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Decomposition and Extension of Abstract Measures in Riesz Spaces

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Summary. - *The aim of these notes is to review some recent developments in the theory of abstract measures taking their values in a Riesz space. The term abstract measure is used here to denote a common abstraction of vector measures and linear operators. The topics considered in this survey are: A common approach to vector measures and linear operators, Jordan and Lebesgue decompositions of abstract measures and their applications to vector measures and linear operators, common extensions of linear operators and of vector measures, and extensions of modular functions. We also propose a number of open problems which may stimulate further research in this area.*

The material of these notes is based on the monograph by Schmidt [54], two papers by Schmidt and Waldschaks [55], [56], and the PhD Thesis of Waldschaks [60].

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1. Introduction

Let Ω be a non–empty set. For a set $A \subseteq \Omega$, let $\chi_A : \Omega \to \{0,1\}$ denote its indicator function.

Let $\mathcal{F}$ be an algebra of subsets of Ω and define

$$\mathbf{D}(\mathcal{F}) \quad := \quad \mathrm{span}\{\chi_A \mid A \in \mathcal{F}\} \ .$$

Under the pointwise defined linear operations and order relation, $\mathbf{D}(\mathcal{F})$ is an ordered (real) vector space such that any two elements have a least upper bound and a greatest lower bound; that is, $\mathbf{D}(\mathcal{F})$ is a vector lattice.

Let $\mathbf{G}$ be a vector space. A map $\varphi : \mathcal{F} \to \mathbf{G}$ is a *vector measure* if the identity

$$\varphi(A+B) \quad = \quad \varphi(A) + \varphi(B)$$

holds for every pair of disjoint sets $A, B \in \mathcal{F}$. Every vector measure $\varphi : \mathcal{F} \to \mathbf{G}$ induces a linear operator $T : \mathbf{D}(\mathcal{F}) \to \mathbf{G}$, given by

$$T\left(\sum_{i=1}^{n} \alpha_i \chi_{A_i}\right) \quad := \quad \sum_{i=1}^{n} \alpha_i \varphi(A_i) \ ,$$

which is called the *elementary integral* with respect to φ or the *representing linear operator* of φ. Conversely, every linear operator $T : \mathbf{D}(\mathcal{F}) \to \mathbf{G}$ induces a vector measure $\varphi : \mathcal{F} \to \mathbf{G}$, given by

$$\varphi(A) \quad := \quad T\chi_A \ .$$

This one–to–one correspondence between vector measures $\mathcal{F} \to \mathbf{G}$ and linear operators $\mathbf{D}(\mathcal{F}) \to \mathbf{G}$ can be used to obtain results on vector measures from corresponding ones on linear operators – provided that suitable results on linear operators are known.

Instead of reducing problems on vector measures to those on linear operators, one can try to develop a common approach to vector measures on an algebra of sets and linear operators on a vector lattice.

An important step into this direction is due to Bauer [6], [7], who observed that algebras of sets as well as vector lattices are distributive lattices and that vector measures on an algebra of sets as well as linear operators on a vector lattice are *valuations* on a distributive lattice in the sense that they satisfy the *modular law*

$$\varphi(a \vee b) + \varphi(a \wedge b) \quad = \quad \varphi(a) + \varphi(b) \ .$$

Bauer's decomposition theory for valuations from a distributive lattice into a vector lattice is directly applicable to vector measures, but its application to linear operators is less immediate and requires additional considerations.

Algebras of sets and vector lattices are examples of Boolean rings and lattice–ordered groups, respectively, and common abstractions of Boolean rings and lattice–ordered groups have been studied by various authors; see Schmidt [51], [53]. A postulate of Rama Rao [45] states that a common abstraction of Boolean rings and lattice–ordered groups should, is order to be useful, *possess as much as possible of the richness of the structures common to both* Boolean rings and lattice–ordered groups. With Rama Rao's postulate in mind, a closer inspection of the structures of algebras of sets and of vector lattices leads to the observation that an additive structure is present not only in vector lattices but also in algebras of sets, where the union and the symmetric difference serve as candidates. Unfortunately, the union of sets does not satisfy the cancellation law and the symmetric difference of sets is not compatible with the natural order given by inclusion. However, this phenomenon vanishes if these binary operations are restricted to *disjoint pairs* of sets, and it is interesting to note that the restrictions of the union and of the symmetric difference to disjoint pairs agree.

These observations suggest to consider distributive lattices which are equipped with a *partial addition* such that order and addition are compatible in the sense that $a \leq b$ implies $a+c \leq b+c$ and such that addition satisfies the cancellation law in the sense that $a+c = b+c$ implies $a = b$, provided that all sums are defined. Such an ordered algebraic structure is that of a (commutative) minimal clan which was introduced by Schmidt [51], [53] and which will be studied in Section 3.

Once a commutative minimal clan $\mathbf{E}$ is given, a very natural common abstraction of vector measures and linear operators is provided by mappings $\varphi : \mathbf{E} \to \mathbf{G}$ satisfying the *additive law*

$$\varphi(a+b) \quad = \quad \varphi(a) + \varphi(b)$$

for all $a, b \in \mathbf{E}$ such that the sum $a+b$ is defined. The decomposition theory for such *additive functions* from a commutative minimal clan into a vector lattice will be developed in Sections 4 and 5 below, and we shall see that the application of the general results on additive functions to vector measures and to linear operators is straightforward *in both cases.* The material of these sections is taken from Schmidt [54], where further results may be found.

Another important topic in the theory of vector measures and linear operators is their extension theory. In Sections 6 and 7 we shall study common extensions of two or more positive or order bounded linear operators or vector measures; these results are due to Schmidt and Waldschaks [55], [56]. The proofs of the results on common extensions of vector measures given there are based on corresponding ones on linear operators, and it is an open problem whether the common approach to vector measures and linear operators based on additive functions on minimal clans is also possible in extension theory.

The major obstacle in studying extensions of vector measures without using representing linear operators is the lack of Hahn–Banach theorems for vector measures. Some results on the extension of a modular function on a Boolean ring were obtained by Waldschaks [60] and will be given in Section 8. We hope that this kind of results can be extended to additive functions on a minimal clan.

2. Riesz Spaces

A *vector lattice* or *Riesz space* is a (real) vector space $\mathbf{G}$ with an order relation $\leq$ such that

(i) the linear operations and the order relation are compatible, that is, for all $x, y, z \in \mathbf{G}$ and $\alpha \in \mathbf{R}_+$, $x \leq y$ implies $x + z \leq y + z$ and $\alpha x \leq \alpha y$, and

(ii) $\langle \mathbf{G}, \leq \rangle$ is a lattice, that is, $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ exist for all $x, y \in \mathbf{G}$.

Throughout this section, let $\mathbf{G}$ be a Riesz space. The set

$$\mathbf{G}_+ \;:=\; \{x \in \mathbf{G} \mid 0 \le x\}$$

is said to be the *positive cone* of $\mathbf{G}$ and the elements of $\mathbf{G}_+$ are said to be *positive*. Two elements $x, y \in \mathbf{G}$ are *disjoint* if $x \wedge y = 0$.

For $x \in \mathbf{G}$, define

$$\begin{aligned} x^+ &:= x \vee 0\,, \\ x^- &:= (-x) \vee 0\,, \\ |x| &:= x \vee (-x)\,. \end{aligned}$$

These elements are called the *positive part*, the *negative part*, and the *modulus* of x, respectively. For $x, z \in \mathbf{G}$, the set

$$[x, z] \;:=\; \{y \in \mathbf{G} \mid x \le y \le z\}$$

is said to be an *order interval.* A set $A \subseteq \mathbf{G}$ is said to be *order bounded* if it is contained in an order interval.

The Riesz space $\mathbf{G}$ has an *order unit* if there exists some $g \in \mathbf{G}_+$ such that

$$\mathbf{G} \;=\; \bigcup\nolimits_{n \in \mathbf{N}} [-ng, ng]\,,$$

it is *Archimedean* if $x \le 0$ holds for all $x \in \mathbf{G}$ satisfying $nx \le y$ for some $y \in \mathbf{G}$ and all $n \in \mathbf{N}$, and it is *order complete* or *Dedekind complete* if every order bounded subset of $\mathbf{G}$ has a supremum and an infimum. Every order complete Riesz space is Archimedean, but the converse is not true.

A family $\{x_\gamma\}_{\gamma \in \Gamma} \subseteq \mathbf{G}$ is *directed* $(\le)$ if for all $\gamma', \gamma'' \in \Gamma$ there exists some $\gamma \in \Gamma$ satisfying $x_{\gamma'} \le x_\gamma$ and $x_{\gamma''} \le x_\gamma$. The Riesz space $\mathbf{G}$ is order complete if and only if $\sup_\Gamma x_\gamma$ exists for every order bounded directed $(\le)$ family $\{x_\gamma\}_{\gamma \in \Gamma} \subseteq \mathbf{G}_+$.

A set $B \subseteq \mathbf{G}$ is *solid* if $x \in B$ holds for all $x \in \mathbf{G}$ satisfying $|x| \le |y|$ for some $y \in B$. A solid vector subspace of $\mathbf{G}$ is said to be an *ideal.* An ideal $B \subseteq \mathbf{G}$ is said to be a *band* if $\sup A \in B$ holds for every set $A \subseteq B$ such that $\sup A$ exists (in $\mathbf{G}$).

For every set $A \subseteq \mathbf{G}$, there exists a smallest band $B(A)$ containing A, and the set

$$A^{\perp} \ := \ \{x \in \mathbf{G} \mid |x| \wedge |z| = 0 \text{ for all } z \in A\}$$

is a band; moreover, $B(A) \subseteq B^{\perp\perp}$.

The Riesz space $\mathbf{G}$ is the *order direct sum* of two bands $B, C \subseteq \mathbf{G}$ if, for each $x \in \mathbf{G}$, there exist unique $y \in B$ and $z \in C$ satisfying $x = y + z$ and $|y| \wedge |z| = 0$.

PROPOSITION 2.1. (RIESZ DECOMPOSITION) *Assume that* $\mathbf{G}$ *is order complete and let* $A \subseteq \mathbf{G}$. *Then* $\mathbf{G}$ *is the order direct sum of* $B(A)$ *and* $A^{\perp}$.

An ideal $B \subseteq \mathbf{G}$ satisfying $B + B^{\perp} = \mathbf{G}$ is said to be a *projection band*. If $\mathbf{G}$ is order complete, then every band is a projection band.

We finally remark that every Riesz space can be identified with a Riesz subspace of an *order complete* Riesz space. Thus, in the discussion of mappings taking their values in a Riesz space $\mathbf{G}$, there is no loss of generality when $\mathbf{G}$ is assumed to be order complete.

For detailed information on Riesz spaces, see Luxemburg and Zaanen [37], Schaefer [47], and Aliprantis and Burkinshaw [2].

3. A Common Abstraction of Boolean Rings and Lattice–Ordered Groups

In the present section we study minimal clans – a common abstraction of Boolean rings and lattice–ordered groups introduced by Schmidt [51], [53] which is the foundation of a common approach to vector measures and linear operators. Minimal clans are distributive lattices equipped with a partial addition which distributes with the lattice operations and which reflects the analogy between suprema of disjoint elements in a Boolean ring and sums of arbitrary elements in a lattice–ordered group – an analogy which has been emphasized by Dinges [25] and which is important in view of the defining properties of vector measures and linear operators.

We first study the general properties of minimal clans. We then characterize Boolean rings and lattice–ordered groups as minimal clans having, respectively, a minimal domain of addition or a maximal set of invertible elements. We complete the discussion of minimal clans with a few comments concerning their axioms and related ordered algebraic structures, and we also show that the additive classes of fuzzy sets introduced by Butnariu [17] are commutative minimal clans which need not be a Boolean ring and cannot be a lattice–ordered group.

Minimal Clans

A *minimal clan* is a set $\mathbf{E}$ with a relation $\mathcal{S} \subseteq \mathbf{E}\times\mathbf{E}$, a map $+ : \mathcal{S} \to \mathbf{E}$, and an order relation $\leq$ such that

(MC–1) there exists an element $0 \in \mathbf{E}$ satisfying $(0, x) \in \mathcal{S}$, $(x, 0) \in \mathcal{S}$, and $0 + x = x = x + 0$ for all $x \in \mathbf{E}$;

(MC–2) for all $x, y, z \in \mathbf{E}$, $(x, y) \in \mathcal{S}$ and $(x + y, z) \in \mathcal{S}$ if and only if $(y, z) \in \mathcal{S}$ and $(x, y + z) \in \mathcal{S}$, and in this case $(x + y) + z = x + (y + z)$;

(MC–3) $x = y$ holds for all $x, y \in \mathbf{E}$ satisfying $u + x + v = u + y + v$ for some $u, v \in \mathbf{E}$ satisfying $(u, x) \in \mathcal{S}$, $(u + x, v) \in \mathcal{S}$, $(u, y) \in \mathcal{S}$, and $(u + y, v) \in \mathcal{S}$;

(MC–4) $u + x + v \leq u + y + v$ holds for all $x, y \in \mathbf{E}$ satisfying $x \leq y$ and for all $u, v \in \mathbf{E}$ satisfying $(u, x) \in \mathcal{S}$, $(u + x, v) \in \mathcal{S}$, $(u, y) \in \mathcal{S}$, and $(u + y, v) \in \mathcal{S}$;

(MC–5) $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ exist for all $x, y \in \mathbf{E}$; and

(MC–6) for all $x, y \in \mathbf{E}$, there exist $u, v \in \mathbf{E}$ satisfying $0 \leq u$, $0 \leq v$, $(u, x) \in \mathcal{S}$, $(x, v) \in \mathcal{S}$, $(u, x \wedge y) \in \mathcal{S}$, $(x \wedge y, v) \in \mathcal{S}$, $u + x = x \vee y = x + v$, and $u + x \wedge y = y = x \wedge y + v$.

Throughout this section, let $\langle \mathbf{E}, \mathcal{S}, +, \leq \rangle$ be a minimal clan.

Two elements $x, y \in \mathbf{E}$ are *summable* if $(x, y) \in \mathcal{S}$, the set $\mathcal{S}$ of all pairs of summable elements of $\mathbf{E}$ is said to be the *domain of addition*, the map $+ : \mathcal{S} \to \mathbf{E}$ is called (*partial*) *addition*, and the (unique) element $0 \in \mathbf{E}$ satisfying $(0, x) \in \mathcal{S}$, $(x, 0) \in \mathcal{S}$, and $0 + x = x = x + 0$ for all $x \in \mathbf{E}$ is said to be the *zero element* of $\mathbf{E}$.

Axiom (MC-3) is the *cancellation property*, and axiom (MC-6) will be referred to as the *difference property*. Indeed, the difference property may be used to define partial left and right subtractions, but this possibility will not to be used in the sequel since it appears to be easier to work with a single partial operation.

For the simplicity of notation, we shall usually write

$$x + y \textit{ has property } \pi$$

instead of the full statememt

$$(x, y) \in \mathcal{S} \quad \textit{and} \quad x + y \textit{ has property } \pi \ ,$$

where π is any property of elements of $\mathbf{E}$.

We first give some further definitions and elementary results concerning the sets of all pairs of summable elements, all invertible elements, all positive elements, and all pairs of disjoint elements.

LEMMA 3.1. *If $u \leq x$, $v \leq y$, and $(x, y) \in \mathcal{S}$, then $(u, v) \in \mathcal{S}$.*

An element $x \in \mathbf{E}$ is *invertible* if there exist $u, v \in \mathbf{E}$ satisfying $u+x = 0 = x+v$; this is equivalent with the existence of some $w \in \mathbf{E}$ satisfying $w + x = 0$ or $0 = x + w$, and this condition is equivalent in turn with the existence of a (unique) element $x^* \in \mathbf{E}$ satisfying $x^* + x = 0 = x + x^*$, which is said to be the *inverse* of x. The set of all invertible elements of $\mathbf{E}$ will be denoted by $\mathbf{E}_*$.

LEMMA 3.2.
(a) *If $u \leq x$ and $x \in \mathbf{E}_*$, then $u \in \mathbf{E}_*$.*
(b) *For all $x \in \mathbf{E}$, $x \vee 0 + x \wedge 0 = x = x \wedge 0 + x \vee 0$.*
(c) *$\mathbf{E}_* \times \mathbf{E} \subseteq \mathcal{S}$ and $\mathbf{E} \times \mathbf{E}_* \subseteq \mathcal{S}$.*
(d) *If $\mathbf{E}_* = \mathbf{E}$, then $\mathcal{S} = \mathbf{E} \times \mathbf{E}$.*
(e) *$\langle \mathbf{E}_*, \mathbf{E}_* \times \mathbf{E}_*, +, \leq \rangle$ is a minimal clan.*

An element $x \in \mathbf{E}$ is *positive* if $0 \leq x$. The set of all positive elements of $\mathbf{E}$ will be denoted by $\mathbf{E}_+$.

LEMMA 3.3. *The following are equivalent:*
(a) $\mathbf{E}_* = \{0\}$.
(b) $\mathbf{E}_+ = \mathbf{E}$.

A minimal clan is *positive* if it satisfies condition (b) of Lemma 3.3.

LEMMA 3.4.
(a) *If* $u \leq x$ *and* $u \in \mathbf{E}_+$, *then* $x \in \mathbf{E}_+$.
(b) $\langle \mathbf{E}_+, (\mathbf{E}_+ \times \mathbf{E}_+) \cap \mathcal{S}, +, \leq \rangle$ *is a positive minimal clan.*

Two elements $x, y \in \mathbf{E}$ are *disjoint* if $x \wedge y = 0$. The set of all pairs of disjoint elements of $\mathbf{E}$ will be denoted by $\mathcal{D}$.

LEMMA 3.5.
(a) *If* $x \wedge y = 0$, *then* $x + y = x \vee y = y + x$.
(b) $\mathcal{D} \subseteq \mathcal{S}$.
(c) *If* $\mathcal{D} = \mathcal{S}$, *then* $\mathbf{E}_+ = \mathbf{E}$.

LEMMA 3.6. *The following are equivalent*:
(a) *For all* $x, y \in \mathbf{E}$, $(x, y) \in \mathcal{S}$ *if and only if* $(y, x) \in \mathcal{S}$, *and in this case* $x + y = y + x$.
(b) *For all* $x, y \in \mathbf{E}$, $(x, y) \in \mathcal{S}$ *if and only if* $(x \vee y, x \wedge y) \in \mathcal{S}$, *and in this case* $x + y = x \vee y + x \wedge y$.

A minimal clan is *commutative* if it satisfies condition (a) of Lemma 3.6.

Condition (b) of Lemma 3.6 is the *modular law* which, in a commutative minimal clan, generalizes assertion (b) of Lemma 3.2 and assertion (a) of Lemma 3.5. In particular, if $\langle \mathbf{E}, \mathcal{S}, +, \leq \rangle$ is commutative, then $(x, y) \in \mathcal{S}$ holds for all $x, y \in \mathbf{E}$ satisfying $x \wedge y \in \mathbf{E}_*$.

We now return to the general case.

LEMMA 3.7.
(a) *For all* $x, y \in \mathbf{E}$, *there exist unique* $u, v \in \mathbf{E}$ *satisfying* $u + x = x \vee y = x + v$.
(b) *For all* $x, y \in \mathbf{E}$, *there exist unique* $u, v \in \mathbf{E}$ *satisfying* $u + x \wedge y = y = x \wedge y + v$.

LEMMA 3.8. $u + x \wedge y = y = x \wedge y + v$ *if and only if* $u + x = x \vee y = x + v$, *and in this case* $u \in \mathbf{E}_+$ *and* $v \in \mathbf{E}_+$.

The following result is the *order cancellation property*:

THEOREM 3.9 (ORDER CANCELLATION PROPERTY). *If* $u + x + v \leq u + y + v$, *then* $x \leq y$.

Proof. Choose first $w \in \mathbf{E}_+$ satisfying $w + u + x + v = u + y + v$. Then we have $u + x \leq w + u + x = u + y$, by the cancellation property.

Choose now $z \in \mathbf{E}_+$ satisfying $u + x + z = u + y$. Then we have $x \leq x + z = y$, as was to be shown. $\square$

COROLLARY 3.10. *If* $u + x \wedge y = y = x \wedge y + v$ *and* $w + x \wedge y = x = x \wedge y + z$, *then* $u \wedge w = 0 = v \wedge z$.

COROLLARY 3.11. *If* $u + x \wedge y = y = x \wedge y + v$ *and* $w + x \wedge y = x = x \wedge y + z$, *then* $u + w = w + u$ *and* $v + z = z + v$.

The following result is the *refinement property*:

THEOREM 3.12 (REFINEMENT PROPERTY). *If* $x_1, x_2, \ldots, x_m \in \mathbf{E}_+$ *and* $y_1, y_2, \ldots, y_n \in \mathbf{E}_+$ *are such that*

$$\sum_{i=1}^{m} x_i \;=\; \sum_{j=1}^{n} y_j \;,$$

then there exist $z_{ij} \in \mathbf{E}_+$ *satisfying*

$$x_i \;=\; \sum_{j=1}^{n} z_{ij}$$

for all $i \in \{1, 2, \ldots, m\}$,

$$y_j \;=\; \sum_{i=1}^{m} z_{ij}$$

for all $j \in \{1, 2, \ldots, n\}$, *and*

$$\left(\sum_{k=i+1}^{m} z_{kj} \right) \wedge \left(\sum_{l=j+1}^{n} z_{il} \right) \;=\; 0$$

for all $i \in \{1, 2, \ldots, m-1\}$ *and* $j \in \{1, 2, \ldots, n-1\}$.

Proof. The assertion is obvious for $m = 1$ or $n = 1$.
Let us first consider the case $m = n = 2$. Define $z_{11} := x_1 \wedge y_1$, choose $z_{12} \in \mathbf{E}_+$ satisfying

$$x_1 \vee y_1 \;=\; y_1 + z_{12} \quad \text{and} \quad x_1 \;=\; x_1 \wedge y_1 + z_{12} \;,$$

and choose $z_{21} \in \mathbf{E}_+$ satisfying

$$x_1 \vee y_1 \;=\; x_1 + z_{21} \quad \text{and} \quad y_1 \;=\; x_1 \wedge y_1 + z_{21} \;.$$

Then we have $z_{11} \in \mathbf{E}_+$, as well as

$$x_1 = z_{11} + z_{12}$$

and

$$y_1 = z_{11} + z_{21} .$$

Now define $z := x_1 + x_2 = y_1 + y_2$. Then we have $z_{11} + z_{12} + z_{21} = x_1 + z_{21} = x_1 \vee y_1 \leq z$, and we may choose $z_{22} \in \mathbf{E}_+$ satisfying

$$\begin{aligned} z_{11} + z_{12} + z_{21} + z_{22} &= z \\ &= x_1 + x_2 \\ &= z_{11} + z_{12} + x_2 , \end{aligned}$$

and thus

$$x_2 = z_{21} + z_{22} ,$$

by the cancellation property. Furthermore, we have $z_{12} + z_{21} = z_{21} + z_{12}$, by Corollary 3.11, hence

$$\begin{aligned} z_{11} + z_{21} + z_{12} + z_{22} &= z \\ &= y_1 + y_2 \\ &= z_{11} + z_{21} + y_2 , \end{aligned}$$

and thus

$$y_2 = z_{12} + z_{22} .$$

Finally, Corollary 3.10 yields

$$z_{21} \wedge z_{12} = 0 .$$

This proves the assertion in the case $m = n = 2$.
The general case now follows by induction. □

We can now prove the *distributive laws*:

Theorem 3.13 (Distributive Laws).

