Rend. Istit. Mat. Univ. Trieste
Vol. XXIX, 55–61 (1997)

# Solvability of a Three–Point Boundary Value Problem below the First Eigenvalue

Franic Ikechukwu Njoku (*)

Summary. - *Assuming only asymptotic conditions on the potential function, we prove the existence of solutions for third order equations whose nonlinearity stays below the first eigenvalue of the associated linear problem.*

In this paper, we are concerned with the solvability of the nonlinear differential equation

$$u''' + f(u) = p(t, u, u') \tag{1}$$

with the boundary condition

$$u(a) = u'(a) = u(b) = 0, \tag{2}$$

or

$$u(a) = u(b) = u'(b) = 0, \tag{3}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $p : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

In the literature, the usual assumptions for the study of the existence of solutions for third order boundary value problems involve conditions on the nonlinearity $f$ like monotonicity or sign conditions or growth restrictions involving the ratio $f(s)/s$. (See for instance, [1-2, 9-10, 12-13]). The purpose of this paper is to obtain existence results for problems (1)-(2) and (1)-(3) which do not require any of

(*) Author's address: Department of Mathematics, University of Nigeria, Nsukka (Nigeria).

such conditions on $f$. Our results are based on a sign assumption on the potential

$$F(u) := \int_0^u f(s)ds.$$

By using topological degree, we shall prove the following

THEOREM 1. *Assume that p is bounded and*

$$\liminf_{s\to\pm\infty} \frac{F(s)}{s^2} \le \frac{\pi^2}{(b-a)^3} \tag{4}$$

*where* $F(u) := \int_0^u f(s)ds$. *Then, problem* (1)-(2) *has a solution.*

THEOREM 2. *Assume that p is bounded and that condition* (4) *holds. Then, problem* (1)-(3) *has a solution.*

The rather weak hypothesis (4) concerning the inferior limits of $F(s)/s^2$ was first introduced in [3]. It was also used in [4-5] for the existence of periodic solutions of second order differential equations, in [6] for elliptic problems and in [11] for parabolic problems. However, the above quoted papers do not include our case which gives a nonresonance result for a non-symmetric problem. We also stress the fact that our proof is not an adaptation of those in [3-6, 11] and indeed, even if the condition (4) is the same like in these papers, we use a different argument.

It is shown in [7-8] that

$$u''' + \lambda u = 0$$

together with the boundary condition (2) or (3) has a sequence of positive real eigenvalues

$$\lambda_1 < \lambda_2 < \dots \quad \text{with} \quad \lambda_1 > \frac{\pi^2}{(b-a)^3}.$$

Therefore, condition (4) means that the nonlinearity is below the first eigenvalue.

*Proof of Theorem 1.* It is well known that problem (1)-(2) can be written as a Hammerstein equation of the form

$$u(t) = \int_a^b G(t,\xi)[p(\xi,u(\xi),u'(\xi)) - f(u(\xi))]d\xi := (\mathcal{K}u)(t)$$

with $G$ a suitable Green function. We can also prove that $\mathcal{K}$ is completely continuous as an operator in $C^2([a,b])$ and fixed points of $\mathcal{K}$ are solutions of the original boundary value problem. Using Leray-Schauder degree theory, we consider the homotopic equation $u = \lambda\mathcal{K}u$ for $0<\lambda\leq 1$, which corresponds to the differential equation

$$u''' + \lambda f(u) = \lambda p(t,u,u'), \quad 0<\lambda\leq 1 \tag{$1_\lambda$}$$

with the associated boundary conditions and find an open bounded subset $\Omega$ of the space $C^2([a,b])$ containing 0, whose boundary does not contain any solutions of $(1_\lambda)$-(2). For this, we claim that there exist two positive constants A, B such that there are no solutions of problem $(1_\lambda)$-(2) with $\max u = B$ or $\min u = -A$ for all $\lambda \in\ ]0,1]$. This then implies that problem (1)-(2) has at least one solution in

$$\Omega := \{u\in C^2([a,b]) : -A < u(t) < B, |u'(t)| + |u''(t)| < C, \forall t\}, \tag{5}$$

where $C$ is a constant that will be fixed at the end of the proof.

