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On Additive Continuous Functions of Figures

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SUMMARY. - *This is an extended summary of results obtained previously by Z. Buczolich and the author [5]. It describes the relationship between derivatives and variational measures of additive continuous functions of figures, and presents a full descriptive definition of a generalized Riemann integral based on figures.*

The set of all real numbers is denoted by $\mathbf{R}$, and the ambient space of this note is $\mathbf{R}^m$ where m is a fixed positive integer. In $\mathbf{R}^m$ we use exclusively the metric induced by the maximum norm $|\cdot|$. The origin of $\mathbf{R}^m$ is denoted by $\mathbf{0}$. For an $x \in \mathbf{R}^m$ and $\varepsilon > 0$, we let

$$U(x,\varepsilon) = \{y \in \mathbf{R}^m : |x-y| < \varepsilon\}$$

and

$$U[x,\varepsilon] = \{y \in \mathbf{R}^m : |x-y| \le \varepsilon\}\,.$$

For $x = (\xi_1,\dots,\xi_m)$ and $y = (\eta_1,\dots,\eta_m)$ in $\mathbf{R}^m$, we set $x \cdot y = \sum_{i=1}^m \xi_i \eta_i$. Note that $|x \cdot y| \le m|x| \cdot |y|$ is the Schwartz inequality with the maximum norm.

The closure, interior, boundary, and diameter of a set $E \subset \mathbf{R}^m$ are denoted by E^-, E°, ∂E, and $d(E)$, respectively. If $A, B \subset \mathbf{R}^m$ and $x \in \mathbf{R}^m$, we let

$$A \triangle B = (A - B) \cup (B - A)$$

and

$$\operatorname{dist}(x, A) = \inf\{|x-y| : y \in A\}\,.$$

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Unless specified otherwise, a number is an extended real number, and a function is an extended real-valued function.

The Lebesgue measure in $\mathbf{R}^m$ is denoted by λ; however, for $E \subset \mathbf{R}^m$, we write $|E|$ instead of $\lambda(E)$. A set $E \subset \mathbf{R}^m$ with $|E| = 0$ is called *negligible*. Sets $A, B \subset \mathbf{R}^m$ are called *nonoverlapping* whenever $A \cap B$ is negligible. Unless specified otherwise, the words "measure" and "measurable" as well as the expresions "almost all," "almost everywhere," and "absolutely continuous" always refer to the Lebesgue measure λ.

The $(m-1)$-dimensional Hausdorff measure in $\mathbf{R}^m$ is denoted by $\mathcal{H}$, and a set $T \subset \mathbf{R}^m$ of σ-finite measure $\mathcal{H}$ is called *thin*. The symbol $\int$ always denotes the Lebesgue integral, with respect to λ or $\mathcal{H}$ as the case may be.

A *cell* is a compact nondegenerate subinterval of $\mathbf{R}^m$, and a *figure* is a finite (possibly empty) union of cells. The family of all figures is denoted by $\mathcal{F}$, and for $A \in \mathcal{F}$, we let $\mathcal{F}_A = \{B \in \mathcal{F} : B \subset A\}$. The *perimeter* and *exterior normal* of a figure A are denoted by $\|A\|$ and ν_A, respectively. Note that $\|A\| = \mathcal{H}(\partial A)$, and that ν_A is defined $\mathcal{H}$-almost everywhere on ∂A. If B and C are figures, then so are $B \cup C$,

$$B \odot C = [(B \cap C)^\circ]^- \qquad \text{and} \qquad B \ominus C = (B - C)^-,$$

and the following inequality holds:

$$\max\{\|B \cup C\|, \|B \odot C\|, \|B \ominus C\|\} \leq \|B\| + \|C\|.$$

The *regularity* of a nonempty figure A is the number

$$r(A) = \frac{|A|}{d(A)\|A\|};$$

if $A = \emptyset$, we let $r(A) = 0$. The usual concept of regularity introduced in [11, Chapter IV, Section 2] is related to $r(A)$ by the inequality $[2mr(A)]^m \leq |A|/[d(A)]^m$ [9, Proposition 12.1.6]. If $r(A) > \eta > 0$, we say the figure A is *η-regular*. A figure C of maximal regularity, i.e., with $r(C) = 1/(2m)$, is a cell called a *cube*.

1. Additive continuous functions

An *additive function* is a real-valued function F defined on the family $\mathcal{F}$ of all figures such that

$$F(B \cup C) = F(B) + F(C)$$

for each pair of nonoverlapping figures B, C.

DEFINITION 1.1. An additive function is *continuous* if given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each figure B with $B \subset U(\mathbf{0}, 1/\varepsilon)$, $\|B\| < 1/\varepsilon$, and $|B| < \eta$.

REMARK 1.2. A distribution function of an additive continuous function is continuous, but the converse is true only in dimension one [10]. Thus it is instructive to describe the topology τ on $\mathcal{F}$ such that additive functions are continuous according to the above definition if and only if they are τ-continuous. On each

$$\mathcal{F}_n = \{B \in \mathcal{F} : B \subset U[\mathbf{0}, n] \text{ and } \|B\| \leq n\}, \quad n = 1, 2, \dots ,$$

define a metric $\rho(B, C) = |B \triangle C|$. Then τ is induced by the largest uniformity ν on $\mathcal{F}$ for which the embeddins $(\mathcal{F}_n, \rho) \hookrightarrow (\mathcal{F}, \nu)$ are uniformly continuous. The topology τ is Hausdorff, separable and sequential, but not metrizable. The sequential completion of $(\mathcal{F}, \nu)$ consists of all bounded Caccioppoli sets [8].