(a) *If* $(x,y) \in \mathcal{S}$ *and* $(x,z) \in \mathcal{S}$, *then* $x + y \vee z = (x+y) \vee (x+z)$ *and* $x + y \wedge z = (x+y) \wedge (x+z)$.
(b) *If* $(x,z) \in \mathcal{S}$ *and* $(y,z) \in \mathcal{S}$, *then* $x \vee y + z = (x+z) \vee (y+z)$ *and* $x \wedge y + z = (x+z) \wedge (y+z)$.

Proof. Choose $v' \in \mathbf{E}_+$ satisfying

$$y \vee z \;=\; z + v' \quad \text{and} \quad y \;=\; y \wedge z + v' ,$$

choose $z' \in \mathbf{E}_+$ satisfying

$$y \vee z \;=\; y + z' \quad \text{and} \quad z \;=\; y \wedge z + z' ,$$

and define $u := x + y \wedge z$. Then we have

$$\begin{aligned} x + y &= x + y \wedge z + v' \\ &= u + v' \end{aligned}$$

and

$$\begin{aligned} x + z &= x + y \wedge z + z' \\ &= u + z' , \end{aligned}$$

as well as $v' \wedge z' = 0$, by Corollary 3.10. Choose now $v'' \in \mathbf{E}_+$ satisfying

$$\begin{aligned} (u+v') \vee (u+z') &= u + v' + v'' \\ u + z' &= (u+v') \wedge (u+z') + v'' \end{aligned}$$

and choose $z'' \in \mathbf{E}_+$ satisfying

$$\begin{aligned} (u+v') \vee (u+z') &= u + z' + z'' \\ u + v' &= (u+v') \wedge (u+z') + z''. \end{aligned}$$

Then we have

$$v' + v'' \;=\; z' + z'' ,$$

by the cancellation property, as well as $v'' \wedge z'' = 0$, by Corollary 3.10. By the refinement property, there exist $z_{ij} \in \mathbf{E}_+$ satisfying

$$v' \;=\; z_{11} + z_{12} \quad \text{and} \quad v'' \;=\; z_{21} + z_{22}$$

as well as

$$z' = z_{11} + z_{21} \quad \text{and} \quad z'' = z_{12} + z_{22} \,.$$

From $0 \le z_{11} \le v' \wedge z' = 0$ and $0 \le z_{22} \le v'' \wedge z'' = 0$ we obtain $z_{11} = 0 = z_{22}$, and thus

$$z' = v'' \,.$$

This yields

$$\begin{aligned} x + y \vee z &= x + y + z' \\ &= u + v' + v'' \\ &= (u + v') \vee (u + z') \\ &= (x + y) \vee (x + z) \,, \end{aligned}$$

as well as

$$\begin{aligned} x + y \wedge z &= u \\ &= (u + v') \wedge (u + z') \\ &= (x + y) \wedge (x + z) \,. \end{aligned}$$

This proves (a).
The proof of (b) is similar. □

The following result shows that every minimal clan is a distributive lattice:

THEOREM 3.14 (DISTRIBUTIVITY).
(a) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
(b) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Proof. Choose $v \in \mathbf{E}_+$ satisfying

$$x \vee (y \wedge z) = y \wedge z + v \quad \text{and} \quad x = x \wedge y \wedge z + v \,,$$

choose $v' \in \mathbf{E}_+$ satisfying

$$x \vee y = x + v' \quad \text{and} \quad y = x \wedge y + v' \,,$$

and choose $v'' \in \mathbf{E}_+$ satisfying

$$x \vee z \;=\; x + v'' \quad \text{and} \quad z \;=\; x \wedge z + v'' \;.$$

Then we have

$$x \wedge y \wedge z + v \wedge v' \wedge v'' \;\leq\; x \wedge y \wedge z \;,$$

hence

$$v \wedge v' \wedge v'' \;=\; 0 \;,$$

by the order cancellation property, and thus

$$v + v' \wedge v'' \;=\; v \vee (v' \wedge v'') \;,$$

by Lemma 3.5. Using the distributive laws, we obtain

$$\begin{aligned}
(x \vee y) \wedge (x \vee z) &= (x + v') \wedge (x + v'') \\
&= x + v' \wedge v'' \\
&= x \wedge y \wedge z + v + v' \wedge v'' \\
&= x \wedge y \wedge z + v \vee (v' \wedge v'') \\
&= (x \wedge y \wedge z + v) \vee (x \wedge y \wedge z + v' \wedge v'') \\
&\leq x \vee ((x \wedge y + v') \wedge (x \wedge z + v'')) \\
&= x \vee (y \wedge z) \;,
\end{aligned}$$

and thus

$$(x \vee y) \wedge (x \vee z) \;=\; x \vee (y \wedge z) \;.$$

This proves (b).
It is well–known that (a) is a consequence of (b); see Birkhoff [15, p. 11]. □

The following result is the *decomposition property*:

THEOREM 3.15 (DECOMPOSITION PROPERTY). *If*

$$x \leq \sum_{i=1}^{n} y_i ,$$

then there exist $x_i \in \mathbf{E}$ *satisfying*

$$x = \sum_{i=1}^{n} x_i$$

and

$$x_i \leq y_i$$

for all $i \in \{1, 2, \ldots, n\}$. *Moreover, if* $x \in \mathbf{E}_+$ *and* $y_1, y_2, \ldots, y_n \in \mathbf{E}_+$, *then the* x_i *can be chosen such that*

$$0 \leq x_i \leq x \wedge y_i$$

holds for all $i \in \{1, 2, \ldots, n\}$.

Proof. Let us first consider the case $n = 2$. Choose $u \in \mathbf{E}_+$ satisfying

$$u + y_2 = x \vee y_2 \quad \text{and} \quad u + x \wedge y_2 = x ,$$

and define $x_1 := u \wedge y_1$. Then we have

$$x_1 \leq y_1 .$$

Choose now $v \in \mathbf{E}_+$ satisfying $u = u \wedge y_1 + v$. Then we have

$$\begin{aligned} x &= u + x \wedge y_2 \\ &= u \wedge y_1 + v + x \wedge y_2 \\ &= x_1 + v + x \wedge y_2 , \end{aligned}$$

and we may define $x_2 := v + x \wedge y_2$. This yields

$$x = x_1 + x_2 ,$$

hence

$$\begin{aligned} x_1 + x_2 &= x \\ &\leq (u + y_2) \wedge (y_1 + y_2) \\ &= u \wedge y_1 + y_2 \\ &= x_1 + y_2 , \end{aligned}$$

by the distributive laws, and thus

$$x_2 \leq y_2 ,$$

by the order cancellation property. Moreover, if x, y_1, y_2 are positive, then the same is true for x_1 and x_2, and it is then clear that in this case

$$0 \leq x_i \leq x \wedge y_i$$

holds for all $i \in \{1,2\}$. This proves the assertion in the case $n = 2$. The general case now follows by induction. □

COROLLARY 3.16. *If $x, y, z \in \mathbf{E}_+$ and $(y,z) \in \mathcal{S}$, then $x \wedge (y+z) \leq x \wedge y + x \wedge z$.*

COROLLARY 3.17. *If $x, y, z \in \mathbf{E}_+$ and $(y,z) \in \mathcal{S}$, then $x \wedge (y+z) = x \wedge (x \wedge y + z) = x \wedge (y + x \wedge z)$.*

For $u, x \in \mathbf{E}$ satisfying $u \leq x$, the set $[u,x] := \{w \in \mathbf{E} | u \leq w \leq x\}$ is said to be the *order interval* with endpoints u and x. As a consequence of the decomposition property, we obtain the following property of order intervals:

THEOREM 3.18. *If $u \leq x$, $v \leq y$, and $(x,y) \in \mathcal{S}$, then $[u,x]+[v,y] = [u+v, x+y]$.*

Proof. Choose $z \in \mathbf{E}_+$ satisfying $x = u + z$, and choose $w \in \mathbf{E}_+$ satisfying $w + v = y$. Then we have

$$[0,z] + [0,w] = [0, z+w] ,$$

by the decomposition property, and thus

$$\begin{aligned} [u,x] + [v,y] &= u + [0,z] + [0,w] + v \\ &= u + [0, z+w] + v \\ &= [u+v, u+z+w+v] \\ &= [u+v, x+y] , \end{aligned}$$

by the order cancellation property. □

We now turn to the discussion of the Jordan decomposition. For $x \in \mathbf{E}$, define

$$\begin{aligned} x^+ &:= x \vee 0 \,, \\ x^- &:= (x \wedge 0)^* \,, \\ |x| &:= x^+ \vee x^- \,. \end{aligned}$$

Note that x^-, and hence $|x|$, is well-defined, by Lemma 3.2, that x^+, x^-, and $|x|$ are positive, and that $|x| = 0$ is equivalent with $x = 0$. The following result is the *Jordan decomposition* in minimal clans:

THEOREM 3.19 (JORDAN DECOMPOSITION). *$x^- + x = x^+ = x + x^-$ and $x^+ \wedge x^- = 0$.*

Proof. By Lemma 3.2, we have $x \wedge 0 + x^+ = x = x^+ + x \wedge 0$, and thus $x^+ = x^- + x$ and $x + x^- = x^+$. Furthermore, using the distributive laws, we obtain

$$\begin{aligned} x^+ \wedge x^- &= (x + x^-) \wedge (0 + x^-) \\ &= x \wedge 0 + x^- \\ &= 0 \,, \end{aligned}$$

which completes the proof. □

COROLLARY 3.20. *$x^- + x^+ = |x| = x^+ + x^-$.*

This follows from the Jordan decomposition and Lemma 3.5.

COROLLARY 3.21. *$x \leq y$ if and only if $x^+ \leq y^+$ and $y^- \leq x^-$.*

Proof. If $x \leq y$ holds, then we have $x^+ \leq y^+$, and from

$$\begin{aligned} y \wedge 0 + y^- &= 0 \\ &= x \wedge 0 + x^- \\ &\leq y \wedge 0 + x^- \end{aligned}$$

we obtain $y^- \leq x^-$, by the order cancellation property.
Conversely, if $x^+ \leq y^+$ and $y^- \leq x^-$ holds, then we have

$$x + y^- \leq x + x^-$$

$$\begin{aligned} &= x^+ \\ &\leq y^+ \\ &= y + y^- , \end{aligned}$$

by the Jordan decomposition, and thus $x \leq y$. □

The following result concerns the uniqueness of the Jordan decomposition:

THEOREM 3.22. *If* $y \wedge z = 0$ *and either* $z + x = y$ *or* $y = x + z$, *then* $y = x^+$ *and* $z = x^-$.

Proof. Let us consider the case $y = x + z$. Since y and z are positive, we have $x^+ \leq y \leq x^- + y$, and thus

$$\begin{aligned} y &= y \wedge (x^- + y) \\ &= y \wedge (x^- + x + z) \\ &= y \wedge (x^+ + z) \\ &= y \wedge x^+ \\ &= x^+ , \end{aligned}$$

by the Jordan decomposition and Corollary 3.17, and it now follows from $x + z = y = x^+ = x + x^-$ that $z = x^-$ holds as well. □

For invertible elements we obtain another result related to the Jordan decomposition:

THEOREM 3.23. *If* $x \in \mathbf{E}_*$, *then* $(x^*)^+ = x^-$, $(x^*)^- = x^+$, *and* $|x^*| = x \vee x^* = |x|$.

Proof. Using the distributive laws and the Jordan decomposition, we obtain

$$\begin{aligned} (x^*)^+ &= x^* \vee 0 \\ &= (x^* + 0) \vee (x^* + x) \\ &= x^* + x^+ \\ &= x^- , \end{aligned}$$

and thus $(x^*)^- = (x^{**})^+ = x^+$. Choose now $v \in \mathbf{E}_+$ satisfying

$$x \vee x^* \;=\; x + v \quad \text{and} \quad x^* \;=\; x \wedge x^* + v\,.$$

Then we have

$$\begin{aligned} x \wedge x^* &\leq x^+ \wedge (x^*)^+ \\ &= x^+ \wedge x^- \\ &= 0\,, \end{aligned}$$

hence

$$\begin{aligned} 0 &= x + x^* \\ &= x + x \wedge x^* + v \\ &\leq x + v \\ &= x \vee x^*\,, \end{aligned}$$

and thus

$$\begin{aligned} |x| &= x^+ \vee x^- \\ &= x^+ \vee (x^*)^+ \\ &= x \vee 0 \vee x^* \vee 0 \\ &= x \vee x^*\,, \end{aligned}$$

and it is then clear that $|x^*| = |x|$ holds as well. □

For invertible elements which commute, we also have the *triangle inequality*:

THEOREM 3.24 (TRIANGLE INEQUALITY). *If $x, y \in \mathbf{E}_*$ and $x+y = y+x$, then $|x+y| \leq |x| + |y| = |y| + |x|$.*

Proof. The identity $x + y = y + x$ yields

$$\begin{aligned} x^* + y^* &= (y + x)^* \\ &= (x + y)^* \\ &= y^* + x^*\,, \end{aligned}$$

and from $x^* + y + x + y^* = x^* + x + y + y^* = 0$ we obtain

$$\begin{aligned} x^* + y &= (x + y^*)^* \\ &= y^{**} + x^* \\ &= y + x^* \,. \end{aligned}$$

Therefore, we have

$$\begin{aligned} (x + y) \vee (x + y)^* &= (x + y) \vee (x^* + y^*) \\ &\leq x \vee x^* + y \vee y^* \,, \end{aligned}$$

and from the distributive laws we obtain

$$\begin{aligned} x \vee x^* + y \vee y^* &= (x + y) \vee (x + y^*) \vee (x^* + y) \vee (x^* + y^*) \\ &= (y + x) \vee (y + x^*) \vee (y^* + x) \vee (y^* + x^*) \\ &= y \vee y^* + x \vee x^* \,. \end{aligned}$$

Now the assertion follows from Theorem 3.23. □

We remark that the commutativity assumption cannot be omitted in the previous result. This is due to the fact that, by Theorem 3.29 below, minimal clans generalize lattice–ordered groups, for which commutativity is equivalent with the validity of the triangle inequality for arbitrary elements; see Birkhoff [15, p. 307].

We conclude this section with some further definitions which will be needed later:

A sequence $\{z_n\}_{n \in \mathbf{N}} \subseteq \mathbf{E}$ is *order bounded* if it is contained in an order interval of $\mathbf{E}$, it is *disjoint* if $z_m \wedge z_n = 0$ holds for all $m, n \in \mathbf{N}$ satisfying $m \neq n$, it *decreases to* 0 if $\inf_{\mathbf{N}} z_n = 0$ and $z_{n+1} \leq z_n$ holds for all $n \in \mathbf{N}$, and it *order converges to* 0 if there exists a sequence $\{z'_n\}_{n \in \mathbf{N}} \subseteq \mathbf{E}_+$ which decreases to 0 and satisfies $|z_n| \leq z'_n$ for all $n \in \mathbf{N}$. For a sequence $\{z_n\}_{n \in \mathbf{N}} \subseteq \mathbf{E}_+$, we shall write

$$\text{o}{-}\sum z_n = z$$

if it is disjoint and $z := \sup_{\mathbf{N}} z_n$ exists,

$$z_n \downarrow 0$$

if it decreases to 0, and

$$\text{o}-\lim z_n \quad = \quad 0$$

if it order converges to 0. Furthermore, for a sequence $\{z_n\}_{n\in\mathbf{N}} \subseteq \mathbf{E}_+$ such that $\sup_{p\in\mathbf{N}} z_{n+p}$ exists for each $n \in \mathbf{N}$ and such that $\inf_{n\in\mathbf{N}} \sup_{p\in\mathbf{N}} z_{n+p}$ exists as well, we define

$$\text{o}-\limsup z_n \quad := \quad \inf_{n\in\mathbf{N}} \sup_{p\in\mathbf{N}} z_{n+p} \ .$$

Then $\text{o}-\limsup z_n = 0$ implies $\text{o}-\lim z_n = 0$.

Boolean Rings

A *Boolean ring* is a set $\mathbf{E}$ with an order relation $\leq$ such that

(BR–1) $x \vee y$ and $x \wedge y$ exist for all $x, y \in \mathbf{E}$;
(BR–2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ holds for all $x, y, z \in \mathbf{E}$;
(BR–3) there exists an element $0 \in \mathbf{E}$ satisfying $0 \leq x$ for all $x \in \mathbf{E}$; and
(BR–4) for all $x, z \in \mathbf{E}$ satisfying $x \leq z$, there exists some $u \in \mathbf{E}$ satisfying $u \wedge x = 0$ and $u \vee x = z$.

Boolean rings were introduced by Stone [58], [59] who called them *generalized Boolean algebras*.

If $\langle \mathbf{E}, \leq \rangle$ is a Boolean ring, then the (unique) element $0 \in \mathbf{E}$ satisfying $0 \leq x$ for all $x \in \mathbf{E}$ is said to be the *least element* of $\mathbf{E}$ and, for $x, z \in \mathbf{E}$ satisfying $x \leq z$, the (unique) element $u \in \mathbf{E}$ satisfying $u \wedge x = 0$ and $u \vee x = z$ is said to be the *relative complement* of x in z and will sometimes be denoted by $z \backslash x$. Two elements x, y are *disjoint* if $x \wedge y = 0$. The set of all pairs of disjoint elements of $\mathbf{E}$ will be denoted by $\mathbf{E}{}^{\perp}\mathbf{E}$.

A Boolean ring $\langle \mathbf{E}, \leq \rangle$ is a *Boolean algebra* if there exists an element $1 \in \mathbf{E}$ satisfying $x \leq 1$ for all $x \in \mathbf{E}$; in this case, the (unique) element $1 \in \mathbf{E}$ satisfying $x \leq 1$ for all $x \in \mathbf{E}$ is said to be the *greatest element* of $\mathbf{E}$.

Detailed information on Boolean rings and Boolean algebras may be found in the books by Abian [1], Halmos [30], and Sikorski [57].

Theorem 3.25. *Let* $\langle \mathbf{E}, \leq \rangle$ *be a Boolean ring with least element* 0. *Then* $\langle \mathbf{E}, \mathbf{E}^{\perp}\mathbf{E}, \vee, \leq \rangle$ *is a positive commutative minimal clan with zero element* 0.

Proof. For all $x, y \in \mathbf{E}$, we have $x \wedge y = 0$ if and only if $y \wedge x = 0$, as well as $x \vee y = y \vee x$. This will simplify the verification of the axioms of minimal clans.
For all $x \in \mathbf{E}$, we have $0 \wedge x = 0$ and $0 \vee x = x$. This proves (MC–1).
For all $x, y, z \in \mathbf{E}$ satisfying $x \wedge y = 0$ and $(x \vee y) \wedge z = 0$, we have $y \wedge z = 0$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = 0$, and $(x \vee y) \vee z = x \vee (y \vee z)$. This proves (MC–2).
For all $u, x, y \in \mathbf{E}$ satisfying $u \wedge x = 0$, $u \wedge y = 0$, and $u \vee x = u \vee y$, we have

$$\begin{aligned} x &= x \wedge (u \vee x) \\ &= x \wedge (u \vee y) \\ &= (x \wedge u) \vee (x \wedge y) \\ &= x \wedge y \end{aligned}$$

and, similarly, $y = x \wedge y$, which yields $x = y$. This proves (MC–3).
Axioms (MC–4) and (MC–5) are obviously satisfied.
For $x, y \in \mathbf{E}$, define $u := (x \vee y) \backslash x$. Then we have $u \wedge x = 0$ and

$$u \vee x = x \vee y \,,$$

and thus $u \wedge (x \wedge y) = 0$ and

$$\begin{aligned} u \vee (x \wedge y) &= (u \vee x) \wedge (u \vee y) \\ &= (x \vee y) \wedge (u \vee y) \\ &= (x \wedge u) \vee y \\ &= y \,. \end{aligned}$$

This proves (MC–6).
Therefore, $\langle \mathbf{E}, \mathbf{E}^{\perp}\mathbf{E}, \vee, \leq \rangle$ is a minimal clan with zero element 0, and it is clear that $\langle \mathbf{E}, \mathbf{E}^{\perp}\mathbf{E}, \vee, \leq \rangle$ is positive and commutative. □

Theorem 3.26. *Let* $\langle \mathbf{E}, \mathcal{S}, +, \leq \rangle$ *be a minimal clan with zero element* 0. *Then the following are equivalent*:

(a) $\langle \mathbf{E}, \leq \rangle$ *is a Boolean ring with least element* 0.
(b) $x + y = x \vee y$ *holds for all* $x, y \in \mathbf{E}$ *satisfying* $(x, y) \in \mathcal{S}$.
(c) $\mathcal{S} \subseteq \mathcal{D}$.
(d) $\mathcal{S} = \mathcal{D}$.

Proof. Suppose first that (a) holds. Consider $x, y \in \mathbf{E}$ satisfying $(x, y) \in \mathcal{S}$ and choose $v \in \mathbf{E}$ satisfying

$$x \vee y \;=\; x + v \quad \text{and} \quad y \;=\; x \wedge y + v\,.$$

By assumption, we have $0 \leq x \wedge y$, hence $v \leq y$, and thus $x \vee y \leq x + y$, and there exists some $u \in \mathbf{E}$ satisfying

$$u \wedge (x \vee y) \;=\; 0 \quad \text{and} \quad u \vee (x \vee y) \;=\; x + y\,.$$

Then we have

$$\begin{aligned} u + x \vee y &= u \vee (x \vee y) \\ &= x + y\,, \end{aligned}$$

by Lemma 3.5, as well as $u \wedge x = 0 = u \wedge y$. By the refinement property, there exist $z_{ij} \in \mathbf{E}$ satisfying

$$u \;=\; z_{11} + z_{12} \quad \text{and} \quad x \vee y \;=\; z_{21} + z_{22}$$

as well as

$$x \;=\; z_{11} + z_{21} \quad \text{and} \quad y \;=\; z_{12} + z_{22}\,.$$

From $0 \leq z_{11} \leq u \wedge x = 0$ and $0 \leq z_{12} \leq u \wedge y = 0$ we obtain $u = z_{11} + z_{12} = 0$, and thus

$$x \vee y \;=\; x + y\,.$$

Therefore, (a) implies (b).
Suppose now that (b) holds. Consider $x, y \in \mathbf{E}$ satisfying $(x, y) \in \mathcal{S}$ and choose $v \in \mathbf{E}_+$ satisfying

$$x \vee y \;=\; x + v \quad \text{and} \quad y \;=\; x \wedge y + v\,.$$

By assumption, we have $x + y = x \vee y = x + v$, hence $y = v$, and thus $x \wedge y = 0$, by the cancellation property. Therefore, (b) implies

(c).
By Lemma 3.5, (c) implies (d).
Suppose now that (d) holds. Obviously, $\langle \mathbf{E}, \leq \rangle$ satisfies (BR–1). By Theorem 3.14, $\langle \mathbf{E}, \leq \rangle$ satisfies (BR–2). By assumption and Lemma 3.5, $\langle \mathbf{E}, \leq \rangle$ satisfies (BR–3). Consider now $x, z \in \mathbf{E}$ satisfying $x \leq z$ and choose $u \in \mathbf{E}_+$ satisfying $u + x = z$. Then we have $u \wedge x = 0$, by assumption, hence $u \vee x = u + x$, by Lemma 3.5, and thus $u \vee x = z$. This means that $\langle \mathbf{E}, \leq \rangle$ satisfies (BR–4). Therefore, (d) implies (a). □

In view of Lemma 3.5, the previous results may be summarized as follows:

COROLLARY 3.27. *Boolean rings are precisely the minimal clans having a minimal domain of addition.*

Let $\langle \mathbf{E}, \leq \rangle$ be an Boolean ring. For the simplicity of notation, we shall usually write

$$x + y \;=\; z$$

instead of the full statement

$$x \wedge y \;=\; 0 \quad \textit{and} \quad x \vee y \;=\; z\,.$$

This is in accordance with Theorem 3.25 and our convention concerning minimal clans.

