Periodic Points
of Small Periods
# of mappings of B-spaces

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Summary. - *B-spaces are a class of uniquely arcwise connected generalized continua, containing trees. In this paper, it is shown that a main result obtained by several authors for existence of periodic points of small periods of mappings of trees is also true for B-spaces.*

1. Introduction

A family of subsets of a set is called a *chain* if it is linearly ordered by inclusion. In the following an arc means a metric arc. A *B-space* X is an arcwise connected Hausdorff space such that every chain of arcs of X is contained in an arc of X. Obviously it is uniquely arcwise connected, i.e., for every pair $\{x, y\}$ of distinct points of X there exists a unique arc $[x, y]$ from x to y. The "B-space" named by Holsztyński derives from Borsuk-Young arcwise connected space [3].

Let S be a space and $f : S \mapsto S$ a mapping (= a continuous function) of S into itself. A point $p \in S$ is called a *periodic point of f of period k* if $p = f^k(p)$ and $p \neq f^i(p)$ for $1 \leq i < k$. The *orbit* of p denoted by $\mathrm{Orb}(p)$ is the set $\{f^i(p) : i = 0, 1, 2, \ldots\}$.

For n points $x_1, x_2, \ldots, x_n$ $(n \geq 3)$ of a B-space, the notation $(x_1 \ x_2 \ldots x_n)$ means that if $x_1 = x_n$, then $x_1 = x_2 = \cdots = x_n$, and

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if $x_1 \neq x_n$, then $x_i \in [x_1, x_n]$ $(1 \leq i \leq n)$ and there exists a homeo-morphism h of $[x_1, x_n]$ onto $[0, 1]$ such that $h(x_1) = 0, h(x_n) = 1$ and $h(x_1) \leq h(x_2) \leq \cdots \leq h(x_n)$. Therefore $(x_1 \ x_2 \ x_3)$ is equivalent to that $x_1 = x_2 = x_3$ or $x_2 \in [x_1, x_3]$.

In this paper we shall prove the following theorem which general-izes a main result for periodic points on trees in $[1, 2, 4]$ to B-spaces.

THEOREM 1.1. *Let X be a B-space and let f be a mapping of X into itself. If there exist $w \in X$ with $f(w) \neq w$ and positive integers r, s such that $(f^r(w) \ w \ f^s(w))$, then f has a periodic point of period $\leq \max\{r, s\}$.*

2. Trees

A *tree T* is a compact, arcwise connected Hausdorff space in which every pair $\{x, y\}$ of distinct points is separated by a third point z, i.e., there are disjoint open sets M, N such that $x \in M$, $y \in N$ and $T \setminus \{z\} \in M \cup N$. A space S is called *locally arcwise connected* if it has a basis consisting of arcwise connected open sets. Equivalently, S is locally arcwise connected if for every point $x \in S$ and any neighborhood U of x there exists a neighborhood V of x such that any $y \in V \setminus \{x\}$ can be joined to x by an arc in U.

LEMMA 2.1. *Every tree T is uniquely arcwise connected.*

LEMMA 2.2. *Every tree T is locally arcwise connected.*

LEMMA 2.3. *Every tree T is a B-space.*

Since a tree is hereditarily unicoherent (e.g., $[7$, Theorem $9.1]$), it is a B-space by $[3$, Proposition $1.12]$. (We owe to the referee this argument.)

PROPOSITION 2.4. *A space X is a tree if and only if it is a compact, locally connected B-space.*

Proof. ($\Rightarrow$) This follows at once from Lemmas 2.2 and 2.3.

($\Leftarrow$) Let x, y be distinct points of X and $z \in [x, y] \setminus \{x, y\}$. Let M be the component of $X \setminus \{z\}$ containing x and N the union of the other components of $X \setminus \{z\}$. Then M, N are open and $y \in N$,

since $X \setminus \{z\}$ is open and X is locally arcwise connected. Obviously $X \setminus \{z\} = M \cup N$ and $M \cap N = \emptyset$. Hence z separates x and y.   □

REMARK 2.5.   Countable comb and Cantor's teepee are B-spaces but not trees.

A point e of a tree T is called an *end point* of T if e is an end point of every arc of T containing e. A *tree in the sense of Nadler* [5] is an acyclic graph, i.e., a continuum (= nonempty, compact, connected metric space) which is the union of finitely many arcs, any two of which are either disjoint or intersect only in one or both of their end points, and which contains no simple closed curve.

LEMMA 2.6. *A tree T is a tree in the sense of Nadler if and only if T has finitely many end points.*

Proof. ($\Rightarrow$) This is obvious.