EXAMPLE 1.3: We give two important examples of additive continuous functions.

1. Let $f \in L^1_{\mathrm{loc}}(\mathbf{R}^m, \lambda)$ [6, Section 1.3], and let $F(A) = \int_A f \, d\lambda$ for each figure A. Then F is an additive continuous function by the absolute continuity of the Lebesgue integral.
2. Let v be a continuous vector field on $\mathbf{R}^m$, and let $F(A) = \int_{\partial A} v \cdot \nu_A \, d\mathcal{H}$ for each figure A. Then F is an additive continuous function, called the *flux* of v [9, Proposition 11.2.8].

LEMMA 1.4. *An additive function F is continuous if and only if the following condition is satisfied: given $\varepsilon > 0$, there is a $\theta > 0$ such that*

$$|F(B)| < \theta |B| + \varepsilon(\|B\| + 1)$$

for each figure $B \subset U[\mathbf{0}, \mathbf{1}/\varepsilon]$.

The proof of Lemma 1.4 is not simple; the interested reader is referred to [9, Proposition 12.8.3].

An additive function *in a figure* A is a real valued function defined on the family $\mathcal{F}_A$ of all subfigures of A such that

$$F(B \cup C) = F(B) + F(C)$$

for each pair of nonoverlapping figures $B, C \subset A$. We say that an additive function F in a figure A is *continuous* if given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each figure $B \subset A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta$.

Let F be a function defined on $\mathcal{F}_A$. Setting

$$(F\lfloor A)(B) = F(A \odot B)$$

for each $B \in \mathcal{F}$ defines a function $F\lfloor A$ on $\mathcal{F}$, called the *canonical extension* of F. It follows immediately that $F\lfloor A$ is an additive function if and only if F is an additive function in A. Moreover, since the condition $B \subset U(\mathbf{0}, 1/\varepsilon)$ is satisfied for all $B \in \mathcal{F}_A$ whenever $\varepsilon > 0$ is sufficiently small, we see that for an additive function F in A, the canonical extension $F\lfloor A$ is continuous if and only if F is continuous.

2. Derivatives

Let $x \in \mathbf{R}^m$, and let F be a real-valued function defined on $\mathcal{F}$. For a positive $\eta < 1/(2m)$, set

$$\underline{D}_\eta F(x) = \sup_{\delta>0}\left[\inf_B \frac{F(B)}{|B|}\right] \quad \text{and} \quad \overline{D}_\eta F(x) = \inf_{\delta>0}\left[\sup_B \frac{F(B)}{|B|}\right]$$

where the infimum and supremum in the brackets are taken over all η-regular figures $B \subset U(x, \delta)$ with $x \in B$. The numbers

$$\underline{D}F(x) = \inf_{0<\eta<\frac{1}{2m}} \underline{D}_\eta F(x) \quad \text{and} \quad \overline{D}F(x) = \sup_{0<\eta<\frac{1}{2m}} \overline{D}_\eta F(x)$$

are called, respectively, the *lower* and *upper derivate* of F at x.

Using an argument similar to [11, Chapter IV, Theorem 4.2], it is easy to show that the functions $\underline{D}_\eta F$, $\overline{D}_\eta F$, $\underline{D}F$, and $\overline{D}F$, defined on $\mathbf{R}^m$ in the obvious way, are measurable and the inequality

$$\underline{D}F \le \underline{D}_\eta F \le \underline{D}_\theta F \le \overline{D}_\theta F \le \overline{D}_\eta F \le \overline{D}F$$

holds for all η, θ with $0 < \eta < \theta < 1/(2m)$.

If $\underline{D}F(x) = \overline{D}F(x) \neq \pm\infty$, we denote this common value by $DF(x)$, and say that F is *derivable* at x; the number $DF(x)$ is called the *derivate* of F at x. If $\overline{D}_\eta|F|(x) < +\infty$ for all positive $\eta < 1/(2m)$, we say that F is *almost derivable* at x (cf. [9, Section 11.7]); in particular, F is almost derivable at x whenever $\overline{D}|F|(x) < +\infty$. The term "almost derivable" is justified by the following theorem.

THEOREM 2.1. *Let F be an additive continuous function, and let E be the set of all $x \in \mathbf{R}^m$ at which F is almost derivable. Then F is derivable at almost all $x \in E$.*

In dimenson one, Thorem 2.1 is a consequence of *Ward's theorem* [11, Chapter IV, Theorem 11.15] or *Stepanoff's theorem* [7, Theorem 3.1.9]. In higher dimensions, however, it requires a rather elaborate proof for which we refer to [5, Theorem 3.3].

Let v be a vector field defined on $\mathbf{R}^m$. We say v is *almost differentiable* at $x \in \mathbf{R}^m$ whenever

$$\limsup_{y \to x} \frac{|v(y) - v(x)|}{|y - x|} < +\infty\,.$$

If E is the set of all $x \in \mathbf{R}^m$ at which v is almost differentiable, then v is differentiable almost everywhere in E by *Stepanoff's theorem* [7, Theorem 3.1.9]. Note that E is measurable whenever v is.

EXAMPLE 2.2: Let F be the flux of a continuous vector field v on $\mathbf{R}^m$, and let $x \in \mathbf{R}^m$. The following facts are easy to prove [9, Lemma 11.7.4].

