$

defines a measure $C_0(F, \cdot)$ on F. Now, this measure can be defined for an arbitrary convex surface. The assumption of constant Gauss curvature can then be replaced by the assumption that $C_0(F, \cdot)$ be proportional to the surface area measure. We shall see that this does in fact lead to a characterization of the ball.

We may also reverse the viewpoint above. Instead of a measurable set β of points on F, we consider a measurable set ω of unit normal vectors, thus ω is a set on the unit sphere S^2. Let $\tau(F,\omega)$ be the set of points of F where the outer unit normal falls in ω. We call $\tau(F,\omega)$ the reverse spherical image of the set ω. Its area,

$$S_2(F,\omega) := \mathcal{H}^2(\tau(F,\omega)),$$

again defines a measure, this time on the sphere. If F is of class C^2 and has positive curvatures, then $S_2(F,\cdot)$ is the indefinite integral of the reciprocal Gauss curvature, hence of the product of the principal radii of curvature, considered as functions of the exterior unit normal vector. The well-known Minkowski problem of differential geometry asks for the existence of a closed convex surface for which the Gauss curvature is prescribed as a function on the spherical image. A version of this problem for general convex surfaces asks for a convex surface for which the measure S_2, the area of the reverse spherical image, is prescribed. We shall see that this problem has a complete solution.

2.1. Support measures and curvature measures

The introduction has given us some idea how a Gaussian curvature measure and its reverse counterpart can be defined for general convex bodies. So far, however, a measure theoretic extension of the mean curvature is not in sight. The consideration of local parallel sets, instead of spherical images, will yield such measures easily. Since we want to have measures on the boundary of a convex body on one hand and measures on normal vectors on the other hand, we shall first introduce a common generalization.

In Chapter 1, we have defined the nearest-point map $p(K,\cdot) : \mathbf{E}^n \to K$ of a convex body K. For $x \in \mathbf{E}^n$ we also define $d(K,x) := \|x - p(K,x)\|$, and for $x \in \mathbf{E}^n \setminus K$,

$$u(K,x) := \frac{x - p(K,x)}{\|x - p(K,x)\|}.$$

Then $u(K,x)$ is an outer unit normal vector to K at x. A pair (x,u), where $x \in \partial K$ and u is an outer unit normal vector of K at x, is called a *support element* of K. The set of all support elements of K is denoted by $\operatorname{Nor} K$ and called the *generalized unit normal bundle* of K. It is a closed subset of the cartesian product $\mathbf{E}^n \times S^{n-1} =: \Sigma$.

We use the notation $\mathcal{B}(T)$ for the σ-algebra of Borel subsets of a topological space T.

Now let $K \in \mathcal{K}^n$. For a Borel set $\eta \in \mathcal{B}(\Sigma)$ and for $\rho > 0$ we define the *local parallel set*

$$M_\rho(K,\eta) := \{x \in \mathbf{E}^n : 0 < d(K,x) \le \rho,\ (p(K,x), u(K,x)) \in \eta\}.$$

EXERCISE 2.1.1: Show that this is a Borel set.

Recall that we denote the Lebesgue measure on $\mathbf{E}^n$ by λ_n. We define

$$\mu_\rho(K,\eta) := \lambda_n(M_\rho(K,\eta)).$$

Clearly this defines a measure $\mu_\rho(K,\cdot)$ on $\mathcal{B}(\Sigma)$. The reason for defining this measure becomes clear if one computes it in the case where ∂K is a regular hypersurface of class C^2 and $\eta = \{(x, u(x)) : x \in \beta\}$, where $\beta \subset \partial K$ is open and $u(x)$ is the unique outer unit normal vector of ∂K at x. Elementary differential geometry then yields that

$$\mu_\rho(K,\eta) = \frac{1}{n} \sum_{j=0}^{n-1} \rho^{n-j} \binom{n}{j} \int_\beta H_{n-1-j}\, d\mathcal{H}^{n-1},$$

where H_i denotes the i-th normalized elementary symmetric function of the principal curvatures. For example, for $n = 3$ we have

$$\mu_\rho(K,\eta) = \frac{1}{3}\rho^3 \int_\beta H_2\, d\mathcal{H}^2 + \rho^2 \int_\beta H_1\, d\mathcal{H}^2 + \rho \int_\beta d\mathcal{H}^2,$$

where now H_2 denotes the Gauss curvature and H_1 is the mean curvature. Thus, $\mu_\rho(K,\eta)$ is a polynomial in ρ, and the coefficients will give us the measures we are looking for. We will, however, not perform this differential-geometric calculation, but rather exhibit the polynomial behaviour in the more elementary case of polytopes.

If P is a polytope (by which we always mean a convex polytope), we denote by $\mathcal{F}_j(P)$ the set of its j-dimensional faces, $j = 0, \dots, n$. For a face F of P, the normal cone $N(P, F)$ of P at F is defined as $N(P, x)$ for an (arbitrary) point x in the relative interior of F.

By decomposing the local parallel set $M_\rho(P, \eta)$ into the sets of points x for which $p(P, x)$ lies in the relative interior of a given face of P, and by applying Fubini's theorem, one finds that

$$\mu_\rho(P, \eta) = \sum_{j=0}^{n-1} \rho^{n-j} \frac{1}{n-j} \sum_{F \in \mathcal{F}_j(P)} \int_F \mathcal{H}^{n-1-j}(N(P, F) \cap \eta_y)\, d\mathcal{H}^j(y), \tag{1}$$

where

$$\eta_y := \{u \in S^{n-1} : (y, u) \in \eta\}$$

is the y-section of the set η.

Exercise 2.1.2: Perform the proof of (1), at least for $n = 2$ and $n = 3$, taking some "obvious" properties of polytopes for granted.

Our next task is to extend the polynomial behaviour of $\mu_\rho(P, \eta)$ to general convex bodies. This is done by approximation. For this purpose, the set $\mathcal{K}^n$ of convex bodies is equipped with the *Hausdorff metric* δ, defined by

$$\begin{aligned} \delta(K, L) &:= \max\left\{\max_{x \in K} \min_{y \in L} \|x - y\|,\ \max_{x \in L} \min_{y \in K} \|x - y\|\right\} \\ &= \min\left\{\alpha \geq 0 : K \subset L + \alpha B^n,\ L \subset K + \alpha B^n\right\}. \end{aligned}$$

It is easy to see that the set of polytopes is dense in the space $(\mathcal{K}^n, \delta)$. Thus we can approximate a given convex body K by a sequence $(K_j)_{j \in \mathbf{N}}$ of polytopes, and we want to show that the corresponding measures $\mu_\rho(K_j, \cdot)$ converge weakly to $\mu_\rho(K, \cdot)$. Weak convergence is denoted by $\stackrel{w}{\to}$. By a familiar characterization of the weak convergence of measures, we have to prove that $K_j \to K$ in $\mathcal{K}^n$ implies

$$\mu_\rho(K, \eta) \leq \liminf_{j \to \infty} \mu_\rho(K_j, \eta) \qquad \text{for open } \eta$$

and

$$\mu_\rho(K, \Sigma) = \lim_{j \to \infty} \mu_\rho(K_j, \Sigma).$$

THEOREM 2.1.3. *If $K_j \to K$ in $\mathcal{K}^n$, then $\mu_\rho(K_j,\cdot) \xrightarrow{w} \mu_\rho(K,\cdot)$.*

Proof. First let $\eta \subset \Sigma$ be open, and let $x \in M_\rho(K,\eta)$ be a point with $d(K,x) < \rho$. The continuity of the nearest-point map, as a map on the product space $\mathcal{K}^n \times \mathbf{E}^n$, implies that $x \in M_\rho(K_j,\eta)$ for almost all j. Thus we have

$$M_\rho(K,\eta) \setminus \partial K_\rho \subset \liminf_{j\to\infty} M_\rho(K_j,\eta)$$

and hence, using $\lambda_n(\partial K_\rho) = 0$,

$$\begin{aligned}
\mu_\rho(K,\eta) &= \lambda_n(M_\rho(K,\eta) \setminus \partial K_\rho) \\
&\le \lambda_n\left(\liminf_{j\to\infty} M_\rho(K_j,\eta)\right) \\
&\le \liminf_{j\to\infty} \lambda_n(M_\rho(K_j,\eta)) \\
&= \liminf_{j\to\infty} \mu_\rho(K_j,\eta).
\end{aligned}$$

Moreover, since the volume is a continuous function on $\mathcal{K}^n$, we get

$$\begin{aligned}
\lim_{j\to\infty} \mu_\rho(K_j,\Sigma) &= \lim_{j\to\infty} [\lambda_n(K_j + \rho B^n) - \lambda_n(K_j)] \\
&= \lambda_n(K + \rho B^n) - \lambda_n(K) \\
&= \mu_\rho(K,\Sigma),
\end{aligned}$$

which completes the proof. □

Theorem 2.1.3 tells us how the measures $\mu_\rho(K,\cdot)$ depend on K. We must also know how $\mu_\rho(K,\eta)$, for fixed η, depends on K.

THEOREM 2.1.4. *For fixed $\rho \ge 0$ and for $\eta \in \mathcal{B}(\Sigma)$, the function $\mu_\rho(\cdot,\eta) : \mathcal{K}^n \to \mathbf{R}$ is Borel measurable.*

EXERCISE 2.1.5: Prove this, along the following lines. In the proof of Theorem 2.1.3 it was shown that $K_j \to K$ implies

$$\liminf_{j\to\infty} \mu_\rho(K_j,\eta) \ge \mu_\rho(K,\eta),$$

if η is open. Why does this imply that $\mu_\rho(K,\eta)$ is measurable for open η? Next, show that the family of all sets $\eta \in \mathcal{B}(\Sigma)$ for which $\mu_\rho(\cdot,\eta)$ is measurable is a Dynkin system (for Dynkin systems, see Bauer [11], §2).

We are now in a position to define generalized curvature measures or *support measures*, as we call them briefly, following a suggestion of S. Glasauer [21].

THEOREM AND DEFINITION 2.1.6. *For every convex body $K \in \mathcal{K}^n$ there exist finite measures $\Theta_0(K,\cdot),\dots,\Theta_{n-1}(K,\cdot)$ on $\mathcal{B}(\Sigma)$ such that, for every $\eta \in \mathcal{B}(\Sigma)$ and every $\rho > 0$, the measure $\mu_\rho(K,\eta)$ of the local parallel set $M_\rho(K,\eta)$ is given by*

$$\mu_\rho(K,\eta) = \frac{1}{n}\sum_{m=0}^{n-1} \rho^{n-m}\binom{n}{m}\Theta_m(K,\eta).$$

The map $\Theta_m : \mathcal{K}^n \times \mathcal{B}(\Sigma) \to \mathbf{R}$ has the following properties. If $K_j \to K$ in $\mathcal{K}^n$, then $\Theta_m(K_j,\cdot) \xrightarrow{w} \Theta_m(K,\cdot)$. For each $\eta \in \mathcal{B}(\Sigma)$, the function $\Theta_m(\cdot,\eta)$ is measurable.

The measure $\Theta_m(K,\cdot)$ is called the m-th support measure of K.

