Rend. Istit. Mat. Univ. Trieste
Suppl. Vol. XXIX, 81–113 (1998)

Some important theorems in measure theory

K.P.S. Bhaskara Rao [(*)]

To Dorothy Maharam Stone with deep admiration

Summary. - *In this monograph I shall give several important theorems in measure theory which are not included in any regular graduate/undergraduate courses in measure theory nor are they normally included in standard text books in measure theory. All these theorems are important and have several applications.*

I shall assume that you know some set theory, some Boolean algebras and some functional analysis. You should definitely know some basic measure theory.

Since my aim is to make you familiar with these theorems and their proofs I make no attempt to give the most general versions. Instead, I confine myself to the simplest possible versions without losing the beauty of the proofs.

Contents

(*) Author's address : Indian Statistical Institute, Bangalore

Notation

We shall list some of the notations used in this monograph.

If A is a subset of a set X then I_A stands for the indicator function of A.

If X is a set then $|X|$ stands for the cardinality of X.

ω stands for the first countable ordinal.

If α is an ordinal, $cf(\alpha)$ stands for the cofinality of α.

If A is a subset of a set X then A^c stands for the complement of the set A.

If μ is a finitely additive measure on a Boolean algebra $\mathcal{B}$ and $\mathcal{C} \subset \mathcal{B}$ is a subalgebra of $\mathcal{B}$ then $\mu|_{\mathcal{C}}$ stands for the restriction of μ to $\mathcal{C}$.

If $\mathcal{A}$ is a σ-field of subsets of a set Ω and μ is a nonnegative bounded countably additive measure on $\mathcal{A}$ then we say that $(\Omega, \mathcal{A}, \mu)$ is *complete* if every $A \subset \Omega$ with the property that there is a $B \in \mathcal{A}$ with $\mu(B) = 0$, belongs to $\mathcal{A}$.

If $(\Omega, \mathcal{A}, \mu)$ is a probability measure space and $\mathcal{B} \subset \mathcal{A}$ is a sub σ-field then for any f which is $\mathcal{A}$-integrable, $\mathcal{E}_\mu(f|\mathcal{B})$ stands for the conditional expectation of f given $\mathcal{B}$.

For a prabability measure space $(\Omega, \mathcal{A}, \mu), \mathcal{L}_\infty(\Omega, \mathcal{A}, \mu)$ stands for the pseudo-normed linear space of all μ-essentially bounded $\mathcal{A}$-measurable functions. $L_\infty(\Omega, \mathcal{A}, \mu)$ stands for the normed linear space of all equivalence classes of $\mathcal{L}_\infty(\Omega, \mathcal{A}, \mu)$ and $L_1(\Omega, \mathcal{A}, \mu)$ is also defined in a similar way.

For a probability measure space $(\Omega, \mathcal{A}, \mu)$, if $\mathcal{B}$ and $\mathcal{C}$ are two sub σ-fields of $\mathcal{A}$ we say that $\mathcal{B}$ and $\mathcal{C}$ are independent if $\mu(B \cap C) = \mu(B)\mu(C)$ for all $B \in \mathcal{B}$ and $C \in \mathcal{C}$.

In a Boolean algebra $\vee, \wedge, '$ stand for the sup, inf and complement and 0 and 1 stand for the zero and one of the Boolean algebra.

If $\mathcal{B}$ is a Boolean algebra $\mathcal{B}^+$ stands for all the nonzero elements of $\mathcal{B}$.

If $\mathcal{B}$ is a Boolean algebra and $b \in \mathcal{B}$ then $\mathcal{B}|_b$ stands for the trace of $\mathcal{B}$ on b, namely, $\mathcal{B}|_b = \{a \wedge b : a \in \mathcal{B}\}$.

If $\mathcal{B}$ is a Boolean algebra and $X \subset \mathcal{B}$ then $ba(X)$ stands for the Boolean algebra generated by X.

If $\{\mathcal{B}_i : i \in I\}$ is an indexed set of Boolean algebras then the direct sum $\sum \mathcal{B}_i : i \in I$ is defined as the Boolean algebra whose elements are $\{\{(b_i) : i \in I\} : b_i \in \mathcal{B}_i$ for all $i \in I\}$ and the Boolean operations are defined in the natural way.

If $\mathcal{B}$ is a Boolean algebra and I is an ideal of $\mathcal{B}$, $\mathcal{B}/I$ stands for the quotient Boolean algebra. If $b \in \mathcal{B}$, $[b]$ stands for the element of $\mathcal{B}/I$ containing b.

A Boolean algebra $\mathcal{B}$ is said to satisfy the countable chain condition if every family of nonzero pairwise disjoint elements of $\mathcal{B}$ is at most countable.

If X is a compact topological space, $C(X)$ stands for the Banach space of all real valued continuous functions with the *sup* norm.

If X is normed linear space then the w^*-topology on X^*, the dual of X, is the weak topology induced by X.

1. Liapounoff's Theorem

This Theorem deals with ranges of measures taking values in R^n. We shall prove some versions for finitely additive measures also.

A countably additive probability measure μ defined on a σ-field $\mathcal{A}$ of subsets of a set Ω is said to be nonatomic if for every $A \in \mathcal{A}$, with $\mu(A) > 0$ there is a $B \in \mathcal{A}$ such that $0 < \mu(B) < \mu(A)$.

Let us straightaway prove a one-dimensional version of the Liapounoff's Theorem.

THEOREM 1.1. *If μ is a countably additive nonatomic probability measure defined on a σ-field $\mathcal{A}$ of subsets of a set Ω then the range of $\mu, Ra(\mu) = \{\mu(A) : A \in \mathcal{A}\}$ is the interval* $[0,1]$

Proof. Let $0 < \alpha < 1$. We shall show that there is an $A \in \mathcal{A}$ with $\mu(A) = \alpha$.

Let $\mathcal{S} = \{A \in \mathcal{A} : \mu(A) \leq \alpha\}$. If $A, B \in \mathcal{S}$, let us say that $A \leq B$ if $\mu(A - B) = 0$. In this class of sets every chain $\{A_i : i \in I\}$ has an upper bound. To see this let $\beta = \sup\ \{\mu(A_i) : i \in I\}$ and realize that there are $A_{i_n} \uparrow$ (in the $\leq$ order) such that $\mu(A_{i_n}) \uparrow \beta$. Then take $A_0 = \cup A_{i_n}$. Since μ is countably additive $\mu(A_0) = \beta$. Clearly, for every $i, A_i \leq A_0$ or $A_0 \leq A_i$. Also, clearly, if $A_0 \leq A_i$ and $\mu(A_i - A_0) > 0$ then $\mu(A_i) = \mu(A_0) + \mu(A_i - A_0) > \mu(A_0) = \beta$ contradicting the definition of β. Hence $A_i \leq A_0$ for all $i \in I$. Using the Hausdorff maximality principle, get a maximal element A^* in $\mathcal{S}$. Of course $\mu(A^*) \leq \alpha$. If $\mu(A^*) = \alpha$ we are done.

From the definition of nonatomicity one easily observes that for any given $A \in \mathcal{A}$ with $\mu(A) > 0$ and $\epsilon > 0$ there is a $B \in \mathcal{A}, B \subset A$ such that $0 < \mu(B) < \epsilon$. Now, get a $B \subset A^{*^c}$ such that $0 < \mu(B) < \alpha - \mu(A^*)$ if $\mu(A^*) < \alpha$. Then $\mu(A^*) < \mu(A^* \cup B) < \alpha$ and so $A^* \cup B \in \mathcal{S}$ contradicting the maximality of A^*.

Thus $\mu(A^*) = \alpha$. The theorem is thus proved. □

Professor Pfeffer informs me that it is possible to prove the above theorem using only the countable dependent choice.

From this theorem we shall give an equivalence to the definition of nonatomicity.

THEOREM 1.2. *A countably additive probability measure defined on a σ-field $\mathcal{A}$ of subsets of a set X is nonatomic if and only if it is strongly continuous, i.e. for every $\epsilon > 0$ there is a partition $\{A_1, \cdots A_n\}$ of $\mathcal{A}$ - sets of Ω such that $0 < \mu(A_i) < \epsilon$ for all i.*

Proof. The 'if' part is easy and the 'only if' part follows from Theorem 1.1 □

For a finitely additive strongly continuous measure defined on a σ-field $\mathcal{A}$ also Theorem 1.1 is true. It needs a different proof. (This is not really true. The above proof also works for the following theorem. We shall anyway give a different proof.)

THEOREM 1.3. *If μ is a finitely additive strongly continuous probability measure defined on a σ-field $\mathcal{A}$, then the range of $\mu, Ra(\mu)$ is the interval [0,1].*

Proof. Let $0 < \alpha < 1$. we shall exhibit a set $A \in \mathcal{A}$ such that $\mu(A) = \alpha$.

Let us first construct two sequences of sets $C_1 \subset C_2 \subset C_3 \cdots \subset \cdots D_3 \subset D_2 \subset D_1$ in $\mathcal{A}$ such that $\mu(D_n - C_n) < \frac{1}{2^n}$ and. $\mu(C_n) \leq \alpha < \mu(D_n)$.

For $\epsilon = 1/2$ get a partition $\{A_1 \cdots A_n\}$ such that $0 < \mu(A_i) < 1/2$ for all i. Let C_1 be a largest possible union of these sets such that $\mu(C_1) \leq \alpha$. Let B_1 be any set in this partition which is not in this union and let $D_1 = C_1 \cup B_1$. Then $\mu(C_1) \leq \alpha < \mu(D_1)$ and $\mu(D_1 - C_1) < \frac{1}{2}$.

To define C_2 and D_2, for $\epsilon = \frac{1}{2^2}$ get a partition of B_1 in $\mathcal{A}$ such that each set in the partition has μ-value less than $\frac{1}{2^2}$ and > 0. Let A_2 be a largest possible union of sets from this partition such that $\mu(A_2) \leq \alpha - \mu(C_1)$. Let B_2 be any set in this partition which is not in the union A_2. Let $C_2 = C_1 \cup A_2$ and $D_2 = C_1 \cup A_2 \cup B_2$. Proceeding in this way we get the desired sequences.

Now, since $\mathcal{A}$ is a σ - field, $A = \cup C_n$ will have the property that $A \in \mathcal{A}$ and $\mu(A) = \alpha$, even though μ is not countably additive. □

Later, we shall generalize theorem 1.3.

Theorem 1.2 can be used to prove several results on nonatomic measures.

EXERCISE 1. *Show that a countably additive probability measure μ on a σ-field $\mathcal{A}$ is nonatomic if and only if there is a countably generated sub σ-field $\mathcal{A}_0 \subset \mathcal{A}$ such that μ on $\mathcal{A}_0$ is nonatomic.*

EXERCISE 2. *Show that the product measure $\mu_1 \times \mu_2$ on $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2)$ is nonatomic if and only if either μ_1 or μ_2 is nonatomic.*

EXERCISE 3. *If μ is a measure on $(\Omega, \mathcal{A})$ and μ is nonatomic on some sub σ-field $\mathcal{B} \subset \mathcal{A}$ then μ on $\mathcal{A}$ is also nonatomic.*

The following exercise is of independent interest.

EXERCISE 4. *If $\mathcal{A}$ is countably generated, a μ on $\mathcal{A}$ is nonatomic if and only if $\mu(A) = 0$ for every atom A of $\mathcal{A}$.*

For finitely additive strongly continuous measures defined on a σ-field the following theorem generalizes Theorem 1.3.

THEOREM 1.4. *Let $\mu_1, \mu_2, \cdots, \mu_n$ be finitely additive strongly continuous probabilities defined on a σ-field $\mathcal{A}$ of subsets of a set Ω. Then the Range $Ra(\mu_1, \cdots \mu_n) = \{(\mu_1(A), \cdots \mu_n(A)) : A \in \mathcal{A})\}$ is a convex subset of R^n.*

Proof. We shall prove this by induction on n. For $n = 1$ this is Theorem 1.3. Assume that the result is true for $n = k$ and we shall prove that the result is true for $n = k + 1$.

1. Define $\tau_1 = \mu_1 + \cdots + \mu_{k+1}, \tau_2 = \mu_2 + \cdots + \mu_{k+1}, \cdots \tau_{k+1} = \mu_{k+1}$. Then clearly $Ra(\mu_1 \cdots \mu_{k+1})$ is convex if and only if $Ra(\tau_1 \cdots \tau_{k+1})$

is convex. Though, $\tau_1 \cdots \tau_{k+1}$ are not probabilities, they are still strongly continuous.

2. If we show that for every $A \in \mathcal{A}$ there is a set $B \subset A, B \in \mathcal{A}$ such that $\tau_i(B) = \frac{1}{2}\tau_i(A)$ for $i = 1, 2, \cdots k+1$ then it would follow that $Ra(\tau_1 \cdots \tau_{k+1})$ is convex,because :

For a given $A \in \mathcal{A}$, by repeated application of the above assertion, for every dyadic rational r between 0 and 1 we can find a set A_r such that $\tau_i(A_r) = r\tau_i(A)$ for all $i = 1, 2, \ \cdots k+1,\ A_0 = \phi,\ A_1 = A$ and $A_r \subset A_s$ if $r < s$ are dyadic rationals. For any real number between 0 and 1 if we define $A_a = \bigcup A_r : r < a$ and r dyadic then $A_a \in \mathcal{A}$ because $\mathcal{A}$ is a σ-field and $\tau_i(A_a) = a\tau_i(A)$ for all $0 \leq a \leq 1$ and $i = 1, 2, \cdots k+1$.

Now given C and D in $\mathcal{A}$ and $0 \leq a \leq 1$, $a\tau_i(C) + (1-a)\tau_i(D)) = \tau_i((C-D)_a \bigcup (C \cap D) \bigcup (D-C)_{1-a})$ for every $i = 1, 2, \cdots k+1$. Thus $Ra(\tau_1 \cdots \tau_{k+1})$ is convex.

3. Thus it suffices to exhibit a B in $\mathcal{A}$ for a given A in $\mathcal{A}$ such that $\tau_i(B) = \frac{1}{2}\tau_i(A)$ for $i = 1, 2, \cdots k+1$. Let us do this.