1. If v is almost differentiable at x, then F is almost derivable at x.
2. If v is differentiable at x, then F is derivable at x and $DF(x) = \operatorname{div} v(x)$.

It will be convenient to *relativize* the concept of derivates. Let $A \in \mathcal{F}$, $x \in A$, and let F be a real-valued function on $\mathcal{F}_A$. For a positive $\eta < 1/(2m)$, set

$$\underline{D}_\eta F_A(x) = \sup_{\delta > 0} \inf_B \frac{F(B)}{|B|}$$

where the infimum is taken over all η-regular figures $B \subset A \cap U(x,\delta)$ with $x \in B$. The number

$$\underline{D}F_A(x) = \inf_{0<\eta<\frac{1}{2m}} \underline{D}_\eta F_A(x)$$

is called the *lower derivate* of F at x *relative* to A. The numbers $\overline{D}_\eta F_A(x)$, $\overline{D}F_A(x)$, and $DF_A(x)$ are defined similarly; the meaning of derivability and almost derivability *relative* to A is obvious.

The connection between derivates and relative derivates is simple. Let $A \in \mathcal{F}$, let F be a real-valued function on $\mathcal{F}_A$, and let $F\lfloor A$ be the canonical extension of F defined at the end of Section 1. If $x \in A^\circ$, then

$$\underline{D}_\eta F_A(x) = \underline{D}_\eta (F\lfloor A)(x)$$

for each positive $\eta < 1/(2m)$. Moreover, $D(F\lfloor A)(x) = 0$ for every $x \in \mathbf{R}^m - A$. While there is no obvious relationship between $\underline{D}_\eta F_A(x)$ and $\underline{D}_\eta (F\lfloor A)(x)$ for $x \in \partial A$, this is irrelevant since ∂A is a thin set.

3. Variations

A *partition* is a collection (possibly empty) $P = \{(A_1,x_1),\dots,(A_p,x_p)\}$ where $A_1,\dots,A_p$ are nonoverlapping figures, and $x_i \in A_i$ for $i = 1,\dots,p$. Given a positive $\eta < 1/(2m)$, a set $E \subset \mathbf{R}^m$, and a nonnegative function on E, we say that P is

1. *η-regular* if each A_i is η-regular;
2. *in* E if $\bigcup_{i=1}^p A_i \subset E$;
3. *anchored* in E if $\{x_1,\dots,x_p\} \subset E$;
4. *δ-fine* if it is anchored in E and $d(A_i) < \delta(x_i)$ for $i = 1,\dots,p$.

A nonnegative real-valued function defined on a set $E \subset \mathbf{R}^m$ is called a *gage* or an *essential gage* (abbreviated as *e-gage*) on E whenever its *null set* $N_\delta = \{x \in E : \delta(x) = 0\}$ is thin or negligible, respectively.

LEMMA 3.1. *Let F be an additive continuous function in a figure A, and let δ be a gage on A. For each positive $\varepsilon < 1/(2m)$ there is an ε-regular δ-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A with*

$$|F(A \ominus \bigcup_{i=1}^{p} A_i)| < \varepsilon$$

.

For the proof of Lemma 3.1, which is not trivial, we refer to [9, Proposition 11.3.7 and Lemma 11.3.4].

Let $E \subset \mathbf{R}^m$, and let F be a real-valued function defined on $\mathcal{F}$. Given a positive $\eta < 1/(2m)$ and a nonnegative function δ defined on E, set

$$V_{\eta,\delta}F(E) = \sup_{P} \sum_{i=1}^{p} |F(A_i)|$$

where the supremum is taken over all η-regular partitions $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ anchored in E that are δ-fine. The *variation* or *essential variation* (abbreviated as *e-variation*) of F on E is the number

$$\sup_{0<\eta<\frac{1}{2m}} \inf_{\delta} V_{\eta,\delta}F(E)$$

where the infimum is taken over all gages or e-gages on E, respectively; it is denoted by $V_*F(E)$ or $V_{e*}F(E)$, respectively. An easy verification reveals that the functions

$$V_*F : E \mapsto V_*F(E) \qquad \text{and} \qquad V_{e*}F : E \mapsto V_{e*}F(E)$$

are metric measures in $\mathbf{R}^m$ (cf. [12, Theorem 3.7] and [9, Section 3.2]), and that the measure $V_{e*}F$ is absolutely continuous.

In dimension one, a concept similar to variation has been introduced in [12]. Versions of the e-variation were studied previously in the real line (see [4] and [2]) and in an abstract measure space (see [1]).

PROPOSITION 3.2. *If F is a real-valued function on $\mathcal{F}$, then $V_{e*}F \leq V_*F$ and the equality occurs whenever V_*F is absolutely continuous.*

Proof. As the inequality is obvious, assume V_*F is absolutely continuous. Seeking a contradiction suppose $V_{e*}F(E) < V_*F(E)$ for an