The following lemma will be needed later:

LEMMA 3.28. *Let $\langle \mathbf{E}, \leq \rangle$ be a Boolean ring.*
(a) *For every disjoint sequence $\{z_n\}_{n \in \mathbf{N}} \subseteq \mathbf{E}$ such that $\sup_{\mathbf{N}} z_m$ exists, there exists a sequence $\{u_n\}_{n \in \mathbf{N}} \subseteq \mathbf{E}$ satisfying $u_n \downarrow 0$ and $z_1 + \ldots + z_n + u_n = \sup_{\mathbf{N}} z_m$ for all $n \in \mathbf{N}$.*
(b) *For every sequence $\{u_n\}_{n \in \mathbf{N}} \subseteq \mathbf{E}$ satisfying $u_n \downarrow 0$, there exists a disjoint sequence $\{z_n\}_{n \in \mathbf{N}} \subseteq \mathbf{E}$ satisfying $u_1 = \sup_{\mathbf{N}} z_m$ and $z_1 + \ldots + z_n + u_n = u_1$ for all $n \in \mathbf{N}$.*

Proof. Consider first a disjoint sequence $\{z_n\}_{n \in \mathbf{N}} \subseteq \mathbf{E}$ such that $\sup_{\mathbf{N}} z_n$ exists. Define $z := \sup_{\mathbf{N}} z_n$ and, for all $n \in \mathbf{N}$, define $v_n := z_1 + \ldots + z_n$ and $u_n := z \setminus v_n$. Then the sequence $\{u_n\}_{n \in \mathbf{N}}$

is decreasing. To see that it decreases to 0, consider a lower bound $u \in \mathbf{E}$ of $\{u_n\}_{n\in\mathbf{N}}$ and define $w := z\backslash u$. Then we have

$$\begin{aligned} z_n &\leq v_n \wedge z \\ &= v_n \wedge (u \vee w) \\ &= (v_n \wedge u) \vee (v_n \wedge w) \\ &\leq (v_n \wedge u_n) \vee w \\ &= w \end{aligned}$$

for all $n \in \mathbf{N}$. This yields $z \leq w$, hence $z = w$, whence $u = 0$, and thus $u_n \downarrow 0$, and it is obvious from the definitions that $z_1 + \ldots + z_n + u_n = z$ holds for all $n \in \mathbf{N}$. This proves (a).

Consider now a sequence $\{u_n\}_{n\in\mathbf{N}} \subseteq \mathbf{E}$ satisfying $u_n \downarrow 0$. Define $u_0 := u_1$ and, for all $n \in \mathbf{N}$, define $z_n := u_{n-1}\backslash u_n$. Then we have

$$z_1 + \ldots + z_n + u_n = u_1$$

for all $n \in \mathbf{N}$. Moreover

$$\begin{aligned} z_n \wedge z_m &\leq z_n \wedge u_{m-1} \\ &\leq z_n \wedge u_n \\ &= 0 \end{aligned}$$

holds for all $m, n \in \mathbf{N}$ satisfying $n + 1 \leq m$, which means that the sequence $\{z_n\}_{n\in\mathbf{N}}$ is disjoint. Consider an upper bound $x \in \mathbf{E}$ of $\{z_n\}_{n\in\mathbf{N}}$, a lower bound $u \in \mathbf{E}$ of $\{x \vee u_n\}_{n\in\mathbf{N}}$, and define $w := (x \vee u_1)\backslash x$. Then we have $x \wedge w = 0$, and thus

$$\begin{aligned} u \wedge w &\leq (x \vee u_n) \wedge w \\ &= (x \wedge w) \vee (u_n \wedge w) \\ &= u_n \wedge w \\ &\leq u_n \end{aligned}$$

for all $n \in \mathbf{N}$. By assumption, this yields $u \wedge w = 0$, hence

$$\begin{aligned} u &= u \wedge (x \vee u_1) \\ &= u \wedge (x \vee w) \\ &= (u \wedge x) \vee (u \wedge w) \\ &= u \wedge x \\ &\leq x\,, \end{aligned}$$

and thus $\inf_{\mathbf{N}} x \vee u_n = x$. Furthermore, we have

$$\begin{aligned} u_1 &= z_1 + \ldots + z_n + u_n \\ &= z_1 \vee \ldots \vee z_n \vee u_n \\ &\leq x \vee u_n \end{aligned}$$

for all $n \in \mathbf{N}$, hence $u_1 \leq x$, and thus $u_1 = \sup_{\mathbf{N}} z_n$. This proves (b). □

Lattice–Ordered Groups

A *lattice–ordered group* is a set $\mathbf{E}$ with a map $+ : \mathbf{E} \times \mathbf{E} \to \mathbf{E}$ and an order relation $\leq$ such that

(LG–1) there exists an element $0 \in \mathbf{E}$ satisfying $0 + x = x$ for all $x \in \mathbf{E}$;
(LG–2) $(x + y) + z = x + (y + z)$ holds for all $x, y, z \in \mathbf{E}$;
(LG–3) for all $x \in \mathbf{E}$, there exists some $w \in \mathbf{E}$ satisfying $w + x = 0$;
(LG–4) $u + x + v \leq u + y + v$ holds for all $x, y \in \mathbf{E}$ satisfying $x \leq y$ and for all $u, v \in \mathbf{E}$; and
(LG–5) $x \vee y$ and $x \wedge y$ exist for all $x, y \in \mathbf{E}$.

Lattice–ordered groups were introduced by Birkhoff [13].

If $\langle \mathbf{E}, \mathbf{E} \times \mathbf{E}, +, \leq \rangle$ is a lattice–ordered group, then the element $0 \in \mathbf{E}$ satisfying $0 + x = x$ for all $x \in \mathbf{E}$ is unique and satisfies also $x = x + 0$ for all $x \in \mathbf{E}$, for all $x \in \mathbf{E}$ there exists a unique element $x^* \in \mathbf{E}$ satisfying $x^* + x = 0 = x + x^*$, and the identities $u + x \wedge y + v = (u + x + v) \wedge (u + y + v)$ and $y^* \wedge x^* = (x \vee y)^*$ hold for all $u, v, x, y \in \mathbf{E}$.

Further information on lattice–ordered groups may be found in the books by Anderson and Feil [3], Bigard, Keimel, and Wolfenstein [12], Birkhoff [14], [15], and Fuchs [28].

THEOREM 3.29. *Let* $\langle \mathbf{E}, \mathbf{E} \times \mathbf{E}, +, \leq \rangle$ *be a lattice–ordered group. Then* $\langle \mathbf{E}, \mathbf{E} \times \mathbf{E}, +, \leq \rangle$ *is a minimal clan satisfying* $\mathbf{E}_* = \mathbf{E}$.

Proof. Axioms (MC–1) through (MC–5) are obviously satisfied.
For $x, y \in \mathbf{E}$, define $u := x \vee y + x^*$ and $v := x^* + x \vee y$. Then we have

$$u + x = x \vee y + x^* + x$$

$$= \quad x \vee y$$

and

$$\begin{aligned} u + x \wedge y &= x \vee y + x^* + x \wedge y + y^* + y \\ &= x \vee y + (y^* \wedge x^*) + y \\ &= x \vee y + (x \vee y)^* + y \\ &= y \,, \end{aligned}$$

and a similar argument yields $x \vee y = x + v$ and $y = x \wedge y + v$. This proves (MC–6).
Therefore, $\langle \mathbf{E}, \mathbf{E} \times \mathbf{E}, +, \leq \rangle$ is a minimal clan, and it is clear that $\langle \mathbf{E}, \mathbf{E} \times \mathbf{E}, +, \leq \rangle$ satisfies $\mathbf{E}_* = \mathbf{E}$. □

THEOREM 3.30. *Let* $\langle \mathbf{E}, \mathcal{S}, +, \leq \rangle$ *be a minimal clan. Then the following are equivalent*:
(a) $\langle \mathbf{E}, \mathcal{S}, +, \leq \rangle$ *is a lattice–ordered group.*
(b) $\mathbf{E}_+ \subseteq \mathbf{E}_*$.
(c) $\mathbf{E} = \mathbf{E}_*$.

Proof. The equivalence of (a) and (c) is obvious, and the equivalence of (b) and (c) follows from Lemma 3.2. □

The previous results may be summarized as follows:

COROLLARY 3.31. *Lattice–ordered groups are precisely the minimal clans having a maximal set of invertible elements.*

We remark that each minimal clan $\langle \mathbf{E}, \mathcal{S}, +, \leq \rangle$ contains a greatest lattice–ordered group, namely $\langle \mathbf{E}_*, \mathbf{E}_* \times \mathbf{E}_*, +, \leq \rangle$. This follows from Lemma 3.2 and Theorem 3.30.

Comments

The problem of developing a common abstraction of Boolean rings and lattice–ordered groups has already been posed by Birkhoff [14, p. 233] a few years after the fundamental papers by Stone [58], [59] and Birkhoff [13] on Boolean rings and lattice–ordered groups had appeared; see also Birkhoff [15, p. 318]. Several solutions to

Birkhoff's problem have been proposed in the past, and it turned out that very different solutions are possible.

A common abstraction of Boolean rings and lattice–ordered groups which is particularly closely related to minimal clans was proposed by Wyler [64] who introduced *symmetric clans*. These are defined in terms of an order relation, a partial subtraction, and an induced partial addition. It can be shown that every symmetric clan is a minimal clan, and that every minimal clan is a symmetric clan having a minimal domain of subtraction. This result explains the name of minimal clans, and it also indicates that minimal clans are free from a certain ambiguity concerning the domain of subtraction in symmetric clans. Another advantage of minimal clans when compared with symmetric clans consists in the fact that their axioms are defined in terms of a single partial operation instead of two. For further details concerning the comparison of minimal clans with symmetric clans and other common abstractions of Boolean rings and lattice–ordered groups, see Schmidt [51], [53].

The following example shows that there exist commutative minimal clans which need not be a Boolean ring and cannot be a lattice–ordered group:

EXAMPLE 3.32: For a nonempty set Ω, let $\mathbf{E}$ denote a collection of functions $\Omega \to [0,1]$ such that

(i) the function 1, given by $1(\omega) := 1$ for all $\omega \in \Omega$, belongs to $\mathbf{E}$;
(ii) for all $x, y \in \mathbf{E}$, the function $x \oplus y$, given by $(x \oplus y)(\omega) := \min\{x(\omega) + y(\omega), 1\}$ for all $\omega \in \Omega$, belongs to $\mathbf{E}$; and
(iii) for all $x, y \in \mathbf{E}$, the function $x \ominus y$, given by $(x \ominus y)(\omega) := \max\{x(\omega) - y(\omega), 0\}$ for all $\omega \in \Omega$, belongs to $\mathbf{E}$.

Define a relation $\mathcal{S} := \{(x,y) \in \mathbf{E}{\times}\mathbf{E} \mid x(\omega){+}y(\omega) \le 1 \text{ for all } \omega \in \Omega\}$, a map $+ : \mathcal{S} \to \mathbf{E}$ by letting $(x+y)(\omega) := x(\omega) + y(\omega)$ for all $\omega \in \Omega$, and an order relation $\le$ by letting $x \le y$ if and only if $x(\omega) \le y(\omega)$ holds for all $\omega \in \Omega$. Then $\langle \mathbf{E}, \mathcal{S}, +, \le \rangle$ is a commutative minimal clan which need not be a Boolean ring and cannot be a lattice–ordered group.

The functions $\Omega \to [0,1]$ considered in Example 3.32 generalize the indicator functions of the ordinary subsets of Ω and are said to be the *membership functions* of the *fuzzy sets* of Ω. Fuzzy sets were introduced by Zadeh [66] and are usually identified with their membership functions. Following Butnariu [17], a collection of fuzzy sets satisfying axioms (i), (ii), and (iii) of Example 3.32 is said to be an *additive class of fuzzy sets*. Every algebra of sets is an additive class of fuzzy sets, but the converse is not true in general; for example, if $\mathbf{E}$ is the collection of all fuzzy subsets of Ω, then $\mathbf{E}$ is an additive class of fuzzy sets containing the fuzzy set $z : \Omega \to [0, 1]$, given by $z(\omega) := 1/2$ for all $\omega \in \Omega$, and this implies that $\mathbf{E}$ cannot be a Boolean ring, by Theorem 3.26. Butnariu [17], [18], [19] developed a fuzzy measure and integration theory and obtained, in particular, a Jordan decomposition of real-valued additive functions on an additive class of fuzzy sets. Since Butnariu's definition of an additive function on an additive class of fuzzy sets is in accordance with our definition of an additive function on a commutative minimal clan, his result is a special case of Theorem 4.3 below, and the other results of Sections 4 and 5 apply to additive functions on an additive class of fuzzy sets as well.

With regard to the subject of these notes, it is remarkable that certain ordered (partial) semigroups were already considered by Riesz [46] in his paper on the Jordan decomposition of linear functionals. In his paper, which is one of the origins of general Riesz space theory, Riesz studied the lattice properties of additive functions on a *fundamental domain*, which he called *linear operations* although they were not assumed to be homogeneous. Later, aiming at a possible unification of measure and integration theory, Dinges [25] replaced the complete addition in fundamental domains by a partial one and thus introduced *Riesz D-semigroups* which were later generalized by Schmidt [49]. Dinges argued that the analogy between suprema of disjoint elements in a Boolean ring and sums of arbitrary elements in a lattice–ordered group was more important than the obvious one concerning the lattice property of Boolean rings and lattice–ordered groups. Minimal clans reflect both of these analogies, and this may help to explain why minimal clans are more suitable for a unified approach to the Jordan decomposition of vector measures and linear operators than other common abstractions of Boolean rings and lattice–ordered groups.

Problems

- Prove or disprove that a minimal clan satisfies the triangle inequality for arbitrary (invertible) elements if and only if it is commutative.
- Prove or disprove that every (commutative) minimal clan contains a greatest Boolean ring.
- Develop an algebraic theory of minimal clans in the spirit of the algebraic theory of lattice–ordered groups.
- Extend the notion of a *Fréchet–Nikodym topology* (*FN–topology*) to (commutative) minimal clans; for detailed information on FN–topologies, see e. g. Fries [27] and the references given there. With regard to topologies on minimal clans, consider also the notion of a *generating set* introduced by Schmidt [54] (see Section 4 below), the notion of a *Nikodym filter* introduced by Constantinescu [22], and the notion of a *lattice uniformity* introduced by Weber [61], [62].

4. The Jordan Decomposition

In this section we study the Jordan decomposition of order bounded additive functions from a commutative minimal clan $\mathbf{E}$ into a Riesz space $\mathbf{G}$.

We first show that every additive function $\mathbf{E} \to \mathbf{G}$ is completely determined by its values on $\mathbf{E}_+$. In the case where $\mathbf{G}$ is an order complete Riesz space, we then show that an additive function $\mathbf{E} \to \mathbf{G}$ has a Jordan decomposition if and only if it is order bounded and that the collection of all order bounded additive functions $\mathbf{E} \to \mathbf{G}$ is an order complete Riesz space under the pointwise defined linear operations and order relation.

Throughout this section, let $\mathbf{G}$ be a Riesz space.

Additive Functions

Let $\mathbf{E}$ be a commutative minimal clan with domain of addition $\mathcal{S}$, partial addition $+ : \mathcal{S} \to \mathbf{E}$, and order relation $\leq$.

For $z \in \mathbf{E}_+$, let $\mathcal{S}(z)$ denote the collection of all pairs of elements $z', z'' \in \mathbf{E}_+$ satisfying $(z', z'') \in \mathcal{S}$ and $z' + z'' = z$. A *partition* of $z \in \mathbf{E}_+$ is a finite sequence $(z_1, z_2, \ldots, z_m)$ of elements $z_1, z_2, \ldots, z_m \in \mathbf{E}_+$ satisfying $(\sum_{i=1}^k z_i, z_{k+1}) \in \mathcal{S}$ for all $k \in \{1, 2, \ldots, m-1\}$ and $\sum_{i=1}^m z_i = z$. By the refinement property (Theorem 3.12), any two partitions of z have a common refinement. The directed family of all partitions of z will be denoted by $\mathcal{P}(z)$.

A function $\varphi : \mathbf{E} \to \mathbf{G}$ is *additive* if $\varphi(x + y) = \varphi(x) + \varphi(y)$ holds for all $x, y \in \mathbf{E}$ satisfying $(x, y) \in \mathcal{S}$. Since $\mathbf{E}$ is a distributive lattice, by Theorem 3.14, and since each additive function $\varphi : \mathbf{E} \to \mathbf{G}$ satisfies $\varphi(0) = 0$ as well as $\varphi(x) + \varphi(y) = \varphi(x \vee y) + \varphi(x \wedge y)$ for all $x, y \in \mathbf{E}$, by the difference property, every additive function on a commutative minimal clan is a normalized valuation on a distributive lattice. Under the pointwise defined linear operations and the order relation $\leq$, given by $\varphi \leq \psi$ if and only if $\varphi(z) \leq \psi(z)$ holds for all $z \in \mathbf{E}_+$, the collection of all additive functions $\mathbf{E} \to \mathbf{G}$ is an ordered vector space which will be denoted by $a(\mathbf{E}, \mathbf{G})$. In the sequel, every collection of additive functions $\mathbf{E} \to \mathbf{G}$ will be considered to be equipped with the linear operations and the order relation inherited from $a(\mathbf{E}, \mathbf{G})$.

Occasionally, the previous definition and others will also be used for functions which are defined on $\mathbf{E}_+$ instead of $\mathbf{E}$. This is justified by the fact that $\mathbf{E}_+$ is again a commutative minimal clan, by Lemma 3.4, and it is also justified by the following *extension lemma* which will be essential in what follows:

LEMMA 4.1. (EXTENSION LEMMA) *Every additive function* $\mathbf{E}_+ \to \mathbf{G}$ *has a unique extension to an additive function* $\mathbf{E} \to \mathbf{G}$.

Proof. Consider an additive function $\tilde{\varphi} : \mathbf{E}_+ \to \mathbf{G}$.
If $\varphi : \mathbf{E} \to \mathbf{G}$ is an additive function extending $\tilde{\varphi}$, then the Jordan decomposition in minimal clans (Theorem 3.19) yields

$$\begin{aligned} \varphi(x) &= \varphi(x^+) - \varphi(x^-) \\ &= \tilde{\varphi}(x^+) - \tilde{\varphi}(x^-) \end{aligned}$$

for all $x \in \mathbf{E}$. Therefore, there exists of most one additive function $\varphi : \mathbf{E} \to \mathbf{G}$ extending $\tilde{\varphi}$.

Consider now $x \in \mathbf{E}$. By the Jordan decomposition in minimal clans, there exist $u, w \in \mathbf{E}_+$ satisfying $u + x = w$, and it follows from the difference property that for all $u', w' \in \mathbf{E}_+$ satisfying $u' + x = w'$ there exist $z, z' \in \mathbf{E}_+$ satisfying $z + w = w \vee w' = z' + w'$, hence $z + u + x = z' + u' + x$, and thus $z + u = z' + u'$, by the cancellation property. Using the additivity of $\tilde{\varphi}$, we obtain

$$\begin{aligned} \tilde{\varphi}(w) - \tilde{\varphi}(u) &= \tilde{\varphi}(z + w) - \tilde{\varphi}(z + u) \\ &= \tilde{\varphi}(z' + w') - \tilde{\varphi}(z' + u') \\ &= \tilde{\varphi}(w') - \tilde{\varphi}(u'). \end{aligned}$$

Therefore, the function $\varphi : \mathbf{E} \to \mathbf{G}$, given by

$$\varphi(x) \;:=\; \tilde{\varphi}(w) - \tilde{\varphi}(u)$$

for all $x \in \mathbf{E}$ and arbitrary $u, w \in \mathbf{E}_+$ satisfying $u + x = w$, is well–defined, and it is evident that

$$\varphi(z) \;=\; \tilde{\varphi}(z)$$

holds for all $z \in \mathbf{E}_+$. Furthermore, it follows from the difference property that for all $x, y \in \mathbf{E}$ satisfying $(x, y) \in \mathcal{S}$ there exist $u, w \in \mathbf{E}_+$ satisfying $u + x + y = (x + y) \vee 0 \vee y = w + y$, and thus $u + x = w$. Using the definition of φ and the additivity of $\tilde{\varphi}$, we obtain

$$\begin{aligned} \varphi(x + y) &= \tilde{\varphi}((x + y) \vee 0 \vee y) - \tilde{\varphi}(u) \\ &= \tilde{\varphi}(w) + \varphi(y) - \tilde{\varphi}(u) \\ &= \varphi(x) + \varphi(y). \end{aligned}$$

Therefore, φ is an additive function extending $\tilde{\varphi}$ to $\mathbf{E}$. □

For $\varphi, \psi \in a(\mathbf{E}, \mathbf{G})$, the supremum and the infimum of φ and ψ in $a(\mathbf{E}, \mathbf{G})$ will be denoted by $\varphi \vee \psi$ and $\varphi \wedge \psi$, respectively, and we define

$$\begin{aligned} \varphi^+ &:= \varphi \vee 0 \\ \varphi^- &:= (-\varphi) \vee 0 \\ |\varphi| &:= \varphi \vee (-\varphi) \end{aligned}$$

The following result gives a sufficient condition for the supremum of two additive functions to exist:

Lemma 4.2. *If* $\varphi, \psi \in a(\mathbf{E}, \mathbf{G})$ *are such that* $\sup_{\mathcal{S}(z)}(\varphi(x) + \psi(y))$ *exists for each* $z \in \mathbf{E}_+$, *then* $\varphi \vee \psi$ *exists in* $a(\mathbf{E}, \mathbf{G})$ *and*

$$(\varphi \vee \psi)(z) \;=\; \sup_{\mathcal{S}(z)}(\varphi(x) + \psi(y))$$

holds for all $z \in \mathbf{E}_+$.

Proof. If $\eta \in a(\mathbf{E}, \mathbf{G})$ majorizes φ and ψ, then

$$\sup_{\mathcal{S}(z)}(\varphi(x) + \psi(y)) \;\leq\; \eta(z)$$

holds for all $z \in \mathbf{E}_+$.
By assumption, the function $\tilde{\mu} : \mathbf{E}_+ \to \mathbf{G}$, given by

$$\tilde{\mu}(z) \;:=\; \sup_{\mathcal{S}(z)}(\varphi(x) + \psi(y))$$

for all $z \in \mathbf{E}_+$, is well–defined, and it is evident that

$$\varphi(z) \;\leq\; \tilde{\mu}(z)$$

and

$$\psi(z) \;\leq\; \tilde{\mu}(z)$$

holds for all $z \in \mathbf{E}_+$. Consider now $z', z'' \in \mathbf{E}_+$ satisfying $(z', z'') \in \mathcal{S}$. For all $(x', y') \in \mathcal{S}(z')$ and $(x'', y'') \in \mathcal{S}(z'')$ we have $(x'+x'', y'+y'') \in \mathcal{S}(z' + z'')$, by the associative and commutative laws, hence

$$\begin{aligned} \varphi(x') + \psi(y') + \varphi(x'') + \psi(y'') \;&=\; \varphi(x' + x'') + \psi(y' + y'') \\ &\leq\; \tilde{\mu}(z' + z'')\,, \end{aligned}$$

and thus

$$\tilde{\mu}(z') + \tilde{\mu}(z'') \;\leq\; \tilde{\mu}(z' + z'')\,.$$

Also, it follows from the refinement property (Theorem 3.12) that for all $(x, y) \in \mathcal{S}(z' + z'')$ there exist $x', x'', y', y'' \in \mathbf{E}_+$ satisfying $(x', x'') \in \mathcal{S}(x)$ and $(y', y'') \in \mathcal{S}(y)$ as well as $(x', y') \in \mathcal{S}(z')$ and $(x'', y'') \in \mathcal{S}(z'')$. This yields

$$\begin{aligned} \varphi(x) + \psi(y) \;&=\; \varphi(x') + \psi(y') + \varphi(x'') + \psi(y'') \\ &\leq\; \tilde{\mu}(z') + \tilde{\mu}(z'')\,, \end{aligned}$$

and thus

$$\tilde{\mu}(z' + z'') \;\; \leq \;\; \tilde{\mu}(z') + \tilde{\mu}(z'') \; .$$

Therefore, $\tilde{\mu}$ is additive, and it now follows from Lemma 4.1 that $\tilde{\mu}$ has a unique extension to an additive function $\mu : \mathbf{E} \to \mathbf{G}$ which majorizes φ and ψ, by the definition of $\tilde{\mu}$, and which actually is the least upper bound of φ and ψ in $a(\mathbf{E}, \mathbf{G})$, by the remark at the beginning of this proof. □

An additive function $\varphi : \mathbf{E} \to \mathbf{G}$ is *positive* if $\varphi(z) \in \mathbf{G}_+$ holds for all $z \in \mathbf{E}_+$, and it is *order bounded* if, for each $z \in \mathbf{E}_+$, the set $\{\varphi(u) | u \in [0, z]\}$ is an order bounded subset of $\mathbf{G}$. Thus, every positive additive function is order bounded. Furthermore, the unique extension $\varphi : \mathbf{E} \to \mathbf{G}$ of an additive function $\tilde{\varphi} : \mathbf{E}_+ \to \mathbf{G}$ is positive or order bounded if and only if the same is true for $\tilde{\varphi}$. The ordered vector space of all order bounded additive functions $\mathbf{E} \to \mathbf{G}$ will be denoted by $oba(\mathbf{E}, \mathbf{G})$.