($\Leftarrow$) Since T is a B-space, it is acyclic. Suppose that T has end points $e_1, \ldots, e_k$. Then $T = \bigcup_{i=1}^{k} [e_1, e_i]$. For if not, there exists a point $p \in T \setminus \bigcup_{i=1}^{k} [e_1, e_i]$. Let $\mathcal{A}$ be the family of arcs one of whose end points is e_1 and containing $[e_1, p]$. Then ordering by inclusion in $\mathcal{A}$ is a partial ordering. Each chain in $\mathcal{A}$ has an upper bound, since T is a B-space. Therefore by Zorn's Lemma $\mathcal{A}$ has at least one maximal element $[e_1, q]$ and q is an end point of T different from e_i $(1 \leq i \leq k)$, a contradiction. By a straightforward induction argument on the number of end points, we can easily show that T is a graph.   □

3. Main Lemma

In this section Y is a locally arcwise connected B-space.

LEMMA 3.1. *Let a, b, c, d be points in a B-space X. If $(a\ b\ c)$, $(b\ c\ d)$ and $b \neq c$, then $(a\ b\ c\ d)$.*

Proof. See [2, IV, p. 36 and Proposition 3.2, p. 37].

LEMMA 3.2. *If $(x\ z\ y)$ for three distinct points x, y, z of Y, then z separates x and y in Y.*

MAIN LEMMA. *Let f be a mapping of Y into itself. If there exist $w \in Y$ with $f(w) \neq w$ and positive integers r, s such that $(f^r(w) \ w \ f^s(w))$, then f has a periodic point of period $\leq \max\{r, s\}$.*

Proof. The idea of our proof is similar to [2, Theorem 5.1]. When $f^r(w) = w$ or $f^s(w) = w$, the point w is a periodic point of period $\leq \min\{r, s\}$, since $f(w) \neq w$.

Suppose that $f^r(w) \neq w \neq f^s(w)$. Then w separates $f^r(w)$ and $f^s(w)$ in Y by Lemma 3.2. Hence $\mathrm{Orb}(w) \setminus \{w\}$ is not contained in a single component of $Y \setminus \{w\}$. Let C be the component of $Y \setminus \{w\}$ containing $f(w)$. Since Y is locally connected by our hypothesis, C is open in Y. Let j be the positive integer such that $f^i(w) \in C$ for each i $(1 \leq i < j)$ and $f^j(w) \in Y \setminus C$. Let Z denote the set $\{t \in Y \setminus C : (f^i(t) \ w \ t \ f^j(t))$ for each i $(1 \leq i < j)\}$. Clearly $w \in Z$.

We prove that Z is closed. If this is not true, then there exists $x \in \bar{Z} \setminus Z$. Clearly $x \neq w$ and $x \in Y \setminus C$. Furthermore $f^j(x) \neq w$. For if not, there exist disjoint open sets U, V such that $w \in U$, $x \in V$, $f^j(V) \subset U$ and U is arcwise connected. Since $x \in \bar{Z}$, there is a point $t \in V \cap Z$. Then we can join w to t by two different arcs in $U \cup [w, t] \cup [t, f^j(t)]$. This is impossible, because Y is uniquely arcwise connected. We show $(f^i(x) \ w \ x)$ for each i $(1 \leq i < j)$. Suppose on the contrary that this is not true. Then $w \notin [f^i(x), x]$ for some i. There exist arcwise connected open sets U, V such that $x \in U$, $f^i(x) \in V$, $w \notin U \cup V$ and $f^i(U) \subset V$. Let $t \in U \cap Z$. From $t \in Z$ it follows that $(f^i(t) \ w \ t \ f^j(t))$ and hence $(f^i(t) \ w \ t)$. Then by uniquely arcwise connectedness of B-space, $[f^i(t), t] \subset V \cup [f^i(x), x] \cup U$ and hence $w \notin [f^i(t), t]$, contrary to $(f^i(t) \ w \ t)$. Similarly we have $(w \ x \ f^j(x))$. Hence $(f^i(x) \ w \ x \ f^j(x))$ by Lemma 2.6 and $x \in Z$, which contradicts $x \in \bar{Z} \setminus Z$. Thus Z is closed.

Now let $\mathcal{A}$ be the collection of arcs of Z with common end point w. We must show $\mathcal{A} \neq \emptyset$. To do this let V be an arcwise connected neighborhood of $f^j(w)$ with $w \notin V$ and let U_i $(1 \leq i < j)$ be a neighborhood of w such that $U_i \cap V = \emptyset$, $f^i(U_i) \subset C$ and $f^j(U_i) \subset V$. Let t be a point different from w such that $[w, t] \subset (\bigcap_{i=1}^{j-1} U_i) \cap [w, f^j(w)]$. Then $[w, t] \in \mathcal{A}$.