Proof. If P is a polytope, we define

$$\binom{n-1}{m}\Theta_m(P,\eta) := \sum_{F\in\mathcal{F}_m(P)} \int_F \mathcal{H}^{n-1-m}(N(P,F)\cap\eta_y)\, d\mathcal{H}^m(y). \tag{2}$$

From (1) we then know that

$$\mu_\rho(P,\eta) = \frac{1}{n}\sum_{m=0}^{n-1} \rho^{n-m}\binom{n}{m}\Theta_m(P,\eta).$$

Writing down this equation for $\rho = 1,\dots,n$, we obtain n linear equations, which can be solved for the Θ's, and we get

$$\Theta_m(P,\eta) = \sum_{k=1}^{n} a_{mk}\mu_k(P,\eta)$$

with coefficients a_{mk}. We define

$$\Theta_m(K,\eta) := \sum_{k=1}^{n} a_{mk}\mu_k(K,\eta) \tag{3}$$

for arbitrary $K \in \mathcal{K}^n$. Then $\Theta_m(K,\cdot)$ is a finite signed measure. If $K_j \to K$ in $\mathcal{K}^n$, then $\Theta_m(K_j,\cdot) \xrightarrow{w} \Theta_m(K,\cdot)$ by Theorem 2.1.3.

Since any $K \in \mathcal{K}^n$ is the limit of a sequence $(K_j)_{j\in\mathbf{N}}$ of polytopes and $\Theta_m(K_j,\cdot) \geq 0$, we deduce that $\Theta_m(K,\cdot)$ is a measure. From (3) and Theorem 2.1.4 it follows that $\Theta_m(\cdot,\eta)$ is measurable. □

EXERCISE 2.1.7: (a) Show that $\Theta_m(K,\cdot)$ is concentrated on the set Nor K of support elements of K (hence the name "support measure").

(b) Show that Θ_m has the additivity (or valuation) property

$$\Theta_m(K_1 \cup K_2,\cdot) + \Theta_m(K_1 \cap K_2,\cdot) = \Theta_m(K_1,\cdot) + \Theta_m(K_2,\cdot)$$

for $K_1, K_2 \in \mathcal{K}^n$ with $K_1 \cup K_2 \in \mathcal{K}^n$. (Hint: Show first a corresponding property for the indicator function of the local parallel set $M_\rho(K,\eta)$.)

We now specialize the support measures, to obtain measures either on sets of boundary points or on sets of unit normal vectors. For this, we put

$$\begin{aligned} C_m(K,\beta) &:= \Theta_m(K,\beta \times S^{n-1}) \qquad \text{for } \beta \in \mathcal{B}(\mathbf{E}^n),\\ S_m(K,\omega) &:= \Theta_m(K,\mathbf{E}^n \times \omega) \qquad \text{for } \omega \in \mathcal{B}(S^{n-1}). \end{aligned}$$

The measure $C_m(K,\cdot)$ is called the m-th *curvature measure* of K, and $S_m(K,\cdot)$ is called the m-th *area measure* of K.

We first have a closer look at the curvature measures $C_m(K,\cdot)$, which are measures concentrated on the boundary of the convex body K. With

$$\begin{aligned} A_\rho(K,\beta) &:= M_\rho(K,\beta \times S^{n-1})\\ &= \{x \in \mathbf{E}^n : 0 < d(K,x) \leq \rho,\ p(K,x) \in \beta\}, \end{aligned}$$

the definition reads

$$\lambda_n(A_\rho(K,\beta)) = \frac{1}{n}\sum_{m=0}^{n-1} \rho^{n-m}\binom{n}{m} C_m(K,\beta).$$

Considering this identity for either $\rho \to 0$ or $\rho \to \infty$, it is easy to guess the meaning of the two measures $C_{n-1}(K,\cdot)$ and $C_0(K,\cdot)$. The set $\sigma(K,\beta)$, the *spherical image of K at β*, is defined as the set of all outer unit normal vectors to K at points of β.

THEOREM 2.1.8. *Let $K \in \mathcal{K}^n$ and $\beta \in \mathcal{B}(\mathbf{E}^n)$. Then*

$$C_0(K,\beta) = \mathcal{H}^{n-1}(\sigma(K,\beta)). \tag{4}$$

If K is n-dimensional, then

$$C_{n-1}(K,\beta) = \mathcal{H}^{n-1}(\beta \cap \partial K). \tag{5}$$

Sketch of the proof. First one has to observe that $\sigma(K,\beta)$, for a Borel set β, is not necessarily a Borel set. However, the following is true. If $\beta_1, \beta_2 \in \mathcal{B}(\mathbf{E}^n)$ are disjoint and if $u \in \sigma(K,\beta_1) \cap \sigma(K,\beta_2)$, then u is a singular normal vector of K, that is, a normal vector at two different boundary points. This notion is dual to the notion of a singular boundary point of K. Dualizing Reidemeister's theorem, one deduces that

$$\mathcal{H}^{n-1}(\sigma(K,\beta_1) \cap \sigma(K,\beta_2)) = 0.$$

This can be used to show that $\sigma(K,\beta)$, for $\beta \in \mathcal{B}(\mathbf{E}^n)$, is $\mathcal{H}^{n-1}$-measurable, and also that the function $\kappa(K,\cdot)$ defined by

$$\kappa(K,\beta) := \mathcal{H}^{n-1}(\sigma(K,\beta)) \qquad \text{for } \beta \in \mathcal{B}(\mathbf{E}^n)$$

is a measure. Next one shows that $K_j \to K$ implies the weak convergence $\kappa(K_j,\cdot) \xrightarrow{w} \kappa(K,\cdot)$. Now the equality $C_0(K,\cdot) = \kappa(K,\cdot)$ follows, since it is true for polytopes and both sides are weakly continuous functions of K.

For the proof of (5), the procedure is similar. One defines

$$\eta(K,\beta) := \mathcal{H}^{n-1}(\beta \cap \partial K) \qquad \text{for } \beta \in \mathcal{B}(\mathbf{E}^n)$$

and then has to show that $K_j \to K$ implies $\eta(K_j,\cdot) \xrightarrow{w} \eta(K,\cdot)$. For the complete proof, see [42], Theorem 4.2.5.

For the remaining curvature measures $C_1(K,\cdot),\dots,C_{n-2}(K,\cdot)$, we can give explicit representations if K is either a polytope or sufficiently smooth. Let P be a polytope and $F \in \mathcal{F}_k(P)$ a k-face of P ($k \in \{0,\dots,n-1\}$). The *external angle* of P at F is defined by

$$\gamma(F,P) := \frac{\mathcal{H}^{n-k-1}(N(P,F) \cap S^{n-1})}{\mathcal{H}^{n-k-1}(S^{n-k-1})}.$$

By specializing (2), we see that

$$C_m(P,\beta) = \frac{n\kappa_{n-m}}{\binom{n}{m}} \sum_{F\in\mathcal{F}_m(P)} \gamma(F,P)\mathcal{H}^m(F\cap\beta),$$

where κ_i denotes the volume of the i-dimensional unit ball. Thus, $C_m(P,\cdot)$ has a simple structure. It is concentrated on the set of m-faces of P, and on each m-face, it is proportional to the m-dimensional Hausdorff (or Lebesgue) measure, and weighted by the external angle of P at that face.

If K is a convex body whose boundary is a regular hypersurface of class C^2, then one can compute that

$$C_m(K,\beta) = \int_{\beta\cap\partial K} H_{n-1-m}\, d\mathcal{H}^{n-1}.$$

This explains the name "curvature measure".

Now we have a closer look at the area measures $S_m(K,\cdot)$. Let $\omega \in \mathcal{B}(S^{n-1})$. With

$$\begin{aligned} B_\rho(K,\omega) &:= M_\rho(K,\mathbf{E}^n\times\omega) \\ &= \{x\in\mathbf{E}^n : 0 < d(K,x) \le \rho,\ u(K,x)\in\omega\} \end{aligned}$$

we see from the definitions that

$$\lambda_n(B_\rho(K,\omega)) = \frac{1}{n}\sum_{m=0}^{n-1} \rho^{n-m}\binom{n}{m} S_m(K,\omega).$$

The measure $S_0(K,\cdot)$ does not depend on K; it is just the spherical Lebesgue measure. As a counterpart to Theorem 2.1.8, we obtain the following result by a similar argument. Here $\tau(K,\omega)$, the *reverse spherical image of* K *at* ω, is the set of all boundary points of K with an outer unit normal vector belonging to ω.

THEOREM 2.1.9. *Let* $K\in\mathcal{K}_0^n$ *and let* $\omega\in\mathcal{B}(S^{n-1})$. *Then*

$$S_{n-1}(K,\omega) = \mathcal{H}^{n-1}(\tau(K,\omega)). \tag{6}$$

If P is a polytope, then

$$S_m(P,\omega) = \frac{1}{\binom{n-1}{m}} \sum_{F\in\mathcal{F}_m(P)} \mathcal{H}^m(F)\mathcal{H}^{n-1-m}(N(P,F)\cap\omega).$$

If the boundary of K is of class C^2 and has everywhere positive curvatures, then

$$S_m(K,\omega) = \int_\omega s_m \, d\mathcal{H}^{n-1}.$$

Here $s_m(u)$ denotes the m-th normalized elementary symmetric function of the principal radii of curvature of ∂K at the unique point with exterior unit normal vector u.

While the two series of measures, the curvature measures and the area measures, exhibit a strong duality, there is one important difference. The curvature measures are concentrated on the boundaries of the convex bodies, whereas the area measures are concentrated on the fixed sphere S^{n-1}. The latter fact allows us to put area measures of different convex bodies in relation. In particular, the following expansion can be proved:

$$S_{n-1}(K+\rho B^n,\cdot) = \sum_{m=0}^{n-1} \rho^{n-1-m} \binom{n-1}{m} S_m(K,\cdot). \tag{7}$$

This shows that all the area measures $S_1(K,\cdot),\dots,S_{n-2}(K,\cdot)$ can be derived from S_{n-1} applied to the "outer parallel body" $K+\rho B^n$. Since S_{n-1} is just the area of the reverse spherical image, this explains the name "area measures". We also note that formula (7) involves the Minkowski (or vector) addition of convex bodies. This observation can be extended considerably and indicates why the area measures and their generalizations are important in the Brunn-Minkowski theory of convex bodies, which relates Minkowski addition to metric notions like volume.

Notes. The curvature measures $C_m(K,\cdot)$ (with a different normalization) were introduced, more generally for sets of positive reach, by Federer [18]. The area measures $S_m(K,\cdot)$ already appeared in the work of Aleksandrov [3] and Fenchel and Jessen [19]. For convex bodies, both types of measures were further studied by Schneider [38]. See [42], Section 4.2 and the Notes therein, for more information, and particularly for full proofs of the results that were only sketched here. The support measures can be expressed as integrals over the generalized normal bundle, involving elementary symmetric functions of generalized principal curvatures. Such representations and related results are due to Zähle [50]; see also Kohlmann [26] and Hug [24].

2.2. Curvature measures and shape

If the curvature measures are to be useful extensions of the differential-geometric curvature functions, there should be close relationships between properties of curvature measures and the local shape of convex bodies. There are clearly such relations in the case of polytopes or of bodies of class C^2. In the case of general convex bodies, they are harder to obtain, but interesting and challenging. Any measure-theoretic property of a curvature measure should correspond to some intuitive geometric property of the convex body. However, even simple questions of this kind are unanswered. The following is an example.

PROBLEM 2.2.1: Is there a simple geometric property of a convex body K that is equivalent to the assumption that the m-th curvature measure $C_m(K,\cdot)$, for some $m \in \{0,\dots,n-2\}$, is absolutely continuous with respect to the $(n-1)$-dimensional Hausdorff measure?

Similarly, one may ask which intuitive geometric conclusions can be drawn from the assumption that $C_m(K,\cdot) \leq aC_{n-1}(K,\cdot)$, for some $m \in \{0,\dots,n-2\}$, with a constant a. For $n=3$, $m=0$ see Aleksandrov [7] and Busemann [15], Section 5; this was recently extended to $n \geq 3$ and general m by Burago and Kalinin [14].