$E \subset \mathbf{R}^m$. There is a positive $\eta < 1/(2m)$ and an e-gage σ on E such that

$$V_{\eta,\sigma}F(E) < c = \inf_{\delta} V_{\eta,\delta}F(E)$$

where the infimum is taken over all gages δ on E. Since the null set N_σ of σ is negligible, $V_*F(N_\sigma) = 0$. Thus given $\varepsilon > 0$, we can find a gage ρ on N_σ so that $V_{\eta,\rho}F(N_\sigma) < \varepsilon$. Define a gage δ on E by setting

$$\delta(x) = \begin{cases} \sigma(x) & \text{if } x \in E - N_\sigma, \\ \rho(x) & \text{if } x \in N_\sigma, \end{cases}$$

and observe that

$$c \le V_{\eta,\delta}F(E) \le V_{\eta,\sigma}F(E - N_\sigma) + V_{\eta,\rho}F(N_\sigma) < V_{\eta,\sigma}F(E) + \varepsilon\,.$$

A contradiction follows from the arbitrariness of ε. □

Let F be a real-valued function defined on $\mathcal{F}$. The *standard variation* of F on a figure A is the number

$$VF(A) = \sup \sum_{k=1}^{n} |F(A_k)|$$

where the supremum is taken over all nonoverlapping collections $\{A_1, \dots, A_n\} \subset \mathcal{F}_A$. If F is additive, a routine argument shows that the function VF, defined on $\mathcal{F}$ in the obvious way, is additive whenever it is real-valued. Note that if F is a real-valued function defined only on $\mathcal{F}_A$, the number $VF(B)$ has still meaning for each figure $B \subset A$, and $(VF)\lfloor A = V(F\lfloor A)$.

LEMMA 3.3. *If F is an additive continuous function, then $V_*F(A) = VF(A)$ for each figure A.*

Proof. Let A be a figure. The function $\delta : x \mapsto \operatorname{dist}(x, \partial A)$ is a gage in A, and every δ-fine partition is a partition in A. Thus $V_{\eta,\delta}F(A) \le VF(A)$ for each positive $\eta < 1/(2m)$, and hence $V_*F(A) \le VF(A)$.

Proceeding towards a contradiction, assume $V_*F(A) < VF(A)$. Then there is a nonoverlapping collection $\{A_1, \dots, A_n\} \subset \mathcal{F}_A$ such that

$$V_*F(A) < \sum_{k=1}^{n} |F(A_k)|\,.$$

Choose a positive $\eta < 1/(2m)$ and find a gage δ on A so that

$$V_{\eta,\delta}F(A) < \sum_{k=1}^{n} |F(A_k)| .$$

Given $\varepsilon > 0$, it follows from Lemma 3.1 that in each A_k there is an η-regular δ-fine partition $P_k = \{(B_1^k, x_1^k), , \dots, (B_{p_k}^k, x_{p_k}^k)\}$ such that

$$\begin{aligned}\sum_{i=1}^{p_k} |F(B_i^k)| \quad &\geq \quad \left| F\left(\bigcup_{i=1}^{p_k} B_i^k\right)\right| = \left|F(A_k) - F\left(A_k \ominus \bigcup_{i=1}^{p_k} B_i^k\right)\right| \\ &> \quad |F(A_k)| - \frac{\varepsilon}{n} .\end{aligned}$$

Since $P = \bigcup_{k=1}^{n} P_k$ is an η-regular δ-fine partition in A,

$$V_{\eta,\delta}F(A) \geq \sum_{k=1}^{n} \sum_{i=1}^{p_k} |F(B_i^k)| > \sum_{k=1}^{n} |F(A_k)| - \varepsilon$$

and a contradiction follows from the arbitrariness of ε. □

EXAMPLE 3.4: Assume $m = 1$. Let C be the Cantor ternary set in $A = [0, 1]$, and let F be an additive continuous function whose distribution function extends the Cantor function in A [9, Example 5.3.11]. Since the function $\delta : x \mapsto \operatorname{dist}(x, C)$ is an e-gage on A and a gage on $A - C$, we have $V_{e*}F(A) = V_*F(A - C) = 0$. On the other hand, it follows from Lemma 3.3 that $V_*F(A) = VF(A) = F(A) = 1$, and so $V_*(C) = 1$.

PROPOSITION 3.5. *If F is a real-valued function on $\mathcal{A}$, then the measures V_*F and $V_{e*}F$ are Borel regular, i.e., the measure of any set $E \subset \mathbf{R}^m$ equals the measure of a Borel set containing E.*

Proof. We prove the lemma for V_*F using the technique of [12, Theorem 3.15]. The proof for $V_{e*}F$ is analogous. Assume $V_*F(E) < +\infty$, choose an $\varepsilon > 0$, and fix a positive $\eta < 1/(2m)$. Find a gage δ on E so that

$$V_{\eta,\delta}F(E) < V_*F(E) + \varepsilon ,$$

and let $E_n = \{x \in E : \delta(x) > 1/n\}$ for $n = 1, 2, \dots$.

We claim $V_{\eta,1/n}F(E_n) = V_{\eta,1/n}F(E_n^-)$. As

$$V_{\eta,1/n}F(E_n) \leq V_{\eta,1/n}F(E_n^-),$$

it suffices to obtain a contradiction by supposing this inequality is sharp. Then there is an η-regular $(1/n)$-fine partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ anchored in E_n^- for which

$$V_{\eta,1/n}F(E_n) < \sum_{i=1}^{p} |F(A_i)|\,.$$

Employing the additivity and continuity of F, it is easy to modify P so that it becomes anchored in E_n and still satisfies the other conditions, a contradiction.

