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The Invariant Subspace Problem: Some Recent Advances

Y. A. Abramovich, C. D. Aliprantis and O. Burkinshaw (*)

Summary. - *This paper is devoted to recent developments regarding the invariant subspace problem for positive operators on Banach lattices. Some of this material was presented by Y. A. Abramovich at "Workshop di Teoria della Misura e Analisi Reale," Grado (Italia) 18–30 September 1995.*

Contents

(*) Authors' Addresses: Y. A. Abramovich and O. Burkinshaw, Department of Mathematical Sciences, IUPUI, Indianapolis, IN 46202, USA; C. D. Aliprantis, Departments of Economics and Mathematics, Purdue University, West Lafayette, IN 47907, USA.

1. Introduction

This paper is devoted to the invariant subspace problem and describes some recent[1] results for positive and close to them operators on Banach lattices. Most of these results have been obtained in a series of papers by the authors [3, 4, 5, 6, 7]. Whenever possible we have included the proofs as well as some other insights from these papers. At the same time this survey is not just the "algebraic" sum of these works. The material is presented in a logical way using the advantage of our present hindsight. There are new theorems, improvements in the old ones, and, last but not least, we have included a number of open problems, all of which seem to be of interest to us and at least some of which, we hope, will be of interest to the reader.

The invariant subspace problem is the following question.

- *Does a continuous linear operator $T\colon X \to X$ on a Banach space have a non-trivial closed invariant subspace?*

A vector subspace is "non-trivial" if it is different from $\{0\}$ and X. A subspace V of X is **T-invariant** if $T(V) \subseteq V$. If V is invariant under every continuous operator that commutes with T, then V is called **T-hyperinvariant**.

If X is a finite dimensional complex Banach space of dimension greater than one, then each non-zero operator T has a non-trivial closed invariant subspace. Indeed, let $\lambda \in \mathbf{C}$ be an arbitrary eigenvalue of T, and so $Tx_0 = \lambda x_0$ for some $x_0 \neq 0$. Then the one-dimensional subspace $\{tx_0\colon\ t \in \mathbf{C}\}$ generated by x_0 is obviously T-invariant. Moreover, if T is not a multiple of the identity operator I, then we can easily produce a non-trivial T-hyperinvariant subspace. Namely, consider the subspace $N_\lambda = \{x \in X\colon\ Tx = \lambda x\}$. Clearly this subspace is closed and non-trivial since $T \neq \lambda I$. It remains to notice that each operator S in the commutant of T leaves this subspace invariant.[2] Indeed, for each $x \in N_\lambda$ we have

[1] This paper was completed in December of 1995 and, accordingly, it covers only the material that was obtained up to then.

[2] The **commutant** $\{T\}'$ of a continuous operator $T\colon X \to X$ on a Banach space is the set of all continuous operators on X which commute with T. That is, $\{T\}' = \{S \in L(X)\colon\ ST = TS\}$, where $L(X)$ is the Banach algebra of all continuous operators on X. Clearly, $\{T\}'$ is a unital subalgebra of $L(X)$.

$TSx = STx = S(\lambda x) = \lambda Sx$, so that $Sx \in N_\lambda$.

On the other hand, if X is non-separable, then the closed vector subspace generated by the orbit $\{x, Tx, T^2x, \dots\}$ of any non-zero vector x is a non-trivial closed T-invariant subspace. Thus, the "invariant subspace problem" is of substance only when X is an infinite dimensional separable Banach space. Accordingly, without any further mention, all Banach spaces under consideration will be assumed to be *infinite dimensional separable real or complex Banach spaces.*

Next, let us outline briefly the contents of this survey. Essentially, it can be divided into four parts. The first part (Sections 1–4) contains some introductory material to familiarize the reader with the problem, its history, and some basic techniques.

The second part (Sections 3–7) starts with operators on ℓ_p-spaces and progresses to operators on more general "discrete" spaces, including Banach spaces with a Schauder basis. A typical result from this section states: *a positive quasinilpotent operator on ℓ_p-space has a non-trivial closed invariant subspace.* The results of this part generalize our work in [3, 5, 6].

The third part (Sections 8–13) represents the main body of this paper. It contains the most general invariant subspace theorems for positive operators which have been obtained so far. Here is a sample result from this part: *Each positive quasinilpotent kernel operator has a non-trivial closed invariant subspace.* The proofs of this and other results are based on the new concept of a compact-friendly operator, introduced and discussed in Section 11. The results of this part extend our work in [4, 7]. We would like to point out that parts 2 and 3 improve upon our work in [3, 4]. This improvement consists in relaxing the commutativity condition. Namely, on many occasions for a pair of positive operators B and S acting on a Banach lattice, we replace the commutativity condition $BS = SB$ by just one of the inequalities $SB \le BS$ or $SB \ge BS$.

The concluding part (Sections 14–16) treats the dual invariant subspace problem, and also the invariant subspace problem for Dunford–Pettis and AM-compact operators. In the last Section 16 the reader will find a list of open problems related to the invariant subspace problem.

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2. Preliminaries and historical comments

For notation and terminology concerning Banach spaces and Banach lattices not explained below, we refer the reader to [12, 14, 53, 61]. The symbol E will denote a real or complex Banach lattice. The basic properties of complex Banach lattices can be found in the monographs [53, pp. 133–138] and [61, pp. 187–208].

In this paper, the word "operator" will be synonymous with "linear operator." Moreover, although most operators under consideration will be continuous, we prefer to emphasize the continuity hypothesis explicitly whenever needed. The reason for this lies in a well known fact that positive operators on Banach lattices are automatically continuous (see [14, Theorem 12.3, p. 175]).

An operator $T\colon E \to E$ is said to be **positive**, in symbols $T \geq 0$, whenever $x \geq 0$ implies $Tx \geq 0$. As usual we write $T > 0$ if $T \geq 0$ and $T \neq 0$. The operators lying in the vector subspace generated by the positive operators are referred to as **regular operators**. In the case of a real Banach lattice an operator is regular if and only if it is the difference of two positive operators.

The main thrust of the approach presented in this survey is to show that an extensive use of the theory of operators on Banach lattices and of their order structure is very helpful in dealing with the invariant subspace problem. Apart from the fact that the order structure enables us to get new results on the existence of invariant subspaces, its presence also guarantees a simple geometric form of the invariant subspaces. As we shall see, the invariant subspaces of positive operators are quite often closed order ideals, which in the case of function spaces are just subspaces of functions vanishing on measurable sets. For this reason, our results are of independent interest even in the non-separable case.

Here are some brief historical remarks (in chronological order) regarding the invariant subspace problem. Several other pertinent results (for example, the Krein–Rutman and Andô–Krieger Theorems) will be mentioned later.

1. In 1950, M. G. Krein [40, Theorem 6.3] proved that the adjoint of a positive operator on a $C(\Omega)$-space, where Ω is a non-trivial compact Hausdorff space, has a a positive eigenvector and, consequently, the operator itself has a non-trivial closed invariant subspace. (A proof of this will be given in Section 3.)
2. In 1954, N. Aronszajn and K. T. Smith [18] proved that compact operators have non-trivial closed invariant subspaces.
3. In 1966, A. R. Bernstein and A. Robinson [22] and subsequently P. R. Halmos [36] established that polynomially compact operators have non-trivial closed invariant subspaces. The proof of Bernstein and Robinson used non-standard analysis, while Halmos' proof was "standard."
4. In 1973, V. I. Lomonosov [42] astounded the mathematical world by proving that every continuous operator which commutes with a non-zero compact operator has a non-trivial closed invariant subspace.
5. It should be pointed out that until the middle of 70's, the invariant subspace problem was phrased a bit stronger than formulated in the introduction. Namely, it asked:

- *Does every continuous linear operator on a (separable) Banach space have a non-trivial closed invariant subspace?*

It was P. Enflo [32] who constructed in 1976 an example of a continuous operator on a Banach space without a non-trivial closed invariant subspace, and thus demonstrated that in this general form the invariant subspace problem has a negative answer. For operators on a Hilbert space, the existence of an invariant subspace is still unknown and is one of the famous unsolved problems in mathematics. Due to the above counterexample, the invariant subspace problem for operators on Banach spaces has been confined to the search for various classes of operators for which one can guarantee the existence of an invariant subspace.
6. In 1978, S. W. Brown [23] proved that subnormal operators on Hilbert spaces have non-trivial closed invariant subspaces.
7. In 1985, C. J. Read [51] presented an example of a continuous operator on ℓ_1 without a non-trivial closed invariant subspace.
8. Let $\mathcal{A} \subseteq L(X)$ be a subalgebra of operators on a Banach space

X. A well known version of the invariant subspace problem is the following question:

- *When does the algebra $\mathcal{A}$ have a non-trivial closed invariant subspace? That is, when does there exist a common (for all operators in $\mathcal{A}$) non-trivial closed invariant subspace?*

The classical Burnside Theorem [50, Corollary 8.6, p.142] asserts that if X is finite-dimensional, then any proper subalgebra of $L(X)$ has a non-trivial closed invariant subspace. In a recent work, V. Lomonosov [43] found a very elegant extension of this result to infinite dimensional spaces. Namely, for a proper and weakly closed subalgebra $\mathcal{A}$ of $L(X)$ he proved the existence of non-zero functionals $x' \in X'$ and $x'' \in X''$ such that for any $A \in \mathcal{A}$ the following estimate holds:

$$|x''(A'x')| \leq d(A, \mathcal{K}(X)),$$

where $d(A, \mathcal{K}(X))$ is the distance from A to the space of compact operators $\mathcal{K}(X)$.

This theorem immediately implies Burnside's Theorem since if $\dim X < \infty$, then $d(A, \mathcal{K}(X)) = 0$ and so $x''(A'x') = 0$ for each $A \in \mathcal{A}$, guaranteeing the existence of a non-trivial closed invariant subspace for $\mathcal{A}$ (see Proposition 14.1). In the same work [43] V. Lomonosov proposed the following conjecture:

- *The adjoint of an arbitrary continuous linear operator on a Banach space has a non-trivial closed invariant subspace.*

We refer to [7, 28, 54, 55] for some results regarding this conjecture. The work done in [7] regarding this conjecture will be discussed in Section 14.

We want to reiterate that this survey is devoted almost entirely to the invariant subspace problem for positive (and close to them) operators on Banach lattices. In particular, we practically do not mention any work on the invariant subspace problem in the vast areas of operators on Hilbert spaces and on various spaces of analytic functions. For a comprehensive account on the history and progress regarding the invariant subspace problem, we refer the reader to [19, 20, 21, 37, 46, 47, 48, 50, 52] and the references therein.

3. Some basic invariant subspace theorems

The purpose of this section is to remind the reader of several well known invariant subspace theorems for operators on Banach spaces. Some of these results will be used repeatedly in our discussion, without reference.

It should be obvious that if a subspace is invariant or hyperinvariant for a continuous operator, then so is its norm closure. Another elementary fact regarding invariant subspaces is presented in the next lemma.

LEMMA 3.1. *If $T\colon X \to X$ is a continuous operator on a Banach space, then its kernel and range are T-hyperinvariant subspaces.*

Proof. Assume $S \in L(X)$ satisfies $ST = TS$. If $Tx = 0$, then from $T(Sx) = S(Tx) = 0$, we see that $S(Ker\,T) \subseteq Ker\,T$, i.e., $Ker\,T$ is a T-hyperinvariant closed subspace. On the other hand, if $x \in X$, then the identity $S(Tx) = T(Sx)$ shows that $S\big(T(X)\big) \subseteq T(X)$, and this means that the range $T(X)$ of the operator T is also T-hyperinvariant. □

COROLLARY 3.2. *If a continuous operator T commutes with another continuous non-zero operator which is either not one-to-one or fails to have a dense range, then T has a non-trivial closed invariant subspace.*

The preceding corollary shows that as far as the invariant subspace problem is concerned, not only can we suppose that the operator is one-to-one and has a dense range but also that every other non-zero operator which commutes with it is one-to-one and has a dense range.

Recall that an **eigenspace** of a continuous operator $T\colon X \to X$ is any closed non-zero subspace of the form $N_\lambda = \{x \in X\colon\ Tx = \lambda x\}$, where λ is an eigenvalue of T. It is obvious that $\lambda = 0$ is an eigenvalue for T if and only if the kernel of T is non-trivial, and in this case, $Ker\,T = N_0$. Clearly, T is not a multiple of the identity if and only if there is no eigenvalue λ such that $N_\lambda = X$. Eigenspaces are the simplest invariant subspaces.

LEMMA 3.3. *If a continuous operator $T\colon X \to X$ is not a multiple of the identity, then every eigenspace of T is a non-trivial T-hyperinvariant closed subspace.*

Proof. Let λ be an eigenvalue of a bounded operator $T\colon X \to X$. By the hypothesis, $N_\lambda = \{x \in X\colon\ Tx = \lambda x\} = Ker\,(T-\lambda) \neq \{0\}$. Since every operator which commutes with T also commutes with $T - \lambda I$, it follows from Lemma 3.1 that N_λ is T-hyperinvariant. □

Next, we shall prove that if the adjoint of an operator has an eigenvector, then the operator has a non-trivial closed invariant subspace.

THEOREM 3.4. *Let $T\colon X \to X$ be a continuous operator on a Banach space which is not a multiple of the identity. If either T or its adjoint T' has an eigenvector, then both T and T' have non-trivial closed hyperinvariant subspaces.*

Proof. Notice that T is not a multiple of the identity if and only if its adjoint T' is not a multiple of the identity. If T has an eigenvalue λ, then by Lemma 3.3 its eigenspace $N_\lambda = \{x \in X\colon\ Tx = \lambda x\}$ is a non-trivial T-hyperinvariant closed subspace.

Now assume that $T'x' = \lambda x'$ for some scalar λ and some non-zero $x' \in X'$. Let $V = \overline{(T-\lambda)(X)}$, the closure of the range of the operator $T - \lambda$. Since $T \neq \lambda I$, we see that $V \neq \{0\}$; and since $x' \neq 0$, the equality $\langle (T-\lambda)x, x'\rangle = \langle x, (T'-\lambda)x'\rangle = 0$ implies that $V \neq X$. Thus, V is a non-trivial closed subspace of X. We claim that V is T-hyperinvariant. Indeed, if $S \in L(X)$ satisfies $ST = TS$ and $x \in X$, then $S[(T-\lambda)x] = (T-\lambda)(Sx) \in V$, which implies that $S(V) \subseteq V$.

To complete the proof notice that if x is an eigenvector of T, then x is also an eigenvector of T'', and so the previous part guarantees that in this case T' also has a non-trivial closed T'-hyperinvariant subspace. □

M. G. Krein [40, Theorem 6.3] has proved the following remarkable result.

THEOREM 3.5 (M. G. KREIN). *The adjoint of an arbitrary positive operator on a $C(\Omega)$-space (where Ω is a compact Hausdorff space) has a positive eigenvector corresponding to a non-negative eigenvalue.*

Proof. Let $T\colon C(\Omega) \to C(\Omega)$ be a positive operator. Consider the set

$$G = \{f \in C(\Omega)'_+ \colon\ f(\mathbf{1}) = 1\},$$

where $\mathbf{1}$ denotes the constant function one on Ω. Clearly, G is a nonempty, convex, and w^*-compact subset of $C(\Omega)'$.

Next, define the mapping $F\colon G \to G$ by

$$F(f) = \tfrac{f+T'f}{[f+T'f](\mathbf{1})} = \tfrac{f+T'f}{1+T'f(\mathbf{1})}\,. \qquad (\star)$$

A straightforward verification shows that F indeed maps the set G back to G and that $F\colon (G, w^*) \to (G, w^*)$ is a continuous function. So, by Tychonoff's Fixed Point Theorem (see, for instance [12, Corollary 14.51, p. 483]) there exists some $\phi \in G$ such that $F(\phi) = \phi$. That is, $\phi + T'\phi = [1 + T'\phi(\mathbf{1})]\phi$, or $T'\phi = [T'\phi(\mathbf{1})]\phi$, which shows that $0 < \phi \in C(\Omega)'_+$ is an eigenvector for T' having the non-negative eigenvalue $T'\phi(\mathbf{1})$. □

COROLLARY 3.6. *Every positive operator on a $C(\Omega)$-space (where Ω is Hausdorff, compact and not a singleton) which is not a multiple of the identity has a non-trivial hyperinvariant closed subspace.*

Proof. Let $T\colon C(\Omega) \to C(\Omega)$ be a positive operator which is not a multiple of the identity. By Theorem 3.5 the adjoint operator T' has an eigenvector, and the conclusion follows from Theorem 3.4. □

Regarding eigenvalues and the spectral radius of a positive compact operator we have the following classical result of M. G. Krein and M. A. Rutman [40], which is an important infinite dimensional generalization of the Perron–Frobenius theorem. We refer to [53, 61] for other proofs and pertinent results concerning the Krein–Rutman theorem. Some relevant results can be also found in [2].

THEOREM 3.7 (KREIN–RUTMAN). *For a positive operator T on a Banach lattice E we have the following.*

a. *The spectral radius of T belongs to the spectrum of T, i.e., $r(T) \in \sigma(T)$.*

b. *If T is also compact with positive spectral radius, then its spectral radius is an eigenvalue having a positive eigenvector, i.e., there exists some $x > 0$ such that $Tx = r(T)x$.*

Recall that a vector subspace V of a vector space X is said to be **complemented** if there exists another subspace W such that $V \oplus W = X$. The vector subspace W is called a **complement** of V. As usual, an operator $P\colon X \to X$ is said to be a **projection** if $P^2 = P$. A projection P is **proper** if $P \neq 0$ and $P \neq I$. It is easy to see that a subspace V is complemented if and only if there exists a projection on X whose range is precisely V. When V is a closed subspace of a Banach space X, the Closed Graph Theorem implies that V has a closed complement if and only if V is the range of a continuous projection on X.

A closed subspace V of a Banach space X is said to be **reducing** a continuous operator $T\colon X \to X$ if V has a closed complement W such that both V and W are T-invariant. The reducing subspaces of an operator are very important and they are characterized in terms of projections as follows.

THEOREM 3.8. *A continuous operator $T\colon X \to X$ on a Banach space has a proper reducing (closed) subspace if and only if there exists a proper continuous projection $P\colon X \to X$ satisfying $TP = PTP$.*

Proof. Assume that a proper closed subspace V reduces T and let $P\colon X \to X$ be a proper continuous projection with $P(X) = V$. Put $W = (I - P)(X)$. Now if $x \in X$, write $x = x_1 + x_2$ with $x_1 \in V$, $x_2 \in W$ and note that $PTPx = PTx_1 = Tx_1 = TPx$.

For the converse, assume that $TP = PTP$ holds for some continuous proper projection P on X. Put $V = P(X)$ and $W = (I-P)(X)$. Clearly, V and W are both closed subspaces, and $V \oplus W = X$. If $x \in V$, then $Tx = TPx = P(TPx) \in V$, and so $T(V) \subseteq V$. On the other hand, if $x \in W$, then $PTx = PTPx = 0$, and so $Tx \in (I - P)(X) = W$, i.e., $T(W) \subseteq W$. ☐

COROLLARY 3.9. *Every continuous operator on a Banach space which commutes with a proper projection has a reducing subspace.*

Proof. If $T\colon X \to X$ is a continuous operator on a Banach space which commutes with a proper continuous projection $P\colon X \to X$, then note that $TP = PT = P^2T = P(PT) = PTP$. $\square$

4. Local quasinilpotence

Recall that a continuous operator $T\colon X \to X$ on a Banach space is said to be **quasinilpotent** if its spectral radius $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ is zero. It is well known that T is quasinilpotent if and only if $\lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0$ for each $x \in X$.

It can easily happen that a continuous operator $T\colon X \to X$ is not quasinilpotent but, nevertheless, $\lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0$ for some $x \neq 0$. In this case we shall say that T is locally quasinilpotent. This property was introduced in [3], where it was found to be useful in the study of the invariant subspace problem.