By the induction hypothesis, get a C in $\mathcal{A}, C \subset A$ such that $\tau_i(C) = \frac{1}{2}\tau_i(A)$ for $i = 1, 2, \cdots k$. If we look at the sets $C_a \bigcup (A - C)_{1-a}$ for $0 \leq a \leq 1$ then $\tau_i((C_a \bigcup (A-C)_{1-a}) = a\tau_i(C) + (1-a)\tau_i(A-C) = a\frac{1}{2}\tau_i(A) + (1-a)\frac{1}{2}\tau_i(A) = \frac{1}{2}\tau_i(A)$ for all $i = 1, 2, \cdots k$.

If $\tau_{k+1}(C) = \frac{1}{2}\tau_{k+1}(A)$ we are done. If not, let us assume that $\tau_{k+1}(C) < \frac{1}{2}\tau_{k+1}(A) < \tau_{k+1}(A-C)$. Also since $\tau_{k+1} \leq \tau_k$, because of the way we have defined $\tau_1, \cdots, \tau_{k+1}$, we get that $\tau_{k+1}(C_a - C_b) \leq \tau_k(C_a - C_b) = (a-b)\tau_k(C)$ and so, $\tau_{k+1}(C_a)$ is a continuous function of a . So $\tau_{k+1}(C_a \cup (A-C)_{1-a})$ is also a continuous function of a taking the values $\tau_{k+1}(C)$ at $a = 1$ and $\tau_{k+1}(A-C)$ at $a = 0$

Thus there is an a_0 such that $\tau_{k+1}(C_{a_0} \cup (A - C)_{(1-a_0)}) = \frac{1}{2}\tau_{k+1}(A)$ since $\tau_{k+1}(C) < \frac{1}{2}\tau_{k+1}(A) < \tau_{k+1}(A-C)$.

Thus $B = C_{a_0} \cup (A-C)_{(1-a_0)}$ is the desired set. □

EXERCISE 5. *However, the range in Theorem 1.4 need not be closed. See example 11.4.8 of my book,* Theory of charges.

EXERCISE 6. *Give an example of two finitely additive probability measures μ_1 and μ_2 defined on σ-fields $\mathcal{A}_1$ and $\mathcal{A}_2$ such that*
(a) $Ra(\mu_1, \mu_2) = [0,1] \times [0,1]$
and

(b) another example such that $Ra(\mu_1, \mu_2)$ = *the diagonal of* $[0,1] \times [0,1]$
and
(c) yet another example such that $Ra(\mu_1, \mu_2) = \{(x,y) : 0 < x < 1, 0 < y < 1\} \cup \{(0,0),(1,1)\}$. *[Note that (c) is connected with Exercise 5.]*

Now we shall prove the Liapounoff's Theorem.

THEOREM 1.5. *Let* $\mu_1, \mu_2, \cdots, \mu_n$ *be countably additive nonatomic probability measures defined on a* σ*-field* $\mathcal{A}$ *of subsets of a set* Ω. *Then the range* $Ra(\mu_1, \cdots, \mu_n)$ *is a compact convex subset of* R^n.

Proof. We shall prove this by induction. For $n = 1$ this is really Theorem 1.1 (Alternately, you can adopt the following proof to the case $n = 1$).

For $n \geq 2$, let $\mu = \mu_1 + \cdots + \mu_n$. Equip $L_\infty(\Omega, \mathcal{A}, \mu)$ with the w^*-topology, i.e., the topology induced by $L_1(\Omega, \mathcal{A}, \mu)$. Define $W = \{g : 0 \leq g \leq 1, g \in L_\infty(\Omega, \mathcal{A}, \mu)\}$. Then W is a w^*-compact subset of $L_\infty(\Omega, \mathcal{A}, \mu)$ by the Banach-Alaoglu Theorem.

Also, W is convex, clearly.

On $L_\infty(\Omega, \mathcal{A}, \mu)$ define a mapping into R^n by $T(g) = (\int g d\mu_1 \cdots \int g d\mu_n)$ for $g \in L_\infty(\Omega, \mathcal{A}, \mu)$. Let us see that this T is continuous. If $\{g_\alpha\}$ is a net in $L_\infty(\Omega, \mathcal{A}, \mu)$ such that $g_\alpha \to g$ where $g \in L_\infty(\Omega, \mathcal{A}, \mu)$ then $\int g_\alpha d\mu_i = \int g_\alpha \frac{d\mu_i}{d\mu} d\mu \to \int g \frac{d\mu_i}{d\mu} d\mu = \int g d\mu$ since $\frac{d\mu_i}{d\mu} \in L_1(\Omega, \mathcal{A}, \mu)$.

So $T(W)$ is a compact convex subset of R^n. So, don't you see that we are in business? We got a compact convex set and we want to show that certain set namely $Ra(\mu_1 \cdots \mu_n)$ is compact convex and so let us show that the two sets are the same.

Let $(a_1, a_2, \cdots, a_n) \in T(W)$. We have to exhibit a $D \in \mathcal{A}$ with $T(I_D) = (a_1, a_2, \cdots, a_n)$. Look at $W_0 = \{h \in W : T(h) = (a_1, a_2 \cdots a_n)\}$. By Krein-Milman Theorem, W_0 has extreme points. let g be an extreme point of W_0. We shall prove that $g = I_D$ a.s. (μ) for some $D \in \mathcal{A}$.

Suppose not. Then for some $\epsilon > 0$ $\mu(\{x : \epsilon \leq g(x) \leq 1 - \epsilon\}) > 0$. This implies that for some i, say $i = 1, \mu_1(\{x : \epsilon \leq g(x) \leq 1 - \epsilon\}) > 0$. Denote the set $\{x : \epsilon \leq g(x) \leq 1 - \epsilon\}$ by Z. Choose an $\mathcal{A}$-measurable subset $A \subset Z$ such that $0 < \mu_1(A) < \mu_1(Z)$.

Using the induction hypothesis get sets B and C from $\mathcal{A}$ such that $B \subset A, C \subset Z - A$, $\mu_i(B) = \frac{1}{2}\mu_i(A)$ and $\mu_i(C) = \frac{1}{2}\mu_i(Z - A)$, for $i = 2, 3, \cdots n$. Next get some real numbers s and t such that

$$s(\mu_1(A) - 2\mu_1(B)) + t(\mu_1(Z - A) - 2\mu_1(C)) = 0$$

and $0 < |s| + |t| < \epsilon$.

Let h be the function $= s(I_A - 2I_B) + t(I_{Z-A} - 2I_C)$. Then $\int h d\mu_i = 0$ for all i:- for $i = 2, \cdots n$ because of the choice of B and C and for $i = 1$ because of the choice of s and t. Let us see that $h \neq 0$ a.s. (μ_1). If $h = 0$ a.s. (μ_1), since at least one of s and t is not zero, say that $s \neq 0$, then $\mu_1(A - B) = 0$ and $\mu_1(B) = 0$ which means that $\mu_1(A) = 0$, a contradiction. Also $|h| < \epsilon$.

Now the functions $g + h$ and $g - h \in W$ because $|h| \leq g \leq 1 - |h|$. Also $g + h, g - h \in W_0$ from the above. Since $h \neq 0$ a.s. (μ), we have that $g + h \neq g \neq g - h$ a.s.(μ). Also, $g = \frac{1}{2}(g + h) + \frac{1}{2}(g - h)$. Thus g is not an extreme point of W_0. Hence $g = I_D$ for some $D \in \mathcal{A}$. $\square$

Let us now generalize the concept of nonatomicity and Liapounoff's theorem.

Let μ be a countably additive probability measure on $(\Omega, \mathcal{A})$ and $\mathcal{B} \subset \mathcal{A}$ be a sub σ-field. We say that a set $A \in \mathcal{A}$ is a $(\mathcal{B}, \mathcal{A})$-atom if $A \cap \mathcal{B} = A \cap \mathcal{A}$ a.s. (μ) . We shall say that μ is $(\mathcal{B}, \mathcal{A})$-nonatomic if there are no $(\mathcal{B}, \mathcal{A})$-atoms of positive measure. In an unpublished paper, several years ago, I studied $(\mathcal{B}, \mathcal{A})$ - nonatomicity. For example,

EXERCISE 7. *A measure μ is $(\mathcal{B}, \mathcal{A})$ -nonatomic if and only if there is a countably generated sub σ-field $\mathcal{C}$ of $\mathcal{A}$ such that μ is nonatomic on $\mathcal{C}$ and $\mathcal{B}$ and $\mathcal{C}$ are independent.*

Liapounoff's theorem can be generalized as follows.

EXERCISE 8. *Let $\mu_1, \mu_2 \cdots \mu_n$ be $(\mathcal{B}, \mathcal{A})$-nonatomic countably additive probability measures. Equip $L_\infty(\Omega, \mathcal{B}, \mu_i)$ with the ω^* topology (i.e., the weak topology induced by $L_1(\Omega, \mathcal{B}, \mu_i)$). Then the set*

$$\{(\mathcal{E}_{\mu_1}(A|\mathcal{B}), \mathcal{E}_{\mu_2}(A|\mathcal{B}), \cdots \mathcal{E}_{\mu_n}(A|\mathcal{B})) : A \in \mathcal{A}\}$$

is a compact convex subset of $L_\infty(\Omega, \mathcal{B}, \mu_1) \times L_\infty(\Omega, \mathcal{B}, \mu_2) \times \cdots \times L_\infty(\Omega, \mathcal{B}, \mu_n)$. [Note: Here $\mathcal{E}_\mu(A/\mathcal{B})$ stands for the conditional expectation of A given $\mathcal{B}$].

We need the result of exercise 8 for the case $n = 1$ later. So, let us prove this.

THEOREM 1.6. *Let μ be a $(\mathcal{B}, \mathcal{A})$-nonatomic countably additive probability measure on $(\Omega, \mathcal{A})$. For every $\mathcal{B}$-measurable function f with $0 \leq f \leq 1$ there is an $A \in \mathcal{A}$ such that $\mathcal{E}_\mu(A|\mathcal{B}) = f$.*

Another version is the following : Let μ be a $(\mathcal{B}, \mathcal{A})$-nonatomic countably additive probability measure on $(\Omega, \mathcal{A})$. Let ν be a countably additive measure on $\mathcal{B}$ (not necessarily a probability) such that $0 \leq \nu(B) \leq \mu(B)$ for all $B \in \mathcal{B}$. Then there is an $A \in \mathcal{A}$ such that $\nu(B) = \mu(A \cap B)$ for all $B \in \mathcal{B}$.

Proof. Let us first prove the second version from the first. Let $f = \frac{d\nu}{d\mu}$. Then clearly $0 \leq f \leq 1$. Then by the first version, there is an $A \in \mathcal{A}$ such that $\mathcal{E}_\mu(A|\mathcal{B}) = f$. So, for any $B \in \mathcal{B}, \nu(B) = \int\limits_B f d\mu = \int\limits_B \mathcal{E}_\mu(A|\mathcal{B}) d\mu = \int\limits_B I_A d\mu = \mu(A \cap B)$.

Now we shall prove the first version. The proof follows the same lines as that of Theorem 1.5.

Equip $L_\infty(\Omega, \mathcal{A}, \mu)$ with the w^*-topology. Let $W = \{g \in L_\infty(\Omega, \mathcal{A}, \mu) : 0 \leq g \leq 1\}$. Then W is a compact convex subset of $L_\infty(\Omega, \mathcal{A}, \mu)$.

The mapping $T : L_\infty(\Omega, \mathcal{A}, \mu) \to L_\infty(\Omega, \mathcal{B}, \mu)$ defined by $T(g) = \mathcal{E}_\mu(g|\mathcal{B})$ is a continuous function (here $L_\infty(\Omega, \mathcal{B}, \mu)$ is equipped with its w^* topology). Hence $T(W)$ is compact and convex. Let us show that $T(W) = \{T(I_D) : D \in \mathcal{A}\}$.

Let $h \in T(W)$. We have to exhibit a $D \in \mathcal{A}$ with $T(I_D) = h$. Let $W_0 = T^{-1}(\{h\})$. W_0 is a compact convex subset of W. By Krein-Milman theorem W_0 has extreme points. Let g be an extreme point of W_0. We shall prove that $g = I_D$ a.s. (μ).

Suppose not. Then for some $\epsilon > 0$ $\mu(\{x : \epsilon \leq g(x) \leq 1 - \epsilon\}) > 0$. Denote $\{x : \epsilon \leq g(x) \leq 1 - \epsilon\}$ by Z. Choose an $\mathcal{A}$-measurable subset A of Z such that for every $B \in \mathcal{B}, \mu(A \Delta (B \cap Z)) > 0$ and $\mu((Z - A) \Delta (B \cap Z)) > 0$. Such a choice is possible since Z is not a $(\mathcal{B}, \mathcal{A})$- atom.

Choose and fix bounded versions of $\mathcal{E}_\mu(A|\mathcal{B})$ and $\mathcal{E}_\mu(Z - A|\mathcal{B})$. Let $u = \mathcal{E}_\mu(A|\mathcal{B})$ and $v = \mathcal{E}_\mu(Z - A|\mathcal{B})$.

Let us define two $\mathcal{B}$-measurable functions s and t such that $|s(w)| \leq \epsilon$ and $|t(w)| \leq \epsilon$ for all $w \in \Omega$ and also $su = tv$. This can be done as follows

$s = \epsilon$ and $t = \epsilon$ on $\{u = 0, v = 0\}$
$s = \epsilon$ and $t = 0$ on $\{u = 0, v \neq 0\}$
$s = 0$ and $t = \epsilon$ on $\{u \neq 0, v = 0\}$
$s = t\frac{v}{u}$ and $t = \epsilon$ on $\{0 < |\frac{v}{u}| \leq 1\}$
and $s = \epsilon$ and $t = s\frac{u}{v}$ on $\{|\frac{v}{u}| > 1\}$

Now we define $h = sI_A - tI_{Z-A}$. Then $T(h) = su - tv = 0$ and $|h| \leq g \leq 1 - |h|$ on Ω. Hence $g - h$ and $g + h \in W_0$. Also $h \neq 0$ a.s. (μ) because $\mu(A) > 0$. Hence $g = \frac{1}{2}(g + h) + \frac{1}{2}(g - h)$ and g is not an extreme point of W_0.