From the claim, we infer

$$\begin{aligned}\inf_{\sigma} V_{\eta,\sigma}F(E_n^-) &\leq V_{\eta,1/n}F(E_n^-) = V_{\eta,1/n}F(E_n) \leq V_{\eta,\delta}F(E_n)\\ &\leq V_{\eta,\delta}F(E) < V_*F(E) + \varepsilon\,,\end{aligned}$$

where the infimum is taken over all gages σ on E. The arbitrariness of η yields $V_*F(E_n^-) \leq V_*F(E) + \varepsilon$, and thus

$$V_*F\left(\bigcup_{n=1}^{\infty} E_n^-\right) = \lim V_*F(E_n^-) \leq V_*F(E) + \varepsilon\,;$$

for $\{E_n^-\}$ is an increasing sequence of closed sets.

Since $E - N_\delta \subset \bigcup_{n=1}^{\infty} E_n^-$, it follows from the arbitrariness of ε that there is a Borel set B such that $E - N_\delta \subset B$ and $V_*F(E) = V_*F(B)$. Now the thin set N_δ is contained in a thin Borel set C [6, Section 2.1, Theorem 1]. As $V_*F(C) = 0$, the lemma is proved. □

Our next result improves on [3, Theorem 3.3]. Its proof is similar to that given in [1, Theorem 1] for an abstract measure space with a derivation base.

THEOREM 3.6. *If F is a real-valued function defined on the family $\mathcal{F}$, then*

$$V_{e*}F(E) = \int_E \overline{D}|F|\,d\lambda$$

for each measurable set $E \subset \mathbf{R}^m$.

Proof. As $V_{e*}F = V_{e*}|F|$, we suppose $F \geq 0$. Select a measurable set $E \subset \mathbf{R}^m$ and note that the integral $I = \int_E \overline{D}F\,d\lambda$ exists (possibly equal to $+\infty$), since $\overline{D}F \geq 0$ is a measurable function.

First we prove the inequality $V_{e*}F(E) \leq I$. If the set $E_\infty = \{x \in E : \overline{D}F(x) = +\infty\}$ has positive measure, then $I = +\infty$ and the inequality holds. If E_∞ is negligible, then $V_{e*}F(E_\infty) = \int_{E_\infty} \overline{D}F \, d\lambda = 0$, and no generality is lost by assuming $E_\infty = \emptyset$. Under this assumption, the measurable sets

$$E_n = \{x \in E \cap U(\mathbf{0}, n) : \overline{D}F(x) < n\}, \;\; n = 1, 2, \dots ,$$

form an increasing sequence whose union is E, and so it suffices to prove the inequality for each E_n.

Consequently, we may assume from the onset $I < +\infty$ and there is an open set $U \subset \mathbf{R}^m$ such that $E \subset U$ and $|U| < +\infty$. Let χ_E be the indicator (characteristic function) of E, and let

$$G(A) = F(A) - \int_A \overline{D}F \cdot \chi_E \, d\lambda = F(A) - \int_{A \cap E} \overline{D}F \, d\lambda$$

for each figure A. Observe the set

$$N = \{x \in E : \overline{D}G(x) \neq 0\}$$

is negligible according to [11, Chapter IV, Theorem 6.3].

Choose an $\varepsilon > 0$ and a positive $\eta < 1/(2m)$, and define an e-gage δ on E as follows: if $x \in N$ let $\delta(x) = 0$, and if $x \in E - N$ select $\delta(x) > 0$ so that $U(x, \delta(x)) \subset U$ and $G(A) < \varepsilon|A|$ for each η-regular figure $A \subset U(x, \delta(x))$ with $x \in A$. Now given an η-regular δ-fine partition $\{(A_1, x_1), \dots , (A_p, x_p)\}$ anchored in E, we obtain

$$\begin{aligned} \sum_{i=1}^{p} F(A_i) &= \sum_{i=1}^{p} \left[G(A_i) + \int_{A_i \cap E} \overline{D}F \, d\lambda \right] \\ &< \sum_{i=1}^{p} \left[\varepsilon|A_i| + \int_{A_i \cap E} \overline{D}F \, d\lambda \right] < \varepsilon|U| + I \end{aligned}$$

and so $V_{\eta,\delta}F(E) \leq \varepsilon|U| + I$. The desired inequality follows from the arbitrarines of η and ε.

Proceeding towards a contradiction, assume $V_{e*}F(E) < I$ and fix an integer $n \geq 1$. For each $x \in E_\infty$ there is a positive $\eta_x < 1/(2m)$ such that given $\theta > 0$, we can find an η_x-regular figure $A \subset U(x, \theta)$

with $x \in A$ and $F(A) > n|A|$. Given an integer $k \geq 1$, let $C_k = \{x \in E_\infty : \eta_x > 1/k\}$, and find an e-gage δ on E_∞ so that

$$V_{1/k,\delta}F(E_\infty) < V_{e*}F(E_\infty) + 1 \leq V_{e*}F(E) + 1 < +\infty\,.$$

The family $\mathcal{C}$ of all $(1/k)$-regular figures A with $d(A) < \delta(x)$ for an $x \in A \cap C_k$ and $F(A) > n|A|$ is a Vitali cover of $C_k - N_\delta$. Using Vitali's covering theorem [11, Chapter IV, Theorem 3.1] and the negligibility of N_δ, find a $(1/k)$-regular δ-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ anchored in C_k such that $F(A_i) > n|A_i|$ for $i = 1, \ldots, p$ and $\sum_{i=1}^{p} |A_i| > |C_k|/2$. It follows

$$\begin{aligned} |C_k| &< 2\sum_{i=1}^{p} |A_i| < \frac{2}{n}\sum_{i=1}^{p} F(A_i) \leq \frac{2}{n} V_{1/k,\delta}F(C_k) \\ &\leq \frac{2}{n} V_{1/k,\delta}F(E_\infty) \leq \frac{2}{n}[V_{e*}F(E) + 1] \end{aligned}$$

and, as $\{C_k\}$ is an increasing sequence whose union is E_∞, we obtain

$$|E_\infty| \leq \frac{2}{n}[V_{e*}F(E) + 1]\,.$$

By the arbitrariness of n, the set E_∞ is negligible. In view of this, we can proceed with the argument assuming the statements made in the third paragraph of this proof, i.e., $I < +\infty$ and there is an open set $U \subset \mathbf{R}^m$ such that $E \subset U$ and $|U| < +\infty$.