DEFINITION 4.1. *A bounded operator $T\colon X \to X$ on a Banach space is said to be* **quasinilpotent at a point** x_0 *if* $\lim_{n\to\infty} \|T^n x_0\|^{\frac{1}{n}} = 0$. *The set of all points at which T is quasinilpotent is denoted by $\mathcal{Q}_T$, i.e.,*

$$\mathcal{Q}_T = \big\{x \in X\colon \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\big\}.$$

It is obvious that the zero vector belongs to $\mathcal{Q}_T$, and also that if T is not one-to-one, then $\mathcal{Q}_T$ is not trivial since each point of the kernel of T belongs to $\mathcal{Q}_T$. It is a bit harder to see that a one-to-one operator can also be quasinilpotent at a non-zero point without being itself a quasinilpotent operator. Examples of this type will be presented in the next section. If $T\colon E \to E$ is an operator on a Banach lattice then, as usual, we let $\mathcal{Q}_T^+ = \mathcal{Q}_T \cap E^+$.

LEMMA 4.2. *If $T\colon X \to X$ is a continuous operator on a Banach space, then the set $\mathcal{Q}_T$ of all points at which T is quasinilpotent is a T-hyperinvariant vector subspace.*

Proof. Clearly, $x \in \mathcal{Q}_T$ implies $\lambda x \in \mathcal{Q}_T$ for each scalar λ. Now let $x, y \in \mathcal{Q}_T$, and fix $\epsilon > 0$. Then, there exists some n_0 such that $\|T^n x\| < \epsilon^n$ and $\|T^n y\| < \epsilon^n$ for all $n \geq n_0$. So, $\|T^n(x+y)\|^{\frac{1}{n}} \leq$

$(\|T^n x\| + \|T^n y\|)^{\frac{1}{n}} < 2\epsilon$ for all $n \geq n_0$. Therefore, $x + y \in \mathcal{Q}_T$, and so $\mathcal{Q}_T$ is a vector subspace.

For the last part, fix $x_0 \in \mathcal{Q}_T$, and let an operator $S\colon X \to X$ satisfy $ST = TS$. Then from

$$\|T^n(Sx_0)\|^{\frac{1}{n}} = \|S(T^n x_0)\|^{\frac{1}{n}} \leq \|S\|^{\frac{1}{n}} \cdot \|T^n x_0\|^{\frac{1}{n}} \xrightarrow[n\to\infty]{} 0,$$

we see that $Sx_0 \in \mathcal{Q}_T$, i.e., $\mathcal{Q}_T$ is S-invariant. □

We collect below a few more simple properties of the vector space $\mathcal{Q}_T$ of quasinilpotent points of a continuous operator $T\colon X \to X$.

- The operator T is quasinilpotent if and only if $\mathcal{Q}_T = X$.
- $\mathcal{Q}_T = \{0\}$ is possible. For instance, every isometry T satisfies $\mathcal{Q}_T = \{0\}$. Notice also that even a compact operator can fail to be locally quasinilpotent at every non-zero vector. For instance, consider the compact positive operator $T\colon \ell_2 \to \ell_2$ defined by $T(x_1, x_2, \dots) = \big(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\big)$. If $x \in \ell_2$ satisfies $x \neq 0$, then pick some k for which $x_k \neq 0$ and note that $\|T^n x\|^{\frac{1}{n}} \geq \frac{1}{k}|x_k|^{\frac{1}{n}}$ for each n, from which it follows that T is not quasinilpotent at x.
- $\mathcal{Q}_T$ can be dense without being equal to X. For instance, the left shift operator $S\colon \ell_2 \to \ell_2$, defined by $S(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$, has this property.
- If $\mathcal{Q}_T \neq \{0\}$ and $\overline{\mathcal{Q}_T} \neq X$, then (by Lemma 4.2) $\overline{\mathcal{Q}_T}$ is a non-trivial closed T-hyperinvariant subspace of X.

The above properties show that as far as the invariant subspace problem is concerned, we need only to consider the two extreme cases: $\mathcal{Q}_T = \{0\}$ and $\overline{\mathcal{Q}_T} = X$.

5. Operators on ℓ_p-spaces

In this section we deal primarily with the classical ℓ_p-spaces over the complex numbers, where $1 \leq p < \infty$. The case $p = \infty$ is excluded for the obvious reason: the space ℓ_∞ is non-separable, and consequently, as explained in the introduction, each continuous operator automatically has a non-trivial closed invariant subspace. Possible generalizations to more general discrete spaces will be discussed briefly at the end of the section. It should be pointed out that Theorem 5.1, the main result of this section, is a special case of a general result which will be established in Section 10. However, in view of the simplicity of the proof in the discrete case, it is useful to present this direct proof as well. Apart from its simplicity it also allows us to obtain an explicit description of the invariant subspaces.

Our presentation in this section is rather self-sufficient in the sense that it is independent of the general theory of Banach lattices. A vector

$$x = (x_1, x_2, \dots) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} \in \ell_p$$

is said to be **positive**, in symbols $x \geq 0$, if its components are non-negative real numbers. The absolute value $|x|$ of a vector x is the vector $|x| = \big(|x_1|, |x_2|, \dots\big)$. The symbol $\mathbf{e}_n$ will denote the vector whose n^{th}-component is one and every other zero.

It is well known that every operator $T\colon \ell_p \to \ell_p$ can be represented by an infinite matrix $[t_{ij}]$, where for each $x = (x_1, x_2, \dots) \in \ell_p$ we have

$$T(x) = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \cdots \\ t_{21} & t_{22} & t_{23} & \cdots \\ t_{31} & t_{32} & t_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \sum_{i=j}^{\infty} t_{1j} x_j \\ \sum_{i=j}^{\infty} t_{2j} x_j \\ \sum_{i=j}^{\infty} t_{3j} x_j \\ \vdots \end{bmatrix}.$$

Recall that an operator $T\colon \ell_p \to \ell_p$ is said to be **positive** if $Tx \geq 0$ holds for all $x \geq 0$. This is equivalent to saying that each entry t_{ij} of the matrix $\big[t_{ij}\big]$ representing T is a non-negative real number. The notation $S \geq T$ for operators S and T simply means that $S - T \geq 0$.

A (continuous) operator $T\colon \ell_p \to \ell_p$ with matrix $[t_{ij}]$ has a **modulus** whenever the matrix of absolute values $\big[|t_{ij}|\big]$ also defines an operator on ℓ_p (being positive this operator is automatically continuous), that is,

$$|T|(x) = \begin{bmatrix} |t_{11}| & |t_{12}| & |t_{13}| & \cdots \\ |t_{21}| & |t_{22}| & |t_{23}| & \cdots \\ |t_{31}| & |t_{32}| & |t_{33}| & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ \vdots \end{bmatrix}.$$

Let us note for further reference that each continuous operator on ℓ_1 or ℓ_∞ has a modulus [14, Theorem 15.3, p. 249]. If a continuous operator $T\colon \ell_p \to \ell_p$ has a modulus, then clearly $|T(x)| \leq |T|\big(|x|\big)$ for each $x \in \ell_p$. Also, it should be noted that if each of two continuous operators $S, T\colon \ell_p \to \ell_p$ has a modulus, then $S+T$, ST, and αT likewise have moduli and $|S+T| \leq |S| + |T|$, $|ST| \leq |S||T|$, and $|\alpha T| = |\alpha||T|$ hold.

A vector subspace J of an ℓ_p-space is said to be an **order ideal** if $|y| \leq |x|$ and $x \in J$ imply $y \in J$. The closure of any order ideal is again an order ideal. It should be noted that any non-trivial closed order ideal J in ℓ_p has the following form: there exists a (unique) non-empty proper subset N_0 of natural numbers such that $J = \big\{x \in \ell_p\colon\ x_n = 0 \text{ for each } n \in N_0\big\}$. In particular, every closed ideal in an ℓ_p-space is complemented.

As mentioned in Section 4, a one-to-one continuous operator can be quasinilpotent at a non-zero vector without being itself quasinilpotent. For instance, the one-to-one operator $T\colon \ell_p \to \ell_p$ defined by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is quasinilpotent at $\mathbf{e}_2$ but fails to be quasinilpotent. Indeed, from $T\mathbf{e}_n = \frac{1}{n}\mathbf{e}_{n+1}$ for all $n \geq 2$, we see that $T^n\mathbf{e}_2 = \frac{1}{(n+1)!}\mathbf{e}_{n+2}$, and hence $\|T^n\mathbf{e}_2\| = \frac{1}{(n+1)!}$, from which it follows that $\lim\limits_{n\to\infty} \|T^n\mathbf{e}_2\|^{\frac{1}{n}} = 0$. On the other hand, $\|T^n\| \geq \|T^n \mathbf{e}_1\| \geq 1$ implies $\|T^n\|^{\frac{1}{n}} \geq 1$ for each n. Therefore, T is not quasinilpotent.

We are now ready to state and prove the main result of this section which is an improvement of our main result in [3]. It implies, in particular, that if a positive operator is quasinilpotent at a non-zero positive vector, then the operator has an invariant subspace.

THEOREM 5.1. *Let* $T\colon \ell_p \to \ell_p$ $(1 \leq p < \infty)$ *be a continuous operator with modulus. If there exists a non-zero positive operator* $S\colon \ell_p \to \ell_p$ *such that*

1. $S|T| \leq |T|S$ *(in particular this holds if* S *commutes with* $|T|$*), and*
2. S *is quasinilpotent at a non-zero positive vector,*

then T *has a non-trivial closed invariant subspace which is an ideal.*

Proof. Assume $S > 0$ satisfies $S|T| \leq |T|S$ and $\lim_{n\to\infty} \big\|S^n x_0\big\|^{\frac{1}{n}} = 0$ for some $x_0 > 0$. We distinguish two cases: $Sx_0 > 0$ and $Sx_0 = 0$.

We begin by assuming that $Sx_0 > 0$. In this case, since S is continuous, an appropriate scaling shows that there exists some k satisfying $x_0 \geq \mathbf{e}_k > 0$ and $S\mathbf{e}_k > 0$.

Now let $P\colon \ell_p \to \ell_p$ denote the natural projection onto the vector subspace generated by $\mathbf{e}_k$. Clearly, $0 \leq Px \leq x$ for each $0 \leq x \in \ell_p$. We claim that

$$P|T|^m S\mathbf{e}_k = 0 \qquad (\star)$$

for each $m \geq 0$. To this end, fix $m \geq 0$ and let $P|T|^m S\mathbf{e}_k = \alpha \mathbf{e}_k$ for some $\alpha \geq 0$. Clearly, $S|T|^m \leq |T|^m S$, and hence we have

$$\begin{aligned} 0 \leq \alpha^n \mathbf{e}_k &= \big(P|T|^m S\big)^n \mathbf{e}_k \leq \big(|T|^m S\big)^n \mathbf{e}_k \\ &= |T|^m S|T|^m S \cdots |T|^m S\mathbf{e}_k \leq |T|^{mn} S^n \mathbf{e}_k \leq |T|^{mn} S^n x_0. \end{aligned}$$

Consequently,

$$0 \leq \alpha \leq \big\||T|\big\|^m \cdot \big\|S^n x_0\big\|^{\frac{1}{n}} \xrightarrow[n\to\infty]{} 0,$$

from which it follows that $\alpha = 0$.

Next, consider the order ideal J generated by the set

$$\big\{|T|^m S\mathbf{e}_k\colon\ m=0,1,\dots\big\}$$

that is,

$$J=\big\{x\in\ell_p\colon\ \exists\,\lambda\ge 0 \text{ and } r\ge 0 \text{ such that } |x|\le\lambda{\textstyle\sum_{i=0}^r}|T|^i S\mathbf{e}_k\big\}.$$

Since $0<S\mathbf{e}_k\in J$, we see that $J\neq\{0\}$. Also, from ($\star$), it follows that

$$\langle\mathbf{e}_k,x\rangle=\langle\mathbf{e}_k,Px\rangle=0\quad\text{for all}\quad x\in J,$$

and consequently $\langle\mathbf{e}_k,x\rangle=0$ for all $x\in\overline{J}$, the norm closure of J. The latter shows that $\overline{J}$ is a non-trivial closed ideal in ℓ_p. We claim that $\overline{J}$ is T-invariant. Indeed, if $x\in J$, then there exist a scalar $\lambda>0$ and some integer $r\ge 0$ such that $|x|\le\lambda\sum_{i=0}^r|T|^iS\mathbf{e}_k$. So, $|T(x)|\le|T|(|x|)\le\lambda\sum_{i=0}^{r+1}|T|^iS\mathbf{e}_k$, and hence $T(x)\in J$. That is, J (and hence $\overline{J}$) is T-invariant.

Now consider the case $Sx_0=0$. Let $\mathcal{A}$ denote the unital algebra generated by $|T|$ in the Banach algebra of all continuous operators on ℓ_p. Also, let

$$J=\big\{x\in\ell_p\colon\ |x|\le|A|x_0 \text{ for some } A\in\mathcal{A}\big\}.$$

Clearly, J is a T-invariant order ideal which is non-zero since $x_0\in J$. To finish the proof, it suffices to show that $\overline{J}\neq\ell_p$. Since S is assumed to be nonzero, the previous claim will be established if we show that the restriction of S to J is identically zero.

To see this, take any element $x\in J$. Pick some $A\in\mathcal{A}$ with $|x|\le|A|x_0$ and then select an integer $r\ge 0$ and a scalar $c>0$ such that $|A|\le c\sum_{i=0}^r|T|^i$. So, from

$$|S(x)|\le S|x|\le S|A|x_0\le S\Big(c\sum_{i=0}^r|T|^i\Big)x_0\le c\sum_{i=0}^r|T|^i(Sx_0)=0,$$

it follows that $S(x)=0$ for all $x\in J$. If $\overline{J}=\ell_p$, then $S=0$, which is a contradiction. Hence, $\overline{J}$ is a non-trivial closed T-invariant subspace, and the proof is complete. $\square$

A simple example of a pair of noncommuting positive operators S and B satisfying the inequality $SB \leq BS$ follows. Let S be the right shift operator on ℓ_p and B be the left shift operator. Then for each $x \in \ell_p$, we clearly have $BSx = x$ and $SBx = (0, x_2, x_3, \dots)$, so that $SB \leq BS$.

We do not know presently if an analogue of the previous result holds provided we replace the inequality $S|T| \leq |T|S$ by the reverse inequality $S|T| \geq |T|S$. However, if we assume additionally that S is quasinilpotent (rather than just being locally quasinilpotent) then such an analogue does hold, and it will be stated next. A direct proof of this result can be obtained by a simple modification of the proof of Theorem 5.1, and is omitted. As a matter of fact, we do not really need this direct proof since Theorem 5.2 is a special case of Theorem 10.2.

THEOREM 5.2. *Let $T\colon \ell_p \to \ell_p$ $(1 \leq p < \infty)$ be a continuous operator with modulus. If there exists a non-zero positive operator $S\colon \ell_p \to \ell_p$ such that*

1. *$S|T| \geq |T|S$, and*
2. *S is quasinilpotent,*

then T has a non-trivial closed invariant subspace which is an ideal.

Our first corollary is perhaps the most striking consequence of Theorem 5.1. It establishes that one can add arbitrary weights to an operator and still be guaranteed that non-trivial closed invariant subspaces exist.

COROLLARY 5.3. *Assume that a positive matrix $A = [a_{ij}]$ defines an operator on an ℓ_p-space $(1 \leq p < \infty)$ which is quasinilpotent at a non-zero positive vector. If $w = \{w_{ij}\colon\ i, j = 1, 2, \dots\}$ is an arbitrary bounded double sequence of complex numbers, then the continuous operator defined by the weighted matrix $A_w = \left[w_{ij} a_{ij}\right]$ has a non-trivial closed invariant subspace.*

Moreover, all these operators A_w have a common non-trivial closed invariant subspace.

Proof. By Theorem 5.1, the operator A has a non-trivial closed invariant order ideal V. Now if $B = [b_{ij}]$ is a matrix whose modulus satisfies $|B| \leq cA$, then from $|Bx| \leq |B|(|x|) \leq cA(|x|)$ it follows that $Bx \in V$ for each $x \in V$, i.e., V is B-invariant. It remains to let $B = A_w$. □

It is worth mentioning that in the preceding corollary our assumption that the weights are bounded is not necessary. It suffices to assume only that the modulus of the matrix A_w defines an operator on ℓ_p.

COROLLARY 5.4. *If the modulus of a continuous operator* $T\colon \ell_p \to \ell_p$ $(1 \leq p < \infty)$ *exists and is quasinilpotent at a non-zero positive vector, then* T *has a non-trivial closed invariant subspace.*

COROLLARY 5.5. *Every positive operator on an* ℓ_p*-space* $(1 \leq p < \infty)$ *which is quasinilpotent at a non-zero positive vector has a non-trivial closed invariant subspace.*

For quasinilpotent positive operators on ℓ_2 Corollary 5.5 was also obtained in [26]. Although Theorem 5.1 and its corollaries are new even for a quasinilpotent operator on ℓ_p, their main attractiveness lies in the fact that we do not really need to know that a positive operator $T\colon \ell_p \to \ell_p$ is quasinilpotent. The only thing needed is the existence of a single vector $x_0 > 0$ for which $\|T^n x_0\|^{\frac{1}{n}} \to 0$. This alone implies that T has a non-trivial closed invariant subspace of a simple geometric form. In view of this, the following important question arises. How can we recognize by "looking at" a matrix $[t_{ij}]$ defining a positive operator $T\colon \ell_p \to \ell_p$ if the set $\mathcal{Q}_T^+$ is non-empty? This question will be considered in the next section. Originally it was addressed in [6], and in our presentation we will follow that work.

We conclude this section with some remarks. First we notice that for a quasinilpotent positive operator T on ℓ_p the existence of a non-trivial closed invariant subspace can also be derived from some results in [1] or [29], and also from the results in the next section. However, a "genuine" explanation as to why the results of this section are true depends on the concept of a compact-friendly operator. This explanation will be given at the end of Section 11 after this new concept is introduced and studied.

We mentioned before that every continuous operator on ℓ_1 has a modulus, i.e., it is a regular operator. Consequently, C. J. Read's example [51] of a continuous operator on ℓ_1 without non-trivial closed invariant subspaces shows that a regular operator on ℓ_p-spaces may fail to have an invariant subspace. Our Theorem 5.1 presents a rather weak additional condition which guarantees that a regular operator on an ℓ_p-space has an invariant subspace. It is still an open problem whether or not each positive operator on ℓ_1 (or ℓ_p with $p < \infty$) has an invariant subspace. For quite a while it seemed plausible that the modulus of the operator constructed by C. J. Read might be a candidate for a counterexample on ℓ_1. However, this is not true. V. Troitsky [57] has shown recently that the modulus of C. J. Read's operator does have a non-trivial closed invariant subspace.

In our previous discussion, we were considering only operators on ℓ_p-spaces. However, since we never used the specifics of the geometry of ℓ_p-spaces, all of our results and proofs remain true for operators on arbitrary discrete Banach lattices, in particular, on the Lorentz and Orlicz sequence spaces. For instance, the following analogue of Theorem 5.1 is true.

THEOREM 5.6. *Let $T\colon E \to E$ be a continuous operator with modulus, where E is a discrete Banach lattice. If there exists a non-zero positive operator $S\colon E \to E$ such that*

1. *$S|T| \le |T|S$ (in particular this holds if S commutes with $|T|$), and*

2. *S is quasinilpotent at a non-zero positive vector,*

then T has a non-trivial closed invariant subspace.

The situation with non-discrete spaces is considerably more complicated and will be discussed in Section 10.

6. Cycles and local quasinilpotence

In this section we continue to deal with operators on ℓ_p-spaces with $1 \le p < \infty$. Our main objective is to study the local quasinilpotence in connection with some other properties that will be introduced below. These properties are strong enough to imply local quasinilpotence, are simple enough to be verified, and therefore imply several invariant subspace results. As usual, $\mathbf{e}_n$ is the vector whose n^{th} coordinate is one and every other is zero; $\mathbf{e}_n$ can be viewed either as a column or row vector. Without any further mention, we shall identify the operators on ℓ_p with the (infinite) matrices representing them. Keep in mind that for an operator $A = [a_{ij}]\colon \ell_p \to \ell_p$ the n^{th}-column of the matrix $[a_{ij}]$ coincides with the column vector $A\mathbf{e}_n$. The proof of the next lemma is trivial and is omitted.

LEMMA 6.1. *A positive operator on an ℓ_p-space is quasinilpotent at some positive vector if and only if it is quasinilpotent at some $\mathbf{e}_j$.*

Let us say that an operator $A = [a_{ij}]\colon \ell_p \to \ell_p$ **essentially shifts** a vector x_0 if, for each k, the k^{th} coordinate of the vector $A^n x_0$ is eventually zero. Equivalently, A essentially shifts x_0 if and only if for each k there exists some n_0 such that $(A^n x_0)_i = 0$ for all $n \geq n_0$ and all $1 \leq i \leq k$.