We shall now give two examples of results where a property of a curvature measure can be translated into geometric information on the convex body. Full proofs are technical and require advanced results from the geometry of convex bodies. We shall, therefore, only give an intuitive description of the main ideas of the proofs.

The first result characterizes the support of the m-th curvature measure. The *support* of a Borel measure is the complement of the largest open set on which the measure vanishes. To formulate the result, we need the notion of m-extreme points. A point x of the convex body K is an *m-extreme* point, for $m \in \{0,\dots,n-1\}$, if there is no $(m+1)$-dimensional ball with centre x contained in K.

THEOREM 2.2.2. *Let $K \in \mathcal{K}^n$ be an n-dimensional convex body and let $m \in \{0,\dots,n-1\}$. The support of the curvature measure*

$C_m(K, \cdot)$ *is the closure of the set of m-extreme points of K.*

To sketch the proof, we denote the support of a measure μ by $\operatorname{supp}\mu$ and the set of m-extreme points of K by $\operatorname{ext}_m K$. Thus the assertion says that

$$\operatorname{supp} C_m(K, \cdot) = \operatorname{cl} \operatorname{ext}_m K,$$

where cl denotes the closure. We put $\beta_m := \partial K \setminus \operatorname{ext}_m K$; this is a Borel set. First we consider the case $m = 0$. By Theorem 2.1.8,

$$C_0(K, \beta_0) = \mathcal{H}^{n-1}(\sigma(K, \beta_0)).$$

Let $u \in \sigma(K, \beta_0)$. If the supporting hyperplane $H(K, u)$ of K with outer normal vector u contains only one point x of K, then $x \in \beta_0 = \partial K \setminus \operatorname{ext}_0 K$, hence x is not an extreme point of K. Thus $H(K, u)$ contains a segment of K, and hence u is a singular normal vector. This implies $\mathcal{H}^{n-1}(\sigma(K, \beta)) = 0$ and therefore

$$C_0(K, \beta_0) = 0.$$

To extend this result to $m > 0$, we use a formula from integral geometry. To formulate it, let $\mathcal{E}_k^n$ denote the space of k-flats (k-dimensional affine subspaces) in $\mathbf{E}^n$ with its usual topology. It carries a rigid motion invariant measure μ_k, which is unique up to a constant factor. Now the Crofton-type formula

$$C_m(K, \beta) = a_{nm} \int_{\mathcal{E}_{n-m}^n} C_0(K \cap E, \beta) \, d\mu_{n-m}(E) \tag{8}$$

is valid for $\beta \in \mathcal{B}(\mathbf{E}^n)$, where a_{nm} is a positive constant. This formula, by the way, gives an interesting interpretation of the curvature measure $C_m(K, \cdot)$ as a mean value of Gaussian curvature measures of intersections with $(n-m)$-flats.

Now for each $E \in \mathcal{E}_{n-m}^n$ we have

$$\beta_m \cap E \subset \partial(K \cap E) \setminus \operatorname{ext}_0(K \cap E),$$

since a point $x \in \beta_m \cap E$ is the centre of an $(m+1)$-dimensional ball contained in K and hence the centre of a segment contained in $K \cap E$. From the result obtained above, but applied in E, we have

$$C_0(K \cap E, \beta_m) = C_0(K \cap E, \beta_m \cap E) = 0.$$

The integral-geometric formula (8) hence gives

$$C_m(K, \partial K \setminus \mathrm{ext}_m K) = 0$$

and thus

$$\mathrm{supp}\, C_m(K, \cdot) \subset \mathrm{cl}\, \mathrm{ext}_m K.$$

For the opposite inclusion, we first observe that

$$C_0(K, \beta) = 0$$

for an open set β implies that no point of $K \cap \beta$ can be an extreme point of K. This observation, however, is not strong enough; we need a quantitative improvement. The following can be proved: Let $\beta \subset \mathbf{E}^n$ be an open ball with centre $x \in K$ and radius ρ. If $C_0(K, \beta) = 0$, then x is the centre of a segment of length $2\rho/n$ contained in K.

Now let $\beta \subset \mathbf{E}^n$ be an open set for which $C_m(K, \beta) = 0$. From (8) we obtain

$$C_0(K \cap E, \beta) = 0 \qquad \text{for } \mu_{n-m}\text{-almost all } E \in \mathcal{E}^n_{n-m}.$$

The fact that this relation holds only almost everywhere is the reason for the necessity of the quantitative result above. Using it, we can now infer the following. If $x \in K \cap \beta$, then there is a number $r > 0$ such that every $(n-m)$-flat through x that meets $\mathrm{int}\, K$ meets K in a set containing a segment of length r with centre x. This implies that x is the centre of an $(m+1)$-dimensional ball contained in K. Thus $x \notin \mathrm{ext}_m K$. This shows that $\beta \cap \mathrm{ext}_m K = \emptyset$ and thus

$$\mathrm{cl}\, \mathrm{ext}_m K \subset \mathrm{supp}\, C_m(K, \cdot),$$

which completes the proof.

Similarly to Theorem 2.2.2, the support of the area measure $S_m(K, \cdot)$ can be characterized: it is the closure of the set of all $(n-1-m)$-extreme unit normal vectors of K. The vector $u \neq 0$ is an *r-extreme normal vector* of K if it is not a positive linear combination of $r+2$ linearly independent normal vectors at the same boundary point of K. This characterization of $\mathrm{supp}\, S_m(K, \cdot)$ is part of a more general, but unproved conjecture. This concerns the *mixed* *area measures*, which can be defined as the coefficients in the polynomial expansion

$$S_{n-1}(\alpha_1 K_1 + \ldots + \alpha_p K_p, \cdot) = \sum_{i_1,\ldots,i_{n-1}=1}^{p} \alpha_{i_1} \cdots \alpha_{i_{n-1}} S(K_{i_1}, \ldots, K_{i_{n-1}}, \cdot),$$

which extends (7).

Open Problem 2.2.3: Determine the support of the mixed area measure $S(K_1, \ldots, K_{n-1}, \cdot)$.

A conjecture is formulated in [42], p. 366. A proof would be of some interest for the theory of mixed volumes. The proof for the special case of

$$S_m(K, \cdot) = S(\underbrace{K, \ldots, K}_{m}, B^n, \ldots, B^n, \cdot)$$

cannot be extended in an obvious way, since it uses an integral-geometric formula involving a rotation invariant measure. It would, therefore, also be of interest to find a proof of Theorem 2.2.2 and its counterpart without using rigid motion invariant integral geometry.

Our second example extends a classical theorem from global differential geometry to arbitrary convex bodies.

Theorem 2.2.4. *Let $K \in \mathcal{K}_0^n$ be a convex body with interior points, and let $m \in \{0, \ldots, n-2\}$. If the curvature measure $C_m(K, \cdot)$ is proportional to the surface area measure $C_{n-1}(K, \cdot)$, then K is a ball.*

If the boundary ∂K of K is of class C^2, the condition of the theorem is equivalent to the condition $H_{n-1-m} = \text{const}$, and classical methods of differential geometry yield the result. For general convex bodies, a different approach is necessary. We shall sketch the principal ideas of such an approach. It makes use of some results from the theory of convex bodies, which will be explained without proof.

After applying a suitable homothety, we may write the assumption of the theorem in the form

$$C_m(K, \cdot) = C_{n-1}(K, \cdot). \tag{9}$$

In the first step, this condition is transformed into one involving area measures instead of curvature measures. This reformulation is possible due to the following facts. Using the support measures, one can show that

$$C_m(K, \tau(K,\omega) \cap \operatorname{reg} K) \leq S_m(K,\omega) \leq C_m(K, \tau(K,\omega)) \tag{10}$$

for $\omega \in \mathcal{B}(S^{n-1})$, where $\operatorname{reg} K$ is the set of regular (= non-singular) boundary points of K. Recall that $\tau(K,\omega)$ is the reverse spherical image. By (5) and Reidemeister's theorem,

$$C_{n-1}(K, \partial K \setminus \operatorname{reg} K) = 0. \tag{11}$$

Now suppose that $\omega \subset S^{n-1}$ is closed. Then $\tau(K,\omega)$ is closed, and

$$\begin{aligned}
S_m(K,\omega) &\leq C_m(K, \tau(K,\omega)) && \text{by (10)}\\
&= C_{n-1}(K, \tau(K,\omega)) && \text{by (9)}\\
&= C_{n-1}(K, \tau(K,\omega) \cap \operatorname{reg} K) && \text{by (11)}\\
&= C_m(K, \tau(K,\omega) \cap \operatorname{reg} K) && \text{by (9)}\\
&\leq S_m(K,\omega). && \text{by (10)}
\end{aligned}$$

This implies

$$\begin{aligned}
S_m(K,\omega) &= C_{n-1}(K, \tau(K,\omega))\\
&= \mathcal{H}^{n-1}(\tau(K,\omega))\\
&= S_{n-1}(K,\omega)
\end{aligned}$$

by (6). Since the equality

$$S_m(K,\omega) = S_{n-1}(K,\omega) \tag{12}$$

holds for all closed sets, it holds for all Borel sets.

The advantage of passing over to the area measures lies in the fact that these are closely related to the theory of mixed volumes of convex bodies. In particular, the coefficients appearing in the *Steiner formula*

$$\lambda_n(K + \epsilon B^n) = \sum_{i=0}^{n} \epsilon^i \binom{n}{i} W_i(K) \tag{13}$$

have the integral representations

$$W_{n-m}(K) = \frac{1}{n} \int_{S^{n-1}} dS_m(K, \cdot), \tag{14}$$

$$W_{n-m-1}(K) = \frac{1}{n} \int_{S^{n-1}} h(K,u)\, dS_m(K,u). \tag{15}$$

Together with (12) this gives the equalities

$$W_{n-m}(K) = W_1(K), \quad W_{n-m-1}(K) = W_0(K). \tag{16}$$

From the Aleksandrov-Fenchel inequalities in the theory of convex bodies one knows that in general

$$\frac{W_1}{W_0} \geq \frac{W_2}{W_1} \geq \dots \geq \frac{W_{n-m}}{W_{n-m-1}}.$$

By (16), for the body K this holds with equality throughout. Equality in one of the Aleksandrov-Fenchel inequalities usually has strong consequences. In our case, we can conclude, using deeper results from convex geometry, that K must be a so-called m-tangential body of a ball. This means that K contains a ball B with the property that each support plane of K that is not a support plane of B contains only $(m-1)$-singular points of K. In the special case $m = 1$ (corresponding to the mean curvature) it follows that every support plane of K not touching B contains only vertices of K. In this case, K is the convex hull of B and a (possibly empty) set of points with the property that the segment connecting two such points meets B. Such a body is called a *cap body* of B. It is now easy to check that such a cap body satisfies (9) only if it is a ball. Also in the cases $m > 1$ we must exploit condition (9) (whereas (12) would not be sufficient) to show that K is a ball. This, however, requires deeper results from convexity, like derivatives of curvature measures and the twice differentiability almost everywhere of convex functions, and we cannot go into the details.

Notes. Theorem 2.2.2 was proved in [38], and its counterpart for the area measures in [37]. The proofs are reproduced in [42], Section 4.6; also the integral geometric formula (8) can be found in [42], or see [45]. The complete proof of Theorem 2.2.4 is found in

[40]. By a different approach, Kohlmann [25], [26], [27], [30] has recently given another proof and has extended the theorem to convex bodies in spaces of constant curvature. Using techniques from geometric measure theory, he succeeded in extending a tool from classical differential geometry, the so-called Minkowski integral formulas, to general convex bodies in space forms. He has also treated related stability problems, [28], [29]. Further information on older related results can be found in [43].