Choose a positive $\eta < 1/(2m)$ and find an e-gage δ on E with $V_{\eta,\delta}F(E) < I$. Making δ smaller, we may assume $N \subset N_\delta$ and $U(x, \delta(x)) \subset U$ for each $x \in E$. Given $\varepsilon > 0$, the family $\mathcal{K}$ of all η-regular figures B with $d(B) < \delta(x)$ for an $x \in B \cap E$ and $G(B) > -\varepsilon|B|$ is a Vitali cover of $E - N_\delta$. Hence there is a disjoint sequence $\{B_i\}$ in $\mathcal{K}$ whose union covers E almost entirely. For $i = 1, 2, \ldots$, select an $x_i \in B_i$ so that $d(B_i) < \delta(x_i)$, and observe that for each integer $p \geq 1$, the collection $\{(B_1, x_1), \ldots, (B_p, x_p)\}$ is an η-regular δ-fine partition in U anchored in E. Thus

$$\begin{aligned} I &= \sum_{i=1}^{\infty} \int_{B_i \cap E} \overline{D}F\, d\lambda = \sum_{i=1}^{\infty} [F(B_i) - G(B_i)] \\ &\leq \lim_{p\to\infty} \sum_{i=1}^{p} F(B_i) + \varepsilon \sum_{i=1}^{\infty} |B_i| \leq V_{\eta,\delta}F(E) + \varepsilon|U| \end{aligned}$$

and a contradiction follows from the arbitrariness of ε. □

COROLLARY 3.7. *An additive continuous function F is derivable almost everywhere in a set $E \subset \mathbf{R}^m$ if and only if E has σ-finite measure $V_{e*}F$.*

Proof. Let $E = \bigcup_{n=1}^{\infty} E_n$ and $V_{e*}F(E_n) < +\infty$ for $n = 1, 2, \ldots$. By Proposition 3.5 there are Borel sets B_n such that $E_n \subset B_n$ and $V_{e*}F(E_n) = V_{e*}F(B_n)$. In view of Theorem 3.6, the function F is almost derivable almost everywhere in each B_n. This and Theorem 2.1 imply F is derivable almost everywhere in E.

Conversely, if F is derivable almost everywhere in E then, up to a negligible set, E is contained in the measurable set B of all $x \in \mathbf{R}^m$ at which F is derivable. Clearly, $D|F|(x) = |DF(x)| < +\infty$ for each $x \in B$. Letting

$$B_n = \{x \in B \cap U(\mathbf{0}, n) : D|F(x)| < n\}$$

for $n = 1, 2, \ldots$, Theorem 3.6 yields

$$V_{e*}F(B_n) = \int_{B_n} D|F|(x)\, d\lambda(x) \leq n|U(\mathbf{0}, n)| < +\infty\,.$$

Since $B = \bigcup_{n=0}^{\infty} B_n$ and $V_{e*}F$ is absolutely continuous, the corollary follows. □

PROPOSITION 3.8. *Let T be a thin set, and let F be an additive continuous function almost derivable at each $x \in \mathbf{R}^m - T$. Then V_*F is σ-finite and absolutely continuous.*

Proof. The function F is derivable almost everywhere by Theorem 2.1. In particular, $V_{e*}F$ is σ-finite according to Corollary 3.7.

Now choose a negligible set $E \subset \mathbf{R}^m$ and a positive $\eta < 1/(2m)$. For $n = 1, 2, \ldots$, let

$$E_n = \{x \in E - T : n - 1 \leq \overline{D}_\eta |F|(x) < n\}$$

and find open sets U_n so that $E_n \subset U_n$ and $|U_n| < \eta 2^{-n}/n$. Given $x \in E_n$ there is a $\delta_n(x) > 0$ such that $U(x, \delta_n(x)) \subset U_n$ and $|F(B)| <$

$n|B|$ for every η-regular figure $B \subset U(x, \delta_n(x))$ with $x \in B$. Since $E - T$ is the disjoint union of the sets E_n, the formula

$$\delta(x) = \begin{cases} \delta_n(x) & \text{if } x \in E_n, \\ 0 & \text{if } x \in E \cap T, \end{cases}$$

defines a gage on E. For an η-regular partition $\{(B_1, x_1), \dots, (B_p, x_p)\}$ anchored in E that is δ-fine, we obtain

$$\begin{aligned} \sum_{i=1}^{p} |F(B_i)| &= \sum_{n=1}^{\infty} \sum_{x_i \in E_n} |F(B_i)| < \sum_{n=1}^{\infty} \sum_{x_i \in E_n} n|B_i| \\ &\le \sum_{n=1}^{\infty} n|U_n| < \sum_{n=1}^{\infty} \eta 2^{-n} = \eta\,. \end{aligned}$$

Thus $V_{\eta,\delta}F(E) \le \eta$, and so $V_*F(E) = 0$ by the arbitrariness of η. An application of Proposition 3.2 completes the proof. □

QUESTION 3.9: Let F be an additive continuous function such that V_*F is absolutely continuous. Is it true that V_*F is σ-finite?