We now turn to the main result of this section:

THEOREM 4.3. *Assume that* $\mathbf{G}$ *is order complete. Then* $oba(\mathbf{E}, \mathbf{G})$ *is an order complete Riesz space. Moreover,*

$$(\varphi \vee \psi)(z) \;=\; \sup_{\mathcal{S}(z)}(\varphi(x) + \psi(y))$$

and

$$(\varphi \wedge \psi)(z) \;=\; \inf_{\mathcal{S}(z)}(\varphi(x) + \psi(y))$$

holds for all $\varphi, \psi \in oba(\mathbf{E}, \mathbf{G})$ *and* $z \in \mathbf{E}_+$, *and*

$$(\sup_\Gamma \varphi_\gamma)(z) \;=\; \sup_\Gamma \varphi_\gamma(z)$$

holds for each directed $(\leq)$ *family* $\{\varphi_\gamma\}_{\gamma \in \Gamma} \subseteq oba(\mathbf{E}, \mathbf{G})$ *having an upper bound in* $oba(\mathbf{E}, \mathbf{G})$ *and for all* $z \in \mathbf{E}_+$.

Proof. To prove that $oba(\mathbf{E}, \mathbf{G})$ is a Riesz space, consider $\varphi, \psi \in oba(\mathbf{E}, \mathbf{G})$. Since $\mathbf{G}$ is order complete, $\sup_{\mathcal{S}(z)}(\varphi(x) + \psi(y))$ exists for all $z \in \mathbf{E}_+$, and it now follows from Lemma 4.2 that $\varphi \vee \psi$ exists in $a(\mathbf{E}, \mathbf{G})$ and that

$$(\varphi \vee \psi)(z) \;\; = \;\; \sup_{\mathcal{S}(z)}(\varphi(x) + \psi(y))$$

holds for all $z \in \mathbf{E}_+$. From this identity we obtain $\varphi \vee \psi \in oba(\mathbf{E}, \mathbf{G})$. Therefore, $oba(\mathbf{E}, \mathbf{G})$ is a Riesz space.
In particular, for all $\varphi, \psi \in oba(\mathbf{E}, \mathbf{G})$, we have $\varphi \wedge \psi = -(-\varphi) \vee (-\psi)$, and thus

$$(\varphi \wedge \psi)(z) \quad = \quad \inf_{\mathcal{S}(z)} (\varphi(x) + \psi(y))$$

for all $z \in \mathbf{E}_+$.
To prove that the Riesz space $oba(\mathbf{E}, \mathbf{G})$ is order complete, consider a directed $(\leq)$ family $\{\varphi_\gamma\}_{\gamma \in \Gamma} \subseteq oba(\mathbf{E}, \mathbf{G})$ such that $0 \leq \varphi_\gamma \leq \psi$ holds for all $\gamma \in \Gamma$ and some $\psi \in oba(\mathbf{E}, \mathbf{G})$. Since $\mathbf{G}$ is order complete, $\sup_\Gamma \varphi_\gamma(z)$ exists for all $z \in \mathbf{E}_+$, and if $\eta \in oba(\mathbf{E}, \mathbf{G})$ satisfies $\varphi_\gamma \leq \eta$ for all $\gamma \in \Gamma$, then

$$\sup_\Gamma \varphi_\gamma(z) \quad \leq \quad \eta(z)$$

holds for all $z \in \mathbf{E}_+$. In particular, the function $\tilde{\varphi} : \mathbf{E}_+ \to \mathbf{G}$, given by

$$\tilde{\varphi}(z) \quad := \quad \sup_\Gamma \varphi_\gamma(z)$$

for all $z \in \mathbf{E}_+$, is well–defined, and it is evident that

$$\varphi_\gamma(z) \quad \leq \quad \tilde{\varphi}(z)$$

holds for all $\gamma \in \Gamma$ and $z \in \mathbf{E}_+$. Consider now $z', z'' \in \mathbf{E}_+$ satisfying $(z', z'') \in \mathcal{S}$. Then we have

$$\begin{aligned} \varphi_\gamma(z' + z'') &= \varphi_\gamma(z') + \varphi_\gamma(z'') \\ &\leq \tilde{\varphi}(z') + \tilde{\varphi}(z'') \end{aligned}$$

for all $\gamma \in \Gamma$, and thus

$$\tilde{\varphi}(z' + z'') \quad \leq \quad \tilde{\varphi}(z') + \tilde{\varphi}(z'') \; .$$

Also, for all $\gamma', \gamma'' \in \Gamma$, there exists some $\gamma \in \Gamma$ satisfying $\varphi_{\gamma'} \vee \varphi_{\gamma''} \leq \varphi_\gamma$. This yields

$$\begin{aligned} \varphi_{\gamma'}(z') + \varphi_{\gamma''}(z'') &\leq \varphi_\gamma(z') + \varphi_\gamma(z'') \\ &= \varphi_\gamma(z' + z'') \\ &\leq \tilde{\varphi}(z' + z'') \; , \end{aligned}$$

and thus

$$\tilde{\varphi}(z') + \tilde{\varphi}(z'') \quad \leq \quad \tilde{\varphi}(z' + z'') \; .$$

Therefore, $\tilde{\varphi}$ is additive, and it now follows from Lemma 4.1 that $\tilde{\varphi}$ has a unique extension to an additive function $\varphi : \mathbf{E} \to \mathbf{G}$ which is positive and hence order bounded, which majorizes each φ_γ, by the definition of $\tilde{\varphi}$, and which actually is the least upper bound of $\{\varphi_\gamma\}_{\gamma \in \Gamma}$ in $oba(\mathbf{E}, \mathbf{G})$, by the remark at the beginning of this part of the proof. Therefore, the Riesz space $oba(\mathbf{E}, \mathbf{G})$ is order complete. □

An additive function $\varphi : \mathbf{E} \to \mathbf{G}$ is *regular* if it is the difference of two positive additive functions, and it has a *Jordan decomposition* if it is the difference of two positive additive functions which are disjoint in $a(\mathbf{E}, \mathbf{G})$.

COROLLARY 4.4. *Assume that* $\mathbf{G}$ *is order complete. Then, for* $\varphi \in a(\mathbf{E}, \mathbf{G})$, *the following are equivalent*:

(a) φ *is order bounded.*

(b) φ *is regular.*

(c) φ *has a Jordan decomposition.*

Moreover, if φ *is order bounded, then its Jordan decomposition is unique and given by* $\varphi = \varphi^+ - \varphi^-$.

The following characterization of order bounded additive functions into an order complete Riesz space will be useful in the sequel:

LEMMA 4.5. *Assume that* $\mathbf{G}$ *is order complete. Then, for* $\varphi \in a(\mathbf{E}, \mathbf{G})$, *the following are equivalent:*

(a) φ *is order bounded.*

(b) $|\varphi|$ *exists in* $a(\mathbf{E}, \mathbf{G})$.

(c) $\sup_{[0,z]} |\varphi(u)|$ *exists for each* $z \in \mathbf{E}_+$.

(d) $\sup_{\mathcal{S}(z)} (\varphi(x) - \varphi(y))$ *exists for each* $z \in \mathbf{E}_+$.

(e) $\sup_{\mathcal{P}(z)} \sum |\varphi(z_i)|$ *exists for each* $z \in \mathbf{E}_+$.

Moreover, if φ *is order bounded, then*

$$|\varphi|(z) \;=\; \sup_{\mathcal{S}(z)} (\varphi(x) - \varphi(y)) \;=\; \sup_{\mathcal{P}(z)} \sum |\varphi(z_i)|$$

holds for all $z \in \mathbf{E}_+$.

Proof. Since $\mathbf{G}$ is order complete, it is obvious that (a) implies (b), by Lemma 4.2, that (b) implies (c), and that (c) implies (a).
If (b) holds, then

$$\sum_{i=1}^{m} |\varphi(z_i)| \quad \leq \quad \sum_{i=1}^{m} |\varphi|(z_i) = |\varphi|(z)$$

holds for all $z \in \mathbf{E}_+$ and $(z_1, z_2, \dots, z_m) \in \mathcal{P}(z)$, and this yields

$$\sup_{\mathcal{P}(z)} \sum |\varphi(z_i)| \quad \leq \quad |\varphi|(z)$$

for all $z \in \mathbf{E}_+$. Therefore, (b) implies (e).
If (e) holds, then

$$\varphi(x) - \varphi(y) \quad \leq \quad \sup_{\mathcal{P}(z)} \sum |\varphi(z_i)|$$

holds for all $z \in \mathbf{E}_+$ and $(x, y) \in \mathcal{S}(z)$, and this yields

$$\sup_{\mathcal{S}(z)} (\varphi(x) - \varphi(y)) \quad \leq \quad \sup_{\mathcal{P}(z)} \sum |\varphi(z_i)|$$

for all $z \in \mathbf{E}_+$. Therefore, (e) implies (d).
If (d) holds, then $|\varphi| = \varphi \vee (-\varphi)$ exists in $a(\mathbf{E}, \mathbf{G})$ and

$$|\varphi|(z) \quad = \quad \sup_{\mathcal{S}(z)} (\varphi(x) - \varphi(y))$$

holds for all $z \in \mathbf{E}_+$, by Lemma 4.2. Therefore, (d) implies (b).
The final assertion is now obvious. □

Vector Measures

Let $\mathbf{E}$ be a Boolean ring.

A function $\varphi : \mathbf{E} \to \mathbf{G}$ is a *vector measure* if $\varphi(x + y) = \varphi(x) + \varphi(y)$ holds for all $x, y \in \mathbf{E}$ satisfying $x \wedge y = 0$. By Theorem 3.25, vector measures are precisely the additive functions $\mathbf{E} \to \mathbf{G}$. Therefore, all results on additive functions on a minimal clan can be read *without any modification* as results on vector measures on a Boolean ring.

For vector measures on a Boolean ring, Theorem 4.3 is due to Bauer [6]; see also Bauer [7], Faires and Morrison [26], Schep [48], Congost Iglesias [21], and Schmidt [50], [52]. In the case $\mathbf{G} = \mathbf{R}$ it is essentially due to Bochner and Phillips [16].

Linear Operators

Let $\mathbf{E}$ be a Riesz space.

A function $T : \mathbf{E} \to \mathbf{G}$ is a *linear operator* or, briefly, an *operator* if $T(x+y) = Tx + Ty$ and $T(\alpha x) = \alpha Tx$ holds for all $x, y \in \mathbf{E}$ and $\alpha \in \mathbf{R}$. By Theorem 3.29, linear operators are precisely the additive functions $\mathbf{E} \to \mathbf{G}$ which, in addition, are homogeneous. The ordered vector space of all linear operators $\mathbf{E} \to \mathbf{G}$ will be denoted by $L(\mathbf{E}, \mathbf{G})$.

PROPOSITION 4.6. *Assume that* $\mathbf{G}$ *is Archimedean. Then every positive additive function* $\mathbf{E} \to \mathbf{G}$ *is a linear operator.*

Proposition 4.6 is due to Kantorovich [32]; for a proof, see also Aliprantis and Burkinshaw [2, Theorem 1.7].

Since every order complete Riesz space is Archimedean, it follows from Proposition 4.6 that a linear operator into an order complete Riesz space is the difference of two positive linear operators if and only if it is the difference of two positive additive functions. Therefore, the application of our general result on additive functions on a commutative minimal clan to linear operators on a Riesz space does *not* involve any additional considerations concerning scalar multiplication.

Let $L^b(\mathbf{E}, \mathbf{G})$ denote the ordered vector space of all order bounded operators $\mathbf{E} \to \mathbf{G}$.

THEOREM 4.7. *Assume that* $\mathbf{G}$ *is order complete. Then* $L^b(\mathbf{E}, \mathbf{G})$ *is an order complete Riesz space.*

Theorem 4.7 follows from Theorem 4.3 and is due to Kantorovich [32]; see also Bauer [6], [7].

By Theorem 4.7, the *order dual* $\mathbf{E}^{\sim} := L^b(\mathbf{E}, \mathbf{R})$ of the Riesz space $\mathbf{E}$ is an order complete Riesz space. This result is also due to Riesz [46].

Comments

A Riesz space $\mathbf{G}$ is a *Banach lattice* if it is a Banach space with norm $\|.\|$ such that $\|x\| \leq \|y\|$ holds for all $x, y \in \mathbf{G}$ satisfying $|x| \leq |y|$.

For a minimal clan $\mathbf{E}$ and an order complete Banach lattice $\mathbf{G}$, there are many other properties of additive functions $\varphi : \mathbf{E} \to \mathbf{G}$ which can be expressed by properties of the range of φ on the order intervals of $\mathbf{E}$ and which define ideals of $oba(\mathbf{E}, \mathbf{G})$; see Schmidt [54].

The properties of an additive function $\varphi : \mathbf{E} \to \mathbf{G}$ which are determined by a property of the range of φ on the order intervals of $\mathbf{E}$ can be considered to be *local* or to be *determined by order*. To each of these properties, there corresponds another property of additive functions $\mathbf{E} \to \mathbf{G}$ which can be considered to be *global* or to be *determined by topology*; these properties of additive functions $\mathbf{E} \to \mathbf{G}$ are defined in terms of a generating set of $\mathbf{E}$:

A subset U of $\mathbf{E}_+$ is *solid* if $[0, u] \subseteq U$ holds for each $u \in U$, and it is a *generating set* of $\mathbf{E}$ if it is solid and if, for each $z \in \mathbf{E}_+$, there exists a partition $(u_1, u_2, \ldots, u_m) \in \mathcal{P}(z)$ satisfying $u_i \in U$ for all $i \in \{1, 2, \ldots, m\}$. In the case where $\mathbf{E}$ is a Boolean ring, the set $\mathbf{E}$ is a generating set of $\mathbf{E}$; in the case where $\mathbf{E}$ is a normed Riesz space, the set $U(\mathbf{E}_+)$ of all positive elements of the closed unit ball is a generating set of $\mathbf{E}$ and it can be shown that the norm of a linear operator from $\mathbf{E}$ into a Banach lattice $\mathbf{G}$ is already determined by its values on $U(\mathbf{E}_+)$.

The concept of a generating set of $\mathbf{E}$ is particularly useful for additive functions $\mathbf{E} \to \mathbf{G}$ when $\mathbf{G}$ is a Banach lattice; for Jordan decompositions of additive functions $\mathbf{E} \to \mathbf{G}$ which have a certain property with respect to a generating set of $\mathbf{E}$, see Schmidt [54].

An automatic linearity result like Proposition 4.6 is also valid for certain additive functions from a Riesz space into a Hausdorff topological vector space; see Constantinescu [22, Corollary 2.12].

Problems

- Extend the results on additive functions on a commutative minimal clan presented in this section to the case of additive functions taking their values in a lattice–ordered commutative semigroup with appropriate order completeness properties; see Schmidt [49], [54].

- Extend the results of Schmidt [54] on additive functions on a commutative minimal clan which are defined by a property related to a generating set to additive functions which are defined by a property related to a suitable generalization of FN-topologies.

5. The Abstract Lebesgue Decomposition

In this chapter we prove several band decompositions of order bounded additive functions from a commutative minimal clan $\mathbf{E}$ into a Riesz space $\mathbf{G}$. Each of these band decompositions is induced by a class of order bounded additive functions which are order continuous with respect to a solid collection of order bounded sequences in $\mathbf{E}_+$. These band decompositions can be summarized under the general notion of an *abstract order Lebesgue decomposition*, which yields band decompositions of the Ogasawara–Yosida–Hewitt type and of the Lebesgue type as special cases.

Throughout this section, let $\mathbf{G}$ be an order complete Riesz space.

Additive Functions

Let $\mathbf{E}$ be a commutative minimal clan.

A collection $\mathcal{N}$ of order bounded sequences in $\mathbf{E}_+$ is *solid* if it contains every sequence $\{z_n\}_{n\in\mathbf{N}} \subseteq \mathbf{E}_+$ for which there exists a sequence $\{z'_n\}_{n\in\mathbf{N}} \in \mathcal{N}$ satisfying $z_n \leq z'_n$ for all $n \in \mathbf{N}$.

Let $\mathcal{N}$ be a solid collection of order bounded sequences in $\mathbf{E}_+$.

An additive function $\varphi : \mathbf{E} \to \mathbf{G}$ is *order $\mathcal{N}$–continuous* if o$-$$\lim \varphi(z_n) = 0$ holds for every sequence $\{z_n\}_{n\in\mathbf{N}}$ in $\mathcal{N}$. The ordered vector space of all order $\mathcal{N}$–continuous additive functions $\mathbf{E} \to \mathbf{G}$ will be denoted by $a^{o\mathcal{N}c}(\mathbf{E},\mathbf{G})$.

Lemma 5.1. *For $\varphi \in oba(\mathbf{E},\mathbf{G})$, the following are equivalent:*
(a) *φ is order $\mathcal{N}$–continuous.*
(b) *$|\varphi|$ is order $\mathcal{N}$–continuous.*

Proof. Assume that (a) holds and consider $\{z_n\}_{n\in\mathbf{N}} \in \mathcal{N}$ and $z \in \mathbf{E}_+$ satisfying $z_n \leq z$ for all $n \in \mathbf{N}$. For all $n \in \mathbf{N}$, choose $w_n \in \mathbf{E}_+$ satisfying $w_n + z_n = z$. Also, for $u \in [0, z]$ and for all $n \in \mathbf{N}$, choose

$u_n \in \mathbf{E}_+$ satisfying $u_n + u \wedge z_n = u$. Then we have, for all $n \in \mathbf{N}$, $u_n + u \wedge z_n = u = u \wedge z = u \wedge (w_n + z_n) \leq u \wedge w_n + u \wedge z_n \leq w_n + u \wedge z_n$, by Corollary 3.16, and thus $u_n \leq w_n$, by the order cancellation property (Theorem 3.9). This yields

$$\begin{aligned} \varphi^+(z_n) &= \varphi^+(z) - \varphi^+(w_n) \\ &\leq \varphi^+(z) - \varphi^+(u_n) \\ &\leq \varphi^+(z) - \varphi(u_n) \\ &= \varphi^+(z) - \varphi(u) + \varphi(u \wedge z_n) \end{aligned}$$

for all $u \in [0, z]$ and $n \in \mathbf{N}$, hence

$$\begin{aligned} \text{o}-\limsup \varphi^+(z_n) &\leq \varphi^+(z) - \varphi(u) + \text{o}-\limsup \varphi(u \wedge z_n) \\ &= \varphi^+(z) - \varphi(u) \end{aligned}$$

for all $u \in [0, z]$, whence

$$\begin{aligned} \text{o}-\limsup \varphi^+(z_n) &\leq \varphi^+(z) + \inf_{[0,z]}(-\varphi(u)) \\ &= \varphi^+(z) - \sup_{[0,z]} \varphi(u) \\ &= 0\,, \end{aligned}$$

and thus

$$\text{o}-\lim \varphi^+(z_n) = 0\,.$$

This means that φ^+ is order $\mathcal{N}$–continuous. Applying the same argument to $-\varphi$ instead of φ, we see that also $\varphi^- = (-\varphi)^+$ and hence $|\varphi| = \varphi^+ + \varphi^-$ is order $\mathcal{N}$–continuous. Therefore, (a) implies (b).
The converse is obvious. □

An additive function $\varphi : \mathbf{E} \to \mathbf{G}$ is *order $\mathcal{N}$–singular* if $|\varphi|$ exists in $a(\mathbf{E}, \mathbf{G})$ and if $\psi = 0$ holds for each $\psi \in a^{o\mathcal{N}c}(\mathbf{E}, \mathbf{G})$ satisfying $0 \leq \psi \leq |\varphi|$. Note that each order $\mathcal{N}$–singular additive function is order bounded. The collection of all order $\mathcal{N}$–singular additive functions $\mathbf{E} \to \mathbf{G}$ will be denoted by $a^{o\mathcal{N}s}(\mathbf{E}, \mathbf{G})$.

LEMMA 5.2. *For $\varphi \in oba(\mathbf{E}, \mathbf{G})$, the following are equivalent:*
(a) *φ is order $\mathcal{N}$–singular.*

(b) $|\varphi| \wedge \psi = 0$ *holds for each positive order $\mathcal{N}$–continuous* $\psi \in a(\mathbf{E}, \mathbf{G})$.
(c) $|\varphi| \wedge |\psi| = 0$ *holds for each order $\mathcal{N}$–continuous* $\psi \in oba(\mathbf{E}, \mathbf{G})$.