In view of the previous section, it suffices for us to obtain a priori bounds for solutions of the parameterized problem

$$\begin{cases} u''' + \lambda f(u) = \lambda p(t,u,u'), \quad 0<\lambda\leq 1 \\ u(a)=0, u'(a)=0, u(b)=0. \end{cases} \tag{6}$$

Assume that $u$ is any solution of problem (6) such that u $\not\equiv 0$ and let

$$t^* \in\ ]a,b[ \text{ and } |u(t^*)| = \max |u(t)|.$$

We first prove the existence of a constant $B$ independent of $u$ and $\lambda$ such that $|u(t^*)| \neq B$. Multiplying the differential equation in (6) by $u'$ and integrating over $[a,t^*]$, we get

$$-\int_a^{t^*} [u''(t)]^2 dt + \lambda F(u(t^*)) = \lambda \int_a^{t^*} p(t,u,u')u' dt$$

which gives

$$\lambda F(u(t^*)) = \int_a^{t^*} [u''(t)^2 + \lambda p(t,u,u')u'(t)] dt.$$

So,

$$\lambda F(u(t^*)) \geq \int_a^{t^*} u''(t)^2 dt - \int_a^{t^*} |p(t,u,u')||u'(t)| dt. \tag{7}$$

But by Poincaré's inequality

$$\sqrt{\int_a^{t^*} u'(t)^2dt} \leq \sqrt{(\frac{t^*-a}{\pi})^2 \int_a^{t^*} u''(t)^2dt} < \frac{b-a}{\pi}\sqrt{\int_a^{t^*} u''(t)^2dt}.$$

Hence,

$$-\sqrt{\int_a^{t^*} u'(t)^2dt} > -\frac{b-a}{\pi}\sqrt{\int_a^{t^*} u''(t)^2dt}. \tag{8}$$

Using Schwarz's inequality, we get that

$$-\int_a^{t^*} |p(t,u,u')||u'(t)|dt \geq -\left(\int_a^{t^*} p(t,u,u')^2dt\right)^{\frac{1}{2}}\left(\int_a^{t^*} u'(t)^2dt\right)^{\frac{1}{2}}.$$

Therefore by (8),

$$\begin{aligned} &-\int_a^{t^*} |p,u,u'||u'(t)|dt \\ &\quad > -\frac{(b-a)}{\pi}\left(\int_a^{b} p(t,u,u')^2dt\right)^{\frac{1}{2}}\left(\int_a^{t*} u''(t)^2dt\right)^{\frac{1}{2}} \end{aligned}$$

Hence,

$$-\int_a^{t^*} |p,u,u'||u'(t)|dt > -\frac{(b-a)}{\pi}||p||_{L^2}\left(\int_a^{t^*} u''(t)^2dt\right)^{\frac{1}{2}}. \tag{9}$$

Substituting (9) in (7), we get

$$\lambda F(u(t^*)) > \int_a^{t^*} u''(t)^2dt - \frac{(b-a)}{\pi}||p||_{L^2}\left(\int_a^{t^*} u''(t)^2dt\right)^{\frac{1}{2}}$$

Since $y^2-\alpha y \geq cy^2-d$ provided $0<c<1$ and $d=\alpha^2/4(1-c)$, we deduce that for all $0<c<1$ there is a constant $d>0$ such that

$$\lambda F(u(t^*)) > c\int_a^{t^*} u''(t)^2dt - d,$$

that is,

$$\int_a^{t^*} u''(t)^2dt < \frac{\lambda}{c}F(u(t^*)) + \frac{d}{c} \tag{10}$$

for all solutions $u$ of $(1_\lambda)$. Now we choose $0 < c < 1$ such that

$$\frac{c\pi^2}{(b-a)^3} > \liminf_{s\to+\infty} \frac{F(s)}{s^2}$$

and fix $\varepsilon$ such that

$$\liminf_{s\to+\infty} \frac{F(s)}{s^2} < \varepsilon < \frac{c\pi^2}{(b-a)^3}.$$

By hypothesis (4), there is $R_n \to \infty$ such that

$$F(R_n) \le \varepsilon R_n^2 \qquad (n \ge 1). \tag{11}$$

We claim the existence of $n$ such that $u(t^*) \ne R_n$ for every solution $u(t)$ of (1) – (2). If not, for all $n$, there is a solution $u_n(t)$ such that $u_n(t_n^*) = R_n$ where $t_n^* = t_n^*(u_n)$. Using

$$[u_n(t^*)]^2 = \left[\int_a^{t^*} u_n'(t)dt\right]^2$$

and applying the Schwarz's inequality on the right hand side of the above, we have

$$[u_n(t^*)]^2 \le (t^* - a)\int_a^{t^*} [u_n'(t)]^2 dt,$$

which by the Poincaré's inequality gives

$$[u_n(t^*)]^2 < \frac{(b-a)^3}{\pi^2} \int_a^{t^*} [u_n''(t)]^2 dt. \tag{12}$$

We then have from (10), (11) and (12) respectively that

$$\begin{aligned}
\int_a^{t_n^*} [u_n''(t)]^2 &< \frac{\lambda}{c} F(u_n(t_n^*)) + \frac{d}{c} \\
&\le \frac{\lambda}{c}\varepsilon[u_n(t_n^*)]^2 + \frac{d}{c} \\
&< \frac{\lambda}{c}\varepsilon\frac{(b-a)^3}{\pi^2}\int_a^{t_n^*} [u_n''(t)]^2 dt + \frac{d}{c}.
\end{aligned}$$