2.3. The Aleksandrov-Fenchel-Jessen theorem

The area measures $S_1(K,\cdot),\dots,S_{n-1}(K,\cdot)$ of a convex body K are measures on the unit sphere S^{n-1}, which is independent of K. This fact makes it possible to compare area measures of different convex bodies and to pose natural uniqueness and existence problems. For example, if $K, L \in \mathcal{K}_0^n$ are convex bodies satisfying

$$S_m(K,\cdot) = S_m(L,\cdot) \tag{17}$$

for some $m \in \{1,\dots,n-1\}$, what does this imply for K and L? To get a feeling for condition (17), let us first consider the case where K and L have boundaries of class C^2 with positive curvatures. In this case, (17) is equivalent to the condition that ∂K and ∂L have the same m-th elementary symmetric function of the principal radii of curvature, at points with parallel (oriented) tangent planes. This means that the support functions of K and L satisfy a certain second order partial differential equation (linear only for $m = 1$, of Monge-Ampère type for $m = n-1$). Hence, the following result can be considered as a global uniqueness theorem for this equation: K and L differ only by a translation. A differential-geometric proof of this result was given by Chern [16]. Remarkably, the more general result referring to (17) and general convex bodies is much older; it was proved independently by Aleksandrov [4] and by Fenchel and Jessen [19].

Theorem 2.3.1 (Aleksandrov-Fenchel-Jessen).
Let $m \in \{1,\dots,n-1\}$ and let $K, L \in \mathcal{K}^n$ be convex bodies of dimension at least $m+1$. If $S_m(K,\cdot) = S_m(L,\cdot)$, then K and L differ only by a translation.

We had to mention this result here, since it is a classical and useful result about the determination of convex bodies by certain measures. The proof, however, requires so much convexity theory that we can only refer to the literature. A slightly more general result is proved in [42].

2.4. Minkowski's existence theorem

The area measure $S_{n-1}(K,\cdot)$ of a convex body K has a simple geometric meaning; recall that $S_{n-1}(K,\omega)$ is the area (= $(n-1)$-dimensional Hausdorff measure) of the reverse spherical image $\tau(K,\omega)$. In particular, if K is a polytope, the measure $S_{n-1}(K,\cdot)$ is concentrated on the finitely many unit normal vectors $u_1,\dots,u_m$ of the facets of K, and $S_{n-1}(K,\{u_i\})$ is the area of the facet with normal vector u_i. It is of considerable interest to know which measures on S^{n-1} are the $(n-1)$-st area measures of convex bodies (with interior points). Two necessary conditions for such measures are easily obtained. First, a special case of formula (15) says that

$$\lambda_n(K) = \frac{1}{n}\int_{S^{n-1}} h(K,u)\,dS_{n-1}(K,u). \tag{18}$$

If we replace K by a translate $K+t$, then $h(K,u)$ is replaced by $h(K,u)+\langle t,u\rangle$, but neither $\lambda_n(K)$ nor $S_{n-1}(K,u)$ is changed. Since this holds for all $t\in\mathbf{E}^n$ we infer that

$$\int_{S^{n-1}} u\,dS_{n-1}(K,u) = 0. \tag{19}$$

For a unit vector v, let K_v be the image of K under orthogonal projection onto the hyperplane through 0 orthogonal to v. For the $(n-1)$-dimensional volume of K_v, it is not difficult to obtain the formula

$$\lambda_{n-1}(K_v) = \frac{1}{2}\int_{S^{n-1}} |\langle u,v\rangle|\,dS_{n-1}(K,u). \tag{20}$$

Exercise 2.4.1: Prove (18) and (20), first for polytopes and then for general convex bodies, using approximation.

Since (20) is not zero, the measure $S_{n-1}(K,\cdot)$ cannot be concentrated on the great subsphere

$$s_v := \{u \in S^{n-1} : \langle u, v\rangle = 0\}.$$

It is a remarkable fact that these two simple necessary conditions for area measures of order $n-1$ are already sufficient.

THEOREM 2.4.2 (MINKOWSKI, ALEKSANDROV AND FENCHEL-JESSEN). *Let φ be a finite measure on $\mathcal{B}(S^{n-1})$ such that*

$$\int_{S^{n-1}} u \, d\varphi(u) = 0 \tag{21}$$

and

$$\varphi(s_v) < \varphi(S^{n-1}) \tag{22}$$

for each great subsphere s_v. Then there is a convex body K with $\varphi = S_{n-1}(K,\cdot)$. The body K is unique up to a translation.

The uniqueness assertion is only a special case of Theorem 2.3.1. For the existence proof one can use that the body K, if it exists, must have a certain minimum property. This information is provided by the theory of mixed volumes. Extending the Steiner formula (13), one has

$$\lambda_n(K+\epsilon L) = \sum_{i=0}^{n} \epsilon^i \binom{n}{i} V(\underbrace{K,\ldots,K}_{n-i},\underbrace{L,\ldots,L}_{i})$$

for convex bodies $K, L \in \mathcal{K}^n$ and $\epsilon > 0$, where the coefficients are so-called mixed volumes. Formula (15) for $m = n-1$ extends to

$$V(K,\ldots,K,L) = \frac{1}{n} \int_{S^{n-1}} h(L,u)\, dS_{n-1}(K,u).$$

Minkowski's inequality from the theory of mixed volumes says that

$$V(K,\ldots,K,L)^n \geq \lambda_n(K)^{n-1}\lambda_n(L),$$

where equality holds if and only if K and L are homothetic. Hence, if there exists a convex body K for which $S_{n-1}(K,\cdot)$ is the given measure φ, then the functional Φ defined by

$$\Phi(L) = \frac{1}{n} \int_{S^{n-1}} h(L,u)\, d\varphi(u), \qquad L \in \mathcal{K}^n,$$

satisfies

$$\Phi(L) \geq \lambda_n(L)^{1/n}\lambda_n(K)^{1-(1/n)},$$

with equality if and only if L and K are homothetic. Therefore, a strategy for proving Minkowski's theorem is to show that the minimum problem

$$\Phi(L) = \min! \quad \text{on } \{L \in \mathcal{K}^n : \lambda_n(L) = 1\}$$

has a solution K and then to show by a variational argument that, in fact, $S_{n-1}(\alpha K, \cdot) = \varphi$ for a suitable factor $\alpha > 0$. In this way, Minkowski [34] gave a proof (in dimension three) for polytopes, and Aleksandrov [5] gave one for general convex bodies. In the latter case, the variational argument requires deeper results from the theory of mixed volumes. It is, therefore, easier to prove the theorem for polytopes first and then to use approximation together with weak continuity. This path was followed by Fenchel and Jessen [19] and also in a further paper by Aleksandrov [6]. We shall now sketch the essential steps of this procedure.

Sketch of a proof of Theorem 2.4.2. The first step is the essential one, but since it is not measure-theoretic, we don't go into details here. One starts with measures with finite support and satisfying the assumptions. Thus, let

$$\varphi = \sum_{i=1}^{N} f_i \delta_{u_i}$$

($\delta =$ Dirac measure), where $u_1, \dots, u_N \in S^{n-1}$ are given unit vectors, not all in a great subsphere, and $f_1, \dots, f_N$ are positive numbers so that

$$\sum_{i=1}^{N} f_i u_i = 0.$$

One considers all polytopes of the form

$$P(\alpha) = \bigcap_{i=1}^{N} \{x \in \mathbf{E}^n : \langle x, u_i \rangle \leq \alpha_i\}$$

with $\alpha := (\alpha_1, \dots, \alpha_N) \in \mathbf{R}^N, \alpha_i \geq 0$, and the subset of $\mathbf{R}^N$ defined by

$$M := \{\alpha \in \mathbf{R}^N : \lambda_n(P(\alpha)) \geq 1\}.$$

The functional Φ defined by

$$\Phi(\alpha) := \frac{1}{n}\sum_{i=1}^{n} \alpha_i f_i, \qquad \alpha_1, \ldots, \alpha_N \geq 0,$$

attains a minimum on M. If this minimum is attained at α^* and its value is μ^{n-1}, one can show by elementary means that the polytope $\mu P(\alpha^*)$ solves Minkowski's problem for the measure φ. With other words,

$$S_{n-1}(\mu P(\alpha^*), \{u_i\}) = f_i \qquad \text{for } i = 1, \ldots, N$$

and

$$S_{n-1}(\mu P(\alpha^*), S^{n-1} \setminus \{u_1, \ldots, u_N\}) = 0.$$

In the second step, the result is now extended by approximation. Let φ be a measure satisfying (21) and (22). We construct a sequence $(\varphi_k)_{k\in\mathbf{N}}$ of measures with finite supports, also satisfying these assumptions and converging weakly to φ. For given $k \in \mathbf{N}$, the sphere S^{n-1} can be decomposed into Borel sets of diameter at most $1/k$ and with spherically convex closure. Let $\Delta_1, \ldots, \Delta_N$ be the sets of this decomposition on which φ does not vanish, and define

$$c_i := \frac{1}{\varphi(\Delta_i)} \int_{\Delta_i} u \, d\varphi(u) \qquad \text{for } i = 1, \ldots, N.$$

If f_i is defined by $c_i = f_i u_i$ with $u_i \in S^{n-1}$, one can easily show that $1 - (2k^2)^{-1} \leq f_i \leq 1$. The measure

$$\varphi_k := \sum_{i=1}^{N} \varphi(\Delta_i) f_i \delta_{u_i}$$

satisfies

$$\int_{S^{n-1}} u \, d\varphi_k(u) = 0.$$

Let g be a continuous real function on S^{n-1}, and let $\epsilon > 0$ be given. For $i \in \{1, \ldots, N\}$ and for $u \in \Delta_i$ we have $\|u_i - u\| \leq 1/k$, hence from the uniform continuity of g it follows that

$$\begin{aligned} |f_i g(u_i) - g(u)| &\leq |f_i - 1|\,|g(u_i)| + |g(u_i) - g(u)| \\ &\leq \frac{1}{2k^2} \max_{v \in S^{n-1}} |g(v)| + |g(u_i) - g(u)| \\ &\leq \epsilon \end{aligned}$$

for $k \geq k_0$ and suitable k_0. This gives

$$\begin{aligned} \left| \int_{S^{n-1}} g \, d\varphi_k - \int_{S^{n-1}} g \, d\varphi \right| &\leq \sum_{i=1}^{N} \int_{\Delta_i} |f_i g(u_i) - g(u)| \, d\varphi(u) \\ &\leq \epsilon \varphi(S^{n-1}). \end{aligned}$$

From this, we conclude that

$$\varphi_k \xrightarrow{w} \varphi \qquad \text{for } k \to \infty.$$

We have to show that φ_k is not concentrated on a great subsphere, if k is sufficiently large. Since φ is not concentrated on a great subsphere, we have

$$\int_{S^{n-1}} \langle u, v \rangle^+ \, d\varphi(u) > 0 \qquad \text{for each } v \in S^{n-1},$$

where $+$ denotes the positive part. By continuity, this integral has a positive lower bound independent of v. Since

$$\lim_{k \to \infty} \int_{S^{n-1}} \langle u, v \rangle^+ \, d\varphi_k(u) = \int_{S^{n-1}} \langle u, v \rangle^+ \, d\varphi(u)$$

uniformly in v (by the estimate above), we conclude that there exist $a > 0$ and $k_1 \in \mathbf{N}$ with

$$\int_{S^{n-1}} \langle u, v \rangle^+ \, d\varphi_k(u) > a$$

for $k \geq k_1$ and $v \in S^{n-1}$. This means that the measure φ_k is not concentrated on a great subsphere if $k \geq k_1$.