Thus the theorem is proved. □

Now that exercise 8 for the case $n = 1$ is proved let us remark that exercise 8 for the general case can be proved by induction. Take $\mu = \mu_1 + \cdots + \mu_n$ and look at the map T from $L_\infty(\Omega, \mathcal{A}, \mu)$ to $L_\infty(\Omega, \mathcal{B}, \mu_1) \times L_\infty(\Omega, \mathcal{B}, \mu_2) \times \cdots \times L_\infty(\Omega, \mathcal{B}, \mu_n)$ defined by $T(g) = (\mathcal{E}_{\mu_1}(g|\mathcal{B}), \cdots \mathcal{E}_{\mu_n}(g/\mathcal{B}))$.

Realize that there exists i, say $i = 1$ such that $\mu_i(Z) > 0$. Then find an $\mathcal{A}$-measurable set $A \subset Z$ as before for μ_1. Now use the induction hypothesis and find $\mathcal{A}$-measurable sets B and C such that $B \subset A, C \subset Z - A$ and for $2 \leq i \leq n, 2\mathcal{E}_{\mu_i}(B|\mathcal{B}) = \mathcal{E}_{\mu_i}(A|\mathcal{B})$ and $2\mathcal{E}_{\mu_i}(C|\mathcal{B}) = \mathcal{E}_{\mu_i}(Z - A|\mathcal{B})$. After fixing some versions of conditional expectations of $\mathcal{E}_{\mu_1}(B|\mathcal{B})$ and $\mathcal{E}_{\mu_1}(C|\mathcal{B})$, define $u = \mathcal{E}_{\mu_1}(A|\mathcal{B}) - 2\mathcal{E}_{\mu_1}(B|\mathcal{B})$ and $v = \mathcal{E}_{\mu_1}(Z - A|\mathcal{B}) - 2\mathcal{E}_{\mu_1}(C|\mathcal{B})$. After defining s and t as in the case $n = 1$ define $h = s(I_A - 2I_B) + t(2I_C - I_{Z-A})$. The rest of the proof is as in the case $n = 1$.

2. Kelley's Theorem

Let us say that a finitely additive measure μ defined on a Boolean algebra $\mathcal{A}$ is strictly positive if $\mu(A) > 0$ whenever $A \neq \phi$ and $A \in \mathcal{A}$. We shall look at the problem of finding necessary and sufficient conditions on a Boolean algebra $\mathcal{A}$ so that there is a strictly positive finitely additive probability measure on $\mathcal{A}$. We shall also look at the existence of strictly positive countably additive measures on complete Boolean algebras.

To study this problem we naturally have to look at properties that can be derived from the existence of a strictly positive finitely additive probability measure μ on a Boolean algebra $\mathcal{A}$. One such property which is useful is given by the following theorem.

THEOREM 2.1. *Let μ be a finitely additive probability measure on a Boolean algebra $\mathcal{A}$. Let $\alpha > 0$. Let $A_1 \cdots A_n$ be sets from $\mathcal{A}$ (possibly with repetitions) such that $\mu(A_i) \geq \alpha$ for all i. Let $i(A_1, \cdots A_n)$ be the largest k such that there exists $i_1 < i_2 < \cdots < i_k$ such that $A_{i_1}, \cap \cdots \cap A_{i_k} \neq \phi$. Then $\frac{i(A_1, \cdots A_n)}{n} \geq \alpha$.*

Proof. This, though looks formidable is quite easy. Look at the function $f = I_{A_1} + \cdots + I_{A_n}$. is a nonnegative integer valued function. Let $S_\ell = \{x : f(x) \geq \ell\}$. Note that $x \in S_\ell$ if and only if there exists $i_1 < i_2 < \cdots < i_\ell$ such that $x \in A_{i_1} \cap \cdots \cap A_{i_\ell}$. Also the largest ℓ for which $S_\ell \neq \phi$ is $\ell = i(A_1, \cdots A_n)$. If we look at the function $g = \sum I_{S_i} : i = 1, \cdots i(A_1, \cdots A_n)$ then $g = f$. This is because S_i's are decreasing and $g(x) = m$ if and only if $x \in S_m$ and $x \notin S_{m+1}$ and this is true if and only if $f(x) = m$. So $\int f d\mu = \int g d\mu$ and so $\sum_{i-1}^{n} \mu(A_i) = \sum \mu(S_i)$ whereas $\sum_{i=1}^{n} \mu(A_i) \geq n\alpha$ and $\sum \mu(S_i) \leq i(A_1 \cdots A_n)$. Hence we have that $\alpha \leq \frac{i(A_1, \cdots A_n)}{n}$. □

If $\mathcal{S} \subset \mathcal{A}^+ (= \{A \in \mathcal{A} : A \notin \phi\})$ let us define the <u>intersection number</u>, $I(\mathcal{S})$ of $\mathcal{S}$ to be $\inf \left\{ \frac{i(A_1, \cdots A_n)}{n} \text{ where } A_1, \cdots A_n \in \mathcal{S} \right\}$. We have proved that if $\mathcal{S} = \{A \in \mathcal{A} : \mu(A) \geq \alpha\}$ then the intersection number of $\mathcal{S}$ is $\geq \alpha$. This is the basic key to Kelley's Theorem.

So, if μ is a strictly positive finitely additive probability measure on a Boolean algebra $\mathcal{A}$ then $\mathcal{A}^+ = \bigcup \mathcal{A}_n$ where $\mathcal{A}_n = \{A \in \mathcal{A}^+ : \mu(A) \geq \frac{1}{n}\}$ and these A_n's have the property that $I(\mathcal{A}_n) \geq \frac{1}{n}$. Thus if $\mathcal{A}$ admits a strictly positive finitely additive probability measure on a Boolean alebra then $\mathcal{A}^+$ can be written as a countable union of subcollections whose intersection numbers are > 0. The converse is the import of Kelley's Theorem.

The basic result is :

THEOREM 2.2. *Let $\mathcal{S}$ be a nonempty subclass of $\mathcal{A}^+$. Then there is a finitely additive probability measure μ on $\mathcal{A}$ such that $\mu(B) \geq I(\mathcal{S})$ for all $B \in \mathcal{S}$.*

Proof. Without loss of generality, using the Stone representaion theorem, assume that $\mathcal{A}$ is the clopen subsets of a compact totally disconnected Hausdorff space X.

Let $C(X)$ be the Banach space of all continuous functions with the supremum norm $\|.\|$. Let us see a relation between the norm and the intersection number.

Let $F = \{I_S : S \in \mathcal{S}\}$ and let G be the convex hull of F. If $g \in G$ we claim that $\|g\| \geq I(\mathcal{S})$. First of all if $g \in G$ is a rational linear combination $\sum_{i=1}^{k} r_i I_{S_i}$ with $\sum r_i = 1$ then in fact, g can be written as $\sum_{i=1}^{k} \frac{n_i}{n} I_{S_i}$ for some integers n_i's and n with $\sum n_i = n$. Then $g = \frac{1}{n} \sum_{i=1}^{n} I_{T_i}$. Where T_i's are only S_j's with each S_j repeated n_j times. The maximum value that g can take is $\frac{i(T_1 \cdots T_n)}{n}$ and $\|g\|$ being this value, $\|g\| \geq I(\mathcal{S})$.

By the same argument as above, note that if $g = \sum_{i=1}^{k} r_i I_{S_i}$ with $\sum_{1}^{k} r_i \geq 1$ then $g = \sum_{i=1}^{k} \frac{n_i}{n} I_{S_i}$ with $\sum_{i=1}^{k} n_i = m \geq n$. So, $\|g\| = \|\frac{1}{n} \sum_{i=1}^{m} I_{T_i}\| = \frac{i(T_1 \cdots T_m)}{n} = \frac{i(T_1 \cdots T_m)}{m} \frac{m}{n} \geq I(\mathcal{S})$.

Now if $g \in G$ is $\sum_{i=1}^{k} t_i I_{S_i}$ with $\sum t_i = 1$ and if $\epsilon > 0$ get rationals $r_i \geq t_i$ for $i = 1, 2, \cdots, k$ such that $|r_i - t_i| < \frac{\epsilon}{k}$ for all i. Let $h = \sum_{i=1}^{k} r_i I_{S_i}$. Then $\|h\| \geq I(\mathcal{S})$ and $\|h - g\| < \epsilon$. So, $I(\mathcal{S}) \leq \|h\| \leq \|h - g\| + \|g\|$. Thus $\|g\| \geq I(\mathcal{S}) - \epsilon$. This being true for every $\epsilon > 0$ we have that $\|g\| \geq I(\mathcal{S})$.

Let H be the open sphere $\{h \in C(X) : \|h\| < I(\mathcal{S})\}$. Let $P = \{p \in C(X) : p \geq 0\}$. Let us show that the function -1 does not belong to the convex set $Q = \{s(g + h) + p : s \geq 0, g \in G, h \in H$ and $p \in P\}$ nor does -1 belong to the closure of Q. We have already shown that if $g \in G$ $g(x_0) = \|g\| \geq I(\mathcal{S})$ at some x_0 and for $h \in H$ $|h(x_0)| \leq I(\mathcal{S})$ and so $(g + h)(x_0) > 0$. And so for any $s > 0$ $(s(g + h) + p)(x_0) > 0$. Thus $-1 \notin Q$. Also, if we take any open sphere of radius $1/4$ around -1 then no element of this sphere belongs to Q. Thus -1 does not belong to the closure of Q. So, by the separation theorem for convex sets there is a nonzero linear functional ϕ such that $\phi(-1) \leq \phi(f)$ for every $f \in Q$.

Since $0 \in Q, \phi(-1) \leq 0$. If $\phi(-1) = 0$ then $\phi(1) = 0$ and also that for every $p \in P$ $\phi(p) \geq 0$. This implies that for $f \geq g$, $\phi(f) \geq \phi(g)$. So for any $f \in C(X)$ since there is a k such that $-k \leq f \leq k$, we have that $0 = \phi(-k) \leq \phi(f) \leq \phi(k) = 0$. Hence $\phi(f) = 0$ for all $f \in C(X)$ which is impossible. Hence $\phi(-1) < 0$. Let us assume without loss of generality that $\phi(-1) = -1$. Now, for every $f \in Q, \phi(f) \geq 0$. Because, if $\phi(f) < 0, \phi(\frac{-2}{\phi(f)} f) = -2$ and $\frac{-2}{\phi(f)} f \in Q$ because $\frac{-2}{\phi(f)}$ is a positive constant. This contradicts that $\phi(-1) \leq \phi(f)$ for all $f \in Q$.

Now if we take a small $\epsilon > 0, I(\mathcal{S}) - \epsilon \in H$ and so also $-I(\mathcal{S}) + \epsilon$. If we take any $g \in G$ we get that $g - I(\mathcal{S}) + \epsilon \in Q$ and so $\phi(g - I(\mathcal{S}) + \epsilon) \geq 0$. Thus $\phi(g) \geq \phi(I(\mathcal{S}) - \epsilon) = I(\mathcal{S}) - \epsilon$ for every $\epsilon > 0$. Hence $\phi(g) \geq I(\mathcal{S})$.

If we define for every $A \in \mathcal{A}, \mu(A) = \phi(I_A)$ then μ is a finitely additive probability measure such that $\mu(A) = \phi(I_A) \geq I(\mathcal{S})$ for every $A \in \mathcal{S}$. □

EXERCISE 9. *Show that Inf*$\{\mu(B) : B \in \mathcal{S}\} = I(\mathcal{S})$ *in the above proof.*

The above theorem helps us give necessary and sufficient conditions for the existence of a strictly positive finitely additive probability measure on a Boolean algebra.

THEOREM 2.3. *There is a strictly positive finitely additive probability measure on a Boolean algebra* $\mathcal{A}$ *if and only if* $\mathcal{A}^+$ *is the union of a countable number of classes* $\mathcal{S}_n$*, each of which has a strictly positive intersection number.*

Proof. The 'only if' part was observed earlier.

For the 'if' part using Theorem 2.2 get finitely additive probability measures μ_n on $\mathcal{A}$ such that $\mu_n(A) \geq I(\mathcal{S}_n)$ for all A in $\mathcal{S}_n$ and for all n. Then $\mu = \sum_{n \geq 1} \frac{1}{2^n} \mu_n$ is a strictly positive finitely additive probability measure on $\mathcal{A}$. □

Now, let us look at the existence of a strictly positive countably additive probability measure on a Boolean σ-algebra. Well, if $\mathcal{A}$ is a Boolean σ-algebra and if μ is a countably additive (even finitely additive) strictly positive probability measure on $\mathcal{A}$ then $\mathcal{A}$ has to be a complete Boolean algebra, because $\mathcal{A}$ satisfies the countable chain condition. So, let us assume that $\mathcal{A}$ is a complete Boolean algebra.

We shall say that a complete Boolean algebra is *weakly countably distributive* (weakly (ω, ω) -distributive in the sense of Sikorski) if for every double sequence $A_{i,j}$ of members of $\mathcal{A}$ such that $A_{i,j+1} \subset A_{i,j}$ for all i and j,

$$\vee\{\wedge\{A_{i,j} : j \geq 1\} i \geq 1\} = \wedge\{\vee\{A_{i,n_i} : i \geq 1\} n_1, n_2, \cdots \geq 1\}$$

The following theorem gives a necessary condition.

THEOREM 2.4. *If there is a strictly, positive countably additive probability measure on a complete Boolean algebra $\mathcal{A}$ then $\mathcal{A}$ is weakly countably distributive.*

Proof. Let $\{A_{i,j} : i \geq 0, j \geq 0\}$ be a double sequence of elements of $\mathcal{A}$ such that $A_{i,j+1} \subset A_{i,j}$ for all i and j. Then clearly

$$\vee\{\wedge\{A_{i,j} : j \geq 1\} i \geq 1\} \leq \wedge\{\vee\{A_{i,n_i} : i \geq 1\}\{n_i\}\}$$

Let C be the left hand side and D be the right hand side. We want to show that $C = D$, consider $D - C = A$. We shall show that $\mu(A) = 0$ if μ is a strictly positive countably additive probability measure on $\mathcal{A}$. This will show that $A = \phi$, i.e., $C = D$.