THEOREM 6.2. *If a continuous operator $A = [a_{ij}]$ on ℓ_1 essentially shifts some $\mathbf{e}_j$, and $\|A\mathbf{e}_n\| \to 0$ as $n \to \infty$, then A is quasinilpotent at $\mathbf{e}_j$.*

Proof. For simplicity we denote the ℓ_1-norm by $\|\cdot\|$, i.e., we write $\|x\|$ instead of $\|x\|_1$ for $x \in \ell_1$. Fix $\epsilon > 0$ and pick some k such that $\|A\mathbf{e}_n\| < \epsilon$ for all $n \geq k$.

Since A essentially shifts $\mathbf{e}_j$ there exists some $m > k$ such that $(A^n\mathbf{e}_j)_i = 0$ for all $n \geq m$ and each $1 \leq i \leq k$. So, if $A^n\mathbf{e}_j = \sum_{i=1}^{\infty} \alpha_i^{(n)}\mathbf{e}_i$, then, in actuality, $A^n\mathbf{e}_j = \sum_{i=k+1}^{\infty} \alpha_i^{(n)}\mathbf{e}_i$ and hence $\|A^n\mathbf{e}_j\| = \sum_{i=k+1}^{\infty} \|\alpha_i^{(n)}|$. Therefore, for $n \geq m$ we have

$$\begin{aligned} \|A^{n+1}\mathbf{e}_j\| &= \Big\|\sum_{i=k+1}^{\infty} \alpha_i^{(n)} A\mathbf{e}_i\Big\| \leq \max_{i>k} \|A\mathbf{e}_i\| \Big(\sum_{i=k+1}^{\infty} |\alpha_i^{(n)}|\Big) \\ &\leq \epsilon\big\|A^n\mathbf{e}_j\big\|\,. \end{aligned}$$

In particular, for $n = m + r$, we have $\|A^n\mathbf{e}_j\| = \|A^{m+r}\mathbf{e}_j\| \leq \epsilon^r\|A^m\mathbf{e}_j\|$, and therefore $\big\|A^n\mathbf{e}_j\big\|^{\frac{1}{n}} \leq \big\|A^m\mathbf{e}_j\big\|^{\frac{1}{n}}\epsilon^{1-\frac{m}{n}}$ for all $n > m$.

Hence, $\limsup_{n\to\infty} \|A^n \mathbf{e}_j\|^{\frac{1}{n}} \leq \epsilon$ for each $\epsilon > 0$, from which it follows that $\|A^n \mathbf{e}_j\|^{\frac{1}{n}} \to 0$. ☐

To illustrate the preceding theorem we consider the same matrix that was used in Section 5:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Clearly, $\|A\mathbf{e}_n\| = \frac{1}{n}$ for each $n \geq 2$, and also A essentially shifts the vector $\mathbf{e}_2$. So, from Theorem 6.2, A is quasinilpotent at $\mathbf{e}_2$.

Also note that an essential shifting alone (i.e., without any additional assumption) does not imply that A is quasinilpotent at some $\mathbf{e}_j$. For instance, consider, the usual forward shift operator on ℓ_2 defined via the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then A essentially shifts every $\mathbf{e}_j$ but, being an isometry, A fails to be locally quasinilpotent.

DEFINITION 6.3. *We shall say that a matrix* $A = [a_{ij}]$ *has a* **cycle starting at column** j*, if there exist* $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ *satisfying the following conditions:*

1. $j_1 = j, j_2 = i_1, j_3 = i_2, \ldots, j_k = i_{k-1}$,
2. $i_k = j_\nu$, *for some* $1 \leq \nu \leq k$, *and*
3. $a_{i_m j_m} \neq 0$ *for all* $1 \leq m \leq k$

Let us notice immediately the following two simple facts:

a. If $a_{ii} \neq 0$ for some i, then (i,i) is a cycle for A starting at column i. In particular, if A does not have a cycle starting at column j, then $a_{jj} = 0$.

b. More generally, if $a_{ij} \neq 0$ and $a_{ii} \neq 0$, then $(i,j),(i,i)$ is a cycle for A starting at column j.

LEMMA 6.4. *Let a non-negative matrix A have a cycle, say*

$$(i_1, j_1), \dots, (i_k, j_k)$$

with $j_1 = j$ and $i_k = j_\nu$ for some $1 \leq \nu \leq k$. Then there exists some constant $c > 0$ such that for each $1 \leq r \leq k$ there exists some $1 \leq m \leq k$ satisfying $A\mathbf{e}_{j_r} \geq c\mathbf{e}_{j_m}$.

Proof. Let $(i_1, j_1), \dots, (i_k, j_k)$ with $j_1 = j$ and $i_k = j_\nu$ for some $1 \leq \nu \leq k$ be a cycle of A starting at column j and let us put $c = \min\{a_{i_r j_r}\colon\ 1 \leq r \leq k\}$. Clearly, $c > 0$.

If $1 \leq r < k$, then $A\mathbf{e}_{j_r} \geq a_{i_r j_r}\mathbf{e}_{i_r} = a_{i_r j_r}\mathbf{e}_{j_{r+1}} \geq c\mathbf{e}_{j_{r+1}}$. For $r = k$, note that $A\mathbf{e}_{j_k} \geq a_{i_k j_k}\mathbf{e}_{i_k} = a_{i_k j_k}\mathbf{e}_{j_\nu} \geq c\mathbf{e}_{j_\nu}$. □

COROLLARY 6.5. *If a non-negative matrix $A\colon \ell_p \to \ell_p$ has a cycle starting at some column j, then A cannot essentially shift $\mathbf{e}_j$.*

The converse of Corollary 6.5 is not true. That is, there exist non-negative matrices that do not essentially shift some $\mathbf{e}_j$ but nevertheless have no cycles starting at column j. Here is an example.

EXAMPLE 6.6. Consider the positive operator $A\colon \ell_p \to \ell_p$ whose action on the basic vectors is as follows:

1. $A\mathbf{e}_1 = \frac{1}{2}\mathbf{e}_3 + \frac{1}{4}\mathbf{e}_5 + \frac{1}{8}\mathbf{e}_7 + \cdots = \left(0, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, \dots\right)$,
2. $A\mathbf{e}_3 = \mathbf{e}_2$,
3. $A\mathbf{e}_{2k} = \mathbf{e}_{2k+2}$ for $k = 1, 2, \dots$, and
4. $A\mathbf{e}_{2k+1} = \mathbf{e}_{2k-1}$ for $k = 2, 3, \dots$.

A moments' thought reveals that A is the following matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Next notice that $\|A^n \mathbf{e}_1\|_p = \bigl(\sum_{i=1}^{\infty} \frac{1}{2^{ip}}\bigr)^{\frac{1}{p}} = c > 0$ holds for each n. This implies that A is not quasinilpotent at $\mathbf{e}_1$. Moreover, observe that $(A^n \mathbf{e}_1)_2 \neq 0$ for all $n \geq 2$, and this shows that A does not essentially shift $\mathbf{e}_1$. Finally, a straightforward verification shows that there is no cycle starting at column 1 (and, as a matter of fact, no cycle starting at any other column either). □

COROLLARY 6.7. *If a non-negative matrix* $A = [a_{ij}]\colon \ell_p \to \ell_p$ *is quasinilpotent at some vector* $\mathbf{e}_j$*, then there is no cycle starting at column* j.

Proof. Assume by way of contradiction that A has a cycle starting at column j, say $(i_1, j_1), \dots, (i_k, j_k)$ with $j_1 = j$ and $i_k = j_\nu$ for some $1 \leq \nu \leq k$.

By Lemma 6.4 there exists a constant $c > 0$ such that for each $1 \leq r \leq k$ there exists some $1 \leq m \leq k$ satisfying $A\mathbf{e}_{j_r} \geq c\mathbf{e}_{j_m}$. Now an easy inductive argument shows that for each n there exists some index m_n such that $1 \leq m_n \leq k$ and $A^n \mathbf{e}_j \geq c^n \mathbf{e}_{j_{m_n}}$. This implies $\|A^n \mathbf{e}_j\|^{\frac{1}{n}} \geq c > 0$ for each n, and so A is not quasinilpotent at $\mathbf{e}_j$, a contradiction. So A has no cycle starting at column j. □

In view of Lemma 6.1, the preceding corollary implies also the following.

COROLLARY 6.8. *If a non-negative matrix* $A = [a_{ij}]\colon \ell_p \to \ell_p$ *has a cycle starting at each column* j *(in particular, if every diagonal entry of* A *is positive), then* A *is quasinilpotent at no positive vector.*

Related to the cycle concept is the notion of a path, which is introduced next.

DEFINITION 6.9. *Let $A = [a_{ij}]$ be a matrix, and let (r,s) and (u,v) be two pairs of natural numbers. We shall say that A has a* **path** *from (r,s) to (u,v) provided there exist $(i_1,j_1),(i_2,j_2),\dots,(i_k,j_k)$ satisfying the following conditions:*

1. *$(i_1,j_1) = (r,s)$ and $(i_k,j_k) = (u,v)$,*
2. *$j_2 = i_1, j_3 = i_2, \dots, j_k = i_{k-1}$, and*
3. *$a_{i_n j_n} \neq 0$ for each $n = 1,2,\dots,k$.*

We shall say that there is a **path** *joining column s to column v if for some r and u there exists a path from (r,s) to (u,v).*

The reader should notice immediately the difference between a path and a cycle: A cycle is automatically a path. But a path $(i_1,j_1),(i_2,j_2),\dots,(i_k,j_k)$ is a cycle if and only if $i_k = j_\nu$ for some $1 \leq \nu \leq k$.

LEMMA 6.10. *Let $A = [a_{ij}] \colon \ell_p \to \ell_p$ be a matrix. If, for some n, s and r we have $(A^n \mathbf{e}_s)_r \neq 0$ and $A\mathbf{e}_r \neq 0$, then there is a path joining column s with column r.*

Proof. The proof is by induction on n. For $n = 1$, we have $(A\mathbf{e}_s)_r = a_{rs} \neq 0$ and $(A\mathbf{e}_r)_\nu = a_{\nu r} \neq 0$ for some ν. Then $(r,s),(\nu,r)$ is a path joining column s to column r. For the induction step, assume that our claim is true for some $n \geq 1$ and suppose that $(A^{n+1}\mathbf{e}_s)_r \neq 0$ and $A\mathbf{e}_r \neq 0$.

Let $A^n\mathbf{e}_s = \sum_{\mu=1}^\infty \lambda_\mu \mathbf{e}_\mu$. Then $A^{n+1}\mathbf{e}_s = \sum_{\mu=1}^\infty \lambda_\mu A\mathbf{e}_\mu$. Since $(A^{n+1}\mathbf{e}_s)_r \neq 0$ there exists some m such that $\lambda_m \neq 0$ and $(A\mathbf{e}_m)_r = a_{rm} \neq 0$. In particular, we have $A\mathbf{e}_m \neq 0$. Moreover, from $A^n\mathbf{e}_s = \sum_{\mu=1}^\infty \lambda_\mu \mathbf{e}_\mu$, we see that $(A^n\mathbf{e}_s)_m \neq 0$. So, by our induction hypothesis, there exists a path

$$(r,s),(i_2,r),\dots,(m,j_{k-1}),(i_k,m)$$

joining column s to column m. Now note that if $(A\mathbf{e}_r)_t = a_{tr} \neq 0$, then

$$(r,s),(i_2,r),\dots,(m,j_{k-1}),(r,m),(t,r)$$

is a path joining column s to column r. □

As noticed before, if a matrix $A = [a_{ij}]$ has no cycle starting at some column j, then the (j,j) entry of A is zero, i.e., $a_{jj} = 0$. Moreover, as we shall see next, in this case every (j,j) entry of each power of A is also zero.

LEMMA 6.11. *If a matrix $A = [a_{ij}]$ defines a bounded operator on some ℓ_p-space and has no cycle starting at some column j, then*

$$(A^n \mathbf{e}_j)_j = 0 \ \text{ for all } \ n = 1, 2, \dots \ .$$

Proof. Assume that A has no cycle starting at column j. In particular, this implies that $(A\mathbf{e}_j)_j = a_{jj} = 0$. Now assume by way of contradiction that $(A^n\mathbf{e}_j)_j \neq 0$ for some n, i.e., assume that the set $J = \{n\colon\ (A^n\mathbf{e}_j)_j \neq 0\}$ is non-empty. Let $k = \min J$. Since $1 \notin J$ we have $k > 1$. Now consider the vector $v = A^{k-1}\mathbf{e}_j = \sum_{m=1}^{\infty} \lambda_m \mathbf{e}_m$. Then $Av = A^k\mathbf{e}_j = \sum_{m=1}^{\infty} \lambda_m A\mathbf{e}_m$, and so

$$\sum_{m=1}^{\infty} \lambda_m (A\mathbf{e}_m)_j = \Big(\sum_{m=1}^{\infty} \lambda_m A\mathbf{e}_m\Big)_j = (A^k\mathbf{e}_j)_j \neq 0 \,.$$

This implies that for some m we have

$$\lambda_m \neq 0 \quad \text{and} \quad (A\mathbf{e}_m)_j = a_{jm} \neq 0 \,.$$

Since $(A^{k-1}\mathbf{e}_j)_m = \lambda_m \neq 0$ and $(A\mathbf{e}_m)_j \neq 0$, Lemma 6.10 guarantees that there is a path joining column j and column m, say

$$(r,j), (i_2, r), \dots, (m, j_{s-1}), (i_s, m) \,.$$

But then $(r,j), (i_2,r), \dots, (m, j_{s-1}), (j,m), (r,j)$ is a cycle starting at column j, a contradiction. Hence $(A^n\mathbf{e}_j)_j = 0$ for all n. □

To illustrate the developed theory we shall present two invariant subspace theorems in terms of cycles of the matrices defining the operators.

THEOREM 6.12. *If a matrix A defines a bounded operator on some ℓ_p-space and has a column with no cycle starting at that column, then A has a non-trivial closed invariant subspace.*

If, in addition, A is a positive operator, then the non-trivial closed A-invariant subspace can be taken to be an order ideal.

Proof. Assume the matrix $A = [a_{ij}]$ defines a bounded operator on some ℓ_p-space and that there exists a column j with no cycle starting there. From Lemma 6.11 we know that $(A^n\mathbf{e}_j)_j = 0$ for all n.

If $A\mathbf{e}_j = 0$, then $V = \{\alpha\mathbf{e}_j\colon\ \alpha \text{ scalar}\}$ is a non-trivial closed invariant ideal for A. So, we can assume that $A\mathbf{e}_j \neq 0$.

Next, let V denote the closed vector subspace generated by the set $\{A\mathbf{e}_j, A^2\mathbf{e}_j, \dots\}$, i.e., $V = \overline{\operatorname{span}\{A\mathbf{e}_j, A^2\mathbf{e}_j, A^3\mathbf{e}_j, \dots\}}$. Since $A\mathbf{e}_j \neq 0$ and $A\mathbf{e}_j \in V$, we see that $V \neq \{0\}$. On the other hand, it follows from $(A^n\mathbf{e}_j)_j = 0$ $(n = 1, 2, \dots)$ that every $x \in V$ satisfies $x_j = 0$ and consequently $V \neq \ell_p$. That is, V is a non-trivial closed subspace of ℓ_p. Clearly, V is A-invariant.

Now assume that A is also non-negative and let W denote the ideal generated by V. That is,

$$W = \Big\{x \in \ell_p\colon\ \exists\ c > 0 \text{ and } m \text{ such that } |x| \leq c\sum_{i=1}^{m} A^i\mathbf{e}_j\Big\}.$$

Finally, if $x \in W$, then pick $c > 0$ and m satisfying $|x| \leq c\sum_{i=1}^{m} A^i\mathbf{e}_j$ and note that

$$|Ax| \leq A|x| \leq c\sum_{i=1}^{m} A^{i+1}\mathbf{e}_j \leq c\sum_{i=1}^{m+1} A^i\mathbf{e}_j\,.$$

This shows that W is A-invariant and hence the closed order ideal generated by V (which is the norm closure $\overline{W}$ of W) is also A-invariant. Now notice that every $x \in \overline{W}$ satisfies $x_j = 0$, and so $\overline{W}$ is a non-trivial A-invariant ideal. □

Since (by Corollary 6.7) every non-negative matrix which is quasinilpotent at some vector $\mathbf{e}_j$ has no cycle starting at column j, our Theorem 6.12 should be viewed as a generalization of the results in Section 5. Now, we are ready to prove a general invariant subspace theorem.

THEOREM 6.13. *If a matrix A defines a regular operator on some ℓ_p-space and has a column with no cycle starting at that column, then A has a non-trivial closed invariant order ideal.*

Proof. Assume that the matrix $A = [a_{ij}]$ has no cycle starting at some column j. Then its modulus matrix $|A| = \big[|a_{ij}|\big]$ does not have a cycle starting at column j either. By Theorem 6.12 there exists a non-trivial closed order ideal V which is $|A|$-invariant. But if $x \in V$, then $|x| \in V$ and so from $|Ax| \leq |A||x| \in V$, we see that $Ax \in V$. That is, V is also A-invariant. □

Since every bounded operator on ℓ_1 has a modulus (see [14, Theorem 15.3, p. 249]), we immediately have the following consequence of the preceding result.

COROLLARY 6.14. *If a matrix A defines a bounded operator on ℓ_1 and has a column with no cycle starting at that column, then A has a non-trivial closed invariant order ideal.*

7. Spaces with a Schauder basis

A subset C of a (real or complex) vector space X is said to be a **cone** whenever $C + C \subseteq C$, $\alpha C \subseteq C$ for each real $\alpha \geq 0$, and $C \cap (-C) = \{0\}$. Every cone C determines a partial order $\geq$ on X by letting $y \geq x$ whenever $y - x \in C$; in particular $C = \{x \in X\colon\ x \geq 0\}$ and the elements of C are referred to as **positive vectors**. The notation $x \leq y$ is, of course, equivalent to $y \geq x$. A (partially) **ordered vector space** (X, C) is a vector space X equipped with a cone C. For a detailed account about cones and partially ordered vector spaces, we refer the reader to [49]. In our presentation in this section we follow [5].

As usual, an operator $T\colon X \to X$ on an ordered vector space (X, C) is said to be **positive** if $Tx \geq 0$ for each $x \geq 0$. For a positive operator T, it follows that $Ty \leq Tx$ whenever $y \leq x$ holds. The notation $T \geq S$ means $T - S \geq 0$, or equivalently $Tx \geq Sx$ for each $x \geq 0$.

Recall that a sequence $\{x_n\}$ in a Banach space X is called a **Schauder basis** (or simply a **basis**) of X if for every $x \in X$ there exists a unique sequence of scalars $\{\alpha_n\}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$ (the convergence of the series is in the norm topology on X). Associated with the basis is the standard sequence of "coefficient functionals" f_n $(n = 1, 2, \dots)$ defined by

$$f_n(x) = \alpha_n \quad \text{for} \quad x = \sum_{i=1}^{\infty} \alpha_i x_i \in X.$$

Obviously each f_n is a linear functional on X, and, as is well known (but not trivial), each of these functionals is continuous, i.e., $f_n \in X'$ for each n. Notice that $f_n(x_m) = \delta_{nm}$.

Every basis $\{x_n\}$ gives rise to a closed cone C defined by

$$C = \Big\{ x = \sum_{n=1}^{\infty} \alpha_n x_n \colon \ \alpha_n \geq 0 \ \text{ for each } \ n = 1, 2, \ldots \Big\}.$$

The cone C will be referred to as the cone generated by the basis $\{x_n\}$. (Exercise: use the continuity of the coefficient functionals to show that C is indeed closed.) Observe that each f_n is automatically positive with respect to the cone generated by the basis $\{x_n\}$. For an extensive discussion concerning the cone generated by a basis see [56]. It should be pointed out that the ordered Banach space (X, C) defined this way is not a Banach lattice in general. (For this to happen the basis generating the cone should be much "nicer" than just a mere Schauder basis. Namely, the basis should be unconditional, and in this case, after an equivalent renorming, we get a discrete Banach lattice with order continuous norm.)

