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Determination of Plane Convex Sets through X-rays

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Summary. - *We prove that, for a fixed positive number δ and two directions in the plane, it is possible to choose a third direction so that if the plane convex sets H and K have the same X-rays along these directions, the measure of their symmetric difference is less than δ.*

1. Introduction

We suppose to have an homogeneous body in the plane, and we consider its X-ray along a fixed direction. The intensity of each beam of the X-ray which traverses the body, depends on how much material it has passed through; so we may consider the X-ray of a set along a fixed direction θ as a function giving the length of each chord of the set parallel to θ.

This paper includes some results about the determination of a plane convex set through its X-rays along a finite number of directions.

We refer to Gardner's recent book, Geometric Tomography [3], for an in-depth description about this kind of problem, and for a detailed bibliography.

The determination of sets through X-ray is of interest in various fields, especially in computerized tomography in medicine.

In [4] Gardner and McMullen proved that there exist more than one convex body with the same X-rays along any set of directions belonging to a subset of the directions of the edges of an affinely

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regular polygon. It follows that convex bodies cannot be determined by any set of fixed three directions.

Here, we assume that the first direction is fixed, while the others are chosen depending on the X-rays along the previous directions. In [3] Gardner proposes the following open question: can a plane convex body be successively determined by its X-rays along three directions? Notice that it is known that, for some convex sets and for every pair of directions, there exists another convex body with the same X-rays along such directions (Theorem 1.2.22 of [3]). Hence an affirmative answer to the above question would be, in some sense, an optimal result.

In this paper we prove a result closely linked to the previous question: for a fixed positive number δ and two distinct directions, it is possible to choose a third direction such that, if two convex sets H and K have the same X-rays along such directions, the distance between H and K, in the Hausdorff metric, is less than δ (Theorem 1). Furthermore (Theorem 2) we prove that, if an upper bound of the diameter of the convex sets H and K is known, two directions are sufficient (the second is chosen successively) to obtain the same result as Theorem 1.

Longinetti in [5] proved that, with fixed n directions ($n \geq 2$), if H and K have the same X-rays along such directions, and l is the length of $\partial(H \cap K)$, then:

$$dist(H, K) \leq \frac{l^2}{8n} \tan \frac{\pi}{n}.$$

Equality occurs if and only if the directions are equally spaced on the unit circle and H is a regular polygon, with n sides, and K is obtained from H by a rotation of $\frac{\pi}{n}$ about its centre.

The comparison between these results points out the advantages of choosing the directions successively.

We also recall that in Theorem 1 of [1] it is proved that, in a special case, a convex body can be successively determined by its X-rays along three directions. This special case occurs if a support line to the convex set, parallel to one of the first two directions, touches the boundary of the convex body at a singular point.

2. Successive determination of plane convex sets through X-rays

Let K be a plane convex body, i.e. a convex, compact set with non-empty interior. Let θ be an oriented line in the plane, o a point belonging to θ, and $\theta^\perp$ the line through o orthogonal to θ. Let a_θ and b_θ be the boundary of the orthogonal projection of K on $\theta^\perp$. For X-ray K_θ of K along the line θ, we mean the function whose value at a point $x \in [a_\theta, b_\theta]$ is the length of the intersection between K and the line through x and parallel to θ. With the notation $(H \triangle K)$ we mean the symmetric difference between H and K, and with λ_2 and λ_1 the usual Lebesgue's measure in $\mathbf{R^2}$ and $\mathbf{R^1}$ respectively.

Our results are expressed in terms of the metric induced by the area of the symmetric difference. We recall that, for convex bodies, this metric is equivalent to the Hausdorff metric.

We prove the following:

THEOREM 1. *Let K be a plane convex body: given a positive number δ and two oriented lines θ_1 and θ_2, there exists a third oriented line θ_3 so that, if H is another plane convex body so that:*

$$H_{\theta_i} \equiv K_{\theta_i}, \ i = 1, 2, 3, \tag{1}$$

then $\lambda_2(H \triangle K) < \delta$.

Proof. First we assume that θ_1 and θ_2 are two orthogonal oriented lines; we consider the coordinate frame $(o, {\theta_1}^\perp, \theta_1)$; let θ_3 be a third oriented line not parallel to θ_1 and θ_2, and suppose that there exists H verifying (1). Let k_1 and h_1 be concave functions and k_2 and h_2 be convex functions so that:

$$K \equiv \{(x, y) : x \in [a_{\theta_1}, b_{\theta_1}], \ k_2(x) \leq y \leq k_1(x)\},$$

$$H \equiv \{(x, y) : x \in [a_{\theta_1}, b_{\theta_1}], \ h_2(x) \leq y \leq h_1(x)\}.$$

From Proposition 2 of [5], it follows that the set $\{\partial H \cap \partial K\}$ contains at least six points. If $H \neq K$, there exist x_1 and x_2, $x_1 < x_2$, $(x_1, x_2) \subset [a_{\theta_1}, b_{\theta_1}]$, so that:

$$h_1(x_i) = k_1(x_i) \ \ i = 1, 2,$$

$$h_1(x) \neq k_1(x) \;\; \forall x \in (x_1, x_2). \tag{2}$$

If α is the angle between the oriented lines θ_1 and θ_3, and $b = \lambda_1[a_{\theta_2}, b_{\theta_2}]$, then:

$$x_2 - x_1 \leq b \tan\alpha. \tag{3}$$

In fact, let r be the line through $(x_2, k_1(x_2))$parallel to θ_3, and x_0 so that $\{r \cap \partial K\} \equiv \{(x_0, k_2(x_0)), (x_2, k_1(x_2))\}$; from (1) it follows that $h_2(x_0) = k_2(x_0)$ and:

$$h_1(x_0) = k_1(x_0). \tag{4}$$

Therefore:

$$x_2 - x_0 = \lambda_1\{r \cap K\} \sin\alpha \leq \frac{b}{\cos\alpha} \sin\alpha = b \tan\alpha.$$

From (2) and (4) it follows that $x_2 - x_1 \leq x_2 - x_0$; hence inequality (3) holds.