OBSERVATION 3.10: An absolutely continuous Borel measure μ in $\mathbf{R}^m$ is σ-finite whenever it is *semi-finite*, i.e., whenever each Borel set A with $0 < \mu(A)$ contains a Borel set B with $0 < \mu(B) < +\infty$.

Proof. By Zorn's lemma, there is a maximal disjoint family $\mathcal{A}$ of Borel sets such that $0 < \mu(A) < +\infty$ for each $A \in \mathcal{A}$. The absolute continuity of μ together with the σ-finiteness of the Lebesgue measure λ imply that $\mathcal{A}$ is a countable family. Since $\mathcal{A}$ is maximal and μ is semi-finite, $\mu(\mathbf{R}^m - \bigcup \mathcal{A}) = 0$ and the observation is proved. □

Observation 3.10 may be helpful in answering Question 3.9. While it is easy to exhibit an absolutely continuous Borel measure μ in $\mathbf{R}^m$ which is not semi-finite [e.g., by letting $\mu = \lim_{n\to\infty}(n\lambda)$], the question is whether such a μ is the restriction of V_*F where F is an aditive continuous function.

EXAMPLE 3.11: Let T be a thin set, and let F be the flux of a continuous vector field v on $\mathbf{R}^m$. If v is almost differentiable at every $x \in \mathbf{R}^m - T$, then V_*F is σ-finite and absolutely continuous (Example 2.2 and Proposition 3.8).

As with the derivates, we *relativize* the concept of variations. Let $A \in \mathcal{F}$, let F be a real-valued function defined on $\mathcal{F}_A$, and let $E \subset A$. Given a positive $\eta < 1/(2m)$ and a nonnegative function δ on E, set

$$V_{\eta,\delta}F_A(E) = \sup_P \sum_{i=1}^{p} |F(A_i)|$$

where the supremum is taken over all η-regular partitions $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ in A anchored in E that are δ-fine. The *variation* of F on E *relative* to A is the number

$$V_* F_A(E) = \sup_{0<\eta<\frac{1}{2m}} \inf_\delta V_{\eta,\delta}F_A(E)$$

where the infimum is taken over all gages δ on E. The *e-variation* $V_{e*}F_A(E)$ of F on E *relative* to A, as well as the measures V_*F_A and $V_{e*}F_A$ in A, are defined in the obvious way.

Let A be a figure, and let F be a real-valued function on $\mathcal{F}_A$. Since the boundary of A is thin and closed, an easy argument shows

$$(V_*F_A)\lfloor A = V_*(F\lfloor A) \qquad \text{and} \qquad (V_{e*}F_A)\lfloor A = V_{e*}(F\lfloor A)\,.$$

From this, Corollary 3.7, and Proposition 3.8, we obtain immediately the following proposition.

PROPOSITION 3.12. *For an additive continuous function F in a figure A the following conditions hold.*

1. *F is derivable relative to A almost everywhere in A if and only if $V_{e*}F_A$ is σ-finite.*
2. *If T is a thin set and F is almost derivable relative to A at every $x \in A^\circ - T$, then V_*F_A is σ-finite and absolutely continuous.*

4. The generalized Riemann integral

DEFINITION 4.1. A real-valued function f defined on a figure A is called *integrable* if there is an additive continuous function F in A satisfying the following condition: given $\varepsilon > 0$, we can find a gage δ on A so that

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon$$

for each ε-regular δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A.

It follows from 3.1 that F, called the *indefinite integral* of f in A, is uniquely determined by f. If f is integrable in A, it is integrable in each figure $B \subset A$, and $F \lceil \mathcal{F}_B$ is the indefinite integral of f in B. The real number $F(A)$ is called the *integral* of f over A, denoted by $\int_A f \, d\lambda$. Since the integral and Lebesgue integral coincide on the intersections of their domains [9, Theorem12.2.2 and 11.4.5], this notation leads to no confusion.

Let A be a figure. As the integral of f over A does not depend on the values f takes in a negligible set [9, Corollary 11.4.7], the concepts of integrability and integral can be readily extended to functions defined almost everywhere in A. We shall assume such an extension has been made, and denote by $\mathcal{R}(A)$ the family of all functions defined almost everywhere in A that are integrable.

PROPOSITION 4.2. *Let $A \in \mathcal{F}$ and $f \in \mathcal{R}(A)$. If F is the indefinite integral of f, then $V_* F_A$ is σ-finite and absolutely continuous.*

Proof. Let $E_n = \{x \in A : |f(x)| \le n\}$ for $n = 1, 2, \dots$, and let E be a negligible subset of A. With no loss of generality, we may assume f is a real-valued function defined on A such that $f(x) = 0$ for each $x \in E$. In particular $A = \bigcup_{n=1}^{\infty} E_n$. Choose a positive $\eta < 1/(2m)$, and find a gage δ on A so that