If possible let $\mu(A) > 0$. Fixing an i, since $A \wedge (\wedge_j A_{i,j}) = 0$, we can get an n_i^0 such that $\mu(A \wedge A_{i,n_i^0}) < \frac{\mu(A)}{2^{i+1}}$. Then $\mu(A \wedge (\vee_i A_{i,n_i^0}) \leq \sum_i \mu(A \wedge A_{i,n_i^0}) < \frac{\mu(A)}{2}$. But since $A \subset D$, we have that $A \subset \vee_i A_{i,n_i^0}$ and so $\mu(A) = \mu(A \wedge (\vee_i A_{i,n_i^0})) < \frac{\mu(A)}{2}$. This is a contradiction. Hence $\mu(A) = 0$. □

The next theorem gives necessary and sufficient conditions.

THEOREM 2.5. *On a complete Boolean algebra $\mathcal{A}$ there is a strictly positive countably additive probability measure if and only if $\mathcal{A}$ is weakly countably distriutive and there is a strictly positive finitely additive probability measure on $\mathcal{A}$.*

Proof. The 'Only if' part is clear from Theorem 2.4.

For the 'if' part, let μ be a strictly positive finitely additive probability measure on $\mathcal{A}$. Then look at the Yosida-Hewitt decomposition of μ as $\mu_c + \mu_p$ where μ_c is countably additive and μ_p is a pure charge as given in Theorem 10.2.1 of *"Theory of Charges"*. Now, by Theorem 10.2.2 of *"Theory of Charges"* again, $\mu_c(A) = \inf \{\lim_{n\to\infty} \mu(A_n) : A_n, n \geq 1 \uparrow A, A_n \in \mathcal{A}, n \geq 1\}$. Let us show that if $A \neq \phi$ then $\mu_c(A) \neq 0$. Suppose that $\mu_c(A) = 0$ and $A \neq \phi$. Then for every $n \geq 0$ there exists $\{B_{n,k} : k \geq 1\} \uparrow A$ as $k \to \infty$ such that $\mu(B_{n,k}) \leq \frac{1}{n}$ for all $k \geq 1$.

Letting $A_{n,k} = A - B_{n,k}$ we have, $A_{n,k} \downarrow \phi$ as $k \to \infty$ and $\mu(A_{n,k}) \geq \mu(A) - \frac{1}{n}$ for all $k \geq 1$. So, $\vee_n \wedge_k A_{n,k} = \phi$. Now if we take any $\{k_n\}, n \geq 1$ then $\mu(A_{n,k_n}) \geq \mu(A) - \frac{1}{n}$ for all n. So $\mu(\vee_{n\geq 1} A_{n,k_n}) = \mu(A)$. Since any way $\vee_{n\geq 1} A_{n,k_n} \leq A$ and since μ is strictly positive we have that $\vee_{n\geq 1} A_{n,k_n} = A$. Since $\{k_n : n \geq 1\}$ is arbitrary, $\wedge\{\vee A_{n,k_n} : n \geq 1\}\{k_n\}\} = A$, contradicting the weak countable distributivity of $\mathcal{A}$. Hence $\mu_c(A) \neq 0$.

Thus μ is a strictly positive countably additive probability measure on $\mathcal{A}$. □

EXERCISE 10. *Find necessary and sufficient conditions on a Boolean algebra $\mathcal{A}$ so that there is a finitely additive bounded signed measure μ on $\mathcal{A}$ such that $\mu(A) \neq 0$ for every non zero $A \in \mathcal{A}$. (Professor Musiał gave me the easy answer to this exercise).*

EXERCISE 11. *Is there a Boolean algebra which does not admit a strictly positive finitely additive probability measure but does admit a strictly nonzero finitely additive bounded signed measure? (If you find the answer to exercise 10 then you can easily find the answer to this exercise also).*

3. Maharam's Theorem on Measure Algebras

If μ is a finitely additive probability on a field $\mathcal{A}$ of subsets of a set Ω then the quotient Boolean algebra $\mathcal{A}/I_\mu$, where I_μ is the ideal of all μ-null sets, i.e., $I_\mu = \{A \in \mathcal{A} : \mu(A) = 0\}$, with the strictly positive finitely additive probability $\tilde{\mu}$ defined by $\tilde{\mu}([A]) = \mu(A)$ is called the measure algebra associated with $(\Omega, \mathcal{A}, \mu)$. Indeed, $\mathcal{A}/I_\mu$

is also a metric space where the distance d is defined by $d([A],[B]) = \mu(A\Delta B)$. Several simple properties of this metric space are listed in the following exrcises. The reference for the following exercises, if necessary, is my paper of 1977 in the list of references.

EXERCISE 12. *If μ is a countably additive probability measure on a σ-field $\mathcal{A}$ of subsets of a set Ω then show that $(\mathcal{A}/I_\mu, d)$ is a complete metric space.*

EXERCISE 13. *Show that $(\mathcal{A}/I_\mu, d)$ is a complete metric space if and only if $\mathcal{A}/I_\mu$ is a complete Boolean algebra and $\tilde{\mu}$ is countably additive on $\mathcal{A}/I_\mu$.*

EXERCISE 14. *If μ is a countably additive probability measure on a σ-field $\mathcal{A}$ of subsets of a set Ω then show that $(\mathcal{A}/I_\mu, d)$ is compact if and only if μ is completely atomic.*

We shall have occasion to invoke this metric space for one of our results.

If μ is a countably additive probability (or nonnegative bounded) measure on a σ-field $\mathcal{A}$ of subsets of a set Ω then the situation becomes interesting and we shall deal with this situation henceforth.

A Boolean algebra $\mathcal{B}$ is said to be complete if for any indexed set $\{b_i : i \in I\}$ of elements from $\mathcal{B}, \vee\{b_i : i \in I\}$ exists.

THEOREM 3.1. *If μ is a countably additive probability measure on a σ-field $\mathcal{A}$ of subsets of a set Ω then the Boolean algebra $\mathcal{A}/I_\mu$ is a complete Boolean algebra and $\tilde{\mu}$ is countably additive.*

Proof. This is really part of exercises 12 and 13. It can be easily shown that I_μ is a σ-ideal in $\mathcal{A}$ and that A/I_μ is a Boolean σ-algebra. It can also be shown that $\mathcal{A}/I_\mu$ satisfies the countable chain condition. Hence $\mathcal{A}/I_\mu$ is a complete Boolean algebra. Proving that $\tilde{\mu}$ is countably additive is not difficult. □

We wish to determine the structure of measure algebras associated with countably additive probability measure spaces. Maharam's Theorem says that all such measure algebras can be built up from some standard simple measure algebras.

Let us call $(\mathcal{A}, \mu)$ a <u>measure algebra</u> if $\mathcal{A}$ is a complete Boolean algebra and μ is a countably additive nonnegative bounded strictly positive measure on $\mathcal{A}$.

EXERCISE 15. *If* $(\mathcal{A}, \mu)$ *is a measure algebra show that the corresponding metric space* $(\mathcal{A}, d)$ *is complete. This can be seen either directly or by using Loomis-Sikorski theorem and exercise 12.*

Now, we need some notions and results from Boolean algebras.

Let $\mathcal{B}$ be a complete Boolean algebra. A subalgebra $\mathcal{C} \subset \mathcal{B}$ is called a complete subalgebra of $\mathcal{B}$ if for every indexed set $\{c_i : i \in I\}$ of elements from $\mathcal{C}$, $\vee\{c_i : i \in I\}$ which exists in $\mathcal{B}$ belongs to $\mathcal{C}$. Of course, for every subset X of a complete Boolean algebra there is a smallest complete subalgebra containing X and we shall call this the complete subalgebra generated by X. We shall start with a small result.

THEOREM 3.2. *If* $\mathcal{A}$ *is a complete Boolean algebra,* $\mathcal{C}$ *is a complete subalgebra of* $\mathcal{A}$ *and* $a \in \mathcal{A}$ *then the complete subalgebra generated by* $\mathcal{C} \cup \{a\}$ *is the Boolean algebra generated by* $\mathcal{C} \cup \{a\}$.

Proof. The Boolean algebra generated by $\mathcal{C} \cup \{a\}$ is simply $\{(c \wedge a) \vee (d \wedge a') : c, d \in \mathcal{C}\}$. Let us show that this is a complete subalgebra of $\mathcal{A}$.

Let $\{(c_i \wedge a) \vee (d_i \wedge a') : i \in I\}$ be an indexed set of elements from this Boolean algebra. Then the $\vee\{c_i \wedge a : i \in I\}$ in $\mathcal{A}$ is simply equal to $(\vee\{c_i : i \in I\} \wedge a)$. Since this is slightly tricky, let us show this.

Let $d = \sup\{c_i \wedge a : i \in I\}$ and $e = \sup\{c_i : i \in I\}$. Then clearly $d \leq e \wedge a$. Now, since $c_i \wedge a \leq d$ for all $i \in I$, we have $c_i' \vee a' \geq d'$ for all $i \in I$. Hence $a \wedge d' \leq a \wedge c_i' \leq c_i'$ for all $i \in I$. So $a \wedge d' \leq \wedge c_i' = e'$. So, $e \leq a' \vee d$. Hence $e \wedge a \leq d \wedge a \leq d$. Thus $d = e \wedge a$.

By using a similar argument for the $\{d_i \wedge a' : i \in I\}$ we conclude that $\vee\{(c_i \wedge a) \vee (d_i \wedge a') : i \in I\}$ in $\mathcal{A}$ belongs to the algebra generated by $\mathcal{C} \cup \{a\}$. □

Another result we require is about the interplay between the metric and the complete subalgebras.

THEOREM 3.3. *Let* $(\mathcal{A}, \mu)$ *be a measure algebra and* d *be the metic induced by* μ.

a) If $\mathcal{C} \subset \mathcal{A}$ is a subalgebra such that $\mathcal{C} = \overline{\mathcal{C}}^d$ then $\mathcal{C}$ is a complete subalgebra.

b) If $\mathcal{C} \subset \mathcal{A}$ is a complete subalgebra then $\mathcal{C}$ is closed in $(\mathcal{A}, d)$.

c) If $\mathcal{B} \subset \mathcal{A}$ is a subalgebra then the complete subalgebra generated by $\mathcal{B}$ is $\overline{\mathcal{B}}^d$.

d) If $(\mathcal{B}, \nu)$ is another measure algebra, $\mathcal{A}' \subset \mathcal{A}, \mathcal{B}' \subset \mathcal{B}$ are subalgebras and if $\phi : \mathcal{A}' \to \mathcal{B}'$ is a measure preserving isomorphism then it can be extended as a measure preserving isomorphism to the complete subalgebra $\mathcal{A}''$ generated by $\mathcal{A}'$ and $\mathcal{B}''$ generated by $\mathcal{B}'$.

Proof. a) Let $\{c_i : i \in I\}$ be elements from $\mathcal{C}$. For any finite subset F of I define $c_F = \vee\{c_i : i \in F\}$. Then $\vee\{c_i : i \in I\}$ in $\mathcal{A}$ is same as $\vee\{c_F : F \text{ finite } \subset I\}$ in $\mathcal{A}$. Let $c_0 = \vee\{c_F : F \text{ finite } \subset I\}$ in $\mathcal{A}$.

Let sup $\{\mu(c_F) : F \text{ finite } \subset I\} = \alpha$. Get finite sets $F_1 \subset F_2 \subset \cdots$, all subsets of I such that $\mu(c_{F_n}) \to \alpha$. Then $d(c_{F_n}, c_{F_m}) = \mu(c_{F_n} \Delta c_{F_m}) = |\mu(c_{F_n}) - \mu(c_{F_m})| \to 0$ as $n, m \to \infty$.

Hence $\{c_{F_n} : n \geq 1\}$ is a Cauchy sequence in the metric space $(\mathcal{C}, d)$. Since $(\mathcal{A}, d)$ is a complete metric space $\mathcal{C}$ being a closed subset of $\mathcal{A}, (\mathcal{C}, d)$ is also a complete metric space. So there is a $c_1 \in \mathcal{C}$ such that $\mu(c_{F_n} \Delta c_1) \to 0$ as $n \to \infty$. We assert that, $c_{F_n} \subset c_1$ for all n. This is true because c_{F_n}'s are increasing and $\mu(c_{F_n} - c_1) \leq \mu(c_{F_n} \Delta c_1) \to 0$. If there is an n_0 such that $\mu(c_{F_{n_0}} - c_1) > 0$, then for all $n \geq n_0$ $\mu(c_{F_n} - c_1) \geq \mu(c_{F_{n_0}} - c_1)$. Hence $c_{F_n} \subset c_1$ for all n. So, $\mu(c_1) = \alpha$. Let us see that $c_1 = \vee\{c_F : F \text{ finite } \subset I\}$ in $\mathcal{C}$. If G finite $\subset I$, then $\mu(c_G \vee c_1)$ is also $= \alpha$. So $c_G \vee c_1 = c_1$. Or $c_G \subset c_1$. Now if there is another $d \in \mathcal{C}, d \subset c_1$ such that $c_F \subset d$ for all finite $F \subset I$ then $\mu(d) \geq \alpha$. Hence $c_1 = d$. Thus $\vee\{c_F : F \text{ finite } \subset I\}$ in $\mathcal{C}$ is c_1. Hence $c_0 \leq c_1$. It follows that $\mu(c_0) = \alpha$. Since $\mu(c_1 - c_0) > 0$ unless $c_1 - c_0 = \phi$, we have that $c_0 = c_1$. Thus $c_0 \in \mathcal{C}$. Thus $\mathcal{C}$ is complete.

b) If $\mathcal{C}$ is a complete subalgebra of $\mathcal{A}$ then μ restricted to $\mathcal{C}$ is countably additive. So the metric d restricted to $\mathcal{C}$ is complete. Hence $\mathcal{C}$ is a closed subset of $\mathcal{A}$.

c) Clearly $\overline{\mathcal{B}}^d$ is a complete subalgebra by a). If $\mathcal{C} \supset \mathcal{B}$ is a complete subalgebra of $\mathcal{A}$ then by b) $\mathcal{C}$ must be closed. Hence $\mathcal{C} \supset \overline{\mathcal{B}}^d$. Thus $\overline{\mathcal{B}}^d$ is the complete subalgebra generated by $\mathcal{B}$.

d) This is really a result in metric spaces. □

The following exercise describes the importance of complete subalgebras.