An operator $T \colon X \to X$ on a Banach space with a basis $\{x_n\}$ is said to be positive (with respect to this basis) if $T(C) \subseteq C$, where C is the cone generated by $\{x_n\}$.

As soon as a basis for a Banach space X is fixed, every operator $T \colon X \to X$ can be identified in the usual manner with an infinite matrix $[t_{ij}]$. In this context, we can also say that an infinite matrix $[t_{ij}]$ defines an operator on X. Note that an operator $T \colon X \to X$ with matrix $[t_{ij}]$ is a positive operator if and only if $t_{ij} \geq 0$ holds for each pair (i, j). If the basis $\{x_n\}$ is also unconditional, then every positive operator is automatically continuous; see [2, Corollary 2.5, p. 4] or [14, Theorem 12.3, p. 175].

We are now ready to extend the results from Section 5 to operators acting on a Banach space with a basis. If a basis is specified, then all notions of positivity are understood with respect to the cone generated by this basis. As we shall show, the order structure determined by a basis implies an analogue of Theorem 5.1. We want to emphasize that since the ordered Banach spaces we are dealing with now are not, in general, Banach lattices, the results in this section need direct proofs and cannot be deduced from similar results valid for Banach lattices.

THEOREM 7.1. *Let $T: X \to X$ be a continuous positive operator on a Banach space with a basis. If T commutes with a non-zero positive operator that is quasinilpotent at a non-zero positive vector, then T has a non-trivial closed invariant subspace.*

Proof. Let $\{x_n\}$ be a basis of the Banach space X and let $\{f_n\}$ be the sequence of coefficient functionals associated with the basis $\{x_n\}$.

Assume that the non-zero positive operator $A: X \to X$ satisfies $TA = AT$ and is quasinilpotent at some non-zero positive vector y_0, i.e., $\lim_{n\to\infty} \|A^n y_0\|^{\frac{1}{n}} = 0$. If $Ay_0 = 0$ then the kernel of A is a non-trivial closed subspace that is invariant under T. Thus, we can suppose that Ay_0 is non-zero. By an appropriate scaling of y_0, we can assume that $0 \le x_k \le y_0$ and $Ax_k \neq 0$ for some k.

Now let $P: X \to X$ denote the continuous projection onto the vector subspace generated by x_k defined by $P(x) = f_k(x)x_k$. Clearly, $0 \le Px \le x$ holds for each $0 \le x \in X$. We claim that

$$PT^m Ax_k = 0 \qquad (\star)$$

for each $m \ge 0$. To see this, fix $m \ge 0$ and let $PT^m Ax_k = \alpha x_k$ for some non-negative scalar $\alpha \ge 0$. Since P is a positive operator and the composition of positive operators is a positive operator, it follows that

$$0 \le \alpha^n x_k = \big(PT^m A\big)^n x_k \le \big(T^m A\big)^n x_k = T^{mn} A^n x_k \le T^{mn} A^n y_0.$$

(The "natural" desire to say that the previous inequality implies the inequality $\alpha^n \|x_k\| \le \|T^{mn} A^n y_0\|$ would be wrong since the norm need not be monotone.) The trick is to use the fact that f_k is a positive linear functional, and so the above inequality yields

$$0 \le \alpha^n = f_k\big(\alpha^n x_k\big) \le f_k\big(T^{mn} A^n y_0\big).$$

Consequently, $0 \le \alpha^n \le \|f_k\| \, \big\|T\big\|^{mn} \cdot \big\|A^n y_0\big\|$, and so

$$0 \le \alpha \le \|f_k\|^{\frac{1}{n}} \big\|T\big\|^m \cdot \big\|A^n y_0\big\|^{\frac{1}{n}}.$$

From $\lim_{n\to\infty} \|A^n y_0\|^{\frac{1}{n}} = 0$, we see that $\alpha = 0$, and thus condition $(\star)$ must be true.

Now consider the linear subspace Y of X generated by the set $\{T^m A x_k\colon\ m = 0, 1, \dots\}$. Clearly, Y is invariant under T and since $0 \neq A x_k \in Y$, we see that $Y \neq \{0\}$. From $(\star)$ it follows now that $f_k(T^m A x_k) x_k = P(T^m A x_k) = 0$, so that $f_k(T^m A x_k) = 0$ for each m. This implies $f_k(y) = 0$ for each $y \in Y$, and consequently $f_k(y) = 0$ for all $y \in \overline{Y}$. The latter shows that $\overline{Y}$ is a non-trivial closed vector subspace of X that is invariant under the operator T, and the proof is complete. □

COROLLARY 7.2. *Let X be a Banach space with a basis. If T is a continuous quasinilpotent positive operator on X, then T has a non-trivial closed invariant subspace.*

As before, we can add arbitrary weights to the matrix representing a quasinilpotent positive operator and still be guaranteed that a non-trivial closed invariant subspace exists.

THEOREM 7.3. *Let X be a Banach space with a basis. Assume that a positive matrix $A = [a_{ij}]$ defines a continuous operator on X that is quasinilpotent at a non-zero positive vector. If for a double sequence $\{w_{ij}\}$ of complex numbers, the weighted matrix $B = [w_{ij} a_{ij}]$ defines a continuous operator B on X, then the operator B has a non-trivial closed invariant subspace.*

Proof. Let $\{x_n\}$ be a basis of the Banach space X, and let $\{f_n\}$ be the sequence of coefficient functionals associated with the basis $\{x_n\}$. Assume that the positive operator A satisfies $\lim_{n\to\infty} \|A^n y_0\|^{\frac{1}{n}} = 0$ for some positive non-zero vector y_0. An appropriate scaling of y_0 shows that there exists some k satisfying $0 \le x_k \le y_0$. If $A x_k = 0$, then an easy argument shows that $B x_k = 0$, and thus the kernel of B is a non-trivial closed invariant subspace (here we assume, of course, that $B \neq 0$). Thus, we can suppose that $A x_k$ is non-zero.

Now let $P\colon X \to X$ denote the positive projection defined by $P(x) = f_k(x) x_k$. Arguing as in the proof of Theorem 7.1, we can establish that $P A^m x_k = 0$ for each $m \ge 1$. In particular, we have $f_k\big(A^m x_k\big) = 0$ for each $m \ge 1$. Consequently, for each $m \ge 1$ and for each positive operator $S\colon X \to X$ satisfying $0 \le S \le A^m$ we have

$$0 \le f_k\big(Sx_k\big) \le f_k\big(A^m x_k\big) = 0. \qquad (\star\star)$$

Next, consider the vector subspace Y generated by the set

$$\big\{Sx_k\colon \ \exists \ S \ \text{ such that } \ 0 \le S \le A^m \ \text{ for some } \ m \ge 1\big\}.$$

Clearly, Y is invariant under each operator R satisfying $0 \le R \le A$. Also, from $(\star\star)$, it follows that

$$f_k(y) = 0$$

for all $y \in \overline{Y}$. The latter shows that $\overline{Y}$ is a non-trivial closed vector subspace of X that is invariant under each operator $R\colon X \to X$ satisfying $0 \le R \le A$.

Next, consider the operator A_{ij} defined by $A_{ij}(x_j) = a_{ij}x_j$ and $A_{ij}(x_m) = 0$ for $m \ne j$. Since the operator satisfies $0 \le A_{ij} \le A$, it follows that $\overline{Y}$ is invariant under each of the operators A_{ij}. Therefore, the vector subspace $\overline{Y}$ is invariant under the operators

$$B_n = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}A_{ij}.$$

However, the sequence of operators $\{B_n\}$ converges in the strong operator topology to B. Therefore, $B(\overline{Y}) \subseteq \overline{Y}$ holds, and thus, the operator B has a non-trivial closed invariant subspace. ☐

COROLLARY 7.4. *Let X be a Banach space with a basis. Assume that a positive matrix $A = [a_{ij}]$ defines a continuous operator on X which is quasinilpotent at a non-zero positive vector. If a continuous operator $T\colon X \to X$ is defined by a matrix $T = \big[t_{ij}\big]$ satisfying $t_{ij} = 0$ whenever $a_{ij} = 0$, then the operator T has a non-trivial closed invariant subspace.*

As mentioned earlier, if a Banach space X has an unconditional basis, then (up to an equivalent norm) X is a discrete Banach lattice. Therefore, some of the results obtained in Section 5 are special cases of the results obtained here.

There exists one more concept of a basis which is weaker than the notion of Schauder basis. We mean the so called Markushevich basis or, to be more precise, a variety of Markushevich bases depending on the additional conditions imposed on the system of vectors (see for example [56]). It would be interesting to investigate to what extent the results of this section can be generalized to positive operators on a Banach space with some type of a Markushevich basis.

We conclude with an important open question. Consider a quasinilpotent operator on a Banach space with a basis. Suppose we do not assume that the operator is positive with respect to this basis. At first glance, it appears that our invariant subspace theorems do not apply. However, if one considers a change of basis, then the operator might become positive with respect to the new basis, and therefore, it would have a non-trivial closed invariant subspace. Here is a simple example that illustrates this point. Consider the operator

$$T = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

on $\mathbb{R}^2$ with the standard basis $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$. If we introduce the new basis $\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_1 - \mathbf{e}_2$, then it is trivial to verify that in this basis the operator T has the matrix representation

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

and, as we see, in this basis the matrix of the same operator T positive. It would be very interesting to find out when a given quasinilpotent operator on a Banach space with a Schauder basis (in particular, on a Hilbert space) can be made positive with respect to some basis. This problem seems to be open even for finite dimensional spaces.

8. Lomonosov's theorem

In 1973, V. I. Lomonosov [42] proved the following remarkable theorem.

THEOREM 8.1 (LOMONOSOV). *If a continuous operator T on a Banach space commutes with an operator S which is not a multiple of* *the identity, and S in turn commutes with a non-zero compact operator, then the operator T has a non-trivial closed hyperinvariant subspace.*

We will prove below a somewhat weaker version of this result using a very simple and elegant proof found by H. M. Hilden and presented in [45]. This version is strong enough to imply immediately the Aronszajn–Smith theorem. The proof of Theorem 8.1 can be found in [42] or [50].

THEOREM 8.2 (LOMONOSOV). *Let $K\colon X \to X$ be a non-zero compact operator on a Banach space. Then K has a non-trivial closed hyperinvariant subspace.*

Proof. If K has any eigenvector x_0, that is, $Kx_0 = \lambda x_0$ for some $\lambda \in \mathbf{C}$, then the closed proper subspace $N_\lambda = \{x \in X : Kx = \lambda x\}$ is K-hyperinvariant.

Therefore we can assume that the operator K has no eigenvectors and so, by the Fredholm theorem, K is quasinilpotent, and thus $r(K) = \lim \|K^n\|^{1/n} = 0$. This implies that $\lim \|(cK)^n\| = 0$ for any scalar c. Since the existence of an invariant subspace for any operator is obviously independent of the norm of that operator, we can assume that $\|K\| = 1$.

For each $x \neq 0$ consider the linear subspace X_x generated by the action of the commutant of K on x, i.e., $X_x = \{Tx : T \in \{K\}'\}$. It is easy to see that $T(X_x) \subset X_x$ for each $T \in \{K\}'$. Therefore, if there exists at least one $x \neq 0$ for which X_x is not dense in X, then the norm closure $\overline{X_x}$ is the required closed nontrivial hyperinvariant subspace of K. To finish the proof we will show that the assumption

$$\overline{X_x} = X \quad \text{for each} \quad x \neq 0 \tag{$\star$}$$

contradicts the condition $r(K) = 0$.

Fix any $x_0 \in X$ with $\|Kx_0\| > 1$ and consider the closed unit ball $U = \{x \in X\colon\ \|x - x_0\| \leq 1\}$ centered at x_0. Since $\|K\| = 1$ and $\|Kx_0\| > 1$, we have

$$0 \notin U = \overline{U} \qquad \text{and} \qquad 0 \notin \overline{K(U)}. \tag{$\star\star$}$$

In view of ($\star$) we know that for each $x \neq 0$ there exists an operator $T \in \{K\}'$ such that $Tx \in U$, i.e., $\|Tx - x_0\| < 1$. Consequently, when

T runs over $\{K\}'$ the open sets $U_T = \{y \in X : \|Ty - x_0\| < 1\}$ cover $X \setminus \{0\}$ and, in particular, the compact set $\overline{K(U)}$. Consequently, we can find a finite number of operators $T_1, \dots, T_n \in \{K\}'$ such that $\overline{K(U)} \subseteq \bigcup_{j=1}^{n} U_{T_j}$. Let $c = \max\{\|T_j\| : 1 \leq j \leq n\}$.

Since $Kx_0 \in K(U)$ there exists j_1 such that $Kx_0 \in U_{T_{j_1}}$, and thus $x_1 = T_{j_1}Kx_0 \in U$. Then $Kx_1 \in K(U)$ and so there is j_2 such that $x_2 = T_{j_2}Kx_1 \in U$. After m steps we produce the indices $j_1, j_2, \dots, j_m \in \{1, 2, \dots, n\}$ such that $x_m = T_{j_m}Kx_{m-1} \in U$. Recalling that each operator T_j commutes with K we can rewrite the previous expression for x_m as follows:

$$\begin{aligned} x_m &= T_{j_m}T_{j_{m-1}} \cdots T_{j_1}K^m x_0 \\ &= (c^{-1}T_{j_m})(c^{-1}T_{j_{m-1}}) \cdots (c^{-1}T_{j_1})(cK)^m x_0 \in U \,. \end{aligned}$$

Since $\|c^{-1}T_{j_k}\| \leq 1$ for each $k = 1, \dots, m$ and since $\|(cK)^m\| \to 0$ as $m \to \infty$ we get $\|x_m\| \to 0$, contrary to the first condition in $(\star\star)$. This contradiction proves that X_x cannot be dense for all $x \neq 0$, and the proof is finished. □

COROLLARY 8.3. *Every continuous operator which commutes with a non-zero compact operator has a non-trivial closed invariant subspace.*

Proof. Let $T \colon X \to X$ be a continuous operator which commutes with a non-zero compact operator K. Now we can apply to K the previous theorem. □

There exists extensive literature devoted to various generalizations of Lomonosov's theorem; see for example [47, 48, 50]. From the multitude of existing results, we will mention here only one, which can be found in several places: [47, Theorem 7.17], [24, Theorem 2] and [33] (see also [38, 39]). This result is of special interest to us since it suggests many possible generalizations to operators on Banach lattices.

THEOREM 8.4. *If $T \colon X \to X$ is a non-scalar continuous operator on a Banach space and there exists a non-zero compact operator $K \colon X \to X$ such that $TK = \lambda KT$ for some scalar λ, then T has a non-trivial closed hyperinvariant subspace.*

9. The dominance property

We start with the definition of a property which is slightly more general than the usual order relation between operators on Banach lattices.

DEFINITION 9.1. *Let $T, B\colon E \to E$ be two operators on a Banach lattice with B positive. We say that the operator T is* **dominated** *by the operator B provided*

$$\big|T(x)\big| \le B\big(|x|\big)$$

for each $x \in E$.

It should be clear that every operator dominated by a positive operator is automatically continuous, and that a positive operator T is dominated by another positive operator B if and only if $0 \le T \le B$. When E is order complete, an operator T is dominated by a positive operator B if and only if T is regular and $|T| \le B$ holds. Here we use the fact that every regular operator T on an order complete Banach lattice has a modulus $|T|$ which is given by Kantorovich's formula

$$|T|(x) = \sup\big\{|Ty|\colon\ |y| \le x\big\},\ x \in E^+.$$

Operators dominated by compact positive operators enjoy many remarkable properties one of which is stated in the next theorem. It was proved in [13] for real Banach lattices; the proof for the complex case can be derived easily from the real one.

THEOREM 9.2 (ALIPRANTIS–BURKINSHAW). *If in the scheme of continuous operators $E \xrightarrow{M_1} F \xrightarrow{M_2} G \xrightarrow{M_3} H$ between (real or complex) Banach lattices each operator M_i is dominated by a compact positive operator, then $M_3M_2M_1$ is a compact operator.*

In particular, if an operator $T\colon E \to E$ on a Banach lattice is dominated by a compact operator, then T^3 is compact.

Recall that a vector subspace J of a Banach lattice E is said to be an (order) **ideal** if $|x| \le |y|$ and $y \in J$ imply that $x \in J$. For each $0 \le u \in E$ the **principal ideal** E_u generated by u is the ideal

$$E_u = \big\{x \in E\colon\ \exists\ \lambda > 0 \text{ such that } |x| \le \lambda u\big\}.$$

A positive element $u \in E$ is called a **quasi-interior point** whenever E_u is norm dense in E, i.e., $\overline{E_u} = E$.

The next result deals with domination properties of multiplication operators and will be useful in our study.

LEMMA 9.3. *For a Banach lattice with a quasi-interior point $u > 0$ the following properties hold.*

1. *For every non-zero element $y \in E_u$ there exists an operator $V\colon E \to E$ which carries y to a non-zero positive vector and V is dominated by the identity operator, i.e.,*

$$V(y) > 0 \quad \textit{and} \quad |V(x)| \le |x| \ \textit{ for all } \ x \in E.$$

2. *For every element v satisfying $0 \le v \le u$ there exists an operator $U\colon E \to E$ which carries u to v and U is dominated by the identity operator, i.e.,*

$$U(u) = v \quad \textit{and} \quad |U(x)| \le |x| \ \textit{ for all } \ x \in E.$$

Proof. Let $u > 0$ be a quasi-interior point in a (real or complex) Banach lattice E. By the well known Kakutani–Krein representation theorem (see for instance [14, Theorem 12.28]), there exists a compact Hausdorff space Ω such that E_u is lattice isomorphic to the space $C(\Omega)$ of all continuous functions on Ω, and the element u corresponds to the constant function one on Ω.

(1) Now fix $y \in E_u$ with $y \neq 0$ and view y as a continuous function on Ω. By scaling appropriately, we can suppose that $\|y\|_\infty = \max\{|y(\omega)|\colon \ \omega \in \Omega\} = 1$. Now consider the function $\overline{y} \in C(\Omega)$ (the complex conjugate of y) and denote by V the multiplication operator defined by the function $\overline{y}$ on $C(\Omega)$ (and hence on E_u). That is, $V(x) = \overline{y}x$ for each $x \in C(\Omega)$. Clearly, $V(y) = |y|^2 > 0$ and $|Vx| \le |x|$ holds for each $x \in E_u$. Since $\overline{E_u} = E$, the (unique) continuous extension of V to E satisfies the desired properties.

(2) Again, as above, we view v as a continuous function on Ω and consider the multiplication operator U defined on $C(\Omega)$ (and hence on E_u) by v, i.e, $U(x) = vx$ for each $x \in C(\Omega)$. Clearly, $U(u) = v$ and $|Ux| \le |x|$ for each $x \in E_u$. The (unique) continuous extension of U to all of E also satisfies $|Ux| \le |x|$ for each $x \in E$. $\square$

We shall close the section by stating two useful results concerning invariant subspaces of operators that are dominated by positive operators. We shall say that an operator T is **polynomially dominated** by a positive operator B whenever there exists a polynomial $p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$ with non-negative coefficients such that $p(B)$ dominates T.

LEMMA 9.4. *Let J be an ideal in a Banach lattice E. If J is invariant under some positive operator $B\colon E \to E$, then J is also invariant under every operator T which is polynomially dominated by B.*

Proof. Let $p(t)$ be a polynomial with non-negative coefficients such that $p(B)$ dominates T. If $x \in J$, then $|Tx| \le p(B)\big(|x|\big) \in J$ implies that $Tx \in J$, that is, J is T-invariant. □

Not every ideal is principal. However, as the next lemma tells us, each closed invariant ideal is "saturated" with invariant principal ideals.

LEMMA 9.5. *If J is a non-trivial closed ideal which is invariant under a positive operator $B\colon E \to E$, then J contains a non-trivial principal ideal which is also invariant under B.*

Proof. By scaling appropriately, we can assume that $\|B\| < 1$. Fix an arbitrary $0 < w \in J$ and let $u = \sum_{n=0}^{\infty} B^n(w)$. Since J is a closed B-invariant ideal, we see that $u \in J$. Therefore, $\overline{E_u} \subseteq J$ and (since $u > 0$) $E_u \neq \{0\}$. It remains to verify that E_u is B-invariant. To see this, take any $x \in E$ satisfying $|x| \le \lambda u$. Then

$$|Bx| \le B\big(|x|\big) \le \lambda B(u) = \lambda \sum_{n=0}^{\infty} B^{n+1}(w) \le \lambda u,$$

and so $Bx \in E_u$. □

10. Invariant subspace theorems for positive operators

We now come to our first basic result which describes a new large class of operators with non-trivial closed invariant subspaces. The proof below utilizes the order structure of Banach lattices and is inspired by M. Hilden's proof of Lomonosov's Theorem presented in Section 8.