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \eta$$

for each η-regular δ-fine partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. If P is anchored in E_n, then

$$\sum_{i=1}^{p} |F(A_i)| \le \sum_{i=1}^{p} |f(x_i)| \cdot |A_i| + \eta < n|A| + \eta \,,$$

and hence $V_{\eta,\delta} F_A(E_n) \le n|A| + \eta$. If P is anchored in E, then $\sum_{i=1}^{p} |F(A_i)| < \eta$, and so $V_{\eta,\delta} F_A(E) \le \eta$. From the arbitrariness of η, we conclude

$$V_* F_A(E_n) \le n|A| \qquad \text{and} \qquad V_* F_A(E) = 0 \,,$$

which proves the proposition. □

THEOREM 4.3. *If F is an additive continuous function in a figure A, then the following conditions are equivalent.*

1. *$V_* F_A$ is σ-finite and absolutely continuous.*

2. *DF_A belongs to $\mathcal{R}(A)$, and F is its indefinite integral.*

Proof. As ($2 \Rightarrow 1$) follows immediately from Proposition 4.2, it suffices to prove ($1 \Rightarrow 2$). By Proposition 3.12, the set E of all $x \in A$ at which $DF_A(x)$ does not exists is negligible. We let

$$f(x) = \begin{cases} DF_A(x) & \text{if } x \in A - E, \\ 0 & \text{if } x \in E, \end{cases}$$

and show that F is the indefinite integral of f. To this end, choose a positive $\varepsilon < 1/(2m)$, and find a gage δ_E on E so that $\sum_{j=1}^{q} |F(B_j)| < \varepsilon$ for each ε-regular δ_E-fine partition $\{(B_1, y_1), \dots, (B_q, y_q)\}$ in A anchored in E; such a gage exists, since $V_* F_A(E) = 0$ by our assumptions. On $A - E$ there is a positive function Δ such that

$$\Big| f(x)|B| - F(B) \Big| < \varepsilon |B|$$

for each $x \in A - E$ and each ε-regular figure $B \subset A \cap U(x, \Delta(x))$ with $x \in B$. Now define a gage δ on A by setting

$$\delta(x) = \begin{cases} \Delta(x) & \text{if } x \in A - E, \\ \delta_E(x) & \text{if } x \in E, \end{cases}$$

and select an ε-regular δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. Then

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \sum_{x_i \in E} |F(A_i)| + \varepsilon \sum_{x_i \notin E} |A_i| < \varepsilon (1 + |A|),$$

and the desired conclusion follows. □

Theorem 4.3 gives the full descriptive definition of the integral (cf. [9, Remark 5.3.6]). It facilitates simple proofs of some important results.

Corollary 4.4. *Let $A \in \mathcal{F}$, and let F be the indefinite integral of $f \in \mathcal{R}(A)$. Then $DF_A(x) = f(x)$ for almost all $x \in A$.*

Proof. By Proposition 4.2 and Theorem 4.3, the derivate $DF_A(x)$ exists for almost all $x \in A$, and F is the indefinite integral of DF_A. The corollary follows from [9, Proposition 6.3.7], which assert that two integrable functions with the same indefinite integral are equal almost everywhere. □

Corollary 4.5. *Let T be a thin set, and let F be an additive continuous function in a figure A that is almost derivable relative to A at each $x \in A^\circ - T$. Then DF_A belongs to $\mathcal{R}(A)$ and F is its indefinite integral.*

This corollary follows immediately from Proposition 3.12 and Theorem 4.3. Its immediate consequence is the following *divergence theorem*.

Theorem 4.6. *Let T be a thin set and let v be a continuous vector field on a figure A. If v is almost differentiable at every $x \in A^\circ - T$, then* $\operatorname{div} v$ *belongs to $\mathcal{R}(A)$ and*

$$\int_A \operatorname{div} v \, d\lambda = \int_{\partial A} v \cdot \nu_A \, d\mathcal{H}.$$

Proof. Since v has a continuous extension to $\mathbf{R}^m$, the flux of v is the indefinite integral of $\operatorname{div} v$ according to Example 2.2 and Corollary 4.5. □

The next proposition contrasts the generalized Riemann and Lebesgue integrals (cf. Theorem 4.3).

Proposition 4.7. *If F is an additive continuous function in a figure A, then the following conditions are equivalent.*

1. *$V_* F_A$ is finite and absolutely continuous.*
2. *DF_A belongs to $L^1(A, \lambda)$, and F is its indefinite Lebesgue integral.*

Proof. Note that $L^1(A, \lambda) \subset \mathcal{R}(A)$ and that the indefinite Lebesgue integral of $f \in L^1(A, \lambda)$ is the indefinite integral of f [9, Theorem12.2.2 and 11.4.5].

(1 ⇒ 2) By Theorem 4.3, the derivate DF_A belongs to $\mathcal{R}(A)$, and F is its indefinite integral. As Theorem 3.6 yields the inequality

$$\int_A |DF_A|\,d\lambda = \int_A D|F|_A\,d\lambda = V_{e*}F_A(A) \le V_*F_A(A) < +\infty\,,$$

the derivate DF_A belongs to $L^1(A,\lambda)$ and the impication follows.

(2 ⇒ 1) If $DF_A \in L^1(A,\lambda)$ and F is its indefinite Lebesgue integral, then V_*F_A is absolutely continuous and finite according to Theorems 4.3 and 3.6, respectively. □

ADDED IN PROOF: Question 3.9 has been answered affirmatively by Zoltán Buczolich and the author in their paper *On Assolute Continuity*, J. Math. Anal. Appl., **222** (1998), pp. 64–78.

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