We can now apply the first step of the proof and deduce that, for $k \geq k_1$, there is a polytope $P_k \in \mathcal{K}_0^n$ with $S_{n-1}(P_k, \cdot) = \varphi_k$. We may assume that $0 \in P_k$. Then all polytopes P_k lie in some fixed ball. This can be seen as follows. Since $\varphi_k(S^{n-1}) \leq \varphi(S^{n-1})$ by the definition of φ_k, the surface areas of the polytopes P_k are bounded by $\varphi(S^{n-1})$. By the isoperimetric inequality, there is a constant b with $\lambda_n(P_k) \leq b$. For $x \in P_k$ and setting $x = \|x\| v$ with $v \in S^{n-1}$, we have

$$h(P_k, u) \geq h(\text{conv}\,\{0, x\}, u) = \|x\| \langle u, v \rangle^+$$

for $u \in S^{n-1}$, hence, using (15) for $m = n-1$,

$$\begin{aligned} b &\geq \lambda_n(P_k) = \frac{1}{n} \int\limits_{S^{n-1}} h(P_k, u)\, dS_{n-1}(P_k, u) \\ &\geq \frac{\|x\|}{n} \int\limits_{S^{n-1}} \langle u, v\rangle^+ \, d\varphi_k(u) \geq \frac{\|x\|}{n} a \end{aligned}$$

and thus $\|x\| \leq nb/a$.

We can now apply the Blaschke selection theorem. It says that in the space $\mathcal{K}^n$ with the Hausdorff metric δ every bounded set is relatively compact; hence, from the bounded sequence $(P_k)_{k \geq k_1}$ we can select a subsequence that converges to some convex body $K \in \mathcal{K}^n$. The weak convergences $\varphi_k \overset{w}{\to} \varphi$ and $S_{n-1}(P_k, \cdot) \overset{w}{\to} S_{n-1}(K, \cdot)$ now imply that $S_{n-1}(K, \cdot) = \varphi$. This completes the proof of Theorem 2.4.2.

2.5. The length measure in the plane

In the case of the plane $\mathbf{E}^2$, Minkowski's existence and uniqueness theorem shows some special features and has, therefore, a number of applications which have no immediate analogue in higher dimensions. It is particularly useful for the treatment of certain decomposition problems and of mappings on convex bodies with additivity properties.

The area measure $S_1(K, \cdot)$ of a convex body $K \in \mathcal{K}^2$ is also called the *length measure* of K. It is well adapted to an important operation for convex bodies, the Minkowski or vector addition, defined by

$$K + L := \{x + y : x \in K,\ y \in L\}.$$

For $K, L \in \mathcal{K}^2$, we have

$$S_1(K + L, \cdot) = S_1(K, \cdot) + S_1(L, \cdot). \tag{23}$$

This is easily seen if K and L are polygons; the general case is obtained by approximation. Similarly one sees that $S_1(\alpha K, \cdot) = \alpha S_1(K, \cdot)$ for $\alpha \geq 0$. These facts can be used to establish an isomorphism between a cone of translation classes of convex bodies in $\mathbf{E}^2$ and a cone of measures.

It is convenient to select from each translation class of bodies in $\mathcal{K}^2$ a particular one, in the following way. For $K \in \mathcal{K}^2$, one defines the *Steiner point* by

$$s(K) := \frac{1}{\pi} \int_{S^1} h(K,u)u \, d\mathcal{H}^1(u).$$

We put

$$\mathcal{K}_s^2 := \{K \in \mathcal{K}^2 : s(K) = 0\}.$$

Observing that $s(K+t) = s(K) + t$, we note that $\mathcal{K}_s^2$ contains precisely one element from each class of translates of a convex body. From the fact that the support function satisfies $h(K+L,\cdot) = h(K,\cdot) + h(L,\cdot)$, we get $s(K+L) = s(K) + s(L)$, hence $\mathcal{K}_s^2$ is closed under Minkowski addition. Since also $s(\alpha K) = \alpha s(K)$ for $\alpha \geq 0$, the set $\mathcal{K}_s^2$ is a convex cone under Minkowski addition and multiplication by nonnegative scalars.

Let us denote by $\mathcal{M}_0(S^1)$ the cone of finite Borel measures on S^1 satisfying

$$\int_{S^1} u \, d\mu(u) = 0.$$

Then the map Φ defined by

$$\begin{array}{rcl} \Phi : \mathcal{K}_s^2 & \to & \mathcal{M}_0(S^1) \\ K & \mapsto & S_1(K,\cdot) \end{array}$$

is bijective. The injectivity follows from the uniqueness part of Minkowski's theorem. That every $\mu \in \mathcal{M}(S^1)$ is attained under Φ, follows from Minkowski's existence theorem if μ is not concentrated on a pair of antipodal points. But if $\mu = a\delta_u + a\delta_{-u}$ with $u \in S^1$ and $a \geq 0$, then $\mu = S_1(K,\cdot)$, where K is the segment of length a orthogonal to u and with centre 0. As $\Phi(K+L) = \Phi(K) + \Phi(L)$ and $\Phi(\alpha K) = \alpha\Phi(K)$ for $\alpha \geq 0$, the map Φ is an isomorphism between the cones $\mathcal{K}_s^2$ and $\mathcal{M}_0(S^1)$. If $\mathcal{K}_s^2$ is equipped with the Hausdorff metric and $\mathcal{M}_0(S^1)$ with the topology of weak convergence, then Φ is continuous.

We shall now give a few examples of results which can be obtained by working with this isomorphism.

Let $K, L \in \mathcal{K}^2$. The body L is said to be a *summand* of K if there exists a convex body $M \in \mathcal{K}^2$ so that $L + M = K$. The body K is called *indecomposable* if every summand of K is homothetic to K (that is, of the form $\alpha K + t$ with $\alpha \geq 0$ and $t \in \mathbf{E}^2$).

THEOREM 2.5.1. *Let $K, L \in \mathcal{K}^2$. Then L is a summand of K if and only if $S_1(L, \cdot) \leq S_1(K, \cdot)$.*

Proof. If $K = L + M$ with $M \in \mathcal{K}^2$, then $S_1(K, \cdot) = S_1(L, \cdot) + S_1(M, \cdot) \geq S_1(L, \cdot)$. Vice versa, if $S_1(L, \cdot) \leq S_1(K, \cdot)$, then $\varphi := S_1(K, \cdot) - S_1(L, \cdot) \in \mathcal{M}_0(S^1)$, hence $\varphi = \Phi(M)$ for some $M \in \mathcal{K}_s^2$, and $M + L = K$. Thus L is a summand of K. □

EXERCISE 2.5.2: Using Theorem 2.5.1, show that $K \in \mathcal{K}^2$ is indecomposable if and only if K is either a triangle or a segment (possibly one-pointed).

Due to its linearity properties, the length measure is a natural tool for the description of certain additive mappings on the space of convex bodies. We quote without proof two results of this type.

A (real) *valuation* on $\mathcal{K}^2$ is a map $f : \mathcal{K}^2 \to \mathbf{R}$ satisfying

$$f(K \cup L) + f(K \cap L) = f(K) + f(L)$$

whenever $K, L, K \cup L \in \mathcal{K}^2$.

Valuations arise often in the theory of convex bodies; classifications of valuations with additional properties are therefore of interest. The following result of this type is a reformulation of a theorem of Hadwiger [23].

THEOREM 2.5.3. *Let f be a valuation on $\mathcal{K}^2$ which is continuous and translation invariant. Then there exist constants a, b and a continuous function $g : S^1 \to \mathbf{R}$ such that*

$$f(K) = a + \int_{S^1} g(u) dS_1(K, u) + b\lambda_2(K)$$

for all $K \in \mathcal{K}^2$.

By an *endomorphism* of $\mathcal{K}^2$ we understand a continuous map $T : \mathcal{K}^2 \to \mathcal{K}^2$ satisfying

$$T(K + L) = T(K) + T(L)$$

for $K, L \in \mathcal{K}^2$ and

$$Tg = gT$$

for every rigid motion g of $\mathbf{E}^2$. Thus an endomorphism of $\mathcal{K}^2$ respects the essential geometric structures of $\mathcal{K}^2$. The following theorem shows, in two different ways, how measures on the unit circle can be used to give a complete description of endomorphisms. For a convenient formulation, we choose an orthonormal basis e_1, e_2 of $\mathbf{E}^2$ and write

$$u = (\cos\theta_u)e_1 + (\sin\theta_u)e_2 \qquad \text{with } 0 \le \theta_u < 2\pi$$

for $u \in S^1$, as well as

$$u(\varphi) = (\cos\varphi)e_1 + (\sin\varphi)e_2 \qquad \text{for } 0 \le \varphi < 2\pi.$$

THEOREM 2.5.4. *Let T be an endomorphism of $\mathcal{K}^2$.*

(a) There exists a continuous, 2π-periodic function $g : \mathbf{R} \to \mathbf{R}$ such that

$$h(TK, u) = \int_{S^1} g(\theta_v - \theta_u)\, dS_1(K, v) + \langle s(K), u\rangle$$

for $K \in \mathcal{K}^2$.

(b) There exists a finite Borel measure ν on $[0, 2\pi)$ such that

$$h(TK, u) = \int_{[0,2\pi)} h(K - s(K), u_{\theta_u+\psi})\, d\nu(\psi) + \langle s(K), u\rangle$$

for $K \in \mathcal{K}^2$.

The function g and the measure ν are essentially unique, that is, unique up to trivial summands. For the proof of Theorem 2.5.4 we refer to Schneider [36].

A second particular aspect of the planar case of Minkowski's existence and uniqueness theorem is the fact that here one has an explicit integral representation of the support function in terms of the length measure. This is given by Theorem 2.5.5, in the proof of which we follow a recent paper of Christina Bauer [10].

Put $N_u := \{v \in S^1 : 0 \le \theta_v < \theta_u\}$; this is the half-open circular arc from e_1 to u. For $u \in S^1$ we write

$$F(K,u) := \{x \in K : \langle x, u\rangle = h(K,u)\};$$

this is the *face* of K with outer normal vector u.

The following theorem expresses the support function of a convex body $K \in \mathcal{K}^2$ in terms of its length measure.

THEOREM 2.5.5. *Let $K \in \mathcal{K}^2$. Determine $x_0 \in \mathbf{E}^2$ so that*

$$F(K, e_1) = \{x_0 + \alpha e_2 : 0 \le \alpha \le S_1(K, \{e_1\})\}.$$

Then

$$h(K,u) = \langle x_0, u\rangle + \int\limits_{N_u} \sin(\theta_u - \theta_v)\, dS_1(K, v) \tag{24}$$

for all $u \in S^1$.

EXERCISE 2.5.6: Prove (24) in the special case where K is a polygon.

Proof of Theorem 2.5.5. Let $K \in \mathcal{K}^2$ and $u \in S^1 \setminus \{e_1\}$. We can easily construct a sequence $(P_i)_{i\in\mathbf{N}}$ of polygons so that $\lim_{i\to\infty} P_i = K$ in the Hausdorff metric and

$$F(P_i, e_1) = F(K, e_1), \qquad F(P_i, u) = F(K, u), \tag{25}$$

thus

$$S_1(P_i, \{e_1\}) = S_1(K, \{e_1\}), \qquad S_1(P_i, \{u\}) = S_1(K, \{u\}) \tag{26}$$

for all $i \in \mathbf{N}$. For $i \in \mathbf{N}$ and a Borel set $\omega \in \mathcal{B}(S^1)$ we define

$$\begin{aligned} \mu_i(\omega) &:= S_1(P_i, \omega \cap \operatorname{int} N_u), \\ \mu(\omega) &:= S_1(K, \omega \cap \operatorname{int} N_u), \end{aligned}$$

where int denotes the interior relative to S^1. By Theorem 2.1.6, $S_1(P_i, \cdot) \xrightarrow{w} S_1(K, \cdot)$ for $i \to \infty$, hence (26) implies that

$$S_1(P_i, \cdot \setminus \{e_1, u\}) \xrightarrow{w} S_1(K, \cdot \setminus \{e_1, u\}).$$

Let $\omega \subset S^1$ be a Borel set whose boundary $\partial\omega$ has measure zero under μ; then the set $\partial(\omega \cap \operatorname{int} N_u)$ has measure zero under $S_1(K, \cdot \setminus \{e_1, u\})$. By a well-known characterization of weak convergence, this implies $\lim_{i\to\infty} \mu_i(\omega) = \mu(\omega)$ and, hence, $\mu_i \xrightarrow{w} \mu$.