THEOREM 10.1. *Let $B\colon E \to E$ be a positive operator on a Banach lattice. Assume that there exists a positive operator $S\colon E \to E$ such that*

1. $SB \le BS$ *(in particular, this holds if S commutes with B),*
2. *S is quasinilpotent at some $x_0 > 0$, i.e., $\lim_{n\to\infty} \|S^n x_0\|^{\frac{1}{n}} = 0$, and*
3. *S dominates a non-zero compact operator.*

Then the operator B has a non-trivial closed invariant subspace. Moreover, we can choose this invariant subspace to be the closure of a principal ideal in E.

Proof. Let B, S, and x_0 satisfy the properties stated in the theorem, and let K be a non-zero compact operator dominated by S, i.e., $|K(x)| \le S(|x|)$ for each $x \in E$. Obviously we can assume that $\|B\| < 1$. Then the series $A = \sum_{n=0}^{\infty} B^n$ defines a positive operator on E, and in view of the first condition on S, the inequalities $SB^k \le B^kS$ and $SA^k \le A^kS$ hold for each k.

For each $x > 0$ we denote by $J[x]$ the principal ideal generated by Ax. That is,

$$J[x] = \big\{y \in E\colon\ |y| \le \lambda Ax \text{ for some } \lambda > 0\big\}.$$

Since $x \in J[x]$, note that $J[x]$ is non-zero. Moreover, we claim that $J[x]$ is B-invariant. Indeed, if $y \in J[x]$, then $|y| \le \lambda Ax$ for some $\lambda > 0$, and so

$$|By| \le B|y| \le \lambda B(Ax) = \lambda \sum_{n=1}^{\infty} B^n x \le \lambda Ax,$$

which implies $By \in J[x]$. So, $\overline{J[x]}$ is a non-zero closed B-invariant ideal.

The proof will be finished, if we show that $\overline{J[x]} \ne E$ for some $x > 0$. To establish this, assume by way of contradiction that

$$\overline{J[x]} = E \quad \text{for each} \quad x > 0. \qquad (\star)$$

Now, we claim that without loss of generality, we can suppose $Kx_0 \neq 0$. To see this, consider the ideal $J[x_0]$. If $K(y) = 0$ for each $0 < y \in J[x_0]$, then $K = 0$ on $J[x_0]$, and consequently by $(\star)$, we get $K = 0$, a contradiction. Therefore, $K(y_0) \neq 0$ for some $0 < y_0 \in J[x_0]$. Since S is quasinilpotent at x_0 and $0 < y_0 \leq kAx_0$ for some k, it is easy to verify that S is also quasinilpotent at y_0. Now, replacing (if necessary) x_0 by y_0, we can assume $K(x_0) \neq 0$.

By scaling, we can also assume $\|K\| = 1$. Also, replacing x_0 by ax_0 for an appropriate scalar $a > 1$, we can suppose that $\|x_0\| > 1$ and $\|Kx_0\| > 1$. Now let $U = \{z \in E\colon\ \|x_0 - z\| \leq 1\}$ be the closed unit ball centered at $x_0 > 0$. By our choice of x_0, we have

$$0 \notin U \qquad \text{and} \qquad 0 \notin \overline{K(U)}. \tag{$\star\star$}$$

By $(\star)$, we know that $\overline{J[|x|]} = E$ for each $x \neq 0$. Hence, for each element $y \geq 0$ the sequence $\big\{y \wedge nA(|x|)\big\}$ is norm convergent and $\lim_{n\to\infty} y \wedge nA(|x|) = y$; see [14, Theorem 15.13]. In particular, for each $x \neq 0$ there exists some n such that $\|x_0 - x_0 \wedge nA(|x|)\| < 1$. Since the function $z \mapsto x_0 \wedge nA(|z|)$ is continuous, we see that the set $\big\{z\colon\ \|x_0 - x_0 \wedge nA(|z|)\| < 1\big\}$ is open for each n. In view of $0 \notin \overline{K(U)}$, the above arguments guarantee that

$$\overline{K(U)} \subseteq \bigcup_{n=1}^{\infty} \big\{z \in E\colon\ \|x_0 - x_0 \wedge nA(|z|)\| < 1\big\}.$$

Since the sets $\big\{z\colon\ \|x_0 - x_0 \wedge nA(|z|)\| < 1\big\}$ are increasing as n increases, the compactness of $\overline{K(U)}$ implies that

$$\overline{K(U)} \subseteq \big\{z \in E\colon\ \|x_0 - x_0 \wedge mA(|z|)\| < 1\big\}$$

for some m. In other words, there exists some fixed m such that $x \in \overline{K(U)}$ implies $x_0 \wedge mA(|x|) \in U$.

In particular, we have $x_1 = x_0 \wedge mA(|Kx_0|) \in U$. Since $K(x_1)$ belongs to $K(U)$, it follows that $x_2 = x_0 \wedge mA(|Kx_1|) \in U$. Proceeding inductively, we obtain a sequence $\{x_n\}$ of positive vectors in U defined by $x_{n+1} = x_0 \wedge mA(|Kx_n|)$. Now we claim that

$$0 \leq x_n \leq m^n A^n S^n(x_0)$$

holds for each n. The proof is by induction. For $n = 1$, we have the inequality $x_1 = x_0 \wedge mA\big(|Kx_0|\big) \le mA(Sx_0)$. For the induction step, recall that $SA^n \le A^nS$ and that if $0 \le x_n \le m^nA^nS^n(x_0)$ holds for some n, then

$$\begin{aligned} 0 &\le x_{n+1} = x_0 \wedge mA(|Kx_n|) \le mA(|Kx_n|) \\ &\le mA(Sx_n) \le m^{n+1}A(SA^nS^n(x_0)) \le m^{n+1}A^{n+1}S^{n+1}(x_0). \end{aligned}$$

Thus, we have $\|x_n\| \le m^n\|A\|^n\|S^nx_0\|$, and so

$$\|x_n\|^{\frac{1}{n}} \le m\|A\| \cdot \|S^nx_0\|^{\frac{1}{n}}$$

for each n. Since $\lim_{n\to\infty} \|S^nx_0\|^{\frac{1}{n}} = 0$, it follows that $\lim_{n\to\infty} \|x_n\|^{\frac{1}{n}} = 0$, and consequently $\lim_{n\to\infty} \|x_n\| = 0$. However, since $\{x_n\} \subset U$, this implies $0 \in \overline{U} = U$, contrary to ($\star\star$), and the proof of the theorem is finished. □

It should be pointed out that Theorem 10.1 generalizes Theorem 4.1 in [4]. The difference is that here we have replaced the commutativity condition $SB = BS$ by the weaker assumption $SB \le BS$. It would be very interesting to investigate in which of the subsequent results we can replace commutativity by some kind of inequality. An interesting paper by V. Caselles [25] may be of use for this purpose.

As in Section 5, we do not know presently if an analogue of the previous result holds true provided that we replace the inequality $SB \le BS$ by the reverse inequality $SB \ge BS$. However, if we assume in addition that S is quasinilpotent (rather than just being locally quasinilpotent) then, as we shall show next, such an analogue does hold. Though the proof is quite similar to that of Theorem 10.1 we present all the details to see the differences.

THEOREM 10.2. *Let $B\colon E \to E$ be a positive operator on a Banach lattice. Assume that there exists a positive operator $S\colon E \to E$ such that*

1. *$SB \ge BS$,*
2. *S is quasinilpotent and*
3. *S dominates a non-zero compact operator.*

Then the operator B has a non-trivial closed invariant subspace. Moreover, we can choose this invariant subspace to be the closure of a principal ideal in E.

Proof. Let B, S satisfy the properties stated in the theorem, and let K be a non-zero compact operator dominated by S, that is, $|K(x)| \leq S(|x|)$ for each $x \in E$.

Obviously we can assume that $\|B\| < 1$. Then the convergent series $A = \sum_{n=0}^{\infty} B^n$ defines a positive operator on E, and in view of the first condition on S, the inequalities $B^k S \leq SB^k$ and $AS^k \leq S^k A$ hold for each k.

For each $x > 0$ we denote by $J[x]$ the principal ideal generated by Ax. That is,

$$J[x] = \{y \in E\colon \ |y| \leq \lambda Ax \text{ for some } \lambda > 0\}.$$

Since $x \in J[x]$, note that $J[x]$ is non-zero. Moreover, we claim that $J[x]$ is B-invariant. Indeed, if $y \in J[x]$, then $|y| \leq \lambda Ax$ for some $\lambda > 0$, and so

$$|By| \leq B|y| \leq \lambda B(Ax) = \lambda \sum_{n=1}^{\infty} B^n x \leq \lambda Ax,$$

which implies $By \in J[x]$. So, $\overline{J[x]}$ is a non-zero closed B-invariant ideal.

The proof will be finished, if we show that $\overline{J[x]} \neq E$ for some $x > 0$. To establish this, assume by way of contradiction that

$$\overline{J[x]} = E \ \text{ for each } \ x > 0. \qquad (\star)$$

Since $K \neq 0$ we can find some $x_0 > 0$ for which $Kx_0 \neq 0$. By scaling, we can assume $\|K\| = 1$. Also, replacing x_0 by ax_0 for an appropriate scalar $a > 1$, we can suppose that $\|x_0\| > 1$ and $\|Kx_0\| > 1$. Now let $U = \{z \in E\colon \ \|x_0 - z\| \leq 1\}$ be the closed unit ball centered at $x_0 > 0$. By our choice of x_0, we have

$$0 \notin U \qquad \text{and} \qquad 0 \notin \overline{K(U)}. \qquad (\star\star)$$

By $(\star)$, we know that $\overline{J[|x|]} = E$ for each $x \neq 0$. Hence, the sequence $\{y \wedge nA(|x|)\}$ converges in norm and $\lim_{n\to\infty} y \wedge nA(|x|) = y$

for each $y \geq 0$. In particular, for each $x \neq 0$ there exists some n such that $\|x_0 - x_0 \wedge nA(|x|)\| < 1$. Since the function $z \mapsto x_0 \wedge nA(|z|)$ is continuous, we see that the set $\{z\colon \|x_0 - x_0 \wedge nA(|z|)\| < 1\}$ is open for each n. In view of $0 \notin \overline{K(U)}$, the above arguments guarantee that

$$\overline{K(U)} \subseteq \bigcup_{n=1}^{\infty} \{z \in E\colon \|x_0 - x_0 \wedge nA(|z|)\| < 1\}.$$

Since the sets $\{z\colon \|x_0 - x_0 \wedge nA(|z|)\| < 1\}$ are increasing as n increases, the compactness of $\overline{K(U)}$ implies that

$$\overline{K(U)} \subseteq \{z \in E\colon \|x_0 - x_0 \wedge mA(|z|)\| < 1\}$$

for some m. In other words, there exists some fixed m such that $x \in \overline{K(U)}$ implies $x_0 \wedge mA(|x|) \in U$.

In particular, we have $x_1 = x_0 \wedge mA(|Kx_0|) \in U$. Since $K(x_1)$ belongs to $K(U)$, it follows that $x_2 = x_0 \wedge mA(|Kx_1|) \in U$. Proceeding inductively, we obtain a sequence $\{x_n\}$ of positive vectors in U defined by $x_{n+1} = x_0 \wedge mA(|Kx_n|)$. Now we claim that

$$0 \leq x_n \leq m^n S^n A^n(x_0)$$

holds for each n. The proof is by induction. For $n = 1$, we have the inequality $x_1 = x_0 \wedge mA\big(|Kx_0|\big) \leq mA(Sx_0) \leq mSA(x_0)$. For the induction step, recall that $AS^k \leq S^k A$ and note that if the inequality $0 \leq x_n \leq m^n S^n A^n(x_0)$ holds for some n, then

$$\begin{aligned} 0 \leq x_{n+1} &= x_0 \wedge mA(|Kx_n|) \leq mA(|Kx_n|) \leq mA(Sx_n) \\ &\leq m^{n+1} A(SS^n A^n(x_0)) = m^{n+1} A(S^{n+1} A^n(x_0)) \\ &\leq m^{n+1} S^{n+1} A^{n+1}(x_0). \end{aligned}$$

Thus for each n we have $\|x_n\| \leq m^n \|S^n\|^{\frac{1}{n}} \cdot \|A\|^n \cdot \|x_0\|$, and so $\|x_n\|^{\frac{1}{n}} \leq m\|S^n\| \cdot \|A\| \cdot \|x_0\|^{\frac{1}{n}}$. Since $\lim\limits_{n\to\infty} \|S^n\|^{\frac{1}{n}} = 0$, it follows that $\lim\limits_{n\to\infty} \|x_n\|^{\frac{1}{n}} = 0$, and consequently $\lim\limits_{n\to\infty} \|x_n\| = 0$. However, since $\{x_n\} \subset U$, this implies $0 \in \overline{U} = U$, contrary to $(\star\star)$, and the proof of the theorem is finished. $\square$

Recall that for a positive operator $B\colon E \to E$ on a Banach lattice its **null ideal** N_B is defined via the formula

$$N_B = \{x \in E\colon\ B(|x|) = 0\}.$$

Clearly, N_B is a closed ideal in E and a vector subspace of the null-space of B. Notice also that $B = 0$ if and only if $N_B = E$. We leave it as a simple exercise to verify that the ideal N_B is invariant under every positive operator commuting with B. A positive operator $B\colon E \to E$ is **strictly positive** if $x > 0$ implies $Bx > 0$. Clearly, a positive operator B is strictly positive if and only if $N_B = \{0\}$.

The next result is a strong companion of the two preceding theorems. It shows that we can "distribute" properties (2) and (3) in Theorem 10.1 (previously imposed exclusively on one of the operators) between the commuting operators.

THEOREM 10.3. *Let $B, S\colon E \to E$ be two commuting non-zero positive operators on a Banach lattice. If one of them is quasinilpotent at a non-zero positive vector and the other dominates a non-zero compact operator, then B and S have a common non-trivial closed invariant ideal.*

Proof. Assume that S is quasinilpotent at some point $x_0 > 0$, and that B dominates a non-zero compact operator K. If S is not strictly positive, then the null ideal N_S is the desired non-trivial common closed invariant subspace. So, suppose that S is strictly positive. Without loss of generality, we can also assume that $\|B + S\| < 1$. Put $A = \sum_{n=0}^{\infty}(B + S)^n$ and let J denote the ideal generated by Ax_0, i.e.,

$$J = \{y \in E\colon\ \exists\, \lambda > 0 \text{ such that } |y| \leq \lambda A x_0\}.$$

Clearly, J is a non-zero ideal that is invariant for $B + S$. Since $0 \leq B, S \leq S + B$, Lemma 9.4 implies immediately that this ideal is invariant under both B and S.

If $\overline{J} \neq E$, then $\overline{J}$ is a non-trivial closed ideal which is invariant under both B and S. So, we consider the case $\overline{J} = E$. In this case, since $K \neq 0$, there exists some $0 < y_0 \in J$ such that $K(y_0) \neq 0$. Clearly, S remains quasinilpotent at y_0.

Since $|K(y_0)| \le B(y_0) \in J$, it follows from Lemma 9.3 that there exists an operator $V\colon E \to E$ satisfying

$$VK(y_0) > 0 \quad \text{and} \quad |V(x)| \le |x| \text{ for each } x \in E.$$

Now consider the compact operator SVK. Since S is strictly positive, it follows that $SVK(y_0) > 0$, and so SVK is non-zero. Moreover, we have

$$|SVK(x)| \le S|VK(x)| \le S(|K(x)|)| \le SB(|x|).$$

Thus, the positive operator SB dominates the non-zero compact operator SVK, is quasinilpotent at y_0, and commutes with the positive operator $S+B$. By Theorem 10.1, there exists a non-trivial closed $(S+B)$-invariant ideal. Clearly, this ideal is invariant under both B and S. □

COROLLARY 10.4. *Let a positive operator $B\colon E \to E$ on a Banach lattice satisfy the following two conditions:*

1. *B is quasinilpotent at a non-zero positive vector, and*
2. *Some power of B dominates a non-zero compact operator.*

Then the operator B has a non-trivial closed invariant ideal.

Proof. Apply Theorem 10.3 to B and $S = B^m$, where B^m dominates a non-zero compact operator. □

The previous results imply immediately the following theorem which (being basically equivalent to Theorem 5.6) also implies the results established in Section 5.

THEOREM 10.5. *If a positive operator $B\colon E \to E$ on a discrete Banach lattice is quasinilpotent at a non-zero positive vector, then B has a non-trivial closed invariant ideal.*

Proof. As E is discrete, there exists a set I, such that E is a subspace of $\mathbf{C}^I$ and $\mathbf{e}_i = \chi_{\{i\}} \in E$ for each $i \in I$. If $B(\mathbf{e}_i) = 0$ for some $i \in I$, then the null ideal N_B is the desired non-trivial closed B-invariant ideal. So, assume that $B(\mathbf{e}_i) > 0$ for each $i \in I$. Take

$x_0 > 0$ at which B is quasinilpotent and let i_0 be any index in I such that $x_0(i_0) > 0$. Consider the operator $K = BP$, where P denotes the standard rank-one projection onto the one-dimensional space spanned by $\mathbf{e}_{i_0}$. Clearly, K is compact and $0 < K \leq B$ holds. Now apply Corollary 10.4. □

11. Compact-friendly operators and invariant subspaces

In the Banach space setting, when we are looking for invariant subspaces of an operator T, Lomonosov's theorem tells us the following: if the commutant of T contains a non-zero compact operator K, then T has a non-trivial closed invariant subspace.

In the Banach lattice setting, when we are looking for invariant subspaces of a positive operator B the order structure provides new "dimensions" of bringing compactness into the picture. In our previous results, we have already seen that compactness enters by means of the inequality $|Kx| \leq B(|x|)$. Now we mention several other ways of how compactness may enter.

i. B may be dominated by a compact operator, or

ii. B may commute with a positive operator which dominates a compact operator, or

iii. B may commute with a positive operator which in turn is dominated by a compact operator.

And the goal of this section is to show that each of these cases is "good enough" to guarantee the existence of a (non-trivial) closed invariant subspace.

To accomplish this, we need to introduce a new notion related to compactness in terms of the order structure. This notion, in spite of some cumbersomeness, is much weaker than compactness and subsumes all the previous cases. This will allow us to extend Theorem 10.1 considerably.

DEFINITION 11.1. *A positive operator* $B\colon E \to E$ *is said to be* **compact-friendly** *if there exists a positive operator in the commutant* *of B that dominates a non-zero operator which in turn is dominated by a positive compact operator.*

That is, B is compact-friendly if and only if there exist three non-zero operators $R, K, C\colon E \to E$ with R, K positive and K compact such that

$$RB = BR, \quad \big|C(x)\big| \le R\big(|x|\big), \quad \text{and} \quad \big|C(x)\big| \le K\big(|x|\big)$$

for each $x \in E$.

Clearly, every power of a compact-friendly operator is itself compact-friendly. Here are some examples of compact-friendly operators.

- Positive compact operators.
- Positive operators commuting with non-zero positive compact operators.
- Positive operators that dominate non-zero compact positive operators.
- Positive operators that are dominated by compact operators.
- Positive kernel operators.

The last three cases are not obvious. Let us discuss them. Assume first that $0 < K_0 \le B$, where K_0 is compact. Then the "triplet" (R, C, K) which is needed to conclude that B is compact-friendly can be defined as follows: $R = B$, $C = K_0$ and $K = K_0$.

Assume now that $0 < B \le K_0$, where K_0 is compact. Then the required "triplet" (R, C, K) is the following: $R = B$, $C = B$ and $K = K_0$.

The proof that each positive kernel operator is compact-friendly will be given in Lemma 12.1.

And now we come to the main result of this section which is an invariant subspace theorem for compact-friendly operators.