Proof. Suppose first that (a) holds and consider $\psi \in a^{o\mathcal{N}c}(\mathbf{E}, \mathbf{G})$ satisfying $0 \leq \psi$. Then we have $0 \leq |\varphi| \wedge \psi \leq \psi$, hence $|\varphi| \wedge \psi$ is order $\mathcal{N}$–continuous, and from $0 \leq |\varphi| \wedge \psi \leq |\varphi|$ and the assumption on φ we obtain $|\varphi| \wedge \psi = 0$. Therefore, (a) implies (b).
Suppose now that (b) holds and consider $\psi \in a^{o\mathcal{N}c}(\mathbf{E}, \mathbf{G})$ satisfying $0 \leq \psi \leq |\varphi|$. Then we have $\psi = |\varphi| \wedge \psi = 0$, by the assumption on φ. Therefore, (b) implies (a).
The equivalence of (b) and (c) is obvious from Lemma 5.1. □

Define $oba^{o\mathcal{N}c}(\mathbf{E}, \mathbf{G}) := oba(\mathbf{E}, \mathbf{G}) \cap a^{o\mathcal{N}c}(\mathbf{E}, \mathbf{G})$. The following result is the *abstract order Lebesgue decomposition* of order bounded additive functions:

THEOREM 5.3. (ABSTRACT ORDER LEBESGUE DECOMPOSITION) *$oba^{o\mathcal{N}c}(\mathbf{E}, \mathbf{G})$ and $a^{o\mathcal{N}s}(\mathbf{E}, \mathbf{G})$ are order complete Riesz spaces and projection bands of $oba(\mathbf{E}, \mathbf{G})$, and $oba(\mathbf{E}, \mathbf{G})$ is the order direct sum of these projection bands.*

Proof. By Lemma 5.1, $oba^{o\mathcal{N}c}(\mathbf{E}, \mathbf{G})$ is an ideal of $oba(\mathbf{E}, \mathbf{G})$.
To prove that $oba^{o\mathcal{N}c}(\mathbf{E}, \mathbf{G})$ is even a band of $oba(\mathbf{E}, \mathbf{G})$, consider a directed ($\leq$) family $\{\varphi_\gamma\}_{\gamma \in \Gamma} \subseteq oba^{o\mathcal{N}c}(\mathbf{E}, \mathbf{G})$ satisfying $0 \leq \varphi_\gamma$ for all $\gamma \in \Gamma$ and such that $\{\varphi_\gamma\}_{\gamma \in \Gamma}$ has a least upper bound, say φ, in $oba(\mathbf{E}, \mathbf{G})$. Consider also a sequence $\{z_n\}_{n \in \mathbf{N}} \in \mathcal{N}$ and $z \in \mathbf{E}_+$ satisfying $z_n \leq z$ for all $n \in \mathbf{N}$. For all $n \in \mathbf{N}$, choose $w_n \in \mathbf{E}_+$ satisfying $w_n + z_n = z$. Then we have

$$\begin{aligned} \varphi(z_n) &= \varphi(z) - \varphi(w_n) \\ &\leq \varphi(z) - \varphi_\gamma(w_n) \\ &\leq \varphi(z) - \varphi_\gamma(z) + \varphi_\gamma(z_n) \end{aligned}$$

for all $\gamma \in \Gamma$ and $n \in \mathbf{N}$, hence

$$\begin{aligned} \text{o}-\limsup \varphi(z_n) &\leq \varphi(z) - \varphi_\gamma(z) + \text{o}-\limsup \varphi_\gamma(z_n) \\ &= \varphi(z) - \varphi_\gamma(z) \end{aligned}$$

for all $\gamma \in \Gamma$, whence

$$\begin{aligned} \text{o}-\limsup \varphi(z_n) &\leq \varphi(z) + \inf_\Gamma(-\varphi_\gamma(z)) \\ &= \varphi(z) - \sup_\Gamma \varphi_\gamma(z) \\ &= 0\,, \end{aligned}$$

and thus

$$\text{o}-\lim \varphi(z_n) \;=\; 0\,,$$

since φ is positive. This proves that φ is order $\mathcal{N}$–continuous. Therefore, $oba^{o\mathcal{N}c}(\mathbf{E},\mathbf{G})$ is a band of $oba(\mathbf{E},\mathbf{G})$.
By Lemma 5.2, we have

$$a^{o\mathcal{N}s}(\mathbf{E},\mathbf{G}) \;=\; oba^{o\mathcal{N}c}(\mathbf{E},\mathbf{G})^\perp\,.$$

Therefore, $a^{o\mathcal{N}s}(\mathbf{E},\mathbf{G})$ is a band of $oba(\mathbf{E},\mathbf{G})$.
The assertion now follows from the Riesz decomposition (Proposition 2.1). □

We now record three special cases of Theorem 5.3:

Let $\mathcal{N}(d)$ denote the collection of all order bounded disjoint sequences in $\mathbf{E}_+$. Then $\mathcal{N}(d)$ is a solid collection of order bounded sequences in $\mathbf{E}_+$.

Corollary 5.4. *$oba^{o\mathcal{N}(d)c}(\mathbf{E},\mathbf{G})$ and $a^{o\mathcal{N}(d)s}(\mathbf{E},\mathbf{G})$ are order complete Riesz spaces and projection bands of $oba(\mathbf{E},\mathbf{G})$, and $oba(\mathbf{E},\mathbf{G})$ is the order direct sum of these projection bands.*

Let $\mathcal{N}(o)$ denote the collection of all sequences $\{z_n\}_{n\in\mathbf{N}} \subseteq \mathbf{E}_+$ satisfying $\text{o}-\lim z_n = 0$. Then $\mathcal{N}(o)$ is a solid collection of order bounded sequences in $\mathbf{E}_+$.

Corollary 5.5. (Order Yosida–Hewitt Decomposition) *$oba^{o\mathcal{N}(o)c}(\mathbf{E},\mathbf{G})$ and $a^{o\mathcal{N}(o)s}(\mathbf{E},\mathbf{G})$ are order complete Riesz spaces and projection bands of $oba(\mathbf{E},\mathbf{G})$, and $oba(\mathbf{E},\mathbf{G})$ is the order direct sum of these projection bands.*

Consider now $\lambda \in oba(\mathbf{E},\mathbf{R})$ and let $\mathcal{N}(\lambda)$ denote the collection of all order bounded sequences $\{z_n\}_{n\in\mathbf{N}} \subseteq \mathbf{E}_+$ satisfying $\lim|\lambda|(z_n) = 0$. Then $\mathcal{N}(\lambda)$ is a solid collection of order bounded sequences in $\mathbf{E}_+$.

Corollary 5.6. (Order Lebesgue Decomposition) *$oba^{o\mathcal{N}(\lambda)c}(\mathbf{E},\mathbf{G})$ and $a^{o\mathcal{N}(\lambda)s}(\mathbf{E},\mathbf{G})$ are order complete Riesz spaces and projection bands of $oba(\mathbf{E},\mathbf{G})$, and $oba(\mathbf{E},\mathbf{G})$ is the order direct sum of these projection bands.*

In the case $\mathbf{G}=\mathbf{R}$, Corollary 5.6 can be improved as follows:

Corollary 5.7. *$oba^{o\mathcal{N}(\lambda)c}(\mathbf{E},\mathbf{R}) = B(\{\lambda\})$ and $a^{o\mathcal{N}(\lambda)s}(\mathbf{E},\mathbf{R}) = \{\lambda\}^{\perp}$.*

Proof. Consider $\varphi \in oba^{o\mathcal{N}(\lambda)c}(\mathbf{E},\mathbf{R}) \cap \{\lambda\}^{\perp}$ and $z \in \mathbf{E}_+$. Then we have

$$\inf\nolimits_{S(z)} \left(|\varphi|(x)+|\lambda|(y)\right) \;=\; \left(|\varphi|\wedge|\lambda|\right)(z) \;=\; 0\,.$$

Thus, for each $n\in\mathbf{N}$, there exist $x_n, y_n \in \mathbf{E}_+$ satisfying $(x_n,y_n)\in S(z)$ and

$$|\varphi|(x_n)+|\lambda|(y_n) \;\leq\; 1/n\,.$$

This yields $\lim|\varphi|(x_n)=0$ and $\lim|\lambda|(y_n)=0$, hence $\lim|\varphi|(y_n)=0$, whence

$$|\varphi|(z) \;=\; \lim|\varphi|(x_n)+\lim|\varphi|(y_n) \;=\; 0\,,$$

and thus $\varphi=0$. Therefore, we have $oba^{o\mathcal{N}(\lambda)c}(\mathbf{E},\mathbf{R})\cap\{\lambda\}^{\perp}=\{0\}$. Since $oba(\mathbf{E},\mathbf{R})$ is the order direct sum of the projection bands $B(\{\lambda\})$ and $\{\lambda\}^{\perp}$, by the Riesz decomposition, it now follows from Corollary 5.6 that $oba^{o\mathcal{N}(\lambda)c}(\mathbf{E},\mathbf{R})$ is the order direct sum of $oba^{o\mathcal{N}(\lambda)c}(\mathbf{E},\mathbf{R})\cap B(\{\lambda\})$ and $oba^{o\mathcal{N}(\lambda)c}(\mathbf{E},\mathbf{R})\cap\{\lambda\}^{\perp}=\{0\}$. Therefore, we have

$$\begin{aligned} B(\{\lambda\}) &\subseteq oba^{o\mathcal{N}(\lambda)c}(\mathbf{E},\mathbf{R}) \\ &= oba^{o\mathcal{N}(\lambda)c}(\mathbf{E},\mathbf{R})\cap B(\{\lambda\}) \\ &\subseteq B(\{\lambda\}), \end{aligned}$$

hence $oba^{o\mathcal{N}(\lambda)c}(\mathbf{E},\mathbf{R}) = B(\{\lambda\})$, and thus $a^{o\mathcal{N}(\lambda)s}(\mathbf{E},\mathbf{R}) = \{\lambda\}^{\perp}$. □

Vector Measures

Let $\mathbf{E}$ be a Boolean ring.

As noticed in the previous section, all results on additive functions on a minimal clan can be read without modification as results on vector measures on a Boolean ring. In some cases, however, the terminology for vector measures differs from that for additive functions. Therefore, we have to reformulate some of the corollaries of the abstract order Lebesgue decomposition for order bounded additive functions in the terminology of vector measures.

A vector measure $\varphi : \mathbf{E} \to \mathbf{G}$ is *order exhaustive* if $\text{o–}\lim \varphi(z_n) = 0$ holds for every order bounded disjoint sequence $\{z_n\}_{n\in\mathbf{N}} \subseteq \mathbf{E}$. Obviously, a vector measure $\mathbf{E} \to \mathbf{G}$ is order exhaustive if and only if it is order $\mathcal{N}(d)$–continuous. The ordered vector space of all order exhaustive vector measures $\mathbf{E} \to \mathbf{G}$ will be denoted by $exa^o(\mathbf{E}, \mathbf{G})$.

Define $obexa^o(\mathbf{E}, \mathbf{G}) := oba(\mathbf{E}, \mathbf{G}) \cap exa^o(\mathbf{E}, \mathbf{G})$. The following result is a reformulation of Corollary 5.4 for vector measures:

COROLLARY 5.8. *$obexa^o(\mathbf{E}, \mathbf{G})$ and $obexa^o(\mathbf{E}, \mathbf{G})^\perp$ are order complete Riesz spaces and projection bands of $oba(\mathbf{E}, \mathbf{G})$, and $oba(\mathbf{E}, \mathbf{G})$ is the order direct sum of these projection bands.*

A vector measure $\varphi : \mathbf{E} \to \mathbf{G}$ is *order countably additive* if $\text{o–}\sum \varphi(z_n) = \varphi(\sup_\mathbf{N} z_n)$ holds for every disjoint sequence $\{z_n\}_{n\in\mathbf{N}} \subseteq \mathbf{E}$ for which $\sup_\mathbf{N} z_n$ exists. The ordered vector space of all order countably additive vector measures $\mathbf{E} \to \mathbf{G}$ will be denoted by $ca^o(\mathbf{E}, \mathbf{G})$.

LEMMA 5.9. *For $\varphi \in a(\mathbf{E}, \mathbf{G})$, the following are equivalent:*
(a) *φ is order countably additive.*
(b) *$\text{o}-\lim \varphi(u_n) = 0$ holds for every sequence $\{u_n\}_{n\in\mathbf{N}} \subseteq \mathbf{E}$ which decreases to 0.*

This follows from Lemma 3.28.

LEMMA 5.10. *For $\varphi \in oba(\mathbf{E}, \mathbf{G})$, the following are equivalent:*
(a) *φ is order countably additive.*
(b) *φ is order $\mathcal{N}(o)$–continuous.*

Proof. Suppose first that (a) holds and consider a disjoint sequence $\{z_n\}_{n\in\mathbf{N}}$ for which $\sup_\mathbf{N} z_n$ exists. Define $u := \sup_\mathbf{N} z_n$ and consider

$(u_1, u_2, \ldots, u_m) \in \mathcal{P}(u)$. Then we have

$$\sum_{n=1}^{k} |\varphi|(z_n) \;=\; |\varphi|\left(\sum_{n=1}^{k} z_n\right) \;\leq\; |\varphi|(u)$$

for all $k \in \mathbf{N}$, and thus

$$\text{o}{-}\sum |\varphi|(z_n) \;\leq\; |\varphi|(u) \; ,$$

and we also have

$$\begin{aligned}
\sum_{i=1}^{m} |\varphi(u_i)| &\leq \sum_{i=1}^{m}\sum_{n=1}^{k} |\varphi(u_i \wedge z_n)| + \sum_{i=1}^{m}\left|\varphi\left(u_i \wedge \left(u \setminus \sum_{n=1}^{k} z_n\right)\right)\right| \\
&\leq \sum_{n=1}^{k} |\varphi|(z_n) + \sum_{i=1}^{m}\left|\varphi\left(u_i \wedge \left(u \setminus \sum_{n=1}^{k} z_n\right)\right)\right| \\
&\leq \text{o}{-}\sum |\varphi|(z_n) + \sum_{i=1}^{m}\left|\varphi\left(u_i \wedge \left(u \setminus \sum_{n=1}^{k} z_n\right)\right)\right| \; ,
\end{aligned}$$

for all $k \in \mathbf{N}$, hence

$$\sum_{i=1}^{m} |\varphi(u_i)| \;\leq\; \text{o}{-}\sum |\varphi|(z_n) \; ,$$

by Lemmas 3.28 and 5.9, and thus

$$|\varphi|(u) \;\leq\; \text{o}{-}\sum |\varphi|(z_n) \; .$$

This yields

$$|\varphi|(u) \;=\; \text{o}{-}\sum |\varphi|(z_n) \; .$$

Therefore, $|\varphi|$ is order countably additive, and it now follows from Lemmas 5.9 and 5.1 that φ is order $\mathcal{N}(o)$–continuous. Therefore, (a) implies (b).

The converse is obvious from Lemma 5.9. □

A vector measure $\varphi : \mathbf{E} \to \mathbf{G}$ is *order purely finitely additive* if $|\varphi|$ exists in $a(\mathbf{E}, \mathbf{G})$ and if $\psi = 0$ holds for each $\psi \in ca^o(\mathbf{E}, \mathbf{G})$ satisfying $0 \leq \psi \leq |\varphi|$. The collection of all order purely finitely additive vector measures $\mathbf{E} \to \mathbf{G}$ will be denoted by $pfa^o(\mathbf{E}, \mathbf{G})$.

LEMMA 5.11. *For $\varphi \in oba(\mathbf{E},\mathbf{G})$, the following are equivalent:*
(a) *φ is order purely finitely additive.*
(b) *φ is order $\mathcal{N}(o)$–singular.*

This follows from Lemma 5.10.

Define $obca^o(\mathbf{E},\mathbf{G}) := oba(\mathbf{E},\mathbf{G}) \cap ca^o(\mathbf{E},\mathbf{G})$. By Lemmas 5.10 and 5.11, the following result is a reformulation of Corollary 5.5:

COROLLARY 5.12. (ORDER YOSIDA–HEWITT–DECOMPOSITION) *$obca^o(\mathbf{E},\mathbf{G})$ and $pfa^o(\mathbf{E},\mathbf{G})$ are order complete Riesz spaces and projection bands of $oba(\mathbf{E},\mathbf{G})$, and $oba(\mathbf{E},\mathbf{G})$ is the order direct sum of these projection bands.*

Corollary 5.12 is due to Bauer [6]; see also Diestel [24], Schep [48], Congost Iglesias [21], and Schmidt [52]. In the case $\mathbf{G} = \mathbf{R}$, it is essentially due to Woodbury [63] and Yosida and Hewitt [65].

A vector measure $\varphi : \mathbf{E} \to \mathbf{G}$ is *order λ–continuous* if $\text{o}-\lim \varphi(z_n) = 0$ holds for every order bounded sequence $\{z_n\}_{n\in\mathbf{N}} \subseteq \mathbf{E}$ satisfying $\lim |\lambda|(z_n) = 0$. The ordered vector space of all order λ–continuous vector measures $\mathbf{E} \to \mathbf{G}$ will be denoted by $a^{o\lambda c}(\mathbf{E},\mathbf{G})$. Define $oba^{o\lambda c}(\mathbf{E},\mathbf{G}) := oba(\mathbf{E},\mathbf{G}) \cap a^{o\lambda c}(\mathbf{E},\mathbf{G})$.

A vector measure $\varphi : \mathbf{E} \to \mathbf{G}$ is *order λ–singular* if $|\varphi|$ exists in $a(\mathbf{E},\mathbf{G})$ and if $\psi = 0$ holds for each $\psi \in a^{o\lambda c}(\mathbf{E},\mathbf{G})$ satisfying $0 \le \psi \le |\varphi|$. The collection of all order λ–singular vector measures $\mathbf{E} \to \mathbf{G}$ will be denoted by $a^{o\lambda s}(\mathbf{E},\mathbf{G})$.

Obviously, a vector measure is order λ–continuous if and only if it is order $\mathcal{N}(\lambda)$–continuous, and it is order λ–singular if and only if it is order $\mathcal{N}(\lambda)$–singular. Therefore, the following result is a reformulation of Corollary 5.6:

COROLLARY 5.13. (ORDER LEBESGUE DECOMPOSITION) *$oba^{o\lambda c}(\mathbf{E},\mathbf{G})$ and $a^{o\lambda s}(\mathbf{E},\mathbf{G})$ are order complete Riesz space and projection bands of $oba(\mathbf{E},\mathbf{G})$, and $oba(\mathbf{E},\mathbf{G})$ is the order direct sum of these projection bands.*

Corollary 5.13 generalizes the order Lebesgue decomposition of vector measures of bounded variation in certain order complete Banach lattices which is contained in a result of Caselles [20].

In the case $\mathbf{G} = \mathbf{R}$, Corollary 5.13 can be improved as follows:

Corollary 5.14. *$oba^{o\lambda c}(\mathbf{E},\mathbf{R}) = B(\{\lambda\})$ and $a^{o\lambda s}(\mathbf{E},\mathbf{R}) = \{\lambda\}^{\perp}$.*

This follows from Corollary 5.7.

In the case where $\mathbf{E}$ is a Boolean algebra, it is easy to see that $\varphi \in oba(\mathbf{E},\mathbf{R})$ is order λ–continuous if and only if for each $\varepsilon \in (0,\infty)$ there exists some $\delta \in (0,\infty)$ such that $|\varphi|(z) < \varepsilon$ holds for all $z \in \mathbf{E}$ satisfying $|\lambda|(z) < \delta$, and that φ is order λ–singular if and only if for each $\varepsilon \in (0,\infty)$ there exists some $z \in \mathbf{E}$ satisfying $|\lambda|(z) < \varepsilon$ and $|\varphi|(1 \setminus z) < \varepsilon$. In this case, and with the previous equivalent definitions of $oba^{o\lambda c}(\mathbf{E},\mathbf{R})$ and $a^{o\lambda s}(\mathbf{E},\mathbf{R})$, Corollary 5.14 was first proven by Bochner and Phillips [16]; see also Bauer [7].

Linear Operators

Let $\mathbf{E}$ be a Riesz space.

As in the case of vector measures, the terminology for linear operators differs in some cases from that for additive functions. Therefore, we have to reformulate the abstract order Lebesgue decomposition and some of its corollaries in the terminology of linear operators.

Let $\mathcal{N}$ be a solid collection of order bounded sequences in $\mathbf{E}_+$, let $L^{o\mathcal{N}c}(\mathbf{E},\mathbf{G})$ denote the ordered vector space of all order $\mathcal{N}$–continuous operators in $L^b(\mathbf{E},\mathbf{G})$, and let $L^{o\mathcal{N}s}(\mathbf{E},\mathbf{G})$ denote the collection of all order $\mathcal{N}$–singular operators $\mathbf{E} \to \mathbf{G}$.

Theorem 5.15 (Abstract Order Lebesgue Decomposition). *$L^{o\mathcal{N}c}(\mathbf{E},\mathbf{G})$ and $L^{o\mathcal{N}s}(\mathbf{E},\mathbf{G})$ are order complete Riesz spaces and projection bands of $L^b(\mathbf{E},\mathbf{G})$, and $L^b(\mathbf{E},\mathbf{G})$ is the order direct sum of these projection bands.*

This follows from Theorem 5.3.

A linear operator $T : \mathbf{E} \to \mathbf{G}$ is *sequentially order continuous* if $\text{o}-\lim Tz_n = 0$ holds for every sequence $\{z_n\}_{n\in\mathbf{N}} \subseteq \mathbf{E}_+$ satisfying $\text{o}-\lim z_n = 0$. The ordered vector space of all sequentially order continuous operators in $L^b(\mathbf{E},\mathbf{G})$ will be denoted by $L^c(\mathbf{E},\mathbf{G})$.

A linear operator $T : \mathbf{E} \to \mathbf{G}$ is *singular* if $|T|$ exists in $L(\mathbf{E},\mathbf{G})$ and if $S = 0$ holds for each $S \in L^c(\mathbf{E},\mathbf{G})$ satisfying $0 \le S \le |T|$. The collection of all singular operators $\mathbf{E} \to \mathbf{G}$ will be denoted by $L^s(\mathbf{E},\mathbf{G})$.

Obviously, a linear operator is sequentially order continuous if and only if it is order $\mathcal{N}(o)$–continuous, and it is singular if and only if it is order $\mathcal{N}(o)$–singular.

COROLLARY 5.16. (SEQUENTIAL OGASAWARA DECOMPOSITION) *$L^c(\mathbf{E},\mathbf{G})$ and $L^s(\mathbf{E},\mathbf{G})$ are order complete Riesz spaces and projection bands of $L^b(\mathbf{E},\mathbf{G})$, and $L^b(\mathbf{E},\mathbf{G})$ is the order direct sum of these projection bands.*

This follows from Theorem 5.15 or Corollary 5.5.

Corollary 5.16 is due to Ogasawara [41]; see also Bauer [6]. In the case $\mathbf{G} = \mathbf{R}$, it is due to Riesz [46].

Consider now $e^\sim \in \mathbf{E}^\sim$.

A linear operator $T \in L(\mathbf{E},\mathbf{G})$ is *sequentially order continuous with respect to* $e^\sim$ if $\mathrm{o}-\lim Tz_n = 0$ holds for every order bounded sequence $\{z_n\}_{n\in\mathbf{N}} \subseteq \mathbf{E}_+$ satisfying $\lim |e^\sim|(z_n) = 0$. The ordered vector space of all linear operators in $L^b(\mathbf{E},\mathbf{G})$ which are sequentially order continuous with respect to $e^\sim$ will be denoted by $L^{e^\sim c}(\mathbf{E},\mathbf{G})$.

A linear operator $T \in L(\mathbf{E}, G)$ is *singular with respect to* $e^\sim$ if $|T|$ exists in $L(\mathbf{E},\mathbf{G})$ and if $S = 0$ holds for each $S \in L^{e^\sim c}(\mathbf{E},\mathbf{G})$ satisfying $0 \le S \le |T|$. The collection of all linear operators in $L(\mathbf{E},\mathbf{G})$ which are singular with respect to $e^\sim$ will be denoted by $L^{e^\sim s}(\mathbf{E},\mathbf{G})$.

Obviously, a linear operator is sequentially order continuous with respect to $e^\sim$ if and only if it is order $\mathcal{N}(e^\sim)$–continuous, and it is singular with respect to $e^\sim$ if and only if it is order $\mathcal{N}(e^\sim)$–singular.

COROLLARY 5.17. (ORDER LEBESGUE DECOMPOSITION) *$L^{e^\sim c}(\mathbf{E},\mathbf{G})$ and $L^{e^\sim s}(\mathbf{E},\mathbf{G})$ are order complete Riesz spaces and projection bands of $L^b(\mathbf{E},\mathbf{G})$, and $L^b(\mathbf{E},\mathbf{G})$ is the order direct sum of these projection bands.*

This follows from Theorem 5.15 or Corollary 5.6.

In the case $\mathbf{G} = \mathbf{R}$, Corollary 5.17 can be improved as follows:

COROLLARY 5.18. *$L^{e^\sim c}(\mathbf{E},\mathbf{R}) = B(\{e^\sim\})$ and $L^{e^\sim s}(\mathbf{E},\mathbf{R}) = \{e^\sim\}^\perp$.*

This follows from Corollary 5.7.