We now use (24) for the polygons P_i (as proved in the exercise) and obtain from (25) and (26) that

$$h(P_i, u) = \langle x_0, u\rangle + \int_{S^1} \sin(\theta_u - \theta_v)\, d\mu_i(v) + \int_{\{e_1\}} \sin(\theta_u - \theta_v)\, dS_1(K, v).$$

For $i \to \infty$, we have $h(P_i, u) \to h(K, u)$ and $\mu_i \xrightarrow{w} \mu$, hence

$$\begin{aligned} h(K,u) &= \langle x_0, u\rangle + \int_{S^1} \sin(\theta_u - \theta_v)\, d\mu(v) + \\ &\quad + \int_{\{e_1\}} \sin(\theta_u - \theta_v)\, dS_1(K, v) \\ &= \langle x_0, u\rangle + \int_{N_u} \sin(\theta_u - \theta_v)\, dS_1(K, v), \end{aligned}$$

which completes the proof of Theorem 2.5.5. ☐

Theorem 2.3.1 for $n = 2$ can be deduced from Theorem 2.5.5. Let $K, L \in \mathcal{K}_0^2$ be convex bodies such that $S_1(K, \cdot) = S_1(L, \cdot)$. With suitable points x_0, y_0 it then follows from (24) that $h(K - x_0, u) = h(L - y_0, u)$ for all $u \in S^1$, which implies $K - x_0 = L - y_0$.

We conclude this section with a few applications of Theorem 2.5.5.

In Theorem 2.5.1 we showed that $S_1(L, \cdot) \le S_1(K, \cdot)$ implies that L is a summand of K; in particular, L can be covered by a translate of K. The latter fact follows also from a weaker assumption:

THEOREM 2.5.7. *Let $K, L \in \mathcal{K}^2$. If there is a point $z_0 \in S^1$ so that*

$$S_1(L, \omega) \le S_1(K, \omega)$$

for all Borel sets $\omega \in \mathcal{B}(S^1)$ with $z_0 \notin \omega$, then L can be covered by a translate of K.

Proof. Let $e_1 := -z_0$ and define the point $x_0 = x_0(K, e_1)$ as in Theorem 2.5.5. After applying suitable translations, we may assume that $x_0(K, e_1) = 0 = x_0(L, e_1)$. For $u \in A := \{v \in S^1 : 0 \leq \theta_v \leq \pi\}$ we have $z_0 \notin N_u$ and thus $S_1(L, \cdot \cap N_u) \leq S_1(K, \cdot \cap N_u)$. Since $\sin(\theta_u - \theta_v) \geq 0$ for $v \in N_u$ with $u \in A$, we deduce from Theorem 2.5.5 that $h(L, u) \leq h(K, u)$ for $u \in A$.

Let $u \in S^1 \setminus A$. Then $z_0 \notin S^1 \setminus N_u$ and thus $S_1(L, \cdot \cap (S^1 \setminus N_u)) \leq S_1(K, \cdot \cap (S^1 \setminus N_u))$. From

$$\int_{S^1} v \, dS_1(v) = 0$$

we get

$$\int_{S^1} \cos\theta_v \, dS_1(v) = 0 = \int_{S^1} \sin\theta_v \, dS_1(v)$$

and thus

$$\int_{S^1} \sin(\theta_u - \theta_v) \, dS_1(v) = 0.$$

Hence, from (24) we have

$$h(K, u) = - \int_{S^1 \setminus N_u} \sin(\theta_u - \theta_v) \, dS_1(K, v).$$

Since $-\sin(\theta_u - \theta_v) \geq 0$ for $v \in S^1 \setminus N_u$ with $u \in S^1 \setminus A$, we get $h(L, u) \leq h(K, u)$.

We have proved that $h(L, u) \leq h(K, u)$ for all $u \in S^1$, which implies $L \subset K$ and thus the assertion of Theorem 2.5.7. □

Theorems 2.5.1, 2.5.5 and 2.5.7 can be used to further investigate the additive structure of $\mathcal{K}^2$. We conclude with describing a recent result of Bauer [10]. For $K, M \in \mathcal{K}^2$, let $\mathcal{S}(K, M)$ be the set of all convex bodies $C \in \mathcal{K}^2$ for which K and M are summands. Clearly, if $K + M$ is a summand of C, then K and M are summands of C, but not necessarily vice versa. We say that the pair $(K, M) \in \mathcal{K}^2 \times \mathcal{K}^2$ is *reduced* if $C \in \mathcal{S}(K, M)$ implies that $K + M$ is a summand of C. The pair (K, M) is called *minimal* if $C \in \mathcal{S}(K, M)$ together with $C \subset K + M$ implies that $K + M$ is a summand of C (and hence $C = K + M$). An equivalent notion was introduced recently, motivated by a question from quasidifferential calculus. On $\mathcal{K}^n \times \mathcal{K}^n$ one can define an equivalence relation $\sim$ by

$$(K, M) \sim (K', M') \Leftrightarrow K + M' = K' + M.$$

Let $\mathcal{E}(K, M)$ denote the corresponding equivalence class of (K, M). The pair (K, M) is a *minimal element of its equivalence class* if $(K', M') \in \mathcal{E}(K, M)$ together with $K' \subset K$ and $M' \subset M$ implies that $(K', M') = (K, M)$. It is easy to see that this is the case if and only if (K, M) is minimal in the sense defined above. For dimension two, it has been proved that every equivalence class contains a minimal element and that this is unique up to translations. (In higher dimensions, neither existence nor uniqueness up to translations are generally satisfied.) The existence proof, however, was nonconstructive, using Zorn's lemma, and the uniqueness proofs used deeper techniques (see [10] for references). In Bauer [10], simple proofs are given using the length measure: For $K, M \in \mathcal{K}^2$, consider the Jordan decomposition

$$S_1(K, \cdot) - S_1(M, \cdot) = \nu^+ - \nu^-$$

and determine $z \in S^1$ and $\xi \geq 0$ so that $\nu^+ + \xi\delta_z \in \mathcal{M}_0(S^1)$, then also $\nu^- + \xi\delta_z \in \mathcal{M}_0(S^1)$. Minkowski's existence theorem yields convex bodies $L^+, L^- \in \mathcal{K}^2$ satisfying

$$S_1(L^+, \cdot) = \nu^+ + \xi\delta_z, \qquad S_1(L^-, \cdot) = \nu^- + \xi\delta_z$$

and $(L^+, L^-) \in \mathcal{E}(K, M)$. It is not difficult to show that (L^+, L^-) is a minimal element of $\mathcal{E}(K, M)$ and that it is unique up to translations.

That the length measure is the appropriate tool here is also shown by the following complete characterization of reduced and minimal pairs in the plane.

THEOREM 2.5.8 (C. BAUER [10]). *Let $K, M \in \mathcal{K}^2$. The pair (K, M) is reduced if and only if the length measures $S_1(K, \cdot)$ and $S_1(M, \cdot)$ are mutually singular.*

The pair (K, M) is minimal if and only if there is a point $z_0 \in S^1$ such that the restrictions of $S_1(K, \cdot)$ and $S_1(M, \cdot)$ to $S^1 \setminus \{z_0\}$ are mutually singular.

Notes. Letac [31] described in greater detail the use of length measures in the study of planar convex bodies; Theorem 2.5.5 appears there (and also, without a complete proof, in Levin [32], p. 81). Theorem 2.5.7 is due to Scholtes [47]. Both theorems were proved in a simpler way by Bauer [10].

3. Zonoids

The connection between measure theory and convex geometry described in this last chapter is of an entirely different nature. A class of convex bodies appearing in measure theory, namely as ranges of nonatomic vector measures, is of considerable interest from a geometric point of view. On the other hand, certain questions on measures arising in stochastic geometry can be answered by constructing such special bodies from the measures and applying to them known results on convex bodies.

A convex body $Z \in \mathcal{K}^n$ is called a *centred zonoid* if its support function can be represented in the form

$$h(Z,u) = \int_{S^{n-1}} |\langle u,v\rangle| \, d\rho(v) \qquad \text{for } u \in S^{n-1}, \tag{27}$$

where ρ is a (real-valued) measure on $\mathcal{B}(S^{n-1})$. Any translate of a centred zonoid is a *zonoid.* With a given measure ρ on $\mathcal{B}(S^{n-1})$ we can associate a convex body Z via (27). Then Z is called the zonoid *generated* by ρ, and ρ is called the *generating measure* of Z. Since

$$\int_{S^{n-1}} |\langle u,v\rangle| \, d\rho(v) = \frac{1}{2} \int_{S^{n-1}} |\langle u,v\rangle| \, [d\rho(v) + d\rho(-v)],$$

we can always assume that ρ is an even measure, that is, $\rho(-A) = \rho(A)$ for $A \in \mathcal{B}(S^{n-1})$.

To get an intuitive interpretation of zonoids, let us first assume that the measure ρ in (27) is concentrated in finitely many points $\pm v_1, \dots, \pm v_m$, where $\rho(\{v_i\}) = \rho(\{-v_i\}) = a_i/2 > 0$. The line segment S_i with endpoints $a_i v_i$ and $-a_i v_i$ has the support function

$$h(S_i,u) = a_i |\langle u, v_i\rangle|,$$

hence

$$\int_{S^{n-1}} |\langle u, v\rangle| \, d\rho(v) = \sum_{i=1}^{m} h(S_i, u) \qquad \text{for } u \in S^{n-1}$$

and therefore

$$Z = S_1 + \ldots + S_m.$$

Thus the zonoid Z generated by a measure ρ with finite support is a finite Minkowski sum of segments. Such a body is called a *zonotope*. It is a polytope with the property that each of its faces has a centre of symmetry. Vice versa, if Z is a polytope with the property that all its two-dimensional faces are centrally symmetric, then Z is a zonotope. Since the two-dimensional faces of a three-dimensional zonotope are centrally symmetric, they are arranged in “equatorial zones”. This explains the name “zonotope”.

From (27) it is easy to see that every zonoid is a limit, in the Hausdorff metric, of a sequence of zonotopes. Conversely, by a suitable compactness argument one can show that every such limit is a zonoid.

3.1. Ranges of vector measures

Let σ be a nonatomic $\mathbf{E}^n$-valued measure (always countably additive) on some measurable space $(X, \mathcal{A})$, and let Z_σ be its range. By Liapounoff's theorem, Z_σ is a compact convex set. First we want to show that Z_σ is a zonoid. This is well-known, and a proof can be found, e.g., in a concise form in Bolker [13]. We find it instructive and useful to reproduce here the argument in a slightly expanded form.

If σ and Z_σ are as above, then $0 = \sigma(\emptyset) \in Z_\sigma$, and for $A \in \mathcal{A}$

$$\sigma(A) - \frac{1}{2}\sigma(X) = -\left[\sigma(X \backslash A) - \frac{1}{2}\sigma(X)\right],$$

hence Z_σ is centrally symmetric with respect to $\frac{1}{2}\sigma(X)$.