EXERCISE 16. *Let* $(\mathcal{A}, \mu)$ *be a measure algebra. If* $\mathcal{B} \subset \mathcal{A}$ *is a subalgebra which is complete by itself and* μ *is countably additive on* $\mathcal{B}$ *then show that* $\mathcal{B}$ *is a complete subalgebra of* $\mathcal{A}$.

We define for any complete Boolean algebra $\mathcal{B}, \tau(\mathcal{B}) = \min\{ |X| : \textit{the complete subalgebra generated by } X \textit{ is } \mathcal{B}\}$. We shall also say that a complete Boolean algebra is *τ-homogeneous* if for every $b \in \mathcal{B}^+, \tau(\mathcal{B}|_b) = \tau(\mathcal{B})$. This definition is especially meaningful because of the following theorem. Note that if $\tau(\mathcal{B}) < \omega$ for a τ-homogeneous Boolean algebra then $\tau(\mathcal{B}) = 0$ and $\mathcal{B} = \{0, 1_{\mathcal{B}}\}$.

THEOREM 3.4. *a) If* $\mathcal{B}$ *is a complete Boolean algebra and* $b \in \mathcal{B}^+$ *then* $\tau(\mathcal{B}|_b) \leq \tau(\mathcal{B})$.

b) Every complete Boolean algebra is isomorphic to the direct sum of complete τ*-homogeneous Boolean algebras.*

Proof. a) Let $X \subset \mathcal{B}$ be such that the complete Boolean algebra generated by X is $\mathcal{B}$. Let $Y = \{x \wedge b : x \in X\}$. Then $Y \subset \mathcal{B}|_b$. Let us see that the complete subalgebra of $\mathcal{B}|_b$ generated by Y is $\mathcal{B}|_b$. If $\mathcal{C} \subset \mathcal{B}|_b$ is a complete subalgebra of $\mathcal{B}|_b$ such that $Y \subset \mathcal{C}$ then $\mathcal{C}^* = \{d \in \mathcal{B}; d \wedge b \in \mathcal{C}\}$ is a complete subalgebra of $\mathcal{B}$ and $X \subset \mathcal{C}^*$. Hence $\mathcal{C}^* = \mathcal{B}$. Also, since $\mathcal{C}^*|_b = \mathcal{C}$, we have that $\mathcal{C} = \mathcal{B}|_b$. Thus the complete subalgebra generated by Y is $\mathcal{B}|_b$. Also $|Y| \leq |X|$. Thus $\tau(\mathcal{B}|_b) \leq \tau(\mathcal{B})$.

b) Let $\mathcal{B}$ be a complete Boolean algebra. Let us first see that given any $b \in \mathcal{B}^+$, there is a $c \in \mathcal{B}^+, c \leq b$ such that $\mathcal{B}|_c$ is τ-homogeneous. If there is no such c, there would exist $c_1 > c_2 > c_3 > \cdots$ all in $(\mathcal{B}|_b)^+$ such that $\tau(\mathcal{B}|_{c_{i+1}}) < \tau(\mathcal{B}|_{c_i})$ for all i. This is impossible because there cannot exist a strictly decreasing sequence of cardinals.

Hence, $\vee\{b \in \mathcal{B}^+ : \mathcal{B}|_b \text{ is } \tau\text{-homogeneous}\} = 1$. If we take a maximal family of pairwise disjoint elements $\{b_i : i \in I\}$ in the class $\{b \in \mathcal{B}^+ : \mathcal{B}|_b \text{ is } \tau\text{-homogeneous}\}$ then we have that $\vee\{b_i : i \in I\} = 1$ and $\mathcal{B}|_{b_i}$ is τ-homogeneous for each $i \in I$. Now, the map ϕ defined by $\phi(b) = (b \wedge b_i : i \in I)$ is an isomorphism between $\mathcal{B}$ and a direct sum of τ-homogeneous Boolean algebras. □

To illustrate the above concepts and for our further use let us look at some examples.

Equip the doubleton set $\{0,1\}$ with its power set σ-field and the $\{1/2, 1/2\}$ probability measure. For any infinite cardinal κ let $\mathcal{B}_\kappa$ be the product σ-field of the κ-fold product $\{0,1\}^\kappa$ of copies of the above measure space and μ_κ be the κ-fold product of the above $\{1/2, 1/2\}$ measure. Here are some properties of this measure space. For $\kappa = 0$ let $\mathcal{B}_\kappa$ be the power set σ-field of a singleton set and $\mu(\text{singleton}) = 1$.

THEOREM 3.5. *For the measure algebra* $\mathcal{B}_\kappa / I_{\mu_\kappa}$

a) $\tau(\mathcal{B}_\kappa / I_{\mu_\kappa}) = \kappa$ *and*

b) $\mathcal{B}_\kappa / I_{\mu_\kappa}$ *is* τ*-homogeneous.*

Proof. This is clear for $\kappa = 0$. Consider the cardinal κ as the initial ordinal corresponding to κ. For each ordinal $\alpha < \kappa$ define $b_\alpha = \{x \in \{0,1\}^\kappa : x_\alpha = 1\}$. Then $b_\alpha \in \mathcal{B}_\kappa$, $\mu_\kappa(b_\alpha) = 1/2$ and for $\alpha \neq \beta, \mu_\kappa(b_\alpha \Delta b_\beta) = 1/2$.

So, if we look at the metric space $\mathcal{B}/I_{\mu_\kappa}$ with the metric d_κ defined by $d_\kappa([b],[c]) = \mu_\kappa(b \Delta c)$ then the set $\{[b_\alpha] : \alpha < \kappa\}$ is a subset of the metric space with distance between any two elements being equal to 1/2.

Now, let $X \subset \mathcal{B}_\kappa / I_{\mu_\kappa}$ be a subset such that the complete subalgebra generated by X is $\mathcal{B}_\kappa / I_{\mu_\kappa}$. But the complete subalgebra generated by X, by Theorem 3.3 (c) is $\overline{ba(X)}^{d_\kappa}$ in the metric space $(\mathcal{B}_\kappa / I_{\mu_\kappa}, d_\kappa)$.

Now, if we look at the open sphere $S_{\frac{1}{4}}([b_\alpha])$ of radius $\frac{1}{4}$ around $[b_\alpha]$ then $S_{\frac{1}{4}}([b_\alpha]) \cap ba(X) \neq \phi$ for every α and for $\alpha \neq \beta$, $S_{\frac{1}{4}}([b_\alpha]) \cap S_{\frac{1}{4}}[b_\beta]) = \phi$,

So, if we pick $[c_\alpha] \in S_{\frac{1}{4}}[b_\alpha] \cap ba(X)$ for every $\alpha < \kappa$ then for $\alpha \neq \beta$ $[c_\alpha] \neq [c_\beta]$. Thus $|ba(X)| \geq \kappa$. But we know that $|ba(X)| = |X|$. Thus $\tau(\mathcal{B}_\kappa / I_{\mu_\kappa}) = \kappa$.

b) For a measure algebra $(\mathcal{A}, \mu)$ we have shown essentially that, if there exist elements $\{a_\alpha : \alpha < \kappa\}$ in $\mathcal{A}$ such that $d(a_\alpha, a_\beta) = r$, a fixed constant > 0 then $\tau(A) \geq \kappa$.

To prove b) we will take a $[b] \in (\mathcal{B}_\kappa / I_{\mu_\kappa})^+$ and show that in $\mathcal{B}_\kappa / I_{\mu_\kappa}|_{[b]}$ there are such elements $\{a_\alpha : \alpha < \kappa\}$. Clearly $b \in \mathcal{B}_\kappa$ and from the properties of the product σ-field $\mathcal{B}_\kappa$ it is clear that there exist $\alpha_1, \alpha_2, \cdots$, countably many indices, such that $b \in$ the σ-field generated by $\{b_{\alpha_i} : i \geq 1\}$. Now, if we take any $\alpha \neq \alpha_i$ for all i then b_α is independent of the σ-field generated by $\{b_{\alpha_i} : i \geq 1\}$. Hence for any $\alpha \neq \alpha_i$ for all i, $\mu(b \wedge b_\alpha) = \mu(b)\mu(b_\alpha)$ and for $\alpha, \beta \neq \alpha_i$ for all i, $\mu((b \cap b_\alpha)\Delta(b \cap b_\beta)) = \mu(b)(\mu(b_\alpha \Delta b_\beta) = \frac{1}{2}\mu(b)$.

Thus $\{[b] \wedge [b_\alpha] : \alpha \neq \alpha_i \text{ for all } i, \alpha < \kappa\}$ serve the purpose of a_α's in the above argument. Thus $\tau(\mathcal{B}_\kappa/I_{\mu_\kappa})|_{[b]}) \geq \kappa$.

But using Theorem 3.4 (a) we conclude that equality holds. □

We are approaching the statement of Maharam's Theorem. Let us first mention a version of Theorem 1.6.

THEOREM 3.6. *Let $(\mathcal{A}, \mu)$ be a measure algebra. Let $\mathcal{B} \subset \mathcal{A}$ be a complete subalgebra. Suppose that μ is $(\mathcal{B}, \mathcal{A})$-nonatomic in the sense that for every $a \in \mathcal{A}^+, \mathcal{A}|_a \neq \mathcal{B}|_a$. Let ν be a bounded countably additive nonnegative measure on $\mathcal{B}$ such that $\nu(b) \leq \mu(b)$ for all $b \in \mathcal{B}$. Then there is an $a \in \mathcal{A}$ such that $\nu(b) = \mu(a \wedge b)$ for all $b \in \mathcal{B}$.*

Proof. Using Loomis-Sikorski theorem $(\mathcal{A}, \mu)$ can be realized as the measure algebra of a measure space $(\Omega, \overline{\mathcal{A}}, \overline{\mu})$. Note that since $\mathcal{B}$ is a complete subalgebra of $\mathcal{A}$, μ is countably additive on $\mathcal{B}$. So $\overline{\mu}$ is countably additive on $\overline{\mathcal{B}} = \{A \in \overline{\mathcal{A}} : [A] \in \mathcal{B}\}$ and $\overline{\mathcal{B}}$ is a sub σ-field of $\overline{\mathcal{A}}$. Also $\overline{\mu}$ is $(\overline{\mathcal{B}}, \overline{\mathcal{A}})$-nonatomic. The measure $\overline{\nu}$ defined from ν is such that $\overline{\nu}(B) \leq \overline{\mu}(B)$ for all $B \in \overline{\mathcal{B}}$. By the second version of Theorem 1.6 we get an $A \in \overline{A}$ such that $\overline{\nu}(B) = \overline{\mu}(B \cap A)$ for all $B \in \mathcal{B}$. If we call $[A] = a$ then we have $\nu(b) = \mu(a \wedge b)$ for all $b \in \mathcal{B}$. □

Let us see a consequence of Theorem 3.6 on extensions of measure preserving homomorphisms.

THEOREM 3.7. *Let $(\mathcal{A}, \mu)$ and $(\mathcal{B}, \nu)$ be measure algebras and $\mathcal{C} \subset \mathcal{A}$ be a complete subalgebra. Let $\phi : \mathcal{C} \to \mathcal{B}$ be a measure preserving homomorphism, i.e., for $c \in \mathcal{C}, \mu(c) = \nu(\phi(c))$. Suppose further that $\mathcal{B}|_b \neq \phi(\mathcal{C})|_b$ for all $b \in \mathcal{B}^+$. Then for any $a \in \mathcal{A}$ these is a measure preserving homomorphism $\overline{\phi}$ from $ba(\mathcal{C} \cup \{a\})$ to $\mathcal{B}$ extending ϕ.*

Proof. There are several results known on extensions of homomorphisms between Boolean algebras — for one of them see A.1 of the

Appendix. The present theorem is a little difficult because we need to extend in a measure preserving way.

Of course, we need to define $\overline{\phi}(a)$ to be an element b of $\mathcal{B}$ such that $\mu(c \wedge a) = \nu(\overline{\phi}(c \wedge a)) = \nu(\phi(c) \wedge b)$ for all $c \in \mathcal{C}$. This will automatically imply that i) $\mu(a) = \nu(b), (ii)$ $\mu(d \wedge a') = \mu(d) - \mu(d \wedge a) = \nu(\phi(d)) - \nu(\phi(d) \wedge b) = \nu(\phi(d) \wedge b')$. and (iii) for $c \leq a, c \in \mathcal{C}, \mu(c) = \mu(c \wedge a) = \nu(\phi(c) \wedge b) = \nu(\phi(c))$ and since $(\mathcal{B}, \nu)$ is a measure algebra $\phi(c) \wedge b = \phi(c)$, or, $\phi(c) \leq b$ and (iv) for $a \leq d, d \in \mathcal{C}, b \leq \phi(d)$.

The above (i) to (iv) say that $\overline{\phi}$ defined by $\overline{\phi}((c \wedge a) \vee (d \wedge a')) = (\phi(c) \wedge b) \vee (\phi(d) \wedge b')$ will be a measure preserving homomorphism (using A.1 of the Appendix).

So, let us obtain an element b in $\mathcal{B}$ such that $\mu(c \wedge a) = \nu(\phi(c) \wedge b)$.

Let us first see that $\phi(\mathcal{C})$ is a complete subalgebra of $\mathcal{B}$. Let $\{\phi(c_i) : i \in I\}$ be an indexed set of elements in $\phi(\mathcal{C})$. For finite subsets $F \subset I$, let $c_F = \vee\{c_i; i \in F\}$.