THEOREM 11.2. *If a non-zero positive operator $B\colon E \to E$ is compact-friendly and is quasinilpotent at some $x_0 > 0$, then B has a non-trivial closed invariant ideal. Moreover, if another positive operator $T\colon E \to E$ commutes with B, then T and B have a common non-trivial closed invariant ideal.*

Proof. Let $B\colon E \to E$ be a non-zero compact-friendly operator on a Banach lattice which is quasinilpotent at some $x_0 > 0$. Also, let $T\colon E \to E$ be another positive operator that commutes with B. Fix three non-zero operators $R, C, K\colon E \to E$ with K compact and satisfying

$$RB = BR, \quad |Cx| \le R(|x|), \quad \text{and} \quad |Cx| \le K(|x|) \ \text{ for each } \ x \in E.$$

Without loss of generality, we can suppose that $\|B + T\| < 1$ and define $A = \sum_{n=0}^{\infty}(B+T)^n$. Clearly, the positive operator A commutes with both B and T and satisfies $Ax \ge x$ for each $x \ge 0$. Also, for each $x > 0$, let $J[x]$ denote the non-zero principal ideal generated by Ax, i.e,

$$J[x] = \big\{y \in E\colon \ |y| \le \lambda Ax \text{ for some } \lambda > 0\big\}.$$

If $\overline{J[x]} \ne E$ for some $x > 0$, then the ideal $\overline{J[x]}$ is a non-trivial closed $(B+T)$-invariant ideal. This ideal $\overline{J[x]}$ is, of course, also invariant under both B and T. So, we can assume that

$$\overline{J[x]} = E$$

for each $x > 0$, i.e., Ax is a quasi-interior point in E for each $x > 0$.[3]

Since $C \ne 0$, there exists some $x_1 > 0$ such that $Cx_1 \ne 0$. Since $A|Cx_1|$ is a quasi-interior point and $|Cx_1| \le A|Cx_1|$ holds, it follows from Lemma 9.3(1), that there exists an operator $V_1\colon E \to E$ dominated by the identity operator such that $x_2 = V_1Cx_1 > 0$. Put $M_1 = V_1C$, and note that M_1 is dominated both by the compact positive operator K and by the operator R.

From $\overline{J[x_2]} = E$ and $C \ne 0$, we see that there exists some element $0 < y \le Ax_2$ such that $Cy \ne 0$. Since Ax_2 is a quasi-interior point, it follows from Lemma 9.3(1) that there is an operator $U\colon E \to E$ dominated by the identity operator such that $UAx_2 = y$. Now note that the element $A|Cy|$ is (again by our hypothesis) a quasi-interior point. Since $|Cy| \le A|Cy|$, it follows from Lemma 9.3(2) that there exists another operator $V_2\colon E \to E$ dominated by the identity operator such that $x_3 = V_2Cy = V_2CUAx_2 > 0$. Let $M_2 = V_2CUA$ and

[3]In the terminology of [1] it means that A is a strongly expanding operator.

note that M_2 is dominated both by the positive compact operator KA and by the operator RA.

If we repeat the preceding arguments with the vector x_2 replaced by x_3, then we obtain one more operator $M_3\colon E \to E$ which satisfies $M_3x_3 > 0$ and which is dominated by both the positive compact operator KA and by the operator RA.

From $M_3M_2M_1x_1 = M_3x_3 > 0$, we see that $M_3M_2M_1$ is a non-zero operator which (by Theorem 9.2) is also compact. Moreover, an easy argument shows that

$$\big|M_3M_2M_1(x)\big| \le RARAR(|x|) \le \big[RARAR + T\big](|x|)$$

for each $x \in E$.

Now consider the non-zero positive operator $S = RARAR + T$. Then B and S commute, S dominates the non-zero compact operator $M_3M_2M_1$, and B is quasinilpotent at x_0. By Theorem 10.3, S and B have a common non-trivial closed invariant ideal. This ideal is invariant under both B and T. □

As we see, both conditions in Theorem 11.2 are imposed on the same operator, i.e., B is assumed to be both compact-friendly and quasinilpotent at some positive vector. It is an open and interesting problem if, as in Theorem 10.3, we can distribute these two properties between two operators. To be precise, let B and T be two commuting positive operators, such that B is compact-friendly and $\mathcal{Q}_T^+ \neq \emptyset$. Does there exist a non-trivial closed B-invariant subspace, or a T-invariant subspace, or, even better, a common invariant subspace? It will be also of interest to find out if we can replace in Theorem 11.2 the commutativity assumption by $BT \le TB$ or $TB \le BT$.

We conclude the section by comparing the results of Sections 10 and 11 with the results of Section 5. All results in Sections 10 and 11 depend on some type of explicitly declared compactness of the operators for which we prove the existence of the invariant subspaces. On the other hand, the results of Section 5 are free of any compactness. However, this difference is only ostensible. There is a "hidden" compactness in the case of discrete spaces and exactly this compactness ties up all the results for discrete and non-discrete Banach lattices alike. Namely, every positive operator on a discrete Banach lattice is automatically compact-friendly, and that is why we did not have to assume any compactness in Section 5. Though the result is very simple, we will formulate it below explicitly. Simultaneously, we wish to point out that this new concept of a compact-friendly operator is not a universal tool to handle the invariant subspace problem of every positive operator. (The situation here is similar to that with Lomonosov's Theorem 8.1 which does not cover all continuous operators [34, 35].) In [8] we exhibit a positive operator which is not compact-friendly but which has a non-trivial invariant subspace.

LEMMA 11.3. *A positive operator on a discrete Banach lattice is compact-friendly.*

Proof. Let E be a discrete Banach lattice and $0 < B\colon E \to E$ be a positive operator. Fix an arbitrary $0 < u \in E$ such that $Bu > 0$ and fix an arbitrary m for which the m^{th} coordinate of Bu is positive.

Consider $K = P_m B$, where P_m denotes the natural band projection on the m^{th} coordinate. Clearly K is a compact positive operator ($K \neq 0$ since $Ku > 0$) which is dominated by B. Thus, B is compact-friendly. □

It is worthwhile to point out that in the above proof we constructed a positive compact operator dominated by B. This is better than just a non-zero compact operator dominated by B and this motivates the following open questions.

Let us say for brevity that a positive operator $B\colon E \to E$ on a Banach lattice E is "**good**" (resp. "**very good**") if there exists a non-zero compact (resp. a non-zero positive compact) operator K which is dominated by B.

- Is every good operator very good?
- Assume that B^2 is good, is then B^2 very good?
- Assume that B^2 is good, is then B (very) good?
- Assume that B^2 is very good, is then B (very) good?

Similar questions can also be asked if we assume that there exists a polynomial p with non-negative coefficients such that $p(B)$ is good or very good.

Finally notice that in general the geometry of the subspace of compact regular operators of the space of all regular operators is a very interesting and important topic, and we refer to [9, 10, 11, 59, 60] for some recent developments in this area.

12. Invariant subspaces for kernel operators

Now let E be an order complete Banach lattice with norm dual E'. Recall that a **rank-one operator** is any operator of the form $\phi \otimes u$, where $\phi \in E'$, $u \in E$ and $\phi \otimes u(x) = \phi(x)u$ for each $x \in E$. Any operator of the form $\sum_{i=1}^{n} \phi_i \otimes u_i$ is known as a **finite rank operator**.

The vector space of all finite rank operators on E is denoted by $E' \otimes E$. The operators in the band $(E' \otimes E)^{\mathrm{dd}}$ generated by $E' \otimes E$ in the Banach lattice of all regular operators are referred to as (abstract) **kernel operators**. The name "kernel operator" comes from the well known result of G. Ya. Lozanovsky [44] that if $E = L_p(\mu)$, then an operator $T\colon L_p(\mu) \to L_p(\mu)$ belongs to $(E' \otimes E)^{\mathrm{dd}}$ if and only if there exists a $\mu \times \mu$-measurable function $T(\cdot,\cdot)$ such that for each $f \in L_p(\mu)$

1. $\int |T(\cdot, s)f(s)|\, d\mu(s) \in L_p(\mu)$ for each $f \in L_p(\mu)$, and
2. $Tf(t) = \int T(t,s)f(s)\, d\mu(s)$ for μ-almost all t.

For details concerning kernel operators see [61, Chapter 13]. The important thing to keep in mind is the following property: If $T\colon E \to E$ is a positive kernel operator, then there exists a net $\{T_\alpha\}$ of positive operators such that $0 \le T_\alpha(x) \uparrow T(x)$ holds for each $x \ge 0$, and each T_α is dominated by a positive finite rank operator. In particular, we have the following result.

LEMMA 12.1. *Every positive kernel operator is compact-friendly.*

We will prove next that the third power of an arbitrary strictly positive kernel operator dominates a non-zero compact positive operator (in the terminology of Section 11 this means that the third power of each strictly positive kernel operator is a very good operator).

LEMMA 12.2. *If $S\colon E \to E$ is a strictly positive kernel operator on an order complete Banach lattice, then for each element $x_0 > 0$ there* *exists a compact positive operator* $K\colon E \to E$ *satisfying* $0 \le K \le S^3$ *and* $Kx_0 > 0$.

Proof. Let $S\colon E \to E$ be a strictly positive kernel operator and fix $x_0 > 0$. So, there exists a net $\{S_\alpha\}$ of positive operators such that $0 \le S_\alpha \uparrow S$ and each S_α is dominated by a positive finite rank operator. Again, by Theorem 9.2, each S_α^3 is a compact operator.

From $S_\alpha x_0 \uparrow Sx_0$ and $Sx_0 > 0$, we see that there exists some index α_1 such that $S_\alpha x_0 > 0$ for each $\alpha \ge \alpha_1$. Similarly, from $S_\alpha(S_{\alpha_1}x_0) \uparrow_\alpha S(S_{\alpha_1}x_0)$ and $S(S_{\alpha_1}x_0) > 0$, we obtain the inequality $S_\alpha(S_{\alpha_1}x_0) > 0$ for all $\alpha \ge \alpha_2 \ge \alpha_1$. Finally, from the facts that $S_\alpha\big(S_{\alpha_2}S_{\alpha_1}x_0\big) \uparrow_\alpha S\big(S_{\alpha_2}S_{\alpha_1}x_0\big)$ and $S\big(S_{\alpha_2}S_{\alpha_1}x_0\big) > 0$, it follows that $S_\alpha\big(S_{\alpha_2}S_{\alpha_1}x_0\big) > 0$ for all $\alpha \ge \alpha_3 \ge \alpha_2 \ge \alpha_1$.

Now note that if $K = S_{\alpha_3}^3$, then $K\colon E \to E$ is a positive compact operator satisfying $0 \le K \le S^3$ and $Kx_0 > 0$. ☐

And now we are ready to present an invariant subspace theorem for kernel operators.

THEOREM 12.3. *Let* $S\colon E \to E$ *be a non-zero positive kernel operator on a Banach lattice and let* $B\colon E \to E$ *be another non-zero positive operator commuting with* S. *If either* S *or* B *is quasinilpotent at a non-zero positive vector, then the operators* S *and* B *have a common non-trivial closed invariant ideal.*

Proof. If the operator S itself is quasinilpotent at a non-zero positive vector, then S is a compact-friendly operator with a non-trivial set $\mathcal{Q}_S^+$ of positive vectors at which S is quasinilpotent. So Theorem 11.2 is applicable and it guarantees the existence of the required invariant subspace.

Assume now that $\mathcal{Q}_B^+$ is non-trivial, i.e., that the operator B is quasinilpotent at a non-zero positive vector. Without loss of generality we can assume that S is strictly positive (since otherwise the null ideal N_S provides at once the required invariant subspace). Therefore, by Lemma 12.2, the operator S^3 dominates a non-zero compact positive operator K. Consider now the operator $B+S+S^3$. Clearly it dominates K and commutes with both S and B. We see now that B is compact-friendly and $\mathcal{Q}_B^+$ is non-trivial. Hence, Theorem 11.2, applied to B and $B+S+S^3$, guarantees the existence of a common non-trivial closed invariant ideal for these two operators.

Since both S and B are dominated by $B+S+S^3$, it follows that this ideal remains invariant under both S and B. □

T. Andô [16] and H. J. Krieger [41] (see also [1, 29]) proved that each positive irreducible integral operator on an L_p-space has a positive spectral radius. In the "invariant subspace" terminology this means that every positive quasinilpotent integral operator has a non-trivial closed invariant subspace. This was the first result on the invariant subspace problem in the framework of Banach lattices.

The next corollary, which follows immediately from Theorem 12.3, improves the Andô–Krieger theorem by replacing the quasinilpotence assumption with quasinilpotence at a single positive vector, and by removing the assumption of the order continuity of the operator; see also [1, 29].

COROLLARY 12.4. *Each positive kernel operator that is quasinilpotent at a non-zero positive vector has a non-trivial closed invariant ideal.*

For the classical L_p-spaces (and, as a matter of fact, for any Banach function space), the preceding corollary yields the following.

COROLLARY 12.5. *Let* $B\colon L_p(\mu) \to L_p(\mu)$ *(where* μ *is* σ*-finite and* $1 \le p < \infty$*) be a positive kernel operator with kernel* $B(\cdot,\cdot)$*, and let* B *be quasinilpotent at a non-zero positive function. Then all kernel operators* $T\colon L_p(\mu) \to L_p(\mu)$ *of the type*

$$Tf(s) = \int w(s,t)B(s,t)f(t)\,d\mu(t), \ \ f \in L_p(\mu),$$

where $w(\cdot,\cdot)$ *is an arbitrary bounded* $\mu \times \mu$*-measurable function, have a common non-trivial closed invariant ideal.*[4]

Proof. Observe that any operator T defined by a kernel $w(\cdot,\cdot)B(\cdot,\cdot)$, where $w(\cdot,\cdot)$ is a bounded $\mu \times \mu$-measurable function, is dominated by a multiple of B, and our conclusion follows from Corollary 12.4 and Lemma 9.4. □

[4]Keep in mind that the closed ideals in $L_p(\mu)$ are the subspaces of the form

$$\{f \in L_p(\mu)\colon\ f(t) = 0 \text{ for } \mu\text{-almost all } t \in D\}$$

for some μ-measurable subset D.

A positive operator $B\colon E \to E$ on an order complete Banach lattice is called a **Harris operator** if some power of B is not disjoint from the band $(E' \otimes E)^{\mathrm{dd}}$ of kernel operators on E. That is, if some power of B dominates a non-zero positive kernel operator. In particular, we see that B is compact-friendly, and therefore the next corollary is an immediate consequence of Theorem 11.2.

COROLLARY 12.6. *Every Harris operator which is quasinilpotent at a non-zero positive vector has a non-trivial closed invariant ideal.*

13. A one-to-one compact positive locally quasinilpotent but not quasinilpotent kernel operator

Our objective here is to present a one-to-one Hilbert–Schmidt positive operator on $L_2[0,1]$ which is locally quasinilpotent at some positive vector but fails to be quasinilpotent. In order to construct such an operator we need two preliminary lemmas.

LEMMA 13.1. *Let $0 \le a < b \le 1$ and $0 \le c < d \le 1$. Then the positive kernel operator $T\colon L_2[c,d] \to L_2[a,b]$ defined by*

$$Tf(x) = \int_c^d \sin(xy) f(y)\, dy, \;\; a \le x \le b,$$

is one-to-one and compact.

Proof. The compactness of T follows from the fact that T is a Hilbert–Schmidt operator. We shall prove that T is one-to-one. The geometric situation is shown in Figure 1, where the kernel $K(x,y) = \sin(xy)$ is considered defined over the rectangle $Q = [a,b] \times [c,d]$.

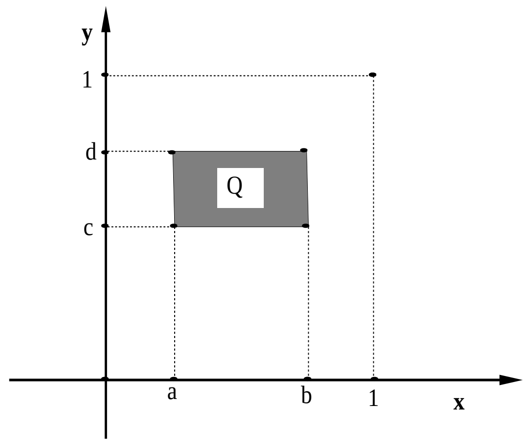

Figure 1

It is easy to see that Tf is a continuous function for each function $f \in L_2[c,d]$. As a matter of fact, it follows from [15, Theorem 20.4] that $Tf \in C^\infty[c,d]$ for each $f \in L_2[c,d]$ and that

$$(Tf)^{(n)}(x) = \int_c^d \left[\tfrac{\partial^n}{\partial x^n}\sin(xy)\right] f(y)\,dy. \qquad (\star)$$

Now assume that $Tf = 0$ for some function $f \in L_2[c,d]$. Then for each n, it follows from $(\star)$ that $\int_c^d y^{2n}\sin(xy)f(y)\,dy = 0$ for almost all $x \in [a,b]$.

The subalgebra $\mathcal{A}$ of $C[c,d]$ consisting of all polynomials with even terms contains the constant function one and separates the points of $[c,d]$. (The polynomial $p(y) = y^2$ is one-to-one on $[c,d]$.) So, by the Stone–Weierstrass theorem, $\mathcal{A}$ is uniformly dense in $C[c,d]$. Consequently, $\mathcal{A}$ is $\|\cdot\|_2$-dense in $L_2[c,d]$. From $(\star)$, we see immediately that $\int_c^d p(y)\sin(xy)f(y)\,dy = 0$ for each $p \in \mathcal{A}$ and almost all $x \in [a,b]$.

It follows that there exists a sequence $\{p_n\}$ of $\mathcal{A}$ satisfying $p_n \to f$ a.e. and $|p_n| \le g$ a.e. for all n and some $g \in L_2[c,d]$ (see [15, p. 207]). So, for almost all $y \in [c,d]$ we have

$$p_n(y)\sin(xy)f(y) \longrightarrow \sin(xy)[f(y)]^2$$

and $\left|p_n(y)\sin(xy)f(y)\right| \le g(y)|f(y)|$. Since $g|f| \in L_1[c,d]$, the Lebesgue Dominated Convergence Theorem implies

$$\int_c^d \sin(xy)[f(y)]^2\,dy = \lim_{n\to\infty}\int_c^d p_n(y)\sin(xy)f(y)\,dy = 0$$

for almost all x. Therefore, $f = 0$ a.e. □

LEMMA 13.2. *Assume that $0 \le a < b \le 1$. Then the positive kernel operator $T\colon L_2[a,b] \to L_2[a,b]$ defined by*

$$Tf(x) = \int_a^b \sin(xy)f(y)\,dy,\ \ a \le x \le b,$$

has a positive spectral radius (and so T is a compact, one-to-one, positive kernel operator which is not quasinilpotent).

Proof. Since the kernel of T is symmetric, it follows that T is a Hermitian operator. Consequently, $r(T) = \|T\|_2 > 0$. □

If for $1 \le p \le \infty$ we consider the operator $T\colon L_p[0,1] \to L_p[0,1]$ defined by

$$Tf(x) = \int_0^1 \sin(xy)f(y)\,dy,\ \ 0 \le x \le 1,$$

then we have the following norm estimates:

$$\begin{aligned}\|T\|_\infty = \|T\mathbf{1}\|_\infty &= \sup_{x\in[0,1]}\int_0^1 \sin(xy)\,dy\\ &= \sup_{x\in[0,1]}\frac{1-\cos x}{x} = 1-\cos 1 \approx 0.4597\end{aligned}$$

and

$$\begin{aligned}\|T\|_2 &\le \Big(\int_0^1\int_0^1 \big[\sin(xy)\big]^2\,dxdy\Big)^{\frac12}\\ &= \Big(\frac12 - \frac14\int_0^1 \frac{\sin 2x}{x}\,dx\Big)^{\frac12} \approx 0.7009\,.\end{aligned}$$

Our next goal is to construct a one-to-one positive compact kernel operator which is quasinilpotent at some positive vectors but fails to be quasinilpotent. The kernel $K(x,y)$ will be defined on the unit square $[0,1]\times[0,1]$ as shown in Figure 2.