Comments

The abstract order Lebesgue decomposition remains valid if the solid collection $\mathcal{N}$ of order bounded sequences in $\mathbf{E}_+$ is replaced by a solid collection of order bounded nets in $\mathbf{E}_+$ (which is the obvious generalization of the notion of a solid collection of order bounded sequences in $\mathbf{E}_+$). The proof is similar to that of Theorem 5.3.

This observation is of interest e. g. for linear operators:

A linear operator $T : \mathbf{E} \to \mathbf{G}$ is *order continuous* or *normal* if o–lim $Tz_\gamma = 0$ holds for every net $\{z_\gamma\}_{\gamma\in\Gamma} \subseteq \mathbf{E}_+$ satisfying o–lim $z_\gamma = 0$. The ordered vector space of all order continuous operators in $L^b(\mathbf{E},\mathbf{G})$ will be denoted by $L^n(\mathbf{E},\mathbf{G})$.

A linear operator $T : \mathbf{E} \to \mathbf{G}$ is *normal singular* if $|T|$ exists in $L(\mathbf{E},\mathbf{G})$ and if $S = 0$ holds for each $S \in L^n(\mathbf{E},\mathbf{G})$ satisfying $0 \le S \le |T|$. The collection of all normal singular operators $\mathbf{E} \to \mathbf{G}$ will be denoted by $L^{ns}(\mathbf{E},\mathbf{G})$.

Thus, replacing the solid collection $\mathcal{N}(o)$ of order bounded sequences $\{z_n\}_{n\in\mathbf{N}} \subseteq \mathbf{E}_+$ satisfying o$-$lim $z_n = 0$ by a solid collection of order bounded nets $\{z_\gamma\}_{\gamma\in\Gamma} \subseteq \mathbf{E}_+$ satisfying o$-$lim $z_\gamma = 0$, we see that Corollary 5.16 remains valid if $L^c(\mathbf{E},\mathbf{G})$ and $L^s(\mathbf{E},\mathbf{G})$ are replaced by $L^n(\mathbf{E},\mathbf{G})$ and $L^{ns}(\mathbf{E},\mathbf{G})$, respectively. The resulting band decomposition is also due to Ogasawara [41]; see also Bauer [6].

On the other hand, it is clear that every solid subset of $\mathbf{E}_+$ can be identified with a solid collection of constant (and hence order bounded) sequences in $\mathbf{E}_+$.

For example, consider $\lambda \in oba(\mathbf{E},\mathbf{R})$. Then the set $U_0(\lambda)$ consisting of all $z \in \mathbf{E}_+$ satisfying $|\lambda|(z) = 0$ is solid and can be identified with the solid collection $\mathcal{N}_0(\lambda)$ of all constant sequences $\{z_n\}_{n\in\mathbf{N}}$ satisfying o$-$lim $|\lambda|(z_n) = 0$ (which is contained in the solid collection $\mathcal{N}(\lambda)$). Thus, replacing $\mathcal{N}(\lambda)$ by $\mathcal{N}_0(\lambda)$ in Corollary 5.6, we obtain a variant of the order Lebesgue decomposition.

We also remark that Theorem 5.3 yields extensions of Corollary 5.6 and its variant mentioned before if $\lambda \in oba(\mathbf{E},\mathbf{R})$ is replaced either by an order bounded additive function $\mathbf{E} \to \mathbf{F}$, where $\mathbf{F}$ is an arbitrary order complete Riesz space, or by a function $\lambda : \mathbf{E}_+ \to$

$\mathbf{R}_+ \cup \{+\infty\}$ satisfying $\lambda(0) = 0$ and $\lambda(x) \leq \lambda(y)$ for all $x, y \in \mathbf{E}_+$ such that $x \leq y$.

Problems

- Find further applications of the abstract order Lebesgue decomposition with respect to a solid collection of order bounded nets.

- Extend the results of this section to additive functions from a minimal clan into a lattice–ordered commutative semigroup; see Pavlakos [42], [43].

6. Common Extensions of Positive Abstract Measures

The following result was proposed by Guy [29]:

PROPOSITION 6.1. *Let Ω be a set, let $\mathcal{M}$ and $\mathcal{N}$ be algebras of subsets of Ω, and let $\mu : \mathcal{M} \to \mathbf{R}$ and $\nu : \mathcal{N} \to \mathbf{R}$ be positive additive set functions. Then the following are equivalent:*
(a) *The inequalities $\mu(A) \leq \nu(B)$ and $\nu(C) \leq \mu(D)$ hold for all $A, D \in \mathcal{M}$ and $B, C \in \mathcal{N}$ satisfying $A \subseteq B$ and $C \subseteq D$.*
(b) *There exists a positive additive set function $\varphi : 2^\Omega \to \mathbf{R}$ satisfying $\varphi(A) = \mu(A)$ for all $A \in \mathcal{M}$ and $\varphi(B) = \nu(B)$ for all $B \in \mathcal{N}$.*

In the present section, we prove a general theorem on the existence of a positive common extension of a family of positive vector measures in an order complete Riesz space. The result yields vector–valued versions of Guy's result and of extension theorems due to Horn and Tarski [31] and Marczewski [39], [40]. Its proof is based on the representation of vector measures by linear operators and on the Hahn–Banach theorem for positive operators in an order complete Riesz space.

Throughout this section, let $\mathbf{G}$ be an order complete Riesz space.

Linear Operators

The following extension theorem is an immediate consequence of the *Hahn–Banach theorem* for positive operators; see Aliprantis and Burkinshaw [2] Theorem 2.8:

PROPOSITION 6.2. *Let* $\mathbf{E}$ *be a Riesz space with order unit* $e \in \mathbf{E}_+$, *let* $\mathbf{F}$ *be a subspace of* $\mathbf{E}$ *satisfying* $e \in \mathbf{F}$, *and let* $S : \mathbf{F} \to \mathbf{G}$ *be a positive operator. Then there exists a positive operator* $T : \mathbf{E} \to \mathbf{G}$ *satisfying* $Tx = Sx$ *for all* $x \in \mathbf{F}$.

For a Riesz space $\mathbf{E}$ and a family $\{\mathbf{E}_\delta\}_{\delta\in\Delta}$ of subspaces of $\mathbf{E}$, let $\Phi(\{\mathbf{E}_\delta\}_{\delta\in\Delta})$ denote the collection of all families $\{x_\delta \in \mathbf{E}_\delta \mid \delta \in \Delta\}$ satisfying $x_\delta \neq 0$ for at most finitely many $\delta \in \Delta$. A family $\{T_\delta : \mathbf{E}_\delta \to \mathbf{G} \mid \delta \in \Delta\}$ of linear operators has a *common extension* if there exists a linear operator $T : \mathbf{E} \to \mathbf{G}$ satisfying $Tx = T_\delta x$ for all $\delta \in \Delta$ and $x \in \mathbf{E}_\delta$.

THEOREM 6.3. *Let* $\mathbf{E}$ *be a Riesz space with order unit* $e \in \mathbf{E}_+$, *let* $\{\mathbf{E}_\delta\}_{\delta\in\Delta}$ *be a family of subspaces of* $\mathbf{E}$ *satisfying* $e \in \operatorname{span}\bigcup_{\delta\in\Delta}\mathbf{E}_\delta$, *and let* $\{T_\delta : \mathbf{E}_\delta \to \mathbf{G} \mid \delta \in \Delta\}$ *be a family of positive operators. Then the following are equivalent*:

(a) *The inequality*

$$0 \leq \sum_{\delta\in\Delta} T_\delta x_\delta$$

holds for every family $\{x_\delta\}_{\delta\in\Delta} \in \Phi(\{\mathbf{E}_\delta\}_{\delta\in\Delta})$ *satisfying* $0 \leq \sum_{\delta\in\Delta} x_\delta$.

(b) *The family* $\{\mathbf{E}_\delta\}_{\delta\in\Delta}$ *has a positive common extension* $T : \mathbf{E} \to \mathbf{G}$.

Proof. Assume that (a) holds and define $\mathbf{F} := \operatorname{span}\bigcup_{\delta\in\Delta}\mathbf{E}_\delta$. Then the map $S : \mathbf{F} \to \mathbf{G}$, given by

$$Sx := \sum_{\delta\in\Delta} T_\delta x_\delta$$

for all $x \in \mathbf{F}$ and arbitrary $\{x_\delta\}_{\delta\in\Delta} \in \Phi(\{\mathbf{E}_\delta\}_{\delta\in\Delta})$ satisfying $x = \sum_{\delta\in\Delta} x_\delta$, is well–defined, linear, and positive. By Proposition 6.2, there exists a positive operator $T : \mathbf{E} \to \mathbf{G}$ satisfying $Tx = Sx$ for all $x \in \mathbf{F}$. Therefore, (a) implies (b).
The converse is obvious. □

Theorem 6.3 is due to Maharam [38].

Vector Measures

Let Ω be a nonempty set.

For a family $\{\mathcal{F}_\delta\}_{\delta\in\Delta}$ of algebras of subsets of Ω, a family $\{\varphi_\delta : \mathcal{F}_\delta \to \mathbf{G} \mid \delta \in \Delta\}$ of vector measures has a *common extension* if there exists a vector measure $\varphi : 2^\Omega \to \mathbf{G}$ satisfying $\varphi(A) = \varphi_\delta(A)$ for all $\delta \in \Delta$ and $A \in \mathcal{F}_\delta$.

THEOREM 6.4. *Let $\{\mathcal{F}_\delta\}_{\delta\in\Delta}$ be a family of algebras of subsets of Ω and let $\{\varphi_\delta : \mathcal{F}_\delta \to \mathbf{G} \mid \delta \in \Delta\}$ be a family of positive vector measures. Then the following are equivalent:*

(a) *The inequality*

$$\sum_{i=1}^{m} \varphi_{\delta(i)}(A_i) \;\leq\; \sum_{i=m+1}^{m+n} \varphi_{\delta(i)}(A_i)$$

holds for all $m,n \in \mathbf{N}$, for all $A_1,\ldots,A_{m+n} \in \bigcup_{\delta\in\Delta}\mathcal{F}_\delta$ satisfying $\sum_{i=1}^{m}\chi_{A_i} \leq \sum_{i=m+1}^{m+n}\chi_{A_i}$, and for all $\delta(1),\ldots,\delta(m+n) \in \Delta$ satisfying $A_i \in \mathcal{F}_{\delta(i)}$ for all $i \in \{1,\ldots,m+n\}$.

(b) *The family $\{\varphi_\delta\}_{\delta\in\Delta}$ has a positive common extension $\varphi : 2^\Omega \to \mathbf{G}$.*

Proof. It is well–known that a vector measure is positive if and only if its representing linear operator is positive.
Assume that (a) holds. For $\delta \in \Delta$, define $\mathbf{E}_\delta := \mathbf{D}(\mathcal{F}_\delta)$ and let $T_\delta : \mathbf{E}_\delta \to \mathbf{G}$ denote the representing linear operator of φ_δ. Then each T_δ is positive.
We claim that

$$0 \;\leq\; \textstyle\sum_{\delta\in\Delta} T_\delta f_\delta$$

holds for every family $\{f_\delta\}_{\delta\in\Delta} \in \Phi(\{\mathbf{E}_\delta\}_{\delta\in\Delta})$ satisfying $0 \leq \sum_{\delta\in\Delta} f_\delta$. Indeed, this is obvious for families of simple functions taking their values in $\mathbf{Z}$, and hence for families of simple functions taking their values in $\mathbf{Q}$.
Consider now an arbitrary family $\{f_\delta\}_{\delta\in\Delta} \in \Phi(\{\mathbf{E}_\delta\}_{\delta\in\Delta})$ satisfying

$$0 \;\leq\; \textstyle\sum_{\delta\in\Delta} f_\delta$$

and let m denote the number of $\delta \in \Delta$ satisfying $f_\delta \neq 0$.
For each $\delta \in \Delta$ and $k \in \mathbf{N}$, choose $f_{\delta,k} \in \mathbf{E}_\delta$ such that each $f_{\delta,k}$ takes its values in $\mathbf{Q}$ and satisfies

$$f_{\delta,k} - \frac{1}{km}\chi_\Omega \;\leq\; f_\delta \;\leq\; f_{\delta,k}$$

as well as $f_{\delta,k} = 0$ whenever $f_\delta = 0$. Then we have, for all $k \in \mathbf{N}$,

$$0 \;\leq\; \sum\nolimits_{\delta\in\Delta} f_{\delta,k}$$

and hence, letting $g := T_\delta\chi_\Omega = \varphi_\delta(\Omega)$ (which is independent of the particular choice of δ),

$$0 \;\leq\; \sum\nolimits_{\delta\in\Delta} T_\delta f_{\delta,k} \;\leq\; \sum\nolimits_{\delta\in\Delta} T_\delta f_\delta + \frac{1}{k}g\ .$$

Since $\mathbf{G}$ is order complete and hence Archimedean, this yields

$$0 \;\leq\; \sum\nolimits_{\delta\in\Delta} T_\delta f_\delta\ ,$$

which proves our claim.
Define now $\mathbf{E} := \mathbf{D}(2^\Omega)$. By what we have shown before and by Theorem 6.3, the family $\{T_\delta\}_{\delta\in\Delta}$ has a positive common extension $T : \mathbf{E} \to \mathbf{G}$, and it is then clear that the vector measure $\varphi : 2^\Omega \to \mathbf{G}$, given by $\varphi(A) := T\chi_A$, is a positive common extension of the familiy $\{\varphi_\delta\}_{\delta\in\Delta}$. Therefore, (a) implies (b).
The converse is obvious. □

In the case $\mathbf{G} = \mathbf{R}$, Theorem 6.4 is equivalent to a result of Lembcke [35].

As a first consequence of Theorem 6.4, we obtain the following vector–valued version of Guy's Proposition 6.1:

COROLLARY 6.5. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be algebras of subsets of Ω and let $\varphi_1 : \mathcal{F}_1 \to \mathbf{G}$ and $\varphi_2 : \mathcal{F}_2 \to \mathbf{G}$ be positive vector measures. Then the following are equivalent:*

(a) *The inequality*

$$\varphi_i(A_i) \;\leq\; \varphi_j(A_j)$$

holds for all $i, j \in \{1, 2\}$ and for all $A_i \in \mathcal{F}_i$ and $A_j \in \mathcal{F}_j$ satisfying $A_i \subseteq A_j$.

(b) *φ_1 and φ_2 have a positive common extension $\varphi : 2^\Omega \to \mathbf{G}$.*

Proof. Assume that (a) holds. Consider $m, n \in \mathbf{N}$ and $B_1, \ldots, B_{m+n} \in \mathcal{F}_1 \cup \mathcal{F}_2$ satisfying

$$\sum_{i=1}^{m} \chi_{B_i} \;\leq\; \sum_{i=m+1}^{m+n} \chi_{\overline{B_i}}$$

and hence

$$\sum_{i=1}^{m+n} \chi_{B_i} \;\leq\; n\chi_\Omega \; .$$

For $i \in \{1, \ldots, m+n\}$, define

$$C_i \;:=\; \begin{cases} B_i & \text{if} \quad B_i \in \mathcal{F}_1 \\ \emptyset & \text{if} \quad B_i \in \mathcal{F}_2 \end{cases}$$

and

$$D_i \;:=\; B_i \backslash C_i \; .$$

Define also

$$c \;:=\; \sum_{i=1}^{m+n} \chi_{C_i}$$

and

$$d \;:=\; \sum_{i=1}^{m+n} \chi_{D_i} \; .$$

Then we have $c \in \mathbf{D}(\mathcal{F}_1)$ and $d \in \mathbf{D}(\mathcal{F}_2)$, as well as

$$c + d \;\leq\; n\chi_\Omega \; .$$

For $k \in \{1, \ldots, n\}$, define

$$M_k \;:=\; \{\omega \in \Omega \mid k\chi_\Omega(\omega) \leq c(\omega)\}$$

and

$$N_k \;:=\; \{\omega \in \Omega \mid k\chi_\Omega(\omega) \leq n\chi_\Omega(\omega) - d(\omega)\} \; .$$

Then we have $M_k \in \mathcal{F}_1$ and $N_k \in \mathcal{F}_2$, as well as $M_k \subseteq N_k$, and thus

$$\varphi_1(M_k) \quad \leq \quad \varphi_2(N_k) \; .$$

This inequality together with

$$\sum_{i=1}^{m+n} \chi_{C_i} \quad = \quad c \quad = \quad \sum_{k=1}^{n} \chi_{M_k}$$

and

$$\sum_{k=1}^{n} \chi_{N_k} \quad = \quad n\chi_\Omega - d \quad = \quad n\chi_\Omega - \sum_{i=1}^{m+n} \chi_{D_i}$$

gives

$$\begin{aligned} \sum_{i=1}^{m+n} \varphi_1(C_i) \quad &= \quad \sum_{k=1}^{n} \varphi_1(M_k) \;\leq\; \sum_{k=1}^{n} \varphi_2(N_k) \\ &= \quad n\varphi_2(\Omega) - \sum_{i=1}^{m+n} \varphi_2(D_i) \; , \end{aligned}$$

hence

$$\sum_{i=1}^{m+n} (\varphi_1(C_i) + \varphi_2(D_i)) \quad \leq \quad n\varphi_2(\Omega) \; ,$$

whence

$$\sum_{i=1}^{m+n} \varphi_{j(i)}(B_i) \quad \leq \quad \sum_{i=m+1}^{m+n} \varphi_{j(i)}(\Omega) \; ,$$

and thus

$$\sum_{i=1}^{m} \varphi_{j(i)}(B_i) \quad \leq \quad \sum_{i=m+1}^{m+n} \varphi_{j(i)}(\overline{B_i}) \; ,$$

for all $j(1), \ldots, j(m+n) \in \{1, 2\}$ satisfying $B_i \in \mathcal{F}_{j(i)}$ for all $i \in \{1, \ldots, m+n\}$. Because of Theorem 6.4, (a) implies (b).
The converse is obvious. □

We now record two further applications of Theorem 6.4:

COROLLARY 6.6. *Let $\mathcal{C}$ be a collection of subsets of Ω satisfying $\emptyset, \Omega \in \mathcal{C}$ and let $\zeta : \mathcal{C} \to \mathbf{G}$ be a set function such that the inequality*

$$\sum_{i=1}^{m} \zeta(C_i) \;\leq\; \sum_{i=m+1}^{m+n} \zeta(C_i)$$

holds for all $m, n \in \mathbf{N}$ and $C_1, \ldots, C_{m+n} \in \mathcal{C}$ satisfying $\sum_{i=1}^{m} \chi_{C_i} \leq \sum_{i=m+1}^{m+n} \chi_{C_i}$. Then there exists a positive vector measure $\varphi : 2^{\Omega} \to \mathbf{G}$ satisfying $\varphi(C) = \zeta(C)$ for all $C \in \mathcal{C}$.

Proof. For $C \in \mathcal{C}$, let $\mathcal{F}_C$ denote the algebra generated by C and define a positive vector measure $\varphi_C : \mathcal{F}_C \to \mathbf{G}$ by letting $\varphi_C(C) := \zeta(C)$ and $\varphi_C(\overline{C}) := \zeta(\Omega) - \zeta(C)$; note that the assumption on ζ yields $\varphi_C(\overline{C}) = \zeta(\overline{C})$ for all $C \in \mathcal{C}$ satisfying $\overline{C} \in \mathcal{C}$. Consider $m, n \in \mathbf{N}$ and $B_1, \ldots, B_{m+n} \in \bigcup_{C \in \mathcal{C}} \mathcal{F}_C$ satisfying

$$\sum_{i=1}^{m} \chi_{B_i} \;\leq\; \sum_{i=m+1}^{m+n} \chi_{\overline{B_i}}$$

and hence

$$\sum_{i=1}^{m+n} \chi_{B_i} \;\leq\; n\chi_{\Omega} \, .$$

Relabelling the B_i if necessary, we obtain $B_1, \ldots, B_p, \overline{B_{p+1}}, \ldots, \overline{B_{m+n}} \in \mathcal{C}$ for some $p \in \{0, 1, \ldots, m+n\}$. Then we have

$$\sum_{i=1}^{p} \chi_{B_i} + (m+n-p)\chi_{\Omega} \;\leq\; \sum_{i=p+1}^{m+n} \chi_{\overline{B_i}} + n\chi_{\Omega} \, ,$$

hence

$$\sum_{i=1}^{p} \zeta(B_i) + (m+n-p)\zeta(\Omega) \;\leq\; \sum_{i=p+1}^{m+n} \zeta(\overline{B_i}) + n\zeta(\Omega) \, ,$$

whence

$$\sum_{i=1}^{m+n} \varphi_{C(i)}(B_i) \;\leq\; n\zeta(\Omega) \, ,$$

and thus

$$\sum_{i=1}^{m} \varphi_{C(i)}(B_i) \quad \leq \quad \sum_{i=m+1}^{m+n} \varphi_{C(i)}(\overline{B_i}) \ ,$$

for all $C(1), \ldots, C(m+n) \in \mathcal{C}$ satisfying $B_i \in \mathcal{F}_{C(i)}$ for all $i \in \{1, \ldots, m+n\}$. The assertion now follows from Theorem 6.4. □

In the case $\mathbf{G} = \mathbf{R}$, Corollary 6.6 is due to Horn and Tarski [31]; see also Lembcke [35].

COROLLARY 6.7. *Let $\mathcal{C}$ be a collection of subsets of Ω such that*

$$\Big(\bigcap\nolimits_{D\in\mathcal{D}} D\Big) \cap \Big(\bigcap\nolimits_{E\in\mathcal{E}} \overline{E}\Big) \neq \emptyset$$

holds for any two disjoint finite subcollections $\mathcal{D}$ and $\mathcal{E}$ of $\mathcal{C}$, and let $\zeta : \mathcal{C} \to \mathbf{G}$ be a set function which maps $\mathcal{C}$ into an order bounded subset of $\mathbf{G}_+$. Then there exists a positive vector measure $\varphi : 2^\Omega \to \mathbf{G}$ satisfying $\varphi(C) = \zeta(C)$ for all $C \in \mathcal{C}$.

Proof. Define $g := \sup_\mathcal{C} \zeta(C)$. For $C \in \mathcal{C}$, let $\mathcal{F}_C$ denote the algebra generated by C and define a positive vector measure $\varphi_C : \mathcal{F}_C \to \mathbf{G}$ by letting $\varphi_C(C) := \zeta(C)$ and $\varphi_C(\overline{C}) := g - \zeta(C)$. Consider $m, n \in \mathbf{N}$ and $B_1, \ldots, B_{m+n} \in \bigcup_{C\in\mathcal{C}} \mathcal{F}_C$ satisfying

$$\sum_{i=1}^{m} \chi_{B_i} \quad \leq \quad \sum_{i=m+1}^{m+n} \chi_{\overline{B_i}}$$

and hence

$$\sum_{i=1}^{m+n} \chi_{B_i} \quad \leq \quad n\chi_\Omega .$$

We now reduce the previous inequality by subtracting

$$\begin{aligned} \chi_{B_i} &= 0 && \text{if} \quad B_i = \emptyset \ , \\ \chi_{B_i} &= \chi_\Omega && \text{if} \quad B_i = \Omega \ , \\ \chi_{B_i} + \chi_{B_j} &= \chi_\Omega && \text{if} \quad B_i = \overline{B_j} \ . \end{aligned}$$

Relabelling the B_i if necessary, we obtain $B_1, \dots, B_p, \overline{B_{p+1}}, \dots, \overline{B_{p+q}} \in \mathcal{C}$ and

$$\sum_{i=1}^{p+q} \chi_{B_i} \;\; \leq \;\; k\chi_\Omega \; ,$$

for suitable $p, q, k \in \mathbf{N}_0$ satisfying $p+q \leq m+n$ and $k \leq n$. The assumption on $\mathcal{C}$ yields $(\bigcap_{i=1}^{p} B_i) \cap (\bigcap_{i=p+1}^{p+q} B_i) \neq \emptyset$. Therefore, we have $p+q \leq k$, hence

$$\sum_{i=1}^{p+q} \varphi_{C(i)}(B_i) \;\; \leq \;\; (p+q)\varphi_{C(i)}(\Omega) \;\; \leq \;\; kg \; ,$$

whence, reversing the reduction made before,

$$\sum_{i=1}^{m+n} \varphi_{C(i)}(B_i) \;\; \leq \;\; ng \; ,$$

and thus

$$\sum_{i=1}^{m} \varphi_{C(i)}(B_i) \;\; \leq \;\; \sum_{i=m+1}^{m+n} \varphi_{C(i)}(\overline{B_i}) \; ,$$

for all $C(1), \dots, C(m+n) \in \mathcal{C}$ satisfying $B_i \in \mathcal{F}_{C(i)}$ for all $i \in \{1, \dots, m+n\}$. The assertion now follows from Theorem 6.4. □

In the case $\mathbf{G} = \mathbf{R}$, Corollary 6.7 is due to Marczewski [39], [40]; see also Lembcke [35].