Let $\sup\{c_F : F \subset I\ F \text{ finite}\} = c_0$. Also let $\sup\{\mu(c_F) : F \subset I, F \text{ finite}\} = \beta$. Then there exists $F_n \uparrow, F_n$ finite $\subset I$ such that $\mu(c_{F_n}) \uparrow \beta$. If we let $c_1 = \vee\{c_{F_n} : n \geq 1\}$ then $\mu(c_1) = \beta$ since μ is countably additive on $\mathcal{C}$. Also for any finite $G \subset I$ $\mu(c_{G \cup F_n}) \uparrow \mu(c_G \cup c_1) \leq \beta$. Hence $\mu(c_G \cup c_1) = \mu(c_1)$. Hence $c_G \subset c_1$. This means that $\vee\{c_F : F \subset I, F \text{ finite}\} \leq c_1$. So, $c_0 = c_1$. Thus $\mu(c_0) = \sup \{\mu(c_F) : F \subset I, F \text{ finite}\}$

If we call $\vee\{\phi(c_i) : i \in I\} = b$ then $b \in \mathcal{B}$ and $b \leq \phi(c_0)$ because $\phi(c_0)$ is an upper bound of $\{\phi(c_i), i \in I\}$. But $\mu(c_0) = \nu(\phi(c_0)) = \sup\{\nu(\phi(c_F)) : F \subset I, F \text{ finite }\} = \sup\{\mu(c_F) : F \subset I, F \text{ finite }\}$. Hence $\nu(b) = \nu(\phi(c_0))$. Hence $b = \phi(c_0) \in \phi(\mathcal{C})$.

Thus $\phi(\mathcal{C})$ is a complete subalgebra of $\mathcal{B}$. On $\phi(\mathcal{C})$ let us define a finitely additive measure η by $\eta(\phi(c)) = \mu(c \wedge a)$. Then $\eta(\phi(c)) \leq \mu(c) = \nu(\phi(c))$. Since ν is countably additive (because $\phi(\mathcal{C})$ is a complete subalgebra of $\mathcal{B}$) on $\phi(\mathcal{C}), \eta$ is also countably additive on $\phi(\mathcal{C})$. Hence by Theorem 3.6 there exists a $b \in \mathcal{B}$ such that $\eta(\phi(c)) = \nu(\phi(c) \wedge b)$. i.e. $\mu(c \wedge a) = \nu(\phi(c) \wedge b)$. As explained earlier this already defines a measure preserving homomorphism $\overline{\phi}$. □

The isomorphism theorem for τ-homogeneous measures algebras is the following.

THEOREM 3.8. *Let* $(\mathcal{A}, \mu)$ *and* $(\mathcal{B}, \nu)$ *be* τ*-homogeneous measure algebras such tht* $\tau(\mathcal{A}) = \tau(\mathcal{B})$ *and* $\mu(1_{\mathcal{A}}) = \nu(1_{\mathcal{B}})$*. Then there is a measure preserving isomorphism between* $(\mathcal{A}, \mu)$ *and* $(\mathcal{B}, \nu)$*.*

Proof. For a τ-homogeneous Boolean algebra $\mathcal{C}$, $\tau(\mathcal{C}) \geq \omega$ if $\tau(\mathcal{C}) \neq 0$. So, let us assume that $\tau(\mathcal{A}) = \kappa \geq \omega$.

Let $a_0, a_1, \cdots, a_\alpha \cdots \alpha < \kappa$ be such that the complete subalgebra generated by this collection is $\mathcal{A}$ and let $b_0, b_1, \cdots, b_\alpha \cdots \alpha < \kappa$ be such that the complete subalgebra generated by this collection is $\mathcal{B}$.

We shall define, using a back and forth argument, elements $a_0^*, a_1^*, \cdots, a_\alpha^* \cdots \alpha < \kappa$ from $\mathcal{A}$, $b_0^*, b_1^*, \cdots, b_\alpha^* \cdots \alpha < \kappa$ from $\mathcal{B}$, complete subalgebras $\mathcal{A}_\alpha$ of $\mathcal{A}$ and $\mathcal{B}_\alpha$ of $\mathcal{B}$ and measure preserving isomorphisms $\phi_\alpha : \mathcal{A}_\alpha \to \mathcal{B}_\alpha$ for each $\alpha < \kappa$.

Since the construction is not difficult, even though we use induction, we shall not list the properties — they will become clear as we go along.

Let $\mathcal{A}_\alpha =$ the complete subalgebra generated by $\{a_\beta : \beta < \alpha\} \cup \{a_\beta^* : \beta < \alpha\}$ and $\mathcal{B}_\alpha =$ the complete subalgebra generated by $\{b_\beta : \beta < \alpha\} \cup \{b_\beta^* : \beta < \alpha\}$. So, let $\mathcal{A}_0 = \{0, 1_{\mathcal{A}}\}, \mathcal{B}_0 = \{0, 1_{\mathcal{B}}\}$ and $\phi_0 : \mathcal{A}_0 \to \mathcal{B}_0$ be the natural isomorphism.

For defining $a_{\alpha+1}^*, b_{\alpha+1}^*$ and $\phi_{\alpha+1}$, look at $\phi_\alpha : \mathcal{A}_\alpha \to \mathcal{B}_\alpha$. Since $\tau(\mathcal{B}_\alpha) \leq |\alpha| < |\kappa|, \mathcal{B}_\alpha|_b \neq \mathcal{B}|_b$ for every $b \in \mathcal{B}^+$. Hence, by Theorem 3.7 there is a measure preserving isomorphism (homomorphism between Boolean algebras with strictly positive bounded measures is an isomorphism if it is measure preserving) $\overline{\phi}_\alpha : ba(\mathcal{A}_\alpha \cup \{a_\alpha\})$ to $ba(\mathcal{B}_\alpha \cup \{b_\alpha^*\})$ for some $b_\alpha^* \in \mathcal{B}$ which extends ϕ_α. Using the same argument again let $\overline{\overline{\phi}_\alpha^{-1}} : ba(\mathcal{B}_\alpha \cup \{b_\alpha^*\} \cup \{b_\alpha\})$ to $ba\{\mathcal{A}_\alpha \cup \{a_\alpha\} \cup \{a_\alpha^*\})$ for some $a_\alpha^* \in \mathcal{A}$ which extends $\overline{\phi}_\alpha^{-1}$. Theorem 3.7 is applicable again because $ba(\mathcal{A}_\alpha \cup \{a_\alpha\})$ and $ba(\mathcal{B}_\alpha \cup \{b_\alpha^*\} \cup \{b_\alpha\})$ are complete subalgebras by Theorem 3.2.

Let $\phi_{\alpha+1} = \overline{\overline{\phi}_\alpha^{-1}}^{-1}$.

For defining ϕ_α for a limit ordinal α assuming that $\phi_\beta : \mathcal{A}_\beta \to \mathcal{B}_\beta$ is already defined for all $\beta < \alpha$ consistently, note that the map $\phi_\alpha^* : \cup\{\mathcal{A}_\beta : \beta < \alpha\}$ to $\cup\{\mathcal{B}_\beta : \beta < \alpha\}$ defined in the natural way from $\{\phi_\beta : \beta < \alpha\}$ is a measure preserving isomorphism. By Theorem 3.3 (d) there is a measure preserving isomorphism $\phi_\alpha : \mathcal{A}_\alpha \to \mathcal{B}_\alpha$ because $\mathcal{A}_\alpha$ is the complete subalgebra generated by $\cup\{\mathcal{A}_\beta : \beta < \alpha\}$ and $\mathcal{B}_\alpha$

is the complete subalgebra generated by $\cup\{\mathcal{B}_\beta : \beta < \alpha\}$.

This will define all the $\mathcal{A}_\alpha$'s, $\mathcal{B}_\alpha$'s and ϕ_α's.

Then ϕ_κ defined in the same way as in the case of limit ordinals will give a measure preserving isomorphism between $\mathcal{A} = \mathcal{A}_\kappa$ and $\mathcal{B} = \mathcal{B}_\kappa$. □

Now we are ready to state and prove Maharam's Theorem.

THEOREM 3.9. *(Maharam's Theorem) Let* $(\mathcal{A}, \mu)$ *be a measure algebra. Then there exist cardinals* $\kappa_1, \kappa_2 \cdots$ *all of them are either* 0 *or infinite and positive reals* $r_1, r_2, r_3 \cdots$ *such that* $\sum\{r_i : i \geq 1\} = \mu(1_{\mathcal{A}})$ *with the property that* $(\mathcal{A}, \mu)$ *and the direct sum* $\sum_{i\geq 1} \left(\mathcal{B}_{\kappa_i}/I_{\mu^*_{\kappa_i}}\right)$ *where* $\mu^*_{\kappa_i} = r_i \mu_{\kappa_i}$ *are isomorphic in a measure preserving way.*

Proof. This is a combination of Theorem 3.4 (b), Theorem 3.5 and Theorem 3.8. Let us see how.

Since $\mathcal{A}$ is a complete Boolean algebra, by Theorem 3.4 (b) $\mathcal{A}$ is isomorphic to the direct sum of $\{\mathcal{A}|_{a_i} : i \in I\}$ for some a_i's in $\mathcal{A}^+$ such that $\mathcal{A}|_{a_i}$ is τ-homogeneous. Since $\mu(a_i) > 0$ for all i, I is countable. We can call $\mu(a_i)$ as r_i for all i. Now each $(\mathcal{A}|_{a_i}, \mu|_{\mathcal{A}|_{a_i}})$ being τ-homogeneous is isomorphic in a measure preserving way to $\left(\mathcal{B}_{\kappa_i}/I_{\mu_{\kappa_i}}, r_i\mu_{\kappa_i}\right)$ by Theorems 3.5 and 3.8. Hence the result. □

4. Von Neumann - Maharam Lifting Theorem

In this chapter I want to tell you about the von Neumann - Maharam Lifting Theorem.

Let $\mathcal{B}$ be a σ-field of subsets of a set Ω and μ be a countably additive probability measure on $\mathcal{B}$. Consider the measure algebra $\mathcal{A} = \mathcal{B}/\mathcal{N}_\mu$. Let π be the natural homomorphism from $\mathcal{B}$ to $\mathcal{B}/I_\mu$ defined by $\pi(b) = [b]$, the equivalence class containing b. The lifting problem asks if we can lift $\mathcal{B}/I_\mu$ to $\mathcal{B}$, in the sense that, is there a subalgebra of $\mathcal{B}$ on which π is 1-1 and whose image is the whole of $\mathcal{B}/I_\mu$ under π? Let us make this more precise.

A homomorphism $f : \mathcal{B}/I_\mu$ to $\mathcal{B}$ is called a lifting of $(\Omega, \mathcal{B}, \mu)$ if $\pi \circ f(a) = a$ for all $a \in \mathcal{B}/I_\mu$.

Every lifting f of a $(\Omega, \mathcal{B}, \mu)$ defines a homomorphism $\theta = f \circ \pi : \mathcal{B} \to \mathcal{B}$ such that $\mu(B \Delta \theta(B)) = 0$ for all $B \in \mathcal{B}$ and whenever

$\mu(B\Delta C) = 0, \theta(B) = \theta(C)$. Also for such a θ, f defined by $f([B]) = \theta(B)$ defines a lifting. Thus there is a correspondence between θ and f. We shall call θ also the lifting of $(\Omega, \mathcal{B}, \mu)$.

We shall show that every complete probability space admits a lifting. But this will be the final result in a sequence of results.

Let us first see that every lifting of $(\Omega, \mathcal{B}, \mu)$ also defines a lifting for $\mathcal{L}^\infty(\Omega, \mathcal{B}, \mu)$, the preudo-normed Banach space of all essentially bounded real valued measurable functions in which we say that $f \sim g$ if $f = g$ a.s. $[\mu]$.

THEOREM 4.1. *Let $\theta : \mathcal{B} \to \mathcal{B}$ be a lifting of $(\Omega, \mathcal{B}, \mu)$. Then there is a linear operator $T : \mathcal{L}^\infty(\Omega, \mathcal{B}, \mu)$ to $\mathcal{L}^\infty(\Omega, \mathcal{B}, \mu)$ which is a lifting that extends θ, i.e.,*

(i) $T(I_B) = I_{\theta(B)}$

(ii) *for all $g \in \mathcal{L}^\infty(\Omega, \mathcal{B}, \mu), Tg \sim g$*

(iii) *if $g \geq 0$ a.s. (μ) and $g \in \mathcal{L}^\infty(\Omega, \mathcal{B}, \mu)$ then $Tg \geq 0$ and,*

(iv) *if $g \sim h$ and $g, h \in \mathcal{L}^\infty(\Omega, \mathcal{B}, \mu)$ then $T(g) = T(h)$.*

Proof. I have to make a comment here. It is clear as to how to define T for simple functions. So, what is the problem? use the usual limiting arguments. If you are careful in verifying the details, this will cause problems. So, I adopt the following technique of Fremlin. See Exercise 18 for an alternate argument.

Let us first realize that for each $f \in \mathcal{L}_\infty$ (short for $\mathcal{L}_\infty(\Omega, \mathcal{B}, \mu)$ we want to pick, or define, one function from $[f] = \{g \in \mathcal{L}_\infty : g \sim f\}$.

What are the methods of defining a function h? One is of course by prescribing the values of the function. In our case we have no hope of doing this. Another method is to prescribe a class of sets $\{A_a : a \in R\}$ which is monotone increasing, to correspond to the sets $h^{-1}(-\infty, a]$, for example. This is what we shall do.

Given $f \in \mathcal{L}_\infty$, look at the sets $\theta(f^{-1}(-\infty, a]) = A_a$, for every $a \in R$. Then $\{A_a : a \in R\}$ is a monotone increasing class of sets and we define the function Tf by $Tf(x) = \inf\{a : x \in A_a\}$. Since f is essentially bounded, and θ is a homomorphism, $Tf(x)$ is well defined.

Of course $Tf(x) = a_0$ if and only if for all $a > a_0, x \in A_a$ and for all $a < a_0, x \notin A_a$. Hence $Tf(x) = a_0$ if and only if for all $\epsilon > 0$ $x \in \theta(\{y : |f(y) - a_0| \leq \epsilon\})$.