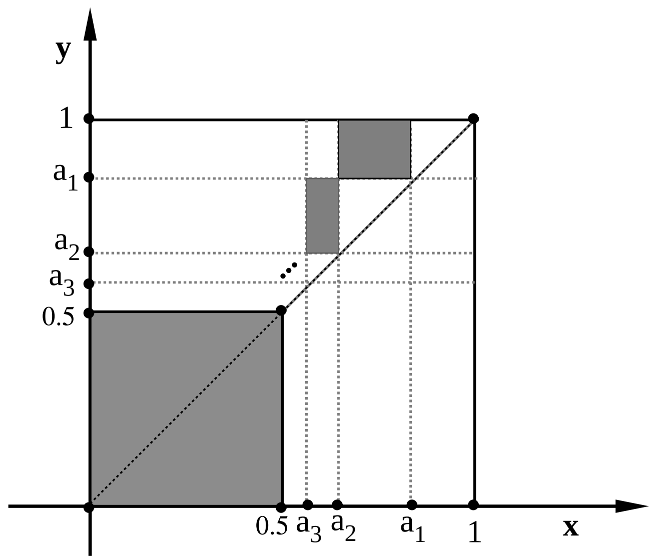

Figure 2

The value $K(x,y)$ will be equal to $\sin(xy)$ if (x,y) lies in any one of the shaded rectangles and will be equal to zero everywhere else.

The points $a_0, a_1, a_2, \ldots$ shown in Figure 2 are defined by

$$a_0 = 1, \quad \text{and} \quad a_n = 1 - \frac{1}{2e}\sum_{k=0}^{n-1}\frac{1}{k!} \quad \text{for} \quad n \geq 1.$$

The length of the interval (a_{n+1}, a_n) is equal to $a_n - a_{n+1} = \frac{1}{n!2e}$.

To avoid introducing some extra notation, we shall denote by K the operator on $L_2[0,1]$ defined be the kernel $K(x,y)$ introduced above. That is, the operator $K\colon L_2[0,1] \to L_2[0,1]$ is defined by

$$Kf(x) = \int_0^1 K(x,y)f(y)\,dy, \quad 0 \leq x \leq 1.$$

THEOREM 13.3. *The kernel operator $K\colon L_2[0,1] \to L_2[0,1]$ is a one-to-one positive compact operator which is quasinilpotent at some positive vectors and fails to be quasinilpotent.*

Proof. The fact that K is one-to-one follows easily from Lemma 13.1. Since the kernel $K(x,y)$ is positive and belongs to $L_2\big([0,1]\times[0,1]\big)$, it follows that the operator K is Hilbert–Schmidt. So, K is a positive compact operator.

To see that K is not quasinilpotent, notice that K leaves $L_2\big[0,\frac{1}{2}\big]$ invariant. (As usual we identify here $L_2\big[0,\frac{1}{2}\big]$ with the closed subspace of $L_2[0,1]$ given by $\big\{f \in L_2[0,1] : f = 0 \text{ a.e. on } \big[\frac{1}{2},1\big]\big\}$.) Moreover, observe that the restriction of K to $L_2\big[0,\frac{1}{2}\big]$ is the kernel operator with kernel $\sin(xy)$. By Lemma 13.2, the restriction of K to $L_2\big[0,\frac{1}{2}\big]$ has positive spectral radius. That is, K is not quasinilpotent when restricted to $L_2\big[0,\frac{1}{2}\big]$. Therefore, K cannot be quasinilpotent.

Now consider the positive function $f = \chi_{[a_1,1]}$. An easy verification shows that $0 \le K^n f \le \chi_{[a_{n+2},a_{n+1}]}$. Consequently,

$$\big(\|K^n f\|_2\big)^{\frac{1}{n}} \le \sqrt{\big(\tfrac{1}{n!2e}\big)^{\frac{1}{n}}}\,,$$

from which it follows that $\lim_{n\to\infty}\big(\|K^n f\|_2\big)^{\frac{1}{n}} = 0$. □

14. The dual invariant subspace problem

In this section we will be concerned with the following conjecture due to Lomonosov [43].

- **Lomonosov's Conjecture:** *The adjoint of a bounded linear operator on a Banach space has a non-trivial closed invariant subspace.*

L. de Branges in [28] initiated a study aimed at proving this conjecture. Specifically, he reduced Lomonosov's conjecture to proving that a certain vector subspace of a natural space of vector functions is not dense with respect to a linear topology introduced in [28]. An alternate approach to this problem will be discussed below. It was suggested in [7], and the presentation in this section follows this paper. We precede this with one important comment.

Consider again C. Read's operator $T\colon \ell_1 \to \ell_1$ which has no non-trivial closed invariant subspace. If this operator were $\sigma(\ell_1, c_0)$-continuous, then it would be the adjoint operator to some operator on c_0, and thus would disprove the above conjecture. However, V. Troitsky [58] has recently shown that the operator T is not $\sigma(\ell_1, c_0)$-continuous, and thus this operator cannot be used to refute Lomonosov's conjecture.

As in the Lomonosov–de Branges approach, we study the invariant subspace problem for algebras of operators. The symbol $\mathcal{A}$ will denote a subalgebra of $L(X)$, the Banach algebra of all bounded linear operators on a Banach space X. The **dual algebra** of $\mathcal{A}$ is the subalgebra of $L(X')$ defined by $\mathcal{A}' = \{T'\colon\ T \in \mathcal{A}\}$.

A subspace V of X is said to be $\mathcal{A}$-**invariant** if V is invariant under every operator of $\mathcal{A}$, i.e., $T(V) \subseteq V$ holds for each $T \in \mathcal{A}$. Notice that a subspace V of X is invariant under an operator T from $L(X)$ if and only if V is invariant under the algebra (unital or not) generated by T in $L(X)$.

The next result is a folklore characterization of the existence of non-trivial closed $\mathcal{A}$-invariant subspaces. As a matter of fact we have already used this characterization, without stating it explicitly, while proving Theorems 5.1 and 7.1.

PROPOSITION 14.1. *A subalgebra $\mathcal{A}$ of $L(X)$ admits a non-trivial closed $\mathcal{A}$-invariant subspace if and only if there exist a nonzero vector $x \in X$ and a nonzero linear functional $x' \in X'$ satisfying $\langle x', Tx\rangle = 0$ for each $T \in \mathcal{A}$.*

Proof. Let V be a non-trivial closed $\mathcal{A}$-invariant subspace. Fix a nonzero vector $x \in V$ and consider the closed subspace generated by the action of $\mathcal{A}$ on x, i.e., $X_x = \overline{\{Tx\colon\ T \in \mathcal{A}\}}$. Clearly, $X_x \subseteq V$, and so $X_x \neq X$. Therefore, there exists some nonzero $x' \in X'$ that annihilates X_x, i.e., we have $\langle x', Tx\rangle = 0$ for all $T \in \mathcal{A}$.

For the converse, assume that $\langle x', Tx\rangle = 0$ holds for all $T \in \mathcal{A}$ and some nonzero vectors $x \in X$ and $x' \in X'$. It easily follows that the closed subspace $X_x = \overline{\{Tx\colon\ T \in \mathcal{A}\}}$ is not norm dense in X. If $X_x \neq 0$, then X_x is a non-trivial closed $\mathcal{A}$-invariant subspace. In the case when $X_x = \{0\}$, note that the non-trivial closed subspace $V = \{\lambda x\colon\ \lambda \in \mathbf{C}\}$ is $\mathcal{A}$-invariant. $\square$

From the identity $\langle x', Tx\rangle = \langle T'x', x\rangle$, it follows immediately that if there exists a non-trivial closed $\mathcal{A}$-invariant subspace, then there is also a non-trivial closed $\mathcal{A}'$-invariant subspace.

In this section we shall denote by S the closed unit ball of X'. That is,

$$S = \{x' \in X' \colon \ \|x'\| \le 1\}.$$

As usual, S will be equipped with its weak* topology, and hence S is a weak* compact subset of X'.

DEFINITION 14.2. *The vector space of all continuous functions from S into X', when both S and X' are equipped with the weak* topology, will be denoted by $C(S, X')$. Occasionally, $C(S, X')$ will also be denoted by Y, i.e., $Y = C(S, X')$.*

Observe that for each $T \in L(X)$ the restriction of the adjoint $T' \colon S \to X'$ is an element of $C(S, X')$. Clearly, the vector space $C(S, X')$ equipped with the norm

$$\|f\| = \sup_{s \in S} \|f(s)\|, \quad f \in C(S, X')$$

is a Banach space.

The Banach space $C(S, X')$ was the main object of study in [43] and [28]. V. I. Lomonosov [43], inspired by L. de Branges' proof of the Stone–Weierstrass theorem [27], characterized the extreme points of the closed unit ball of the norm dual of $C(S, X')$. Subsequently, L. de Branges [28] presented a deep analysis of the behavior of these extreme points and obtained an abstract version of the Stone–Weierstrass theorem. The Lomonosov–de Branges analysis will be employed later on in our characterization of the invariant subspace problem.

As always, the symbol $C(S)$ denotes the Banach space of all continuous complex valued functions defined on S. It is worth mentioning that each function $\alpha \in C(S)$ defines an "action" (or equivalently, an operator) on $C(S, X')$ via the formula

$$(\alpha f)(s) = \alpha(s) f(s), \quad f \in C(S, X'), \ s \in S.$$

This is, of course, the multiplication operator determined by α. Clearly,

$$\|\alpha f\| \le \|\alpha\|_\infty \|f\|.$$

In algebraic terminology this means that $C(S,X')$ is a $C(S)$-module and our discussion can be formulated in terms of modules. However, we shall not use this terminology any further.

The second norm dual of X will be denoted by X''. Every element $x'' \in X''$ and every $s \in S$ give rise to a continuous linear functional $x'' \otimes s$ on $C(S,X')$ via the formula

$$\big\langle x'' \otimes s, f\big\rangle = (x'' \otimes s)(f) = \big\langle x'', f(s)\big\rangle = x''\big(f(s)\big), \quad f \in C(S,X').$$

Clearly, $x'' \otimes s$ is a norm continuous linear functional on $C(S,X')$. The vector space generated by the set $\{x'' \otimes s\colon\ x'' \in X'' \text{ and } s \in S\}$ in the norm dual of $Y = C(S,X')$ will be denoted by $Y^{\#}$, i.e.,

$$Y^{\#} = \Big\{\sum_{i=1}^{n} x_i'' \otimes s_i\colon\ x_i'' \in X'' \text{ and } s_i \in S \text{ for each } i = 1,\dots,n\Big\}.$$

Obviously, $Y^{\#}$ separates the points of Y, and so $\langle Y, Y^{\#}\rangle$ with its natural duality is a dual system. Apart from the norm topology, we shall consider on the Banach space $C(S,X')$ two other easily accessible topologies. They are defined as follows.

1. The topology τ_w on $C(S,X')$ is the locally convex topology generated by the set $\big\{\rho_{x'',s}\colon\ x'' \in X'' \text{ and } s \in S\big\}$ of seminorms, where
$$\rho_{x'',s}(f) = \big|(x'' \otimes s)(f)\big| = \big|\langle x'', f(s)\rangle\big|$$
for each $f \in C(S,X')$. Note that τ_w is simply the weak topology $\sigma(Y, Y^{\#})$.

2. The topology τ_s on $C(S,X')$ is the locally convex topology generated by the set of seminorms $\big\{\rho_s\colon\ s \in S\big\}$, where
$$\rho_s(f) = \big\|f(s)\big\|$$
for each $f \in C(S,X')$.

The introduced topologies are similar to the usual weak and strong operator topologies on $L(X)$ and this justifies our choice of the subscripts w and s. Moreover, in analogy with the classical weak and strong operator topologies [31, Theorem 4, p. 477], the topologies τ_w and τ_s have the same continuous linear functionals. That is, they are both consistent with the dual system $\langle Y, Y^{\#}\rangle$. The details follow.

THEOREM 14.3. *The locally convex topology τ_s is consistent with the dual system $\langle Y, Y^\# \rangle$. That is, we have the inclusions*

$$\sigma(Y, Y^\#) \subseteq \tau_s \subseteq \tau(Y, Y^\#),$$

where as usual $\tau(Y, Y^\#)$ denotes the Mackey topology.

Proof. Clearly, we have $\sigma(Y, Y^\#) \subseteq \tau_s$. So, we must establish only that $\tau_s \subseteq \tau(Y, Y^\#)$.

To this end, fix some element $s \in S$. It suffices to show that the set $\{f \in Y\colon\ \rho_s(f) = \|f(s)\| \leq 1\}$ is a $\tau(Y, Y^\#)$-neighborhood of zero. Let U'' denote the closed unit ball of X''. Next, note that the operator $R\colon \big(X'', \sigma(X'', X')\big) \to \big(Y^\#, \sigma(Y^\#, Y)\big)$, defined by $Rx'' = x'' \otimes s$, is continuous. Since U'' is a $\sigma(X'', X')$-compact set, it follows that the convex circled set $D = R(U'') = \{x'' \otimes s\colon\ x'' \in U''\}$ is $\sigma(Y^\#, Y)$-compact. So, its polar

$$\begin{aligned} D^{\mathrm{o}} &= \{f \in Y\colon\ |\langle x'' \otimes s, f\rangle| = |x''(f(s))| \leq 1 \text{ for all } x'' \in U''\} \\ &= \{f \in Y\colon\ \|f(s)\| \leq 1\} \end{aligned}$$

is a $\tau(Y, Y^\#)$-neighborhood of zero, and the proof is finished. □

It should be clear that the preceding theorem can be reformulated as follows.

THEOREM 14.4. *For a linear functional ϕ defined on $C(S, X')$ the following statements are equivalent.*

1. $\phi = \sum_{i=1}^n x_i'' \otimes s_i$, *where $s_1, \ldots, s_n \in S$ and $x_1'', \ldots, x_n'' \in X''$.*
2. *ϕ is τ_w-continuous.*
3. *ϕ is τ_s-continuous.*

Now the standard duality theory yields the following result.

COROLLARY 14.5. *The topologies τ_w and τ_s on $C(S, X')$ have the same closed convex sets.*

We are now ready to establish that if a subspace $\mathcal{M}$ of $C(S, X')$ is $C(S)$-invariant, then $\mathcal{M}$ satisfies a nice separation property in the sense that elements outside of the closure of $\mathcal{M}$ can be separated by a linear functional of the form $x'' \otimes s$.

LEMMA 14.6. *Let $\mathcal{M}$ be a vector subspace of $C(S,X')$ which is invariant under multiplication by elements of $C(S)$. Then an element $f_0 \in C(S,X')$ does not belong to the τ_s-closure of $\mathcal{M}$ if and only if there exist $x'' \in X''$ and $s \in S$ such that*

$$\big\langle x'' \otimes s, f_0 \big\rangle = 1 \quad \textit{and} \quad \big\langle x'' \otimes s, f \big\rangle = 0$$

for all $f \in \mathcal{M}$.

Proof. The "only if" part needs verification. So, suppose that f_0 does not belong to the τ_s-closure of $\mathcal{M}$. Then, there exists some τ_s-continuous linear functional ϕ on $C(S,X')$ such that $\phi(f_0) \neq 0$ and $\phi(f) = 0$ for each $f \in \mathcal{M}$. By Theorem 14.4, $\phi = \sum_{i=1}^n x_i'' \otimes s_i$, where $s_i \neq s_j$ for $i \neq j$. From $\phi(f_0) = \sum_{i=1}^n \langle x_i'', f_0(s_i)\rangle \neq 0$, it follows that there exists some k satisfying $x_k'' \neq 0$ and $\langle x_k'', f_0(s_k)\rangle \neq 0$. We can suppose $\langle x_k'', f_0(s_k)\rangle = 1$.

Next, by Urysohns' Lemma, pick some $\alpha \in C(S)$ such that $\alpha(s_k) = 1$ and $\alpha(s_i) = 0$ for $i \neq k$. Since $\alpha f \in \mathcal{M}$ for each $f \in \mathcal{M}$, we have

$$\big\langle x_k'' \otimes s_k, f \big\rangle = \big\langle x_k'', f(s_k) \big\rangle = \sum_{i=1}^n \big\langle x_i'', \alpha(s_i) f(s_i) \big\rangle = \phi(\alpha f) = 0$$

for all $f \in \mathcal{M}$, and the proof is finished. □

Now we are ready to prove our first necessary and sufficient condition for the existence of a common non-trivial closed invariant subspace of X' for all the adjoint operators T' with $T \in \mathcal{A}$. As we shall see, this condition is closely related to the properties of the vector subspace generated in $C(S,X')$ by the collection of functions $\{\alpha T' \colon\ \alpha \in C(S), T \in \mathcal{A}\}$.

THEOREM 14.7. *For an arbitrary subalgebra $\mathcal{A}$ of $L(X)$, the following two statements are equivalent.*

1. *There exists a non-trivial closed $\mathcal{A}'$-invariant subspace.*
2. *There exists an operator $B \in L(X)$ such that B' does not belong to the τ_s-closure in $C(S,X')$ of the vector space generated by the set*

$$\{\alpha T' \colon\ \alpha \in C(S) \textit{ and } T \in \mathcal{A}\}.$$

Proof. Let $\mathcal{M}$ denote the vector subspace generated in $C(S, X')$ by the collection of functions $\{\alpha T' : \ \alpha \in C(S) \text{ and } T \in \mathcal{A}\}$, and let $\overline{\mathcal{M}}$ denote the closure of $\mathcal{M}$ in the topology τ_s.

(1) $\Longrightarrow$ (2) By Proposition 14.1 there exist non-zero $x'' \in X''$ and $s \in X'$ satisfying $\langle x'', T's \rangle = 0$ for each $T \in \mathcal{A}$. Without loss of generality we can suppose that $s \in S$.

Pick $b' \in X'$ and $b \in X$ such that $\langle x'', b' \rangle = 1$ and $\langle b, s \rangle = 1$. We claim that the rank-one operator $B = b' \otimes b \in L(X)$ satisfies $B' \notin \overline{\mathcal{M}}$.

To see this, note that

$$\big\langle x'' \otimes s, \alpha T' \big\rangle = \alpha(s) \big\langle x'', T's \big\rangle = 0$$

for each $T \in \mathcal{A}$ and all $\alpha \in C(S)$. That is, the τ_s-continuous linear functional $x'' \otimes s$ vanishes on $\mathcal{M}$. On the other hand, the relation

$$\big\langle x'' \otimes s, B' \big\rangle = \big\langle x'' \otimes s, b \otimes b' \big\rangle = \langle x'', b' \rangle \langle b, s \rangle = 1,$$

implies that $B' \notin \overline{\mathcal{M}}$.

(2) $\Longrightarrow$ (1) Pick some $B \in L(X)$ such that $B' \notin \overline{\mathcal{M}}$. Since $\mathcal{M}$ is invariant under multiplication by elements of $C(S)$, it follows from Lemma 14.6 that there exist $x'' \in X''$ and $s \in S$ such that

$$\langle x'', B's \rangle = 1 \quad \text{and} \quad \langle x'', T's \rangle = 0$$

for all $T \in \mathcal{A}$. Since $\langle x'', B's \rangle = 1$, it follows that $x'' \neq 0$ and $s \neq 0$, and by Proposition 14.1 the proof is finished. ☐

Our next goal is to obtain a similar characterization for the invariant subspace problem in terms of the norm topology. To accomplish this, we need to introduce the following class of completely continuous functions.

DEFINITION 14.8. *A function $f \in C(S, X')$ is said to be* **completely continuous** *if it is continuous for the weak* topology on S and the norm topology on X'.*

The vector subspace of all completely continuous functions of $C(S, X')$ will be denoted by $\mathcal{K}(S, X')$.

In other words, a function $f \in C(S, X')$ is completely continuous if and only if $s_\alpha \xrightarrow{w^*} s$ in S implies $\|f(s_\alpha) - f(s)\| \to 0$. Observe that if $T\colon X \to X$ is a compact operator, then $T'\colon S \to X'$ is completely continuous, i.e., $T' \in \mathcal{K}(S, X')$.

Clearly, $\mathcal{K}(S, X')$ is a normed closed subspace of $C(S, X')$. As a vector subspace of $C(S, X')$, the space $\mathcal{K}(S, X')$ inherits the three topologies considered on $C(S, X')$; the norm topology, the τ_w-topology, and the τ_s-topology. It is obvious that neither τ_w nor τ_s is consistent with the norm topology on $\mathcal{K}(S, X')$. Nevertheless, for $C(S)$-invariant subspaces of $\mathcal{K}(S, X')$ the situation is different. Using the Lomonosov–de Branges technique, we are now ready to characterize the closures of the $C(S)$-invariant subspaces of $\mathcal{K}(S, X')$ under these topologies.