Hence,

$$\int_a^{t^*} [u_n''(t)]^2 dt < \frac{\lambda}{c} \varepsilon \frac{(b-a)^3}{\pi^2} \int_a^{t_n^*} [u_n''(t)]^2 dt + \frac{d}{c}. \tag{13}$$

Since,

$$\frac{\varepsilon}{c} \frac{(b-a)^3}{\pi^2} < 1 \text{ and } \int_a^{t_n^*} [u_n''(t)]^2 dt \to \infty,$$

inequality (13) is false for large $n$. Hence, we have proved that there is $n$ such that $\max u \neq R_n$ and we take $B = R_n$. In a completely similar manner, using (4), we find a sequence $S_n \to -\infty$ such that $\lambda F(S_n) \leq \varepsilon S_n^2$ and prove that $\min u \neq S_n$ for $n$ sufficiently large. Hence, we have a constant $A > 0$ such that $\min u \neq -A$, for any solution of (6). Finally, define

$$\rho = \max\{|f(s)| : -A \leq s \leq B\}$$

and, from (6) we have that

$$||u'''||_{L^2} \leq \rho\sqrt{b-a} + ||p||_{L^2}.$$

By Rolle's theorem, we know that there is $\tilde{t} \in ]a, t^*[$ such that $u''(\tilde{t}) = 0$, and using the fact that

$$u''(t) = u''(\tilde{t}) + \int_{\tilde{t}}^t u'''(s) ds$$

we see that $||u''||_\infty$ is bounded. It follows that $||u'||_\infty$ is bounded too and therefore, there is a constant $C > 0$ (independent of $\lambda$ and $u$) such that

$$|u'(t)| + |u''(t)| < C, \quad \text{for all} \quad t \in [a, b].$$

Thus, taking $\Omega$ as in (5), we have that (6) has no solutions on $\partial\Omega$. Then, Leray–Schauder theorem gives the conclusion. □

*Proof of Theorem 2.* The proof of Theorem 2 follows similar arguments as that of Theorem 1 with the necessary modifications, so it is omitted.

Acknowledgement - The author is grateful to the International Centre for Theoretical Physics, Trieste for hospitality and the CDE's travel support to the ICTP. Also, he wishes to thank P. Omari, G. Vidossich and F. Zanolin for their help.

## References

[1] Aftabizadeh A.R., Gupta C.P. and Xu J.M., *Existence and uniqueness theorems for three-point boundary value problems*, Siam J. Math. Anal. **20** (1989), 716–726.

[2] Agarwal R.P., *Existence-uniqueness and iterative methods for third order boundary value problems*, J. Comp. Appl. Math. **17** (1987), 271–289.

[3] Fernandes M.L.C., Omari P. and Zanolin F., *On the solvability of a semilinear two-point Bvp around the first eigenvalue*, Differential Integral Eq. **2** (1989), 63–79.

[4] Fernandes M.L.C. and Zanolin F., *Periodic solutions of a second order differential equation with a one-sided growth restrictions on the restoring term*, Arch. Math. **51** (1988), 151–163.

[5] Fonda A., *Periodic solutions of scalar second order differential equations with a singularity*, Acad. Royale de Belgique, Belgique, 1993.

[6] Fonda A., Gossez J.P. and Zanolin F., *On a nonresonance condition for a semilinear elliptic problem*, Diff. and Int. Equations **4** (1991), 945–951.

[7] Gantmacher F. and Krein M., *Oscillation matrices and kernels and small vibrations of mechanical systems*, AEC-tr 4481 Dept. Commerce Washington 25 D.C. (1961).

[8] Gaudenzi M. and Habets P., *Fucik spectrum for a third order equation*, Univ. Cath. de Louvain, preprint no. 39 (1994).

[9] Granas A., Guenther R. and Lee J., *Nonlinear boundary value problems for ordinary differential equations*, Polish Acad. of Sci., Poland (1985).

[10] Gregus M., *Third Order Linear Differential Equations*, Dordrecht, the Netherlands, 1987.

[11] Grossinho M.D.R. and Omari P., *A Hammerstein-type result for a semilinear parabolic problem*, Univ. de Lisboa, preprint (1995).

[12] Murty K.N. and Prasad B.D.C.N., *Three-point boundary value problems existence and uniqueness*, Yokohama Math. J. **29** (1981), 101–105.

[13] Omari P. and Zanolin F., *On forced nonlinear oscillations in $n$-th order differential systems with geometric conditions*, Nonlinear Analysis TMA **8** (1984), 723–748.

Received May 5, 1997.