If $Tf(x) = a_0$ and $Tg(x) = b_0$ then for all $\epsilon > 0$ $x \in \theta(\{y : |f(y) - a_0| \leq \epsilon\}) \cap \theta(\{y : |g(y) - b_0| \leq \epsilon\})$. Hence $x \in \theta(\{y : |f(y) + g(y) - (a_0 + b_0)| \leq 2\epsilon\})$ for all $\epsilon > 0$ because the later set contains the former intersection. Hence $T(f + g)(x) = a_0 + b_0$. That $T(rf)(x) = rT(f)(x)$ is clear. Thus $T : \mathcal{L}_\infty \to \mathcal{L}_\infty$ is linear. Also, if $f \sim g$ then $\theta(f^{-1}[-\infty, a]) = \theta(g^{-1}(-\infty, a])$ for all a, because $f^{-1}(-\infty, a] \sim g^{-1}(-\infty, a] a.s.(\mu)$. Thus $Tf = Tg$.

If at a point $x, Tf(x) \neq f(x)$ then there is a rational number r such that $Tf(x) < r < f(x)$ or $Tf(x) > r > f(x)$. In the first case $x \in \theta(A_r) - A_r$ and in the later case $x \in A_r - \theta(A_r)$. In any case $x \in \theta(A_r) \Delta A_r$. Thus $\{x : Tf(x) \neq f(x)\} = \cup\{\theta(A_r)\Delta A_r : r$ is a rational number$\}$ and this set $\in \mathcal{B}$ and has μ measure equal to *zero*. Thus $Tf \sim f$.

The above also implies that $Tf \in \mathcal{L}_\infty$, if you have completeness of $(\Omega, \mathcal{B}, \mu)$. But we do not have it. So realize that $Tf(x) < a \Rightarrow x \in \theta(A_a) \Rightarrow Tf(x) \leq a$. Hence $Tf(x) < a$ if and only if there is a rational $r < a$ such that $Tf(x) < r$ if and only if there is a rational $r < a$ such that $x \in \theta(A_r)$. Thus $\{x : Tf(x) < a\} = \cup\{\theta(A_r) : r < a$ and r rational$\}$. Thus Tf is measurable and $Tf \in \mathcal{L}_\infty$ because of the above paragraph.

Clearly, if $f \geq 0$ then $A_a = \phi$ for all $a < 0$. Hence $\theta(A_a) = \phi$ for all $a < 0$. Thus $Tf \geq 0$. Lastly, if $f = I_B$ then $A_a = X$ for all $a \geq 1, A_a = B^c$ for all $0 \leq a < 1$ and $A_a = \phi$ for all $a < 0$. Hence $\theta(A_a) = X$ for all $a \geq 1, \theta(A_a) = \theta(B^c)$ for all $0 \leq a < 1$ and $\theta(A_a) = \phi$ for all $a < 0$. Thus $Tf = I_{\theta(B)}$.

Thus T has all the required properties. □

EXERCISE 17. *Show that T above also satisfies $T(fg) = T(f)T(g)$ [Hint: If $Tf(x) = a_0$ and $Tg(x) = b_0$ then $x \in \theta(\{y : |f(y) - a_0| < \frac{\epsilon}{||g||_\infty})$ and $x \in \theta(\{y : |g(y) - b_0)| < \frac{\epsilon}{a_0}\})$. So $x \in \theta(\{y : |fg(y) - a_0 b_0)| < 2\epsilon\})$]*

EXERCISE 18. *Regarding the remark at the beginning of the proof of Theorem 4.1, define $T(f)$ for nonnegative $f \in \mathcal{L}_\infty(\Omega, \mathcal{B}, \mu)$ by taking a sequence of simple functions $f_n \uparrow f$ and then by defining*

$T(f) = \lim \uparrow T(f_n)$ *which exists because* T *is monotone on simple functions. Show that* T *is well defined. Then extend* T *in a natural way to the whole of* $\mathcal{L}_\infty(\Omega, \mathcal{B}, \mu)$. (I should confess that I have not really worked out this exercise).

We shall also need a special case of the Martingale Convergence Theorem from probability theory. We shall give a simple proof of this special case since I do not assume your knowledge of any deep probability theory.

THEOREM 4.2. *Let* $\mathcal{B}$ *be a* σ*-field of subsets of a set* Ω *and* μ *be a countably additive probability measure on* $\mathcal{B}$*. Let* $\mathcal{C}_n$ *be sub* σ*-fields of* $\mathcal{B}$ *such that* $\mathcal{C}_n \subset \mathcal{C}_m$ *for all* $n \leq m$*. Let* $\mathcal{C} = \sigma(\cup \mathcal{C}_n)$*. Let* f *be a* $\mathcal{C}-$*integrable (i.e.,* $\mathcal{C}$*-measurable and* μ*-integrable) function and let* $g_n = \mathcal{E}(f|\mathcal{C}_n)$*. Then* $g_n \to f$ *a.s.* (μ)*.*

Proof. We shall show that $\mu(\{x : \limsup g_n(x) > f(x)\}) = 0$. Similarly one can show also that $\mu(\{x : \liminf g_n(x) < f(x)\}) = 0$.

First observe that for every h which is $\mathcal{C}$-integrable and for every $\epsilon > 0$ there is an m and a g which is $\mathcal{C}_m$-integrable such that $\int |f - g| d\mu < \epsilon$. This is easily seen by first observing that for every $C \in \mathcal{C}$ and $\epsilon > 0$ there is an m and a $D \in \mathcal{C}_m$ such that $\mu(C \Delta D) < \epsilon$ (How do we do this? Look at all the $C \in \mathcal{C}$ with this property and show that this class contains $\cup\{\mathcal{C}_n : n \geq 1\}$ and that this class is a σ-field). Then look at $\mathcal{F} = \{f : f$ is integrable, f is $\mathcal{C}$-measurable and for every $\epsilon > 0$ there is an m and a g which is $\mathcal{C}_m$-integrable such that $\int |f - g| d\mu < \epsilon\}$. Then clearly, $I_C \in \mathcal{F}$ for all $C \in \mathcal{C}$ from the above. Also $\mathcal{F}$ is a linear space. Also, if $\{f_n, n \geq 1\}$ are in $\mathcal{F}$, f is $\mathcal{C}$-integrable and $\int |f_n - f| d\mu \to 0$ then f is also in $\mathcal{F}$. Hence $\mathcal{F}=$ all $\mathcal{C}$-integrable functions. Hence the result.

Now, let $\epsilon > 0$ and $\eta > 0$ be fixed. From the above observation find an m and a $\mathcal{C}_m$-integrable g such that $\int |f - g| d\mu < \epsilon\eta$. Let

$$
\begin{aligned}
D &= \{x : \sup_{n \geq m} g_n(x) > g(x) + \epsilon\} \\
E &= \{x : g(x) > f(x) + \epsilon\} \quad \textit{and} \\
F &= \{x : \limsup_{n \to \infty} g_n(x) > f(x) + 2\epsilon\}
\end{aligned}
$$

Then clearly $F \subset D \cup E$ and $\epsilon\mu(F) \leq \epsilon\mu(D) + \epsilon\mu(E)$. Now, $\epsilon\mu(D) = \sum_{n \geq m} \epsilon\mu(D_n)$ where $D_n = \{x : g_n(x) > g(x) + \epsilon$ but $g_k(x) \leq g(x) + \epsilon$ for $k = m, m+1, \cdots, n-1\}$. From the definition of D_n, we have $\epsilon\mu(D_n) \leq \int_{D_n} g_n d\mu - \int_{D_n} g d\mu = \int_{D_n} (f-g)d\mu$. Hence $\epsilon\mu(D) \leq \int_D (f-g)d\mu$. Also, $\epsilon\mu(E) \leq \int_E (g-f)d\mu$. Hence $\epsilon\mu(F) \leq \int_D (f-g)d\mu + \int_E (g-f)d\mu \leq \int |f-g|d\mu \leq \epsilon\eta$. [Observe that for any integrable h, $\int_A hd\mu + \int_B (-h)d\mu = \int_{A-B} hd\mu + \int_{B-A} -hd\mu \leq \int |h|d\mu$]. Thus we have $\mu(F) \leq \eta$ for all η and ϵ positive. Thus $\mu(F) = 0$ for all $\epsilon > 0$.

Thus $\mu(\{x : \limsup g_n(x) > f(x)\}) = 0$. Hence the Theorem. □

We are now ready to prove the Lifting Theorem.

THEOREM 4.3. *Let $\mathcal{B}$ be a σ-field of subsets of a set Ω and μ be a countably additive probability measure on $\mathcal{B}$. Let further $\mathcal{B}$ be complete with respect to the μ-null sets. Then $(\Omega, \mathcal{B}, \mu)$ has a lifting.*

Proof. Let $(\mathcal{A} = \mathcal{B}/I_\mu, \mu)$ be the measure algebra of $(\Omega, \mathcal{B}, \mu)$. Of course $\mathcal{A}$ is a complete Boolean algebra. Enumerate the elements of $\mathcal{A}$ as $a_0, a_1, a_2, \cdots, a_\alpha \cdots \alpha < \xi$ for some initial ordinal ξ. For $\alpha \leq \xi$, let $\mathcal{A}_\alpha$ be the complete Boolean subalgebra of $\mathcal{A}$ generated by $\{a_\beta : \beta < \alpha\}$. I hope that you have no confusion for $\alpha = 0$.

Let $\pi : \mathcal{B} \to \mathcal{B}/I_\mu = \mathcal{A}$ be the natural homomorphism. Not only that π is a homomorphism, but also that $\pi\left(\bigcup_{i=1}^{\infty} B_i\right) = \vee_{i=1}^{\infty} \pi(B_i)$ for any sequence $\{B_i : i \geq 1\}$ in $\mathcal{B}$. Since $\mathcal{B}$ is complete with respect to μ we have that $I_\mu = \{A : A \subset B$ for aome $B \in \mathcal{B}$ such that $\mu(B) = 0\}$ and that $I_\mu \subset \mathcal{B}$.

Let us first observe that for every $\alpha \leq \xi, \mathcal{A}_\alpha$ arises from a $\mathcal{B}_\alpha \subset \mathcal{B}$. For this, let us define $\mathcal{B}_\alpha = \{B \in \mathcal{B} : \pi(B) \in \mathcal{A}_\alpha\}$. Then, since $\mathcal{A}_\alpha$ is σ-complete, $\mathcal{B}_\alpha$ is a sub σ-field of $\mathcal{B}$, $I_\mu \subset \mathcal{B}_\alpha, \pi(\mathcal{B}_\alpha) = \mathcal{A}_\alpha$ and $(\mathcal{A}_\alpha, \mu)$ is the measure algebra of $(\Omega, \mathcal{B}_\alpha, \mu)$. Also, for any limit ordinal $\alpha, \mathcal{B}_\alpha = \sigma\left(\bigcup_{\beta<\alpha} \mathcal{B}_\beta\right)$. To show this, let $\mathcal{B}_\alpha^* = \sigma\left(\bigcup_{\beta<\alpha} \mathcal{B}_\beta\right)$. Then $\pi(\mathcal{B}_\alpha^*)$ is a σ- complete subalgebra of $\mathcal{A}$ (note the difference: not just a σ-complete Boolean algebra which is a subalgebra of $\mathcal{A}$) and $\pi(\mathcal{B}_\alpha^*) \supset \mathcal{A}_\beta$ for all $\beta < \alpha$. Also, since $\pi(\mathcal{B}_\alpha^*)$ is a subalgebra of $\mathcal{A}, \pi(\mathcal{B}_\alpha^*)$ also satisfies the countable chain condition. Hence $\pi(\mathcal{B}_\alpha^*)$

is a complete subalgebra of $\mathcal{A}$. Thus $\pi(\mathcal{B}_\alpha^*) = \mathcal{A}_\alpha$. Also, $\pi(\mathcal{B}_\alpha) = \mathcal{A}_\alpha$ and $\mathcal{B}_\alpha \supset \mathcal{B}_\alpha^*$. Also, $I_\mu \subset \mathcal{B}_\alpha^*$. Now, if $A \in \mathcal{B}_\alpha$, there is an $A^* \in \mathcal{B}_\alpha^*$ such that $\pi(A) = \pi(A^*)$. This means that $\mu(A\Delta A^*) = 0$, i.e., $A\Delta A^* \in I_\mu \subset \mathcal{B}_\alpha^*$. So $A = A\Delta A^* \Delta A^* \in \mathcal{B}_\alpha^*$. Thus $\mathcal{B}_\alpha = \mathcal{B}_\alpha^*$.

With the above observations in place let us start the proof of the Lifting theorem.

We shall, for each $\alpha \leq \xi$, define a homomorphism $f_\alpha : \mathcal{A}_\alpha \to \mathcal{B}_\alpha$ such that $\pi \circ f_\alpha(a) = a$ for every $a \in \mathcal{A}_\alpha$ and also such that f_β extends f_α if $\alpha < \beta \leq \xi$. We shall do this by transfinite induction.

Since $\mathcal{A}_0 = \{0, 1\}$ we define $f_0(0) = \phi$ and $f_0(1) = \Omega$. We shall define $f_{\alpha+1}$, knowing f_α, as follows. The idea is to use the homomorphism extension theorem $A.1$ from the Appendix. By Theorem 3.2, $\mathcal{A}_{\alpha+1} = ba(\mathcal{A}_\alpha \cup \{a_\alpha\})$. Let $\underline{a_\alpha} = \sup\{a \in \mathcal{A}_\alpha : a \leq a_\alpha\}$ and $\overline{a_\alpha} = \inf\{a \in \mathcal{A}_\alpha : a_\alpha \leq a\}$. Both of these exist and $\underline{a_\alpha} \leq a_\alpha \leq \overline{a_\alpha}$. We need to define $f_{\alpha+1}(a_\alpha)$ to be an element B of $\mathcal{B}_{\alpha+1}$ such that $\pi(B) = a_\alpha$ and such that $f_{\alpha+1}(\underline{a_\alpha}) \leq B \leq f_{\alpha+1}(\overline{a_\alpha})$ Take any $C \in \mathcal{B}_{\alpha+1}$ such that $\pi(C) = a_\alpha$. Then correct C by taking $B = (C \cup f_\alpha(\underline{a_\alpha})) \cap f_\alpha(\overline{a_\alpha})$. This B serves the purpose of $f_{\alpha+1}(a_\alpha)$. By $A.1$ of the Appendix there is a homomorphism $f_{\alpha+1} : \mathcal{A}_{\alpha+1} \to \mathcal{B}_{\alpha+1}$ extending f_α such that $\pi \circ f_{\alpha+1}(a_\alpha) = a_\alpha$. Since $\mathcal{A}_{\alpha+1} = ba(\mathcal{A}_\alpha \cup \{a_\alpha\})$ it follows that $\pi \circ f_{\alpha+1}(a) = a$ for all a in $\mathcal{A}_{\alpha+1}$.