Let $x \in Z_\sigma$, say $x = \sigma(B)$ with $B \in \mathcal{A}$. Then

$$\tau(A) := \sigma(A \setminus B) - \sigma(A \cap B) \qquad \text{for } A \in \mathcal{A}$$

defines a nonatomic vector measure τ on $\mathcal{A}$ satisfying

$$\begin{aligned}\tau(A \triangle B) &= \sigma(A) - \sigma(B) = \sigma(A) - x,\\ \sigma(A \triangle B) &= \tau(A) + \sigma(B) = \tau(A) + x\end{aligned}$$

for $A \in \mathcal{A}$. It follows that $Z_\sigma - x = Z_\tau$. For this reason we may assume in the following, without loss of generality, that $\sigma(X) = 0$ and, hence, $Z_\sigma = -Z_\sigma$.

Now let μ be a real-valued Borel measure on the sphere S^{n-1} and define

$$\tilde{\mu}(A) := \int_A x \, d\mu(x) \qquad \text{for } A \in \mathcal{B}(\mathbf{E}^n).$$

(We may assume that μ is defined on all of $\mathbf{E}^n$, but concentrated on S^{n-1}. The vector-valued integral can be defined coordinate-wise.) Then $\tilde{\mu}$ is an $\mathbf{E}^n$-valued measure on $\mathcal{B}(\mathbf{E}^n)$, possibly with atoms. By K_μ we denote the closed convex hull of the range of $\tilde{\mu}$.

LEMMA 3.1.1. *If $K_\mu = -K_\mu$, then*

$$h(K_\mu, u) = \frac{1}{2} \int_{S^{n-1}} |\langle u, x\rangle| \, d\mu(x) \qquad \text{for } u \in \mathbf{E}^n.$$

Proof. For $u \in \mathbf{E}^n \setminus \{0\}$, let

$$H_u^+ := \{x \in \mathbf{E}^n : \langle x, u\rangle \geq 0\}.$$

Let $A \in \mathcal{B}(S^{n-1})$. Since $\langle x, u\rangle \geq 0$ for $x \in H_u^+$ and $\langle x, u\rangle \leq 0$ for $x \in -H_u^+$, we get

$$\begin{aligned}\langle \tilde{\mu}(A), u\rangle &= \int_A \langle u, x\rangle \, d\mu(x) \leq \int_{A\cap H_u^+} \langle u, x\rangle \, d\mu(x)\\ &\leq \int_{S^{n-1}\cap H_u^+} \langle u, x\rangle \, d\mu(x)\\ &= \langle \tilde{\mu}(H_u^+), u\rangle.\end{aligned}$$

By the definitions of K_μ and of the support function, this gives

$$h(K_\mu, u) = \langle \tilde{\mu}(H_u^+), u\rangle = \int_{H_u^+} \langle u, x\rangle \, d\mu(x).$$

Since $K_\mu = -K_\mu$, it follows that

$$\begin{aligned} 2h(K_\mu, u) &= h(K_\mu, u) + h(K_\mu, -u) \\ &= \int\limits_{H_u^+} \langle u, x \rangle \, d\mu(x) + \int\limits_{H_{-u}^+} \langle -u, x \rangle \, d\mu(x) \\ &= \int\limits_{S^{n-1}} |\langle u, x \rangle| \, d\mu(x), \end{aligned}$$

which completes the proof of Lemma 3.1.1. □

After this preparation, let Z_σ be the range of the nonatomic vector measure σ. Let $|\sigma|$ be the total variation measure of σ and

$$f := \frac{d\sigma}{d|\sigma|}$$

the Radon-Nikodym derivative, then $f : X \to S^{n-1}$ $|\sigma|$-almost everywhere. (The properties of the total variation measure of a vector measure used here can be proved similarly as the corresponding ones in the case of complex measures; see, e.g., Chapter 6 of Rudin [35].) Hence, the image measure of $|\sigma|$ under f,

$$\mu := |\sigma| \circ f^{-1},$$

is a measure on S^{n-1}, and we can define K_μ as above. Denoting by $\mathbf{1}_A$ the indicator function of A and by id the identity mapping on S^{n-1}, we have for $A \in \mathcal{B}(S^{n-1})$

$$\begin{aligned} \tilde{\mu}(A) &= \int\limits_{S^{n-1}} \mathbf{1}_A(x) x \, d\mu(x) = \int\limits_{S^{n-1}} \mathbf{1}_A \cdot \mathrm{id} \, d\mu \\ &= \int\limits_X (\mathbf{1}_A \cdot \mathrm{id}) \circ f \, d|\sigma| = \int\limits_X (\mathbf{1}_A \circ f) f \, d|\sigma| \\ &= \int\limits_X \mathbf{1}_A \circ f \, d\sigma = \int\limits_{S^{n-1}} \mathbf{1}_A \, d(\sigma \circ f^{-1}), \end{aligned}$$

thus $\tilde{\mu} = \sigma \circ f^{-1}$. Since $\tilde{\mu}(A) = \sigma(f^{-1}(A))$, each value of $\tilde{\mu}$ is also a value of σ, hence $K_\mu \subset Z_\sigma$.

To show the opposite inclusion, we estimate the support function of Z_σ. Let $A \in \mathcal{A}$ and $x \in S^{n-1}$. For $\omega \in X$ we have

$$\langle f(\omega), x\rangle \geq 0 \Leftrightarrow f(\omega) \in H_x^+ \Leftrightarrow \mathbf{1}_{f^{-1}(H_x^+)}(\omega) = 1,$$

hence

$$\mathbf{1}_A \langle f, x\rangle \leq \mathbf{1}_{f^{-1}(H_x^+)} \langle f, x\rangle,$$

which yields

$$\begin{aligned} \langle \sigma(A), x\rangle &= \int_X \mathbf{1}_A \langle f, x\rangle \, d|\sigma| \leq \int_X \mathbf{1}_{f^{-1}(H_x^+)} \langle f, x\rangle \, d|\sigma| \\ &= \langle \sigma(f^{-1}(H_x^+)), x\rangle = \langle \tilde{\mu}(H_x^+), x\rangle \\ &= h(K_\mu, x). \end{aligned}$$

This implies $h(Z_\sigma, x) \leq h(K_\mu, x)$. Since $x \in S^{n-1}$ was arbitrary, $Z_\sigma \subset K_\mu$ and thus $Z_\sigma = K_\mu$.

Since we may assume that $Z_\sigma = -Z_\sigma$, Lemma 3.1.1 gives

$$h(Z_\sigma, u) = \frac{1}{2} \int_{S^{n-1}} |\langle u, x\rangle| \, d\mu(x) \qquad \text{for } u \in \mathbf{E}^n,$$

thus Z_σ is a centred zonoid.

Vice versa, if (27) is satisfied, we refer to Bolker [13] for the construction of a nonatomic vector measure σ satisfying $Z = Z_\sigma$.

We add two further observations on ranges of vector measures. If the $\mathbf{E}^n$-valued measure σ on $(X, \mathcal{A})$ is nonatomic, then its range Z_σ is convex. Under which stronger condition on σ is the range strictly convex?[1] (A convex body is called *strictly convex* if its boundary does not contain a segment.) Extending the method used above, we can give the following answer.

THEOREM 3.1.2. *The range of the nonatomic $\mathbf{E}^n$-valued measure σ on $(X, \mathcal{A})$ is strictly convex if and only if for every set $A \in \mathcal{A}$ with $\sigma(A) \neq 0$ there are n measurable subsets $A_1, \dots, A_n \subset A$ such that $\sigma(A_1), \dots, \sigma(A_n)$ are linearly independent.*

[1] This question was asked by Carlo Mariconda in a discussion at the Workshop. We are not aware of an answer given in the previous literature.

Proof. Suppose that the range Z_σ is not strictly convex. Then there is a vector $e \in S^{n-1}$ such that the face

$$F(Z_\sigma, e) := Z_\sigma \cap H(Z_\sigma, e)$$

contains more than one point. We may assume that $Z_\sigma = -Z_\sigma$, then there is a real even measure μ on the sphere S^{n-1} so that

$$h(Z_\sigma, u) = \frac{1}{2} \int_{S^{n-1}} |\langle u, v\rangle| \, d\mu(v) \qquad \text{for } u \in \mathbf{E}^n.$$

By directional differentiation, one can obtain a representation for the support function of the face $F(Z_\sigma, e)$: if

$$\begin{aligned} s_e &:= \{x \in S^{n-1} : \langle x, e\rangle = 0\}, \\ S_e^+ &:= \{x \in S^{n-1} : \langle x, e\rangle > 0\}, \end{aligned}$$

then

$$h(F(Z_\sigma, e), u) = \frac{1}{2} \int_{s_e} |\langle u, v\rangle| \, d\mu(v) + \left\langle \int_{S_e^+} v \, d\mu(v), u \right\rangle \tag{28}$$

for $u \in \mathbf{E}^n$; see [42], Lemma 3.5.5. Since $F(Z_\sigma, e)$ is not one-pointed, we must have $\mu(s_e) > 0$. As in the proof given above, we have

$$\mu = |\sigma| \circ f^{-1} \qquad \text{with } f = \frac{d\sigma}{d|\sigma|}.$$

Therefore, the set $f^{-1}(s_e) \in \mathcal{A}$ satisfies $|\sigma|(f^{-1}(s_e)) > 0$ and hence contains a subset $A \in \mathcal{A}$ with $\sigma(A) \neq 0$. For any $B \in \mathcal{A}$ with $B \subset A$ we have $\langle f(\omega), e\rangle = 0$ for $\omega \in B$ and hence

$$\langle \sigma(B), e\rangle = \int_B \langle f, e\rangle \, d|\sigma| = 0.$$

Thus, for any measurable sets $B_1, \dots, B_n \subset A$, the vectors $\sigma(B_1), \dots, \sigma(B_n)$ are linearly dependent.

Conversely, suppose that there exists a set $A \in \mathcal{A}$ with $\sigma(A) \neq 0$ and such that

$$\operatorname{lin}\{\sigma(B) : B \in \mathcal{A}, B \subset A\} =: L \neq \mathbf{E}^n.$$

The range of the restriction of σ to A is a zonoid Z' in L. Let Z'' be the range of the restriction of σ to $X \setminus A$. Then $Z_\sigma = Z' + Z''$. If e is a unit vector orthogonal to L, then

$$F(Z_\sigma, e) = F(Z', e) + F(Z'', e) = Z' + F(Z'', e).$$

Thus the face $F(Z_\sigma, e)$ of Z_σ contains a translate of Z'. Since $\sigma(A) \neq 0$, the zonoid Z' contains more than one point, hence Z_σ is not strictly convex. This completes the proof of Theorem 3.1.2. □

Finally, we remark that formula (28) also permits to prove the following result.