Comments

For more than two positive vector measures, a positive common extension need not exist even if Guy's condition is satisfied for any two of them; see Bhaskara Rao and Bhaskara Rao [8, Example 3.6.3].

Problems

- Find a variant of Guy's condition for arbitrary families of positive vector measures which implies the existence of a positive common extension.
- Extend the results of this section to positive additive functions on commutative minimal clans.

7. Common Extensions of Order Bounded Abstract Measures

The following proposition summarizes two results of Lipecki [36]:

PROPOSITION 7.1. *Let Ω be a set, let $\mathcal{M}$ and $\mathcal{N}$ be algebras of subsets of Ω, and let $\mu : \mathcal{M} \to \mathbf{R}$ and $\nu : \mathcal{N} \to \mathbf{R}$ be bounded additive set functions satisfying $\mu(A) = \nu(A)$ for all $A \in \mathcal{M} \cap \mathcal{N}$. If either*

(a) *$\mathcal{M}$ or $\mathcal{N}$ is finite, or*

(b) *for any two partitions $\mathcal{G} \subseteq \mathcal{M}$ and $\mathcal{H} \subseteq \mathcal{N}$ there exist $G' \in \mathcal{G}$ and $H' \in \mathcal{H}$ satisfying $G' \cap H \neq \emptyset$ for all $H \in \mathcal{H}$ and $G \cap H' \neq \emptyset$ for all $G \in \mathcal{G}$,*

then there exists a bounded additive set function $\varphi : 2^{\Omega} \to \mathbf{R}$ satisfying $\varphi(A) = \mu(A)$ for all $A \in \mathcal{M}$ and $\varphi(A) = \nu(A)$ for all $A \in \mathcal{N}$.

In the present section, we prove a general result on the existence of an order bounded common extension of two vector measures in an order complete Riesz space under a general condition on the algebras which contains the two conditions of Lipecki's results as special cases. As in the case of common extensions of positive vector measures, the proof of this result is based on the representation of vector measures by linear operators and a new result on the existence of an order bounded common extension of two order bounded operators which may be of independent interest.

Throughout this section, let $\mathbf{G}$ be an order complete Riesz space.

Linear Operators

For a vector space $\mathbf{E}$, a map $P : \mathbf{E} \to \mathbf{G}$ is *sublinear* if $P(x + y) \leq P(x) + P(y)$ and $P(\alpha x) = \alpha P(x)$ holds for all $x, y \in \mathbf{E}$ and $\alpha \in \mathbf{R}_+$. The following result is the *Hahn–Banach theorem* for linear operators; for its proof, see Aliprantis and Burkinshaw [2, Theorem 2.1]:

PROPOSITION 7.2. *Let $\mathbf{E}$ be a vector space, let $\mathbf{F}$ be a subspace of $\mathbf{E}$, and let $S : \mathbf{F} \to \mathbf{G}$ be a linear operator. If there exists a sublinear map $P : \mathbf{E} \to \mathbf{G}$ satisfying $Sx \leq P(x)$ for all $x \in \mathbf{F}$, then there exists a linear operator $T : \mathbf{E} \to \mathbf{G}$ satisfying $Tx = Sx$ for all $x \in \mathbf{F}$ and $Tx \leq P(x)$ for all $x \in \mathbf{E}$.*

Let $\mathbf{E}$ be a Riesz space and let $\mathbf{E}_1$ and $\mathbf{E}_2$ be Riesz subspaces of $\mathbf{E}$. If $\mathbf{E}$ has an order unit $e \in \mathbf{E}_+$, then $\mathbf{E}_1$ and $\mathbf{E}_2$ have a *controlling constant* if there exists some $\alpha \in \mathbf{R}_+$ such that for all $x \in \text{span}(\mathbf{E}_1 \cup \mathbf{E}_2)$ satisfying $|x| \leq e$ there exist $x_1 \in \mathbf{E}_1$ and $x_2 \in \mathbf{E}_2$ satisfying $x = x_1 + x_2$ and $|x_1| \vee |x_2| \leq \alpha e$. Two linear operators $T_1 : \mathbf{E}_1 \to \mathbf{G}$ and $T_2 : \mathbf{E}_2 \to \mathbf{G}$ have a *common extension* if there exists a linear operator $T : \mathbf{E} \to \mathbf{G}$ satisfying $Tx = T_1 x$ for all $x \in \mathbf{E}_1$ and $Tx = T_2 x$ for all $x \in \mathbf{E}_2$.

THEOREM 7.3. *Let $\mathbf{E}$ be an Archimedean Riesz space with order unit $e \in \mathbf{E}_+$, let $\mathbf{E}_1$ and $\mathbf{E}_2$ be Riesz subspaces of $\mathbf{E}$ satisfying $e \in \mathbf{E}_1 \cap \mathbf{E}_2$ and having a controlling constant, and let $T_1 : \mathbf{E}_1 \to \mathbf{G}$ and $T_2 : \mathbf{E}_2 \to \mathbf{G}$ be order bounded operators. Then the following are equivalent:*

(a) *The identity $T_1 x = T_2 x$ holds for all $x \in \mathbf{E}_1 \cap \mathbf{E}_2$.*

(b) *T_1 and T_2 have an order bounded common extension $T : \mathbf{E} \to \mathbf{G}$.*

Proof. Assume that (a) holds and define $\mathbf{F} := \text{span}(\mathbf{E}_1 \cup \mathbf{E}_2)$. Then the map $S : \mathbf{F} \to \mathbf{G}$, given by

$$Sx \quad := \quad T_1 x_1 + T_2 x_2$$

for all $x \in \mathbf{F}$ and arbitrary $x_1 \in \mathbf{E}_1$ and $x_2 \in \mathbf{E}_2$ satisfying $x = x_1 + x_2$, is well–defined and linear. Furthermore, since $\mathbf{E}$ is Archimedean, the Minkowski functional $\rho : \mathbf{E} \to \mathbf{R}_+$, given by

$$\rho(x) \quad := \quad \inf \{\lambda \in \mathbf{R}_+ \mid |x| \leq \lambda e\} \ ,$$

satisfies $\rho(x) = 0$ if and only if $x = 0$, as well as $|\rho(x)^{-1} x| \leq e$ for all $x \in \mathbf{E} \backslash \{0\}$. Let $\alpha \in \mathbf{R}_+$ be a controlling constant of $\mathbf{E}_1$ and $\mathbf{E}_2$, and define

$$u \quad := \quad 2\alpha \left(|T_1| e \vee |T_2| e \right) \ .$$

Then the map $P : \mathbf{E} \to \mathbf{G}$, given by

$$P(x) \quad := \quad \rho(x)\, u \ ,$$

is sublinear, and we claim that it also satisfies $Sx \leq P(x)$ for all $x \in \mathbf{F}$.

Indeed, for all $x \in \mathbf{F}$ satisfying $\rho(x) = 1$, we have $|x| \le e$, hence there exist $x_1 \in \mathbf{E}_1$ and $x_2 \in \mathbf{E}_2$ satisfying $x = x_1 + x_2$ and $|x_1| \vee |x_2| \le \alpha e$, and this yields

$$\begin{aligned} Sx &= T_1 x_1 + T_2 x_2 \\ &\le |T_1||x_1| + |T_2||x_2| \\ &\le 2\alpha\,(|T_1|e \vee |T_2|e) \\ &= u \\ &= P(x)\,. \end{aligned}$$

It now follows from Proposition 7.2 that there exists a linear operator $T : \mathbf{E} \to \mathbf{G}$ satisfying $Tx = Sx$ for all $x \in \mathbf{F}$ and $Tx \le P(x)$ for all $x \in \mathbf{E}$. Therefore, we have $Tx = T_1 x$ for all $x \in \mathbf{E}_1$ and $Tx = T_2 x$ for all $x \in \mathbf{E}_2$, as well as $|Tx| \le P(x) \le P(e)$ for all $x \in [-e, e]$, which means that T is order bounded. Therefore, (a) implies (b). The converse is obvious. □

Theorem 7.3 is related to a result of Ptak [44] on common extensions of linear functionals on closed subspaces of a Banach space.

Vector Measures

Let Ω be a nonempty set.

Let $\mathcal{G}$ and $\mathcal{H}$ be partitions of Ω and consider $G, G' \in \mathcal{G}$. For $k \in \mathbf{N}$, a finite sequence $(G_1, \ldots, G_k)$ with $G_i \in \mathcal{G}$ for all $i \in \{1, \ldots, k\}$ is an $(\mathcal{H}, k)$*–bridge*, or simply an $\mathcal{H}$*–bridge*, *from* G *to* G' if

(i) $G = G_1$ and $G_k = G'$,

(ii) $G_i \ne G_j$ holds for all $i, j \in \{1, \ldots, k\}$ satisfying $1 \le |i - j| \le k - 2$, and

(iii) for each $i \in \{1, \ldots, k-1\}$ there exists some $H \in \mathcal{H}$ satisfying $G_i \cap H \ne \emptyset$ and $G_{i+1} \cap H \ne \emptyset$.

The sets G and G' are $(\mathcal{H}, k)$*–equivalent* if there exists an $(\mathcal{H}, k)$–bridge from G to G', and they are $\mathcal{H}$*–equivalent* if they are $(\mathcal{H}, k)$–equivalent for some $k \in \mathbf{N}$; in this case we shall write $G \sim_{\mathcal{H}} G'$.

LEMMA 7.4. *Let* $\mathcal{G}$ *and* $\mathcal{H}$ *be partitions of* Ω. *Then* $\sim_{\mathcal{H}}$ *is an equivalence relation on* $\mathcal{G}$.

Proof. It is immediate from the definitions that the relation $\sim_{\mathcal{H}}$ is reflexive and symmetric. To see that it is also transitive, consider $G, G', G'' \in \mathcal{G}$ satisfying $G \sim_{\mathcal{H}} G'$ and $G' \sim_{\mathcal{H}} G''$. Obviously, $G \sim_{\mathcal{H}} G''$ holds whenever at least two of the sets G, G', G'' are identical. Let us now assume that these sets are all distinct and consider an $\mathcal{H}$–bridge $(G_1, \ldots, G_k)$ from G to G' and an $\mathcal{H}$–bridge $(G'_1, \ldots, G'_{k'})$ from G' to G''. Since $G_k = G' = G'_1$, there exists a smallest $i \in \{1, \ldots, k\}$ satisfying $G_i = G'_j$ for some $j \in \{1, \ldots, k'\}$, and this j is unique since, by assumption, the sets $G'_1, \ldots, G'_k$ are all distinct. Define $k'' := i + k' - j$ and, for all $h \in \{1, \ldots, k''\}$, define

$$G''_h \;:=\; \begin{cases} G_h & \text{if} \quad h \in \{1, \ldots, i\} \\ G'_{h-i+j} & \text{if} \quad h \in \{i+1, \ldots, k''\} \end{cases}$$

Then $(G''_1, \ldots, G''_{k''})$ is an $\mathcal{H}$–bridge from G to G'', and we thus have $G \sim_{\mathcal{H}} G''$. Therefore, $\sim_{\mathcal{H}}$ is also transitive. □

Let $\mathcal{M}$ and $\mathcal{N}$ be algebras of subsets of Ω. The algebras $\mathcal{M}$ and $\mathcal{N}$ are *weakly independent* if for any two partitions $\mathcal{G} \subseteq \mathcal{M}$ and $\mathcal{H} \subseteq \mathcal{N}$ there exist $G' \in \mathcal{G}$ and $H' \in \mathcal{H}$ satisfying $G' \cap H \neq \emptyset$ for all $H \in \mathcal{H}$ and $G \cap H' \neq \emptyset$ for all $G \in \mathcal{G}$; see Lipecki [36]. Two algebras $\mathcal{M}$ and $\mathcal{N}$ have a *bound on bridges* if there exists some $k \in \mathbf{N}$ such that for any two partitions $\mathcal{G} \subseteq \mathcal{M}$ and $\mathcal{H} \subseteq \mathcal{N}$ either any two $\mathcal{H}$–equivalent sets in $\mathcal{G}$ are $(\mathcal{H}, k')$–equivalent for some $k' \in \{1, \ldots, k\}$ or any two $\mathcal{G}$–equivalent sets in $\mathcal{H}$ are $(\mathcal{G}, k')$–equivalent for some $k' \in \{1, \ldots, k\}$.

LEMMA 7.5. *If either*

(a) *$\mathcal{M}$ or $\mathcal{N}$ is finite or*

(b) *$\mathcal{M}$ and $\mathcal{N}$ are weakly independent,*

then $\mathcal{M}$ and $\mathcal{N}$ have a bound on bridges.

The verification of Lemma 7.5 is immediate.

EXAMPLE 7.6: Define $\Omega := (-\mathbf{N}) \cup \mathbf{N}$, let $\mathcal{M}$ denote the smallest algebra in 2^{Ω} containing $-\mathbf{N}$ and all singletons $\{n\}$ with $n \in \mathbf{N}$, and let $\mathcal{N}$ denote the smallest algebra in 2^{Ω} containing $\mathbf{N}$ and all singletons $\{-n\}$ with $n \in \mathbf{N}$. Then the algebras $\mathcal{M}$ and $\mathcal{N}$ are both infinite and they are not weakly independent, but they do have a bound on bridges.

The following lemma is the key to the application of the extension theorem for order bounded operators to order bounded vector measures:

LEMMA 7.7. *If $\mathcal{M}$ and $\mathcal{N}$ have a bound on bridges, then the Riesz subspaces $\mathbf{D}(\mathcal{M})$ and $\mathbf{D}(\mathcal{N})$ of $\mathbf{D}(2^\Omega)$ have a controlling constant.*

Proof. Let $k \in \mathbf{N}$ be a bound on bridges of $\mathcal{M}$ and $\mathcal{N}$, and define $\alpha := 2k-1$. We shall show that α is a controlling constant of $\mathbf{D}(\mathcal{M})$ and $\mathbf{D}(\mathcal{N})$.
Consider $g \in \mathbf{D}(\mathcal{M})$ and $h \in \mathbf{D}(\mathcal{N})$ satisfying $|g+h| \leq \chi_\Omega$, and choose partitions $\mathcal{G} \subseteq \mathcal{M}$ and $\mathcal{H} \subseteq \mathcal{N}$ satisfying

$$g \;=\; \sum_{G \in \mathcal{G}} \gamma_G \chi_G \quad \text{and} \quad h \;=\; \sum_{H \in \mathcal{H}} \eta_H \chi_H$$

for suitable $\gamma_G, \eta_H \in \mathbf{R}$. Without loss of generality, we may assume that any two $\mathcal{H}$–equivalent sets in $\mathcal{G}$ are $(\mathcal{H}, k')$–equivalent for some $k' \in \{1, \ldots, k\}$. Let $\mathcal{G}_1, \ldots, \mathcal{G}_l$ denote the equivalence classes of $\mathcal{G}$ with respect to $\sim_\mathcal{H}$.
For $p \in \{1, \ldots, l\}$, choose $G_p \in \mathcal{G}_p$ and define

$$M_p \;:=\; \bigcup_{G \in \mathcal{G}_p} G \quad \text{and} \quad N_p \;:=\; \bigcup_{H \in \mathcal{H},\, H \cap M_p \neq \emptyset} H\,.$$

Then we have $M_p \subseteq N_p$. Moreover, for $H \in \mathcal{H}$ satisfying $H \cap M_p \neq \emptyset$, there exists some $G' \in \mathcal{G}_p$ satisfying $H \cap G' \neq \emptyset$, and for each $G \in \mathcal{G}$ satisfying $H \cap G \neq \emptyset$ we have $G \sim_\mathcal{H} G'$ and hence $G \in \mathcal{G}_p$, and this yields

$$H \;\subseteq\; \bigcup_{G \in \mathcal{G},\, H \cap G \neq \emptyset} G \;\subseteq\; \bigcup_{G \in \mathcal{G}_p} G \;=\; M_p.$$

Therefore, we also have $N_p \subseteq M_p$, hence $N_p = M_p$, and thus $M_p \in \mathcal{M} \cap \mathcal{N}$.
Define now

$$g' \;:=\; \sum_{G \in \mathcal{G}} \gamma_G \chi_G - \sum_{p=1}^{l} \gamma_{G_p} \chi_{M_p}$$

and

$$h' \ := \ \sum_{H\in\mathcal{H}} \eta_H \chi_H + \sum_{p=1}^{l} \gamma_{G_p} \chi_{M_p} \ .$$

Then we have $g' \in \mathbf{D}(\mathcal{M})$ and $h' \in \mathbf{D}(\mathcal{N})$, as well as $g + h = g' + h'$ and hence

$$|g' + h'| \ \leq \ \chi_\Omega \ .$$

For $G \in \mathcal{G}$, choose $p \in \{1, \ldots, l\}$ such that $G \in \mathcal{G}_p$ and let $(G'_1, \ldots, G'_{k'})$ be an $(\mathcal{H}, k')$–bridge from G to G_p such that $k' \leq k$. For each $i \in \{1, \ldots, k' - 1\}$, there exists some $H \in \mathcal{H}$ satisfying $G'_i \cap H \neq \emptyset$ as well as $H \cap G'_{i+1} \neq \emptyset$, hence $|g'(\omega_i)| \leq |h'(\omega)| + 1$ for all $\omega_i \in G'_i$ and $\omega \in H$ as well as $|h'(\omega)| \leq |g'(\omega_{i+1})| + 1$ for all $\omega \in H$ and $\omega_{i+1} \in G'_{i+1}$, and thus $|g'(\omega_i)| \leq |g'(\omega_{i+1})| + 2$ for all $\omega_i \in G'_i$ and $\omega_{i+1} \in G'_{i+1}$. Since $G = G'_1$, $G'_{k'} = G_p$, and $g'\chi_{G_p} = 0$, this yields $|g'\chi_G| \leq 2(k' - 1)\chi_G$.
Therefore, we have $|g'| \leq 2(k' - 1)\chi_\Omega$, hence $|h'| \leq (2k' - 1)\chi_\Omega$, and thus

$$|g'| \vee |h'| \ \leq \ \alpha\chi_\Omega \ ,$$

which completes the proof. □

The following example shows that the converse of Lemma 7.7 is false:

EXAMPLE 7.8: Define $\Omega := \mathbf{N}$ and $\mathcal{M} = \mathcal{N} := 2^\Omega$. Then $\mathcal{M}$ and $\mathcal{N}$ do not have a bound on bridges, but $\mathbf{D}(\mathcal{M})$ and $\mathbf{D}(\mathcal{N})$ have a controlling constant.

Two vector measures $\mu : \mathcal{M} \to \mathbf{G}$ and $\nu : \mathcal{N} \to \mathbf{G}$ have a *common extension* if there exists a vector measure $\varphi : 2^\Omega \to \mathbf{G}$ satisfying $\varphi(A) = \mu(A)$ for all $A \in \mathcal{M}$ and $\varphi(A) = \nu(A)$ for all $A \in \mathcal{N}$.

THEOREM 7.9. *Assume that* $\mathbf{D}(\mathcal{M})$ *and* $\mathbf{D}(\mathcal{N})$ *have a controlling constant, and let* $\mu : \mathcal{M} \to \mathbf{G}$ *and* $\nu : \mathcal{N} \to \mathbf{G}$ *be order bounded vector measures. Then the following are equivalent:*

(a) *The identity* $\mu(A) = \nu(A)$ *holds for all* $A \in \mathcal{M} \cap \mathcal{N}$.

(b) μ *and* ν *have an order bounded common extension* $\varphi : 2^\Omega \to \mathbf{G}$.

Proof. It is well–known that a vector measure is order bounded if and only if its representing linear operator is order bounded.
Let $T_\mu : \mathbf{D}(\mathcal{M}) \to \mathbf{G}$ and $T_\nu : \mathbf{D}(\mathcal{N}) \to \mathbf{G}$ denote the representing linear operators of μ and ν, respectively. By Theorem 7.3, T_μ and T_ν have an order bounded common extension $T : \mathbf{D}(2^\Omega) \to \mathbf{G}$, and it is then clear that the vector measure $\varphi : 2^\Omega \to \mathbf{G}$, given by $\varphi(A) := T\chi_A$, is an order bounded common extension of μ and ν. □

By Lemmas 7.5 and 7.7 and Example 7.6, Theorem 7.9 unifies and extends the results of Lipecki [36] summarized in Proposition 7.1. It seems to be an open problem to find a condition which is not only sufficient but also necessary for the existence of an order bounded common extension of order bounded vector measures $\mu : \mathcal{M} \to \mathbf{G}$ and $\nu : \mathcal{N} \to \mathbf{G}$.

Comments

One might conjecture that two positive vector measures having an order bounded common extension do have a common extension which is positive. The following example shows that this conjecture is false:
EXAMPLE 7.10: Define $\Omega := \{1,2,3\}$, let $\mathcal{M} := \{\emptyset, \{1\}, \{2,3\}, \Omega\}$ and $\mathcal{N} := \{\emptyset, \{1,2\}, \{3\}, \Omega\}$, and let $\mu : \mathcal{M} \to \mathbf{R}$ and $\nu : \mathcal{N} \to \mathbf{R}$ be the unique additive set functions satisfying $\mu(\{1\}) = \nu(\{3\}) = 2/3$ and $\mu(\{2,3\}) = \nu(\{1,2\}) = 1/3$. Then the algebras $\mathcal{M}$ and $\mathcal{N}$ are finite and weakly independent and hence have a bound on bridges, the additive set functions μ and ν are positive and hence have an order bounded common extension $\varphi : 2^\Omega \to \mathbf{R}$, and no common extension of μ and ν is positive.

Let us finally mention a nice result for the case $\mathbf{G} = \mathbf{R}$:

The vector measures μ and ν have a *bound on chains* if the supremum

$$\sup \textstyle\sum_{i=1}^{n} |\eta_i(C_i) - \eta_{i-1}(C_{i-1})| \,,$$

taken over all increasing sequences $\{C_i\}_{i\in\{0,1,\ldots,n\}} \subseteq \mathcal{M} \cup \mathcal{N}$ satisfying

$C_0 = \emptyset$ and $C_n = \Omega$ and with

$$\eta_i(C_i) \quad := \quad \begin{cases} \mu(C_i) & \text{if} \quad C_i \in \mathcal{M} \\ \nu(C_i) & \text{if} \quad C_i \in \mathcal{N}\,, \end{cases}$$

is finite.