If α is a limit ordinal, let us define f_α, knowing $f_\beta : \beta < \alpha$, in the following way, considering two cases.

In case $cf(\alpha) > \omega$, let us see that $\mathcal{A}_\alpha = \cup\{\mathcal{A}_\beta : \beta < \alpha\}$. Indeed, if we take $a_1, a_2, \cdots$ from $\cup\{\mathcal{A}_\beta : \beta < \alpha\}$, since $cf(\alpha) > \omega$ there exists a β_0 such that $a_i \in \mathcal{A}_{\beta_0}$ for all i. Since $\mathcal{A}_{\beta_0}$ is a complete subalgebra of $\mathcal{A}, \vee\{a_i : i \geq 1\} \in \mathcal{A}_{\beta_0}$. Thus $\cup\{\mathcal{A}_\beta : \beta < \alpha\}$ is a σ-complete subalgebra of $\mathcal{A}$. Also $\bigcup\{\mathcal{A}_\beta : \beta < \alpha\}$ being a subalgebra of $\mathcal{A}$, satisfies the countable chain condition. Hence $\cup\{\mathcal{A}_\beta : \beta < \alpha\}$ is a complete subalgebra of $\mathcal{A}$ (Note: We are not exactly using the result: a σ-complete Boolean algebra which satisfies the countable chain condition is complete. We are using a similar result; a σ-complete subalgebra of a complete Boolean algebra which satisfies the countable chain condition is a complete subalgebra). Also $\cup\{\mathcal{A}_\beta : \beta < \alpha\} \supset \{a_\beta : \beta < \alpha\}$. Hence $\mathcal{A}_\alpha = \cup\{\mathcal{A}_\beta : \beta < \alpha\}$. We can now define $f_\alpha(a) = f_\beta(a)$ if $a \in \mathcal{A}_\beta$. This f_α is an unambiguously defined homomorphism satisfying all the conditions.

In case $cf(\alpha) = \omega$, we proceed as follows. Let $\alpha_1, \alpha_2 \cdots$ be an increasing sequence of ordinals increasing to α. We are going to define $f_\alpha(a)$ for all $a \in \mathcal{A}_\alpha$ at the same time (unlike the previous cases). For $a \in \mathcal{A}_\alpha$ let $B \in \mathcal{B}_\alpha$ be such that $\pi(B) = a$. Look at the sequence of conditional expectations $\{\mathcal{E}(I_B|\mathcal{B}_{\alpha_n}) : n \geq 1\}$. They are of course defined only $\mu-$almost surely. Let $g_n^a = T_{\alpha_n}(\mathcal{E}(I_B|\mathcal{B}_{\alpha_n}))$. The T_{α_n}'s are the liftings obtained from Theorem 4.1 starting with the liftings f_{α_n}'s. By Theorem 4.2 $\mathcal{E}(I_B|\mathcal{B}_{\alpha_n}) \longrightarrow I_B$ $a.s.(\mu)$ as $n \to \infty$. Since $T_{\alpha_n}(f) \sim f$ $a.s.(\mu)$ for every f, $g_n^a = T_{\alpha_n}(\mathcal{E}(I_B|\mathcal{B}_{\alpha_n})) \longrightarrow I_B$ $a.s.(\mu)$.

If we look at $B^* = \{x : g_n^a(x) \to 1\}$ and $B_* = \{x : g_n^a(x) \to 0\}$ then $\pi(B^*) = a = \pi(B_*^c)$ and $B^* \subset B_*^c$. We shall define a set somewhere between B^* and B_*^c as $f_\alpha(a)$.

Now, let, for $x \in \Omega, \mathcal{F}_x = \{a \in \mathcal{A}_\alpha : g_n^a(x) \to 1\}$. Let us see that $\mathcal{F}_x$ is a filter. Of course $\Omega \in \mathcal{F}_x$. If $a \in \mathcal{F}_x$ and $c \in \mathcal{A}_\alpha$ is such that $c \geq a$ and $D, B \in \mathcal{B}_\alpha$ are such that $\pi(D) = c$ and $\pi(B) = a$ then $I_D \geq I_B$ $a.s.(\mu)$. Because of the properties of the conditional expectation and because of the clever way we had demanded that (iii) be satisfied in Theorem 4.1 we have that $1 \geq g_n^c = T_{\alpha_n}(\mathcal{E}(I_D|\mathcal{B}_{\alpha_n})) \geq T_{\alpha_n}(\mathcal{E}(I_B|\mathcal{B}_{\alpha_n})) = g_n^a$ and so $g_n^c(x) \to 1$. Hence $c \in \mathcal{F}_x$. Also, if $a, c \in \mathcal{A}_\alpha$ and if $D, B \in \mathcal{B}_\alpha$ are such that $\pi(D) = c$ and $\pi(B) = a$ then $\pi(B \cap D) = a \wedge c$. So $g_n^{a \wedge c} = T_{\alpha_n}(I_{B \cap D}|\mathcal{B}_{\alpha_n}) = T_{\alpha_n}(I_B + I_D - I_{B \cup D}|\mathcal{B}_{\alpha_n}) = g_n^a + g_n^c - g_n^{a \vee c}$. If $a, c \in \mathcal{F}_x$ then $g_n^a(x), g_n^c(x)$ and $g_n^{a \vee c}(x)$ all $\to 0$. And so $a \wedge c \in \mathcal{F}_x$. Thus each of the $\mathcal{F}_x$'s is a filter in $\mathcal{A}_\alpha$.

Extend these filters to maximal filters $\mathcal{G}_x \supset \mathcal{F}_x$ in $\mathcal{A}_\alpha$. Define $f_\alpha(a) = \{x \in \Omega : a \in \mathcal{G}_x\}$ for $a \in \mathcal{A}_\alpha$.

For $a, b \in \mathcal{A}_\alpha$, by the properties of ultrafilters we know that $a \vee b \in \mathcal{G}_x$ if and only if either a or $b \in \mathcal{G}_x, a \wedge b \in \mathcal{G}_x$ if and only if a and b both $\in \mathcal{G}_x$ and for any $a \in \mathcal{A}_\alpha$, either $a \in \mathcal{G}_x$ or $a' \in \mathcal{G}_x$ and exactly one of them holds. Thus f_α is a homomorphism from $\mathcal{A}_\alpha$ to $\mathcal{P}(\Omega)$ the power set of Ω.

If $x \in B^*$ then $a \in \mathcal{F}_x$ (and so $a \in \mathcal{G}_x$ and consequently $x \in f_\alpha(a)$). Also, if $x \in B_*$ then $g_n^a(x) \to 0$. Since $g_n^{a'}(x) + g_n^a(x) = 1, g_n^{a'}(x) \to 1$ and so $a' \in \mathcal{F}_x \subset \mathcal{G}_x$. Hence $x \in f_\alpha(a') = \Omega - f_\alpha(a)$. Thus if $x \in f_\alpha(a)$ then $x \in B_*^c$. Thus $B^* \subset f_\alpha(a) \subset B_*^c$. Since $\pi(B^*) = \pi(B_*^c) = \pi(B) = a$ as remarked earlier, by the completeness of the measure space $(\Omega, \mathcal{B}, \mu)$, we have that f_α is a homomorphism into $\mathcal{B}$.

In case $a \in \mathcal{A}_\beta$ for some $\beta < \alpha$, take an $\alpha_{n_0} < \alpha$ such that $\beta \leq \alpha_{n_0} < \alpha$. Then for $n \geq n_0, T_{\alpha_n}(\mathcal{E}(I_B|\mathcal{B}_{\alpha_n})) = T_{\alpha_n}(I_B)$ which, by Theorem 4.1 (i) is equal to $I_{\theta(B)}$, and this in turn is equal to $f_\beta(a)$. Thus f_α extends each of f_β for all $\beta < \alpha$.

Thus the proof is complete. □

Let me be very optimistic in the following exercise. In the above proof we have used the hypothesis that $(\Omega, \mathcal{B}, \mu)$ is complete. Is this hypothesis necessary?

EXERCISE 19. *Give an example of a σ-field $\mathcal{B}$ of subsets of a set Ω and a countably additive probability measure μ on $\mathcal{B}$ such that $(\Omega, \mathcal{B}, \mu)$ does not admit any lifting. Note that $\mathcal{B}$ is not given to be μ-complete.(This is an open problem.)*

We shall now make some comments about the question of lifting for finitely additive measures — let me call them charges.

In general, for nonnegative bounded charges, liftings need not exit. Here is an example due to Maharam and Erdös. We refer to *"Theory of charges"* for the proof.

Let $\Omega = \{1, 2, 3, \cdots\}$ and $\mathcal{A} = \mathcal{P}(\Omega)$. We say that a nonnegative charge μ on $\mathcal{A}$ is a density charge if for $A \in \mathcal{A}$ $\mu(A) = \lim\limits_{n\to\infty} \frac{\#(A\cap\{1,2,\cdots,n\})}{n}$ whenever the right side limit exists. Maharam and Erdös have shown that for any density charge μ on $(\Omega, \mathcal{A})$ there is no lifting. See chapter 12 of *"Theory of charges"* for a proof.

Let me again be very optmistic and give you the following exercise.

EXERCISE 20. *If μ is a strongly continuous probability charge on $(\Omega, \mathcal{A})$ as above, show that $(\Omega, \mathcal{A}, \mu)$ does not admit a lifting. [Note: A charge μ is said to be strongly continuous if for every $\epsilon > 0$ there exists a finite partition of Ω such that μ of each of the sets in the partition is $\leq \epsilon$.](I do not know how to do this, but I suspect this result to be true.)*

Appendix

A.1 **Extention theorem for homomorphisms.** Let $\mathcal{A}$ and $\mathcal{B}$ be two Boolean algebras. Let ϕ be a Boolean homomorphism from a

Boolean sub algebra $\mathcal{C}$ of $\mathcal{A}$ into $\mathcal{B}$. Let a be an element of $\mathcal{A}$. If b is an element of $\mathcal{B}$ such that whenever $x \leq a \leq y$ in $\mathcal{A}$, $\phi(x) \leq b \leq \phi(y)$ then, there is a homomorphism $\overline{\phi}$ defined on $ba(\mathcal{C} \cup \{a\})$ such that $\overline{\phi}$ extends ϕ and $\overline{\phi}(a) = b$.

A.2 **Stone representation theorem.** If $\mathcal{A}$ is a Boolean algebra there is a compact totally disconnected Hausdorff space X such that $\mathcal{A}$ is isomorphic to the field of clopen subsets of X.

A.3 **Loomis-Sikorski theorem.** If $\mathcal{A}$ is a Boolean σ-algebra there is a σ-field $\mathcal{B}$ of subsets of a set X and a σ-ideal I in $\mathcal{B}$ such that $\mathcal{A}$ is isomorphic to $\mathcal{B}/I$

A.4 **Banach-Alaoglu theorem.** A norm closed ball in X^* is w^*-compact where X is a normed linear space.

A.5 **Separation theorem for convex sets.** If A is a closed convex subset of a normed linear space X and if x is an element of X which is not in A then there is a non-zero real linear functional ϕ on X such that $\phi(x) \leq \phi(y)$ for all $y \in A$.

A.6 **Krein-Milman theorem.** If K is a compact convex set in a normed linear space then K is the closed convex hull of the set of its extreme points.

In the following section there is a list of papers and books which I have consulted in the preparation of this monograph. I should make a special mention of the paper by Fremlin which I have consulted more than the others. The reader who is interested in further study is advised to look at Fremlin's paper.

The following is the list of papers and books which I have consulted in the preparation of this monograph. I should make a special mention of the paper by Fremlin which I have consulted more than the others. The reader who is interested in further study is advised to look at Fremlin's paper.

References

[1] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Topological properties of charge algebras*, Math. Pures et appl. **Tome XXII** (1977), no. 3, 363–375.

[2] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of charges, a study of finitely additive measures*, Academic Press., 1983.
[3] K. P. S. Bhaskara Rao and R. Subramanian, *On the metric space of sub σ-algebras of a probability space.*
[4] K. P. S. Bhaskara Rao and R. Subramanian, *On the metric space of sub σ- algebras of a probability space*, 1973.
[5] Dinculeanu, *Vector measures*, Pergamon Press, 1967.
[6] D. H. Fremlin, Handbook of Boolean Algebras, ch. Measure Algebras,22, pp. 877–979, Elsevier Science Publishers, 1989, pp. 877–979.
[7] A. Horn and A. A. Tarski, *Measures on boolean algebras*, Trans. Amer. Math. Soc. **64** (1948), 467–497.
[8] K. K. Jacobs, *Measure and integral*, Academic Press, 1978.
[9] J. L. Kelley, *Measures on boolean algebras*, Pacific J. Math. **9** (1959), 1165–1177.
[10] J. Lindenstrauss, *A short proof of liapounoff's convexity theorem*, J. Math. Mech **15** (1965), 971–972.
[11] D. Maharam, *An algebraic characterization of measure algebras*, Annals of Mathematics **2** (1947), no. 48, 154–167.
[12] D. Maharam, *On a theorem of von neumann*, Proc. Amer. Math. Soc. **9** (1958), 702–707.
[13] D. Maharam, *Finitely additive measures on integers*, Sankhya Series A **38** (1976), 44–59.
[14] J. Neveu, *Atomes conditionnels d'espaces de probabilite et theorie de l'information*, Symposium on probability methods in analysis, lecture notes in mathematics, Springer-Verlag, 1967, pp. 256–271.
[15] R. Sikrski, *Boolean algebras*, Springer, 1964.
[16] Ionescu A. Tulcea and Ionescu C. Tulcea, *Topics in the theory of lifting*, Springer, 1969.

Received September 9, 1998.