THEOREM 14.9. *For a vector subspace $\mathcal{M}$ of $\mathcal{K}(S, X')$ which is invariant under multiplication by elements of $C(S)$ the following statements are equivalent.*

1. *The vector space $\mathcal{M}$ is τ_w-closed in $\mathcal{K}(S, X')$.*
2. *The vector space $\mathcal{M}$ is τ_s-closed in $\mathcal{K}(S, X')$.*
3. *The vector space $\mathcal{M}$ is norm closed.*

Proof. Clearly, $(1) \Longrightarrow (2) \Longrightarrow (3)$. It remains to establish the implication $(3) \Longrightarrow (1)$.

To this end, let $f_0 \in \mathcal{K}(S, X')$ belong to the τ_w-closure of $\mathcal{M}$. To show that $f_0 \in \mathcal{M}$, it suffices to prove that f_0 belongs to the norm closure of $\mathcal{M}$. By the Hahn–Banach Theorem, this will be established if we verify that for a norm continuous linear functional ϕ on $C(S, X')$ the identity $\langle \phi, f \rangle = 0$ for all $f \in \mathcal{M}$ implies that $\langle \phi, f_0 \rangle = 0$. So, let $\langle \phi, f \rangle = 0$ for all $f \in \mathcal{M}$. We can assume that ϕ has norm one.

For each $\alpha \in C(S)$, we define on $Y = C(S, X')$ the continuous linear functional ϕ_α by $\phi_\alpha(f) = \phi(\alpha f)$ where $f \in C(S, X')$. Since $\mathcal{M}$ is invariant under multiplication by elements of $C(S)$, it follows that $\langle \phi_\alpha, f \rangle = 0$ for all $f \in \mathcal{M}$. This certainly implies that $\langle \theta, f \rangle = 0$ for all $\theta \in V$ and $f \in \mathcal{M}$, where V is the weak* closed subspace generated by the ϕ_α in the norm dual Y'. We denote by $\mathcal{U}$ the intersection of V with the closed unit ball of Y'. Obviously we have $\phi = \phi_{\mathbf{1}} \in \mathcal{U}$.

Let ψ be any extreme point of $\mathcal{U}$. By the characterization of extreme points in [28, Theorem 1], it follows that there exist an element $s \in S$ and an element $x'' \in X''$ such that

$$\psi(f) = \big\langle x'' \otimes s, f \big\rangle = \big\langle x'', f(s) \big\rangle$$

holds for every $f \in \mathcal{K}(S, X')$. In particular, $\psi(f_0) = \big\langle x'', f_0(s) \big\rangle$ and $\psi(f) = \big\langle x'', f(s) \big\rangle = 0$ for all $f \in \mathcal{M}$. Since f_0 belongs to the τ_w-closure of $\mathcal{M}$, it follows that $\psi(f_0) = \langle x'', f_0(s) \rangle = 0$.

That is, we have proved that $\psi(f_0) = 0$ for each extreme point ψ of $\mathcal{U}$. Since $\phi \in \mathcal{U}$ and since, by the Krein–Milman theorem, $\mathcal{U}$ is the weak* closed convex hull of its extreme points, we can conclude that $\phi(f_0) = 0$. This completes the proof. □

We are now ready to state the main result of this section which considerably improves Theorem 14.7 by replacing the topology τ_s with the norm topology.

THEOREM 14.10. *If $\mathcal{A}$ is a subalgebra of $L(X)$, then the following statements are equivalent.*

1. *There exists a non-trivial closed $\mathcal{A}'$-invariant subspace.*

2. *There are operators $B, K \in L(X)$ with K compact such that the operator $B'K'$ does not belong to the norm closure in $C(S, X')$ of the vector space $\mathcal{M}$ generated by the set*

$$\big\{\alpha T'K' \colon\ \alpha \in C(S) \text{ and } T \in \mathcal{A}\big\}.$$

Proof. (1) $\Longrightarrow$ (2) By Proposition 14.1, there exist non-zero $x'' \in X''$ and $s \in X'$ satisfying $\langle x'', T's \rangle = 0$ for each $T \in \mathcal{A}$. We can suppose that $s \in S$. Now, pick elements $b \in X$ and $b' \in X'$ such that $\langle s, b \rangle = 1$ and $\langle x'', b' \rangle = 1$. Next, consider the rank-one operators $K = s \otimes b$ and $B = b' \otimes b$, and note that $K's = \langle s, b \rangle s = s$ and $B's = \langle s, b \rangle b' = b'$. We claim that $B'K' \notin \overline{\mathcal{M}}$, the norm closure of $\mathcal{M}$.

To see this, note that the norm continuous linear functional $\phi = x'' \otimes s$ on $C(S, X')$ satisfies

$$\big\langle \phi, \alpha T'K' \big\rangle = \alpha(s)\big\langle x'', T'K's \big\rangle = \alpha(s)\big\langle x'', T's \big\rangle = 0$$

for each $T \in \mathcal{A}$ and all $\alpha \in C(S)$. That is, ϕ vanishes on $\mathcal{M}$. On the other hand, $\phi(B'K') = \langle x'', B'K's \rangle = \langle x'', b' \rangle = 1$ shows that $B'K' \notin \overline{\mathcal{M}}$.

(2) $\Longrightarrow$ (1) Assume that $B, K \in L(X)$ satisfy the stated properties. Clearly, $B'K' \in \mathcal{K}(S, X')$ and $\mathcal{M} \subseteq \mathcal{K}(S, X')$. Also, $\mathcal{M}$ is $C(S)$-invariant.

Since $B'K'$ does not belong to the norm closure of $\mathcal{M}$, it follows from Theorem 14.9 that $B'K'$ is not in the τ_w-closure of $\mathcal{M}$ in $\mathcal{K}(S, X')$. Therefore, by Lemma 14.6, there exist $x'' \in X''$ and $s \in S$ satisfying

$$\big\langle x'', B'K's \big\rangle = 1 \quad \text{and} \quad \big\langle x'', T'K's \big\rangle = 0$$

for all $T \in \mathcal{A}$. The former condition implies that $K's \neq 0$, and hence the latter condition shows that Proposition 14.1 is applicable to $\mathcal{A}'$. □

The approach suggested by L. de Branges in [28] was aimed at the dual invariant subspace problem. As we shall see next, our arguments above allow us to obtain also a necessary and sufficient condition for the existence of a common non-trivial closed invariant subspace for the algebra $\mathcal{A}$ itself. It is interesting to notice that the only difference is that we must apply the compact operator on the left, as opposed to the multiplication on the right in the preceding theorem.

THEOREM 14.11. *For a subalgebra $\mathcal{A}$ of $L(X)$ the following statements are equivalent.*

1. *There exists a non-trivial closed $\mathcal{A}$-invariant subspace.*
2. *There are operators $B, K \in L(X)$ with K compact such that the operator $K'B'$ does not belong to the norm closure in $C(S, X')$ of the subspace generated by the set*

$$\big\{\alpha K'T' \colon\ \alpha \in C(S) \text{ and } T \in \mathcal{A}\big\}.$$

Proof. (1) $\Longrightarrow$ (2) By Proposition 14.1 there exist non-zero $x \in X$ and $x' \in X'$ satisfying $\langle x', Tx \rangle = 0$ for each $T \in \mathcal{A}$. We can suppose that $x' \in S$.

Take any $b' \in X'$ such that $\langle b', x \rangle = 1$, and then consider the rank-one operator $K = b' \otimes x \in L(X)$. Clearly $Kx = \langle b', x \rangle x = x$. Next choose any element b in X such that $\langle x', b \rangle = 1$. Define now the operator $B = b' \otimes b \in L(X)$. Clearly, the adjoint operator B' satisfies $B'x' = \langle x', b \rangle b' = b'$.

The vector space generated by $\{\alpha K'T' \colon\ \alpha \in C(S), T \in \mathcal{A}\}$ will be denoted by $\mathcal{N}$. We claim that the operator $K'B'$ is not in the norm closure of $\mathcal{N}$. To see this, note that the linear functional $\phi = x \otimes x'$ is norm continuous on $C(S, X')$ and satisfies

$$\langle \phi, \alpha K'T' \rangle = \alpha(x')\langle x, K'T'x' \rangle = \alpha(x')\langle TKx, x' \rangle = \alpha(x')\langle Tx, x' \rangle = 0$$

for each $T \in \mathcal{A}$ and all $\alpha \in C(S)$. That is, $\phi = x \otimes x'$ vanishes on $\mathcal{N}$. On the other hand, the equality $B'x' = b'$ yields

$$\phi(K'B') = \langle x, K'B'x' \rangle = \langle x, K'b' \rangle = \langle Kx, b' \rangle = \langle x, b' \rangle = 1,$$

which shows that $K'B'$ does not belong to the norm closure of $\mathcal{N}$.

(2) $\Longrightarrow$ (1) Assume that the operators B and K satisfy the stated properties. Again, let $\mathcal{N}$ denote the vector space generated by the collection of functions $\{\alpha K'T' \colon\ \alpha \in C(S), T \in \mathcal{A}\}$ in $C(S, X')$. Clearly, $K'B' \in \mathcal{K}(S, X')$ and $\mathcal{N} \subseteq \mathcal{K}(S, X')$. Also, $\mathcal{N}$ is $C(S)$-invariant.

Since the compact operator $K'B'$ does not belong to the norm closure of $\mathcal{N}$, it follows from Theorem 14.9 that $K'B'$ is not in the τ_s-closure of $\mathcal{N}$ in $\mathcal{K}(S, X')$. Consequently, by Lemma 14.6, there exist $x'' \in X''$ and $s \in S$ such that

$$\langle x'', K'B's \rangle = 1 \quad \text{and} \quad \langle x'', K'T's \rangle = 0$$

for all $T \in \mathcal{A}$. Since $\langle x'', K'B's \rangle = 1$, it follows that $x_0 = K''x'' \neq 0$. Moreover, the compactness of K implies that $x_0 = K''x'' \in X$. Therefore, for each $T \in \mathcal{A}$, we have

$$\langle Tx_0, s \rangle = \langle TK''x'', s \rangle = \langle K''x'', T's \rangle = \langle x'', K'T's \rangle = 0,$$

and by Proposition 14.1 the proof is finished. □

We close the section by emphasizing once again the following remarkable fact that has been present throughout our discussion: as soon as we start asking about the existence of a non-trivial closed $\mathcal{A}$-invariant subspace of X (or about the existence of a non-trivial closed $\mathcal{A}'$-invariant subspace of X'), a compact operator emerges though the given algebra $\mathcal{A}$ need not be connected with compactness in any way!

We refer to [7] for some applications of the above results, to [30] for additional properties of spaces of weakly continuous functions from a compact space to a Banach space and to [54, 55] for some interesting further developments.

15. Invariant subspaces for other classes of operators

In this section, we shall briefly indicate how one can prove the existence of invariant subspaces for several other classes of operators on Banach lattices.

Recall that an operator $T\colon E \to E$ on a Banach lattice is said to be AM-**compact**, provided T maps order bounded sets onto norm precompact sets, i.e., $T[a,b]$ is norm precompact for each order interval $[a,b]$; see [61, p. 505]. Clearly, each compact operator is AM-compact but an AM-compact operator need not be compact. (Since the order intervals in any ℓ_p-space ($1 \le p < \infty$) are norm compact, every continuous operator on ℓ_p is AM-compact but, of course, not every continuous operator is compact.)

If we follow the proofs of Theorems 10.1 and 10.3 and Corollary 10.4 closely, we shall see that the domination of a non-zero compact operator can be replaced by the domination of a non-zero AM-compact operator.

For instance, the details of the necessary changes to the proof of Theorem 10.1 are as follows. In this case, we cannot claim that the set $\overline{K(U)}$ is compact, but we certainly have that the set $\overline{K(U \cap [0, x_0])}$ is compact. The inclusion

$$\overline{K(U \cap [0, x_0])} \subseteq \bigcup_{n=1}^{\infty} \{z \in E\colon \ \|x_0 - x_0 \wedge nA(|z|)\| < 1\}$$

obviously remains true, and so, in view of the compactness of the set

$\overline{K(U \cap [0, x_0])}$, there exists some m such that

$$K(U \cap [0, x_0]) \subseteq \{z \in E\colon \ \|x_0 - x_0 \wedge mA(|z|)\| < 1\}.$$

Now note that the sequence $\{x_n\}$, defined inductively by

$$x_1 = x_0 \wedge mA\big(|Kx_0|\big) \ \text{ and } \ x_{n+1} = x_0 \wedge mA\big(|Kx_n|\big), \ n = 1, 2, \ldots,$$

satisfies $x_n \in U \cap [0, x_0]$ for each n. The rest of the proof remains the same.

Thus, we can state Theorem 10.1 in the following slightly more general form.

THEOREM 15.1. *Let $B\colon E \to E$ be a positive operator on a Banach lattice. Assume that there exists a positive operator $S\colon E \to E$ such that*

1. *$SB \leq BS$ (in particular, this holds if S and B commute),*
2. *S is quasinilpotent at some $x_0 > 0$, i.e., $\lim_{n\to\infty} \|S^n x_0\|^{\frac{1}{n}} = 0$, and*
3. *S dominates a non-zero AM-compact operator.*

Then the operator B has a non-trivial closed invariant subspace. Moreover, we can choose this invariant subspace to be the closure of a principal ideal in E.

Similarly, one can prove the following slightly more general versions of Theorem 10.3 and Corollary 10.4.

THEOREM 15.2. *Let $B, S\colon E \to E$ be two commuting non-zero positive operators on a Banach lattice. If one of them is quasinilpotent at a non-zero positive vector and the other dominates a non-zero AM-compact operator, then B and S have a common non-trivial closed invariant ideal.*

COROLLARY 15.3. *Let a positive operator $B\colon E \to E$ on a Banach lattice satisfy the following two conditions:*

1. *B is quasinilpotent at a non-zero positive vector, and*

2. *Some power of B dominates a non-zero AM-compact operator.*

Then the operator B has a non-trivial closed invariant ideal.

Another important class of operators to which our results can be applied is that of Dunford–Pettis operators. Recall that an operator $T\colon X \to X$ is said to be **Dunford–Pettis** if $\|Tx_n\| \to 0$ for each sequence $\{x_n\}$ in X that converges weakly to zero. For these operators the following result is true.

THEOREM 15.4. *Every positive Dunford–Pettis operator on a Banach lattice which is quasinilpotent at a non-zero positive vector has a non-trivial closed invariant ideal.*

Proof. Let $B\colon E \to E$ be a positive Dunford–Pettis operator on a Banach lattice which is quasinilpotent at a non-zero positive vector. We can assume that B is strictly positive—otherwise N_B is a non-trivial closed invariant ideal. Then B carries order intervals onto relatively weakly compact sets [14, Theorem 19.12, p. 339], and so B^2 carries order intervals onto norm precompact sets. That is, B^2 is a non-zero AM-compact operator. Since B commutes with B^2, our conclusion follows from Theorem 15.1. □

If by analogy with compact-friendly operators we introduce two similar concepts by replacing in Definition 11.1 the compact operators by AM-compact operators or Dunford–Pettis operators respectively, then, it seems plausible that we should be able to generalize much of the theory described in this survey to these two new classes of operators. We do not pursue this direction here, leaving it as a promising open venue for future research.

The last result which we are about to formulate is not directly related to the invariant subspace problem but, somehow, it seems relevant to our discussion. Let again $T\colon X \to X$ be a continuous operator on a Banach space. A vector $x \in X$ is called **cyclic** if the linear space generated by the orbit $\{x, Tx, T^2x, \dots\}$ of x is norm dense in X. It is obvious that T has a closed non-trivial invariant subspace if and only if there is a non-zero vector which is not cyclic.

We need two more definitions closely related to the previous one. A vector $x \in X$ is called **hypercyclic** (for an operator T) if the orbit itself $\{x, Tx, T^2x, \dots\}$ is dense in X. Similarly, $x \in X$ is called **supercyclic** (for an operator T) if the set $\{cT^n x : c \in \mathbf{C}, n \in \mathbb{N}\}$ is dense in X.

An operator $T\colon X \to X$ is said to be cyclic (resp. hypercyclic, or supercyclic) if there exists a cyclic (resp. hypercyclic, or supercyclic) vector for T.

If we look at the right shift operator S on any ℓ_p-space with $p < \infty$, then clearly S is a cyclic operator (for example, $\mathbf{e}_1$ is a cyclic vector) but the second power S^2 is not cyclic. A similar problem for hypercyclic and supercyclic operators has been open for quite a while. And only recently S. Ansari [17] has found a very elegant and complete solution to this problem.

THEOREM 15.5 (ANSARI). *If a vector $x \in X$ is hypercyclic (resp. supercyclic) for a continuous operator $T\colon X \to X$, then x is also hypercyclic (resp. supercyclic) for T^n for every n.*

Finally, we mention that all the results in this paper remain true if we replace $\lim\limits_{n\to\infty} \|T^n x_0\|^{\frac{1}{n}} = 0$ by $\liminf\limits_{n\to\infty} \|T^n x_0\|^{\frac{1}{n}} = 0$ in Definition 4.1.

16. Open problems and remarks

For the convenience of the reader we have collected here the open problems which were posed in the course of this work. The first number in each reference indicates the number of the section where this problem was mentioned. The second number if present is used just to distinguish between the consecutive problems within the same section.

5.1. How can we recognize by "looking at" a matrix $[t_{ij}]$ defining a positive operator $T : \ell_p \to \ell_p$ if the set $\mathcal{Q}_T^+$ is non-empty?

5.2. It is an open problem whether or not each positive operator on ℓ_1 (or ℓ_p with $p < \infty$) has an invariant subspace.

6.1. There exists a concept of a basis which is weaker than the notion of Schauder basis. We mean the so called Markushevich basis (see for example [56]). It would be interesting to investigate to what extent the results of Section 7 can be generalized to positive operators on a Banach space with some kind of a Markushevich basis.

6.2. Consider a quasinilpotent operator on a Banach space with a basis. Suppose we do not assume that the operator is positive with respect to this basis, and thus we cannot apply directly our results on the existence of invariant subspaces of positive operators. However, if one considers a change of basis, then the operator might very well become positive with respect to the new basis, and therefore, it would have a non-trivial closed invariant subspace.

When is a given quasinilpotent operator on a Banach space with a Schauder basis (in particular, on a Hilbert space) positive with respect to some basis?

9.1. We do not know presently if an analogue of Theorem 10.1 is true provided we replace the inequality $SB \leq BS$ by the reverse inequality $SB \geq BS$. In other words, does Theorem 10.2 remain true if the operator S is assumed to be merely locally quasinilpotent at a positive vector instead of being quasinilpotent?

9.2. In the present formulation, Theorem 10.1 generalizes Theorem 4.1 in [4]. The difference is that here we have replaced the commutativity condition $SB = BS$ by the weaker assumption $SB \leq BS$. It would be very interesting to investigate in which results one can replace commutativity by some kind of inequality. In particular, whether one can replace the commutativity assumption in Theorem 11.2 by an inequality $BT \leq TB$ or $TB \leq BT$.

11.1. As we have seen, both conditions in Theorem 11.2 are imposed on the same operator, i.e., B is assumed to be both compact-friendly and quasinilpotent at some positive vector. It is an open and very interesting question if, in analogy with Theorem 10.3, we can distribute these two properties between the two operators. To be precise, let B and T be two commuting positive operators, such that B is compact-friendly and $\mathcal{Q}_T^+ \neq \emptyset$. Does there exist a non-trivial closed B-invariant subspace, or a T-invariant subspace, or even a common invariant subspace?

11.2. Let us say that a positive operator $B\colon E \to E$ on a Banach lattice E is "good" (resp. "very good") if there exists a non-zero compact (resp. a non-zero *positive* compact) operator K which is dominated by B.

- Is every good operator very good?
- Assume that B^2 is good, is then B^2 very good?

- Assume that B^2 is good, is then B (very) good?
- Assume that B^2 is very good, is then B (very) good?

Similar questions can also be asked if we assume that there exists a polynomial p with non-negative coefficients such that $p(B)$ is good or very good.

14. Each hypercyclic vector is obviously supercyclic. It would be interesting to find conditions on the vector and/or the operator under which a supercyclic vector is hypercyclic.

S. Ansari's Theorem 15.5 on hypercyclic and supercyclic vectors is true for any Banach space. It would be interesting to investigate if some additional specific features can be found in the case of regular operators on Banach lattices.

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