THEOREM 3.1.3. *Let Z_σ be the range of the nonatomic $\mathbf{E}^n$-valued measure σ on $(X, \mathcal{A})$. If y is an exposed boundary point of Z_σ, then a set $A \in \mathcal{A}$ with $\sigma(A) = y$ is uniquely determined, up to sets of $|\sigma|$-measure zero.*

Proof. That y is an exposed boundary point of Z_σ means that there exists a unit vector e so that $F(Z_\sigma, e) = \{y\}$. From (28) it then follows that

$$\mu(s_e) = 0 \tag{29}$$

and

$$y = \int_{S_e^+} v \, d\mu(v).$$

Here μ and $\tilde{\mu}$ are derived from σ as before, in particular

$$\tilde{\mu}(A) = \int_A v \, d\mu(v) \qquad \text{for } A \in \mathcal{B}(S^{n-1})$$

and $\tilde{\mu} = \sigma \circ f^{-1}$, hence

$$y = \tilde{\mu}(S_e^+) = \sigma(A) \qquad \text{with } A := f^{-1}(S_e^+).$$

Suppose that $B \in \mathcal{A}$ is another set with $y = \sigma(B)$. Write $\overline{A} := A \cup f^{-1}(s_e)$. For $\omega \in B \setminus \overline{A}$ we have $\langle f(\omega), e \rangle < 0$. Since $|\sigma|(f^{-1}(s_e)) = 0$ by (29), we deduce that

$$\langle \sigma(B \setminus A), e \rangle = \int_{B \setminus \overline{A}} \langle f, e \rangle \, d|\sigma| \leq 0. \tag{30}$$

For $\omega \in B \cap A$ we have $\langle f(\omega), e\rangle > 0$, hence

$$\langle \sigma(B \cap A), e\rangle = \int_{B\cap A} \langle f, e\rangle \, d|\sigma| \le \int_A \langle f, e\rangle \, d|\sigma| = \langle \sigma(A), e\rangle = \langle y, e\rangle. \tag{31}$$

This gives

$$\langle y, e\rangle = \langle \sigma(B), e\rangle = \langle \sigma(B \setminus A), e\rangle + \langle \sigma(B \cap A), e\rangle \le \langle y, e\rangle.$$

Therefore we have equality in (30), which gives $|\sigma|(B \setminus \overline{A}) = 0$ and hence $|\sigma|(B \setminus A) = 0$, and equality in (31), which gives $|\sigma|(A \setminus B) = 0$. This completes the proof of Theorem 3.1.3. □

3.2. An application of zonoids

In the following, we want to give an example for the application of zonoids to a certain problem on measures arising in Stochastic Geometry. We consider infinite systems of random hyperplanes in $\mathbf{E}^n$ and study their intersections.

A few preliminary explanations are necessary. Let T be a locally compact, second countable topological space. Let M be the family of all locally finite subsets of T (M is locally finite, if $\operatorname{card}(M \cap C) < \infty$ for every compact set $C \subset T$). Let $\mathcal{M}$ be the σ-algebra on M generated by all functions f_B, $B \in \mathcal{B}(T)$, where

$$f_B(M) := \operatorname{card}(M \cap B) \qquad \text{for } M \in \mathsf{M}.$$

A (simple) *point process* on T is a measurable map X from some probability space $(\Omega, \mathcal{A}, \mathbf{P})$ into the measurable space $(\mathsf{M}, \mathcal{M})$. Let X be such a point process. Let

$$\Theta(B) := \mathbf{E} \operatorname{card}(X \cap B) \qquad \text{for } B \in \mathcal{B}(T),$$

where $\mathbf{E}$ denotes mathematical expectation. Then Θ is a measure, called the *intensity measure* of X; we assume that it is locally finite (i.e., finite on compact sets). The point process X is called a *Poisson process* if each counting variable $\operatorname{card}(X \cap B)$, $B \in \mathcal{B}(T)$, has a Poisson distribution, that is, if

$$\mathbf{P}(\operatorname{card}(X \cap B) = k) = e^{-\Theta(B)} \frac{\Theta(B)^k}{k!} \qquad \text{for } k \in \mathbf{N}_0.$$

Poisson processes are, for several reasons, the simplest and most interesting class of point processes.

Now we consider the space $\mathcal{E}^n_{n-1}$ of hyperplanes in $\mathbf{E}^n$. With the usual topology, it is a locally compact, second countable space, so that the foregoing can be applied. Let X be a stationary Poisson process on $\mathcal{E}^n_{n-1}$. Here "stationary" means that the intensity measure Θ of X, which is a measure on $\mathcal{B}(\mathcal{E}^n_{n-1})$, is invariant under translations. Since Θ is assumed to be locally finite and translation invariant, it can be decomposed in the following form. We parametrize the hyperplanes by writing

$$H_{u,\tau} := \{x \in \mathbf{E}^n : \langle x, u \rangle = \tau\}$$

for $u \in S^{n-1}$ and $\tau \in \mathbf{R}$. Then it can be shown that there exists a uniquely determined finite even measure ρ on the sphere S^{n-1} so that

$$\int_{\mathcal{E}^n_{n-1}} f \, d\Theta = \int_{S^{n-1}} \int_{-\infty}^{\infty} f(H_{u,\tau}) \, d\tau \, d\rho(u)$$

for every Θ-integrable real function f. The measure ρ describes the frequency of hyperplanes with given directions in the process X. More precisely, for a hyperplane H let $u(H)$ be one of its unit normal vectors. For a symmetric Borel set $\omega \subset S^{n-1}$ we then have

$$\rho(\omega) = \frac{1}{2} \mathbf{E} \operatorname{card} \{H \in X : H \cap B^n \neq \emptyset, \ u(H) \in \omega\}.$$

The number

$$\gamma := \rho(S^{n-1}) = \frac{1}{2} \mathbf{E} \operatorname{card} \{H \in X : H \cap B^n \neq \emptyset\}$$

is called the *intensity* of the hyperplane process X.

The hyperplanes in a realization of X determine lower-dimensional flats by intersections. We want to measure the density of such intersections by a number. Let $k \in \{2, \dots, n\}$. In each realization of X, we form all intersections of any k hyperplanes in general position. Let X_k be the set of all $(n-k)$-flats obtained in this way. We call

$$\gamma_k := \frac{1}{\kappa_k} \mathbf{E} \operatorname{card} \{F \in X_k : F \cap B^n \neq \emptyset\}$$

the k-th *intersection density* of the hyperplane process X.

If the intensity γ of X is given, the intersection density γ_k depends on the probability measure $\gamma^{-1}\rho$, the *direction distribution* of X. We may ask for which direction distributions the k-th intersection density γ_k becomes maximal (the minimum, of course, is zero). This question can be answered with the help of zonoids.

First, we need an explicit expression for the intersection density γ_k. One finds that

$$\gamma_k = \frac{1}{k!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} [u_1, \dots, u_k] \, d\rho(u_1) \cdots d\rho(u_k),$$

where $[u_1, \dots, u_k]$ denotes the k-dimensional volume of the parallelepiped spanned by the vectors $u_1, \dots, u_k$. Next, one associates with the stationary Poisson hyperplane process X the *Matheron zonoid* Z, defined by its support function

$$h(Z, u) = \frac{1}{2} \int_{S^{n-1}} |\langle u, v \rangle| \, d\rho(v), \qquad u \in \mathbf{E}^n. \tag{32}$$

We need an expression for the volume $\lambda_n(Z)$ if Z is defined by (32). Suppose, first, that

$$\rho = \sum_{i=1}^{k} a_i (\delta_{v_i} + \delta_{-v_i}),$$

thus

$$h(Z, \cdot) = \sum_{i=1}^{k} a_i |\langle \cdot, v_i \rangle|$$

and

$$Z = S_1 + \ldots + S_k \qquad \text{with } S_i := \operatorname{conv}\{a_i v_i, -a_i v_i\}.$$

It is easy to see that

$$\lambda_n(S_1 + \ldots + S_k) = \sum_{1 \le i_1 < \ldots < i_n \le k} \lambda_n(S_{i_1} + \ldots + S_{i_n}).$$

On the other hand,

$$\begin{aligned}\sum_{1\le i_1<\ldots<i_n\le k} &\lambda(S_{i_1}+\ldots+S_{i_n})\\ &= \frac{2^n}{n!}\sum_{i_1,\ldots,i_n=1}^{k}[v_{i_1},\ldots,v_{i_n}]a_{i_1}\cdots a_{i_n}\\ &= \frac{1}{n!}\int_{S^{n-1}}\cdots\int_{S^{n-1}}[u_1,\ldots,u_n]\,d\rho(u_1)\cdots d\rho(u_n).\end{aligned}$$

By approximation, we obtain for general (even) measures ρ the result

$$\lambda_n(Z)=\frac{1}{n!}\int_{S^{n-1}}\cdots\int_{S^{n-1}}[u_1,\ldots,u_n]\,d\rho(u_1)\cdots d\rho(u_n). \tag{33}$$

We are interested in $\lambda_n(Z+\epsilon B^n)$ for $\epsilon\ge 0$. But the ball B^n is also a zonoid, since

$$h(B^n,\cdot)=\frac{1}{2}\int_{S^{n-1}}|\langle\cdot,v\rangle|\,d\lambda_s(v)$$

with $\lambda_s(A):=\mathcal{H}^{n-1}(A)/\kappa_{n-1}$ for $A\in\mathcal{B}(S^{n-1})$. Hence,

$$h(Z+\epsilon B^n,\cdot)=\frac{1}{2}\int_{S^{n-1}}|\langle\cdot,v\rangle|\,d(\rho+\epsilon\lambda_s).$$

Using formula (33) for $Z+\epsilon B^n$ instead of Z, we see that for the zonoid Z the functionals W_i defined by the Steiner formula (13) (the so-called quermassintegrals or Minkowski functionals) have an integral representation in terms of the generating measure, namely

$$\begin{aligned}W_i(Z) \;=\; &\frac{1}{n!}\int_{S^{n-1}}\cdots\int_{S^{n-1}}[u_1,\ldots,u_n]\\ &d\rho(u_1)\cdots d\rho(u_{n-i})\,d\lambda_s(u_{n-i+1})\cdots d\lambda_s(u_n).\end{aligned}$$

The integrations with respect to λ_s can be carried out, and one obtains

$$W_i(Z)=\frac{i!\kappa_i}{n!}\int_{S^{n-1}}\cdots\int_{S^{n-1}}[u_1,\ldots,u_{n-i}]\,d\rho(u_1)\cdots d\rho(u_{n-i}). \tag{34}$$

Thus, the intensity $\gamma = \gamma_1$ and the intersection densities of the hyperplane process X are essentially quermassintegrals of its Matheron zonoid:

$$\gamma_k = \frac{\binom{n}{k}}{\kappa_{n-k}} W_{n-k}(Z) \qquad \text{for } k = 1, \dots, n.$$

From the theory of convex bodies it is known that

$$W_{n-1}(Z)^k \geq \kappa_n^{k-1} W_{n-k}(Z),$$

with equality if and only if Z is a ball. The zonoid Z is a ball if and only if its generating measure ρ is rotation invariant. For the stationary hyperplane process X this is equivalent to the rigid motion invariance of its intensity measure. Such a hyperplane process is called *isotropic*. Thus it has turned out that among the stationary Poisson hyperplane processes of given intensity $\gamma > 0$, precisely the isotropic ones have maximal k-th intersection density, for $k = 2, \dots, n$.

Notes. Special zonoids (though not under this name) were already treated by Blaschke [12], pp. 154–157; also special cases of formula (34) appear there. For later generalizations of such formulas, see [42], Section 5.3, and the references given there in Note 1.

The paper of Bolker [13] collects various equivalent characterizations of zonoids, among them that as ranges of nonatomic vector measures. Later surveys on zonoids, mainly from the geometric point of view, were given by Schneider and Weil [44] and by Goodey and Weil [22].

The Matheron zonoid was first used (under the name of *Steiner compact*) by Matheron [33]. The extremum property of isotropic hyperplane processes just shown was observed by Thomas [48], and in an essentially equivalent form for finitely many random hyperplanes by Schneider [41]. Further applications of associated zonoids, which generalize the Matheron zonoid, are surveyed in Schneider and Wieacker [46], Section 6, and in Weil and Wieacker [49], Sections 6 and 7.

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Received December 30, 1995.