The order by inclusion in $\mathcal{A}$ is a partial order. We show that $\mathcal{A}$ has a maximal element. By definition of B-space, every chain $\{\alpha_\lambda\}$ of $\mathcal{A}$ is contained in an arc α of Y. Therefore $\overline{\bigcup_\lambda \alpha_\lambda}$ is a subarc of α,

one of whose end points is w. It is contained in Z since Z is closed. Consequently it belongs to $\mathcal{A}$ and is an upper bound of the chain $\{\alpha_\lambda\}$. Thus by Zorn's Lemma $\mathcal{A}$ has a maximal element β.

Let e be the end point of β different from w. Then we have $f^j(e) = e$. To show this, suppose on the contrary that $f^j(e) \neq e$. As $\beta \subset Z$, e is a point of Z and hence $(f^i(e) \; w \; e \; f^j(e))$ $(1 \leq i < j)$. For each i $(1 \leq i < j)$, we can find mutually exclusive open sets U, V, W such that $e \in U \subset Y \setminus C$, $f^i(e) \in V \subset C$, $f^j(e) \in W$, $f^i(U) \subset V$, $f^j(U) \subset W$ and both of V and W are arcwise connected. Then let $t \in U$ be a point different from e such that $[e, t] \subset U \cap [e, f^j(e)]$. For every $u \in [e, t]$, we have $f^i(u) \in V$, $f^j(u) \in W$, $[f^i(u), u] \subset V \cup [f^i(e), u]$ and $[u, f^j(u)] \subset [u, f^j(e)] \cup W$. Accordingly it follows that $(f^i(u) \; w \; u)$ and $(w \; u \; f^j(u))$. Since $w \neq u$, we have $(f^i(u) \; w \; u \; f^j(u))$ by Lemma 3.1. Since $u \in Y \setminus C$, $u \in Z$ and hence $[w, t] \in \mathcal{A}$ as u is an arbitrary point of $[e, t]$, which contradicts that $[w, e]$ is a maximal element of $\mathcal{A}$. Thus we conclude that $f^j(e) = e$.

Since $e \in Z$, $(f^i(e) \; w \; e \; f^j(e))$ holds. Therefore $f(e) \neq e$. For if not, we have $f^i(e) = f^j(e) = e$ and $w = e$, a contradiction. Thus e is a periodic point of f of period $\leq j$.

It remains to show that $j \leq \max\{r, s\}$. Suppose on the contrary that $\{f^r(w), f^s(w)\} \subset C$. Since C is arcwise connected, $[f^r(w), f^s(w)] \subset C$. This contradicts that w separates $f^r(w)$ and $f^s(w)$. Therefore we have only to consider three cases: (1) $\{f^r(w), f^s(w)\} \subset Y \setminus C$, (2) $f^r(w) \in C$, $f^s(w) \in Y \setminus C$ and (3) $f^s(w) \in C$, $f^r(w) \in Y \setminus C$. For every case we see $j \leq \max\{r, s\}$ by the definition of j. $\square$

4. Proof of Theorem 1.1 and Corollary 4.1

Proof of Theorem 1.1. An *arc component* of a space S is a maximal arcwise connected subset of S. *Young's arc topology* [8] is the topology with the arc components of open sets of the given topology as a basis.

Let τ be the topology of X in the theorem and let λ be Young's arc topology on X. Then (X, λ) is a locally arcwise connected B-space.

Moreover $f : (X, \lambda) \mapsto (X, \tau)$ is continuous, since λ is finer than τ. Since (X, τ) is Hausdorff, the image of any arcwise connected

subset under $f : (X, \lambda) \mapsto (X, \tau)$ is arcwise connected. Hence $f :$ $(X, \lambda) \mapsto (X, \lambda)$ is continuous.

According to the Main Lemma, $f : (X, \lambda) \mapsto (X, \lambda)$ has a periodic point of period $\leq \max\{r, s\}$, which is also a periodic point of $f :$ $(X, \tau) \mapsto (X, \tau)$ of the same period. $\quad\square$

COROLLARY 4.1. ([1, 4]) *Let T be a tree with m end points and let f be a mapping of T into itself. If f has a periodic point u of period $n > m$, then it has a periodic point of period $\leq m$.*

Proof. Put $S = \bigcup_{\nu \in \mathrm{Orb}(u)}[u, \nu]$. Then it is a subtree of T each of whose end points belongs to $\mathrm{Orb}(u)$. The number of end points of any subtree of T does not exceed m. Therefore there exists $w \in \mathrm{Orb}(u)$ which is not an end point of S. Then since w is a cut point of S, we can find two end points p, q of S belonging to different components of $S \setminus \{w\}$. Hence we have $(p \ w \ q)$. We can easily see that there exist integers r, s such that $0 < r < n$, $0 < s < n$, $p = f^r(w)$ and $q = f^s(w)$. Thus according to the theorem, f has a periodic point of period $< n$.

Inductively we may continue this process up to get our assertion. $\quad\square$

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