Basile, Bhaskara Rao, and Shortt [5] proved the following result:

PROPOSITION 7.11. *Let $\mu : \mathcal{M} \to \mathbf{R}$ and $\nu : \mathcal{N} \to \mathbf{R}$ be order bounded vector measures such that the identity $\mu(A) = \nu(A)$ holds for all $A \in \mathcal{M} \cap \mathcal{N}$. Then the following are equivalent:*

(a) *μ and ν have a bound on chains.*

(b) *μ and ν have an order bounded common extension $\varphi : 2^\Omega \to \mathbf{R}$.*

This result is an appropriate analogue of Guy's Proposition 6.1 since it characterizes the existence of a common extension of two (real–valued) vector measures in terms of the vector measures and not in terms of the algebras. It is an open problem whether the result can be extended to the general case.

Further results on common extensions of vector measures may be found in the papers by Basile and Bhaskara Rao [10] and by Bhaskara Rao and Shortt [10], [11].

Problems

- Find a condition which is necessary and sufficient for the existence of an order bounded extension $\varphi : 2^\Omega \to \mathbf{G}$ of two order bounded vector measures $\mu : \mathcal{M} \to \mathbf{G}$ and $\nu : \mathcal{N} \to \mathbf{G}$ such that the identity $\mu(A) = \nu(A)$ holds for all $A \in \mathcal{M} \cap \mathcal{N}$.

- Find a condition under which every order bounded common extension $\varphi : 2^\Omega \to \mathbf{G}$ of two positive vector measures $\mu : \mathcal{M} \to \mathbf{G}$ and $\nu : \mathcal{N} \to \mathbf{G}$ is positive.

- Extend the results of this section to more than two linear operators or vector measures.

- Extend the results of this section to order bounded additive functions on commutative minimal clans.

8. Extensions of Abstract Measures

In this section, we study the existence of a modular extension of a modular map from a Boolean subring of a Boolean algebra to a larger Boolean subring or the Boolean algebra itself.

Throughout this section, let $\mathbf{A}$ be a Boolean algebra and let $\mathbf{G}$ be an order complete Riesz space.

The Results

If $\mathbf{B}$ is a Boolean subring of $\mathbf{A}$, then a map $\beta : \mathbf{B} \to \mathbf{G}$ is

– *supermodular*, if it satisfies $\beta(0) = 0$ and if

$$\beta(x \vee y) + \beta(x \wedge y) \quad \geq \quad \beta(x) + \beta(y)$$

holds for all $x, y \in \mathbf{B}$, it is

– *submodular*, if $-\beta$ is supermodular, and it is

– *modular*, if it is supermodular and submodular.

For a map $\beta : \mathbf{B} \to \mathbf{G}$ and a Boolean subring $\mathbf{C}$ of $\mathbf{A}$ satisfying $\mathbf{C} \subseteq \mathbf{B}$, let $\beta|_{\mathbf{C}}$ denote the restriction of β to $\mathbf{C}$.

Let $\mathbf{M}$ and $\mathbf{S}$ be Boolean subrings of $\mathbf{A}$ satisfying $\mathbf{M} \subseteq \mathbf{S}$ and such that $\mathbf{M}$ *majorizes* $\mathbf{S}$ in the sense that for each $x \in \mathbf{S}$ there exists some $y \in \mathbf{M}$ satisfying $x \leq y$. (Note that every Boolean subalgebra of $\mathbf{A}$ majorizes every Boolean subring of $\mathbf{A}$.)

LEMMA 8.1. *Assume that $\varphi : \mathbf{S} \to \mathbf{G}$ is submodular and that $\psi : \mathbf{S} \to \mathbf{G}$ is supermodular such that $\varphi \leq \psi$. If $\varphi|_{\mathbf{M}} = \psi|_{\mathbf{M}}$, then $\varphi = \psi$.*

The proof of Lemma 8.1 is straightforward.

For a map $\mu : \mathbf{M} \to \mathbf{G}$, a map $\sigma : \mathbf{S} \to \mathbf{G}$ is said to be an *extension* of μ to $\mathbf{S}$ if it satisfies $\sigma|_{\mathbf{M}} = \mu$.

The following result characterizes modular extensions of a modular map $\mu : \mathbf{M} \to \mathbf{G}$ as maximal or minimal elements in the collections of all supermodular or submodular extensions of μ, respectively:

THEOREM 8.2. *Assume that $\mu : \mathbf{M} \to \mathbf{G}$ is modular. Then, for $\sigma : \mathbf{S} \to \mathbf{G}$, the following are equivalent:*

(a) *σ is a modular extension of μ.*

(b) *σ is a maximal element in the collection of all supermodular extensions of μ.*
(c) *σ is a minimal element in the collection of all submodular extensions of μ.*

Proof. It is sufficient to prove the equivalence of (a) and (b). To this end, let $\Psi(\mu)$ denote the collection of all supermodular extensions of μ to $\mathbf{S}$.
Assume first that (a) holds. Then we have $\sigma \in \Psi(\mu)$. For each $\psi \in \Psi(\mu)$ satisfying

$$\sigma \leq \psi ,$$

we have

$$\sigma|_{\mathbf{M}} = \psi|_{\mathbf{M}} .$$

Since σ is submodular while ψ is supermodular, Lemma 8.1 yields

$$\sigma = \psi .$$

Therefore, (a) implies (b).
Assume now that (b) holds. For $z \in \mathbf{S}$, define a map $\psi_z : \mathbf{S} \to \mathbf{G}$ by letting

$$\psi_z(x) := \sigma(x \vee z) + \sigma(x \wedge z) - \sigma(z) .$$

If we can show that $\psi_z \in \Psi(\mu)$ holds for all $z \in \mathbf{S}$, then the obvious inequality

$$\sigma \leq \psi_z$$

together with the maximality of $\sigma \in \Psi(\mu)$ yields

$$\sigma = \psi_z$$

for all $z \in \mathbf{S}$, and this implies that σ is modular.
Let us now prove that indeed $\psi_z \in \Psi(\mu)$ holds for all $z \in \mathbf{S}$. Consider $z \in \mathbf{S}$. It is easy to see that ψ_z is supermodular. To see that ψ_z is also an extension of μ, consider $x \in \mathbf{M} \subseteq \mathbf{S}$. Since $x \vee z \in \mathbf{S}$ and since $\mathbf{M}$ majorizes $\mathbf{S}$, there exists some $y \in \mathbf{M}$ satisfying $x \vee z \leq$

y. Then $\{0, y\}$ and $\mathbf{M} \cap [0, y]$ are Boolean subrings of $\mathbf{A}$ satisfying $\{0, y\} \subseteq \mathbf{M} \cap [0, y]$ and such that $\{0, y\}$ majorizes $\mathbf{M} \cap [0, y]$. Because of $\mu = \sigma|_{\mathbf{M}} \leq \psi_z|_{\mathbf{M}}$, we have

$$\mu|_{\mathbf{M}\cap[0,y]} \quad \leq \quad \psi_z|_{\mathbf{M}\cap[0,y]} \ ,$$

and it is easy to see that

$$\mu|_{\{0,y\}} \quad = \quad \psi_z|_{\{0,y\}} \ .$$

Since $\mu|_{\mathbf{M}\cap[0,y]}$ is submodular while $\psi_z|_{\mathbf{M}\cap[0,y]}$ is supermodular, Lemma 8.1 yields

$$\mu|_{\mathbf{M}\cap[0,y]} \quad = \quad \psi_z|_{\mathbf{M}\cap[0,y]} \ ,$$

and this implies

$$\mu(x) \quad = \quad \psi_z(x) \ .$$

Since $x \in \mathbf{M}$ was arbitrary, we see that ψ_z is an extension of μ. We have thus shown that $\psi_z \in \Psi(\mu)$.
Therefore, (b) implies (a). □

The next result characterizes the existence of a modular extension of a modular map $\mu : \mathbf{M} \to \mathbf{G}$ in terms of the existence of a supermodular or submodular extension of μ:

THEOREM 8.3. *Assume that $\mu : \mathbf{M} \to \mathbf{G}$ is modular. Then the following are equivalent*:
(a) *μ has a modular extension to $\mathbf{S}$.*
(b) *μ has a supermodular extension to $\mathbf{S}$.*
(c) *μ has a submodular extension to $\mathbf{S}$.*

Moreover,
(d) *if $\varphi : \mathbf{S} \to \mathbf{G}$ is a supermodular extension of μ, then μ has a modular extension $\sigma : \mathbf{S} \to \mathbf{G}$ such that $\varphi \leq \sigma$, and*
(e) *if $\varphi : \mathbf{S} \to \mathbf{G}$ is a submodular extension of μ, then μ has a modular extension $\sigma : \mathbf{S} \to \mathbf{G}$ such that $\sigma \leq \varphi$.*

Proof. It is sufficient to prove the equivalence of (a) and (b), as well as (d).

Obviously, (a) implies (b).
Assume now that (b) holds and consider a supermodular extension $\varphi : \mathbf{S} \to \mathbf{G}$ of μ. Let $\Psi(\mu)$ denote the collection of all supermodular extensions of μ to $\mathbf{S}$ and define

$$\Psi(\mu, \varphi) \quad := \quad \{\psi \in \Psi(\mu) \mid \varphi \leq \psi\} \; .$$

Using Zorn's lemma, we shall show that $\Psi(\mu, \varphi)$ contains a maximal element σ. Then σ is also a maximal element of $\Psi(\mu)$, and Theorem 8.2 implies that σ is a modular extension of μ; moreover, since $\sigma \in \Psi(\mu, \varphi)$, we have $\varphi \leq \sigma$.
To prove that $\Psi(\mu, \varphi)$ contains a maximal element, consider a chain $\Psi \subseteq \Psi(\mu, \varphi)$.
Consider $x \in \mathbf{S}$. Since $\mathbf{M}$ majorizes $\mathbf{S}$, there exists some $y \in \mathbf{M}$ satisfying $x \leq y$, and this yields, for all $\psi \in \Psi(\mu, \varphi)$,

$$\begin{aligned} \psi(x) + \varphi(y \setminus x) &\leq \psi(x) + \psi(y \setminus x) \\ &\leq \psi(y) \\ &= \mu(y) \; , \end{aligned}$$

and thus

$$\psi(x) \quad \leq \quad \mu(y) - \varphi(y \setminus x) \; .$$

Since $\mathbf{G}$ is order complete, it follows that $\sup_{\Psi} \psi(x)$ exists.
By the previous argument, the map $\varrho : \mathbf{S} \to \mathbf{G}$, given by

$$\varrho(x) \quad := \quad \sup\nolimits_{\Psi} \psi(x) \; ,$$

is well–defined, and it follows from the assumption that Ψ is a chain in $\Psi(\mu, \varphi)$ that ϱ is a supermodular extension of μ satisfying $\varphi \leq \varrho$. This means that ϱ is an upper bound of Ψ in $\Psi(\mu, \varphi)$. Now Zorn's lemma yields the existence of a maximal element in $\Psi(\mu, \varphi)$.
Therefore, (b) implies (a).
The previous part of the proof includes the proof of (d). □

Theorem 8.3 can be specialized to positive modular maps:

COROLLARY 8.4. *Assume that* $\mu : \mathbf{M} \to \mathbf{G}$ *is modular and positive. Then the following are equivalent*:

(a) *μ has a positive modular extension to* $\mathbf{S}$.
(b) *μ has a positive supermodular extension to* $\mathbf{S}$.

Moreover,

(c) *if $\varphi : \mathbf{S} \to \mathbf{G}$ is a positive supermodular extension of μ, then μ has a positive modular extension $\sigma : \mathbf{S} \to \mathbf{G}$ such that $\varphi \leq \sigma$.*

As an application of Theorem 8.3, we obtain a result on the existence of a modular minorant or majorant of a supermodular or submodular map, respectively:

COROLLARY 8.5.

(a) *If $\varphi : \mathbf{A} \to \mathbf{G}$ is supermodular, then there exists a modular $\sigma : \mathbf{A} \to \mathbf{G}$ satisfying $\varphi \leq \sigma$ and $\varphi(1) = \sigma(1)$.*
(b) *If $\varphi : \mathbf{A} \to \mathbf{G}$ is submodular, then there exists a modular $\sigma : \mathbf{A} \to \mathbf{G}$ satisfying $\sigma \leq \varphi$ and $\sigma(1) = \varphi(1)$.*

Moreover, if φ is positive and supermodular, then σ can be chosen to be positive as well.

Proof. Define $\mathbf{M} := \{0, 1\}$ and $\mathbf{S} := \mathbf{A}$, and let $\mu := \varphi|_{\mathbf{M}}$. Now the assertion follows from Theorem 8.3. □

Comments

In economic game theory, the elements of Ω are interpreted as *agents*, the sets from the algebra $\mathcal{F}$ are interpreted as *possible coalitions* among the agents, and the elements of $\mathbf{R}^n$ are interpreted as *commodity bundles.* Consequently, a map $\varphi : \mathcal{F} \to \mathbf{R}^n$ is interpreted as an *income function* such that, for every coalition $A \in \mathcal{F}$, the vector $\varphi(A)$ describes the income of the coalition A; in particular, the *total income* or *wealth* of the economy is $\varphi(\Omega)$. With this interpretation, Corollary 8.5 asserts, in particular, that every positive supermodular income function φ can be replaced by a positive modular income function σ such that the income of every coalition increases (in the weak sense) while the total income of the economy remains unchanged. For a fixed total income and a modular income function, the income of an arbitrary coalition cannot be increased in the strict sense without strictly decreasing the income of at least one other coalition; this means that modular income functions are optimal in the Pareto sense. In other words, Corollary 8.5 extends a classical core theorem; see Delbaen [23] and Kindler [33], [34].

Problems

- The results of this section extend without difficulty to the case where **G** is only a lattice–ordered commutative group. Is it possible to replace **G** by a lattice–ordered commutative semigroup?
- Try to extend the results of this section to the case where the Boolean algebra and its Boolean subrings are replaced by commutative minimal clans and where modular maps are replaced by additive functions. By Lemma 3.6, every additive function on a commutative minimal clan is modular. Some additional effort will probably be needed to characterize additive extensions of additive functions.

References

[1] A. Abian, *Boolean Rings*, Branden Press, Boston, 1976.
[2] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, New York – London, 1985.
[3] M. Anderson and T. Feil, *Lattice–Ordered Groups*, Reidel, Dordrecht – Boston, 1988.
[4] A. Basile and K. P. S. Bhaskara Rao, *Common extensions of group-valued charges*, Boll. Unione Mat. Ital. **5** (1991), 157–162.
[5] A. Basile, K. P. S. Bhaskara Rao, and R. M. Shortt, *Bounded common extensions of charges*, Proc. Amer. Math. Soc. **121** (1994), 137–143.
[6] H. Bauer, *Eine Rieszsche Bandzerlegung im Raum der Bewertungen eines Verbandes*, Sitzungsber. (1953), 89–117.
[7] H. Bauer, *Reguläre und singuläre Abbildungen eines distributiven Verbandes in einen vollständigen Vektorverband, welche der Funktionalgleichung $f(x \vee y) + f(x \wedge y) = f(x) + f(y)$ genügen*, J. Reine Angew. Math. **194** (1955), 141–179.
[8] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges*, Academic Press, New York – London, 1983.
[9] K. P S. Bhaskara Rao and M. Shortt, R., *Group–valued charges: Common extensions and the infinite Chinese*, Proc. Amer. Math. Soc. **113** (1991), 965–972.

[10] K. P. S. Bhaskara Rao and R. M. Shortt, *Common extensions for homomorphisms and group–valued charges*, Preprint (1991).

[11] K. P. S. Bhaskara Rao and R. M. Shortt, *Common extension of a family of group–valued, finitely additive measures*, Coll. Math. **63** (1992), 85–88.

[12] A. Bigard, K. Keimel, and S. Wolfenstein, *Groupes et Anneaux Réticulés*, Lecture Notes in Mathematics, vol. 608, Springer, Berlin – Heidelberg – New York, 1977.

[13] G. Birkhoff, *Lattice-ordered groups*, Ann. of Math. **43** (1942), 298–331.

[14] G. Birkhoff, *Lattice Theory*, (second) revised ed., Amer. Math. Soc., Providence, Rhode Island, 1948.

[15] G. Birkhoff, *Lattice Theory*, third (new) ed., Amer. Math. Soc., Providence, Rhode Island, 1967.

[16] S. Bochner and R. S. Phillips, *Additive set functions and vector lattices*, Ann. of Math. **42** (1941), 316–324.

[17] D. Butnariu, *Additive fuzzy measures and integrals I*, J. Math. Anal. Appl. **93** (1983), 436–452.

[18] D. Butnariu, *Decompositions and range for additive fuzzy measures*, Fuzzy Sets and Systems **10** (1983), 135–155.

[19] D. Butnariu, *Additive fuzzy measures and integrals III*, J. Math. Anal. Appl. **125** (1987), 288–303.

[20] V. Caselles, *A characterization of weakly sequentially complete Banach lattices*, Math. Z. **190** (1985), 379–385.

[21] M. Congost Iglesias, *Medidas y probabilidades en estructuras ordenadas*, Stochastica **5** (1981), 45–68.

[22] C. Constantinescu, *Some properties of spaces of measures*, Atti Sem. Mat. Fis. Univ. Modena **35** (1989), 1–286.

[23] F. Delbaen, *Convex games and extreme points*, J. Math. Anal. Appl. **45** (1974), 210–233.

[24] J. Diestel, *Abstract–valued additive set functions of locally finite variations*, Notices Amer. Math. Soc. **17** (1970), 657.

[25] H. Dinges, *Zur Algebra der Maßtheorie*, Bull. Greek Math. Soc. **19** (1978), 25–97.

[26] B. Faires and T. J. Morrison, *The Jordan decomposition of vector-valued measures*, Proc. Amer. Math. Soc. **60** (1976), 139–143.

[27] G. Fries, *Untersuchung von Submaßen und Inhalten mit Hilfe von FN Topologien*, Dissertation Universität Mannheim (1989).

[28] L. Fuchs, *Teilweise Geordnete Algebraische Strukturen*, Vandenhoeck & Ruprecht, Göttingen, 1966.

[29] D. L. Guy, *Common extensions of finitely additive probability measures*, Portugaliae Math. **20** (1961), 1–5.

[30] P. R. Halmos, *Lectures on Boolean Algebras*, Springer, Berlin – Heidelberg – New York, 1974.
[31] A. Horn and A. Tarski, *Measures in Boolean algebras*, Trans. Amer. Math. Soc. **64** (1948), 467–497.
[32] L. Kantorovich, *Linear operations in semi-ordered spaces I*, Mat. Sbornik (N. S.) **7** (1940), 209–279.
[33] J. Kindler, *Supermodular and tight set functions*, Math. Nachr. **134** (1987), 131–147.
[34] J. Kindler, *The sigma–core of convex games and the problem of measure extension*, Manuscripta Math. **66** (1989), 97–108.
[35] J. Lembcke, *Gemeinsame Urbilder endlich additiver Inhalte*, Math. Ann. **198** (1972), 239–258.
[36] Z. Lipecki, *On common extensions of two quasi-measures*, Czech. Math. J. **36** (1986), 489–494.
[37] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, North Holland, Amsterdam – London, 1971.
[38] D. Maharam, *Consistent extensions of linear functionals and of probability measures*, Proc. Sixth Berkeley Symp. Math. Stat. Probab. (Berkeley), vol. 2, University of California Press, 1972, pp. 127–147.
[39] E. Marczewski, *Indépendance d'ensembles et prolongement de mesures*, Coll. Math. **1** (1947–48), 122–132.
[40] E. Marczewski, *Ensembles indépendants et leurs applications à la théorie de la mesure*, Fund. Math. **35** (1948), 13–28.
[41] T. Ogasawara, *Some general theorems and convergence theorems in vector lattices*, J. Sci. Hiroshima Univ. **14** (1949), 14–25.
[42] P. K. Pavlakos, *The Lebesgue decomposition theorem for partially ordered semigroup-valued measures*, Proc. Amer. Math. Soc. **71** (1978), 207–211.
[43] P. K. Pavlakos, *On the space of lattice semigroup–valued set functions*, Measure Theory (Oberwolfach 1981), Lecture Notes in Mathematics, vol. 945, Springer, Berlin – Heidelberg – New York, 1982, pp. 291–295.
[44] V. Ptak, *Simultaneous extensions of two functionals*, Czech. Math. J. **19** (1969), no. 94, 553–566.
[45] V V. Rama Rao, *On a common abstraction of Boolean rings and lattice ordered groups I*, Monatshefte Math. **73** (1969), 411–421.
[46] F. Riesz, *Sur quelques notions fondamentales dans la théorie générale des opérations linéaires*, Ann. of Math. **41** (1940), 174–206.
[47] H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer, Berlin – Heidelberg – New York, 1974.
[48] A. R. Schep, *Order continuous components of operators and measures*, Proc. Kon. Nederl. Akad. Wetensch. **81** (1978), 110–117.

[49] K. D. SCHMIDT, *A general Jordan decomposition*, Arch. Math. **38** (1982), 556–564.

[50] K. D. SCHMIDT, *On the Jordan decomposition for vector measures*, Probability in Banach Spaces IV (Oberwolfach 1982), Lecture Notes in Mathematics, vol. 990, Springer, Berlin – Heidelberg – New York, 1983, pp. 198–203.

[51] K. D. SCHMIDT, *A common abstraction of Boolean rings and lattice-ordered groups*, Comp. Math. **54** (1985), 51–62.

[52] K. D. SCHMIDT, *Decompositions of vector measures in Riesz spaces and Banach lattices*, Proc. Edinburgh Math. Soc. **29** (1986), 23–39.

[53] K. D. SCHMIDT, *Minimal clans: A class of ordered partial semigroups including Boolean rings and lattice-ordered groups*, Semigroups: Theory and Applications (Oberwolfach 1986, Lecture Notes in Mathematics, vol. 1320, Springer, Berlin – Heidelberg – New York, 1988, pp. 300–341.

[54] K. D. SCHMIDT, *Jordan Decompositions of Generalized Vector Measures*, Pitman Research Notes in Mathematics Series **214** (1989).

[55] K. D. SCHMIDT AND G. WALDSCHAKS, *Common extensions of positive vector measures*, Portugaliae Math. **48** (1991), 155–164.

[56] K. D. SCHMIDT AND G. WALDSCHAKS, *Common extensions of order bounded vector measures*, Measure Theory (Oberwolfach 1990), Rend. Circ. Mat. Palermo Ser. II, vol. 28 Suppl., 1992, pp. 117–124.

[57] R. SIKORSKI, *Boolean Algebras*, 3rd ed., Springer, Berlin – Heidelberg – New York, 1969.

[58] M. H. STONE, *Postulates for Boolean algebras and generalized Boolean algebras*, Amer. J. Math. **57** (1935), 703–732.

[59] M. H. STONE, *The theory of representations for Boolean algebras*, Trans. Amer. Math. Soc. **40** (1936), 37–111.

[60] G. WALDSCHAKS, *Fortsetzungen modularer Abbildungen mit Werten in einer verbandsgeordneten Halbgruppe*, Dissertation Universität Mannheim (1995).

[61] H. WEBER, *Uniform lattices I: A generalization of topological Riesz spaces and topological Boolean rings*, Ann. Mat. Pura Appl. **160** (1991), 347–370.

[62] H. WEBER, *Uniform lattices II: Order continuity and exhaustivity*, Ann. Mat. Pura Appl. **165** (1993), 133–158.

[63] M. A. WOODBURY, *A decomposition theorem for finitely additive set functions. Preliminary report.*, Bull. Amer. Math. Soc. **56** (1950), 171–172.

[64] O. WYLER, *Clans*, Comp. Math. **17** (1966), 172–189.

[65] K. YOSIDA AND E. HEWITT, *Finitely additive measures*, Trans. Amer. Math. Soc. **72** (1952), 46–66.

[66] L. A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338–353.

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