Next we observe that:

$$\lambda_2(H \triangle K) = \int_{a_{\theta_1}}^{b_{\theta_1}} \mid k_1(x) - h_1(x) \mid + \mid k_2(x) - h_2(x) \mid dx.$$

Therefore, from (1), it follows:

$$\lambda_2(H \triangle K) = 2\int_{a_{\theta_1}}^{b_{\theta_1}} \mid k_1(x) - h_1(x) \mid dx.$$

Let I be the set defined as follows:

$$I = \{x : x \in [a_{\theta_1}, b_{\theta_1}], h_1(x) \neq k_1(x)\}.$$

Since h_1 and k_1 are continuous, I is open, and then it is the union of countable many disjoint open intervals I_i, $i \in \mathbf{N}$. Let a_i, b_i be the endpoints of I_i, we can write:

$$\lambda_2(H \triangle K) = 2\sum_i \int_{a_i}^{b_i} \mid k_1(x) - h_1(x) \mid dx. \tag{5}$$

Furthermore, from (3), it follows that:

$$b_i - a_i \leq b \tan\alpha. \tag{6}$$

Since h_1 and k_1 are two concave functions, they have exactly one maximum in $[a_{\theta_1}, b_{\theta_1}]$. Let ξ and η be the points where h_1 and k_1 respectively attain their maximum. We suppose that ξ and η belong to I; let i_0 and j_0 be two integers such that $a_{i_0} \leq \xi \leq b_{i_0}$ and $a_{j_0} \leq \eta \leq b_{j_0}$. Therefore, if $i \in N$, $i \neq \{i_0, j_0\}$, h_1 and k_1 are monotone functions in $[a_i, b_i]$. Without loss of generality, we assume that h_1 and k_1 are non-decreasing functions. This implies that the graphs of the functions h_1 and k_1 in $[a_i, b_i]$ are contained in the triangle whose vertices are:

$$[(a_i, k_1(a_i)), (a_i, k_1(b_i)), (b_i, k_1(b_i))].$$

Hence, for (6):

$$\begin{aligned} \int_{a_i}^{b_i} \mid k_1(x) - h_1(x) \mid dx &\leq \frac{(k_1(b_i) - k_1(a_i))(b_i - a_i)}{2} \\ &\leq \frac{(k_1(b_i) - k_1(a_i))}{2} b \tan \alpha. \end{aligned} \tag{7}$$

For $i = i_0$ and $i = j_0$:

$$\int_{a_i}^{b_i} \mid k_1(x) - h_1(x) \mid dx \leq [b \tan \alpha] \frac{b}{2}. \tag{8}$$

If ξ, $\eta \notin I$, then (7) holds for every interval (a_i, b_i) belonging to I.

Because of the concavity of the function k_1, for any partition $\{y_i\}$ of $[a_{\theta_1}, b_{\theta_1}]$, we have:

$$\sum_{i=0}^{\infty} \mid k_1(y_{i+1}) - k_1(y_i) \mid \leq 2[max k_1 - min k_1] \leq 2b.$$

From (5), (7) and (8), it follows that:

$$\begin{aligned} \lambda_2(H \triangle K) &\leq 2\{\sum_{i=0}^{n} \frac{b}{2} \tan \alpha \mid k_1(b_i) - k_1(a_i) \mid + 2[\frac{b^2}{2} \tan \alpha]\} \\ &\leq 2\{b^2 \tan \alpha + b^2 \tan \alpha\} = 4b^2 \tan \alpha. \end{aligned} \tag{9}$$

Then, if θ_3 verifies:

$$\alpha < \arctan \frac{\delta}{4b^2},$$

we get $\lambda_2(H \triangle K) < \delta$.

Now we assume that θ_1 and θ_2 are arbitrary oriented distinct lines. Let P be the convex quadrangle bounded by the support lines to K, parallel to θ_1 and θ_2. Among the rectangles having two sides parallel to θ_1 and containing P, let R be the smallest. Let b be the length of the side of R parallel to θ_1. Since $R \supset K$, $b > \lambda_1[a_{\theta_1{}^\perp}, b_{\theta_1{}^\perp}]$. Then, if we choose θ_3 so that:

$$\alpha < \arctan \frac{\delta}{4b^2},$$

it follows that $\lambda_2(H \triangle K) < \delta$ and this completes the proof. □

In the foregoing proof, we used the X-ray along the oriented line θ_2 only to determine an upper bound of the widths of K and H in the direction $\theta_1{}^\perp$. On the other hand, such upper bound is trivially known, if we know an a priori bound of the diameters of K and H. Then we get the following:

THEOREM 2. *Let K and H be two plane convex bodies and suppose that an upper bound of their diameters is known. Given a positive number δ and an oriented line θ_1, there exists a second oriented line θ_2 such that, if:*

$$H_{\theta_i} \equiv K_{\theta_i}, \ i = 1, 2,$$

then $\lambda_2(H \triangle K) < \delta$.

REFERENCES

[1] BOCCONI A., *Ricostruzione Tomografica di anelli convessi*, Pubblicazione dell'Istituto di Analisi Globale e Applicazioni (C.N.R.), Firenze, 1996.

[2] EDELSBRUNNER H. and SKIENA S.S., *Probing convex polygons with X-rays*, SIAM J. Computing **17** (1988), 870–882.

[3] GARDNER R.J., *Geometric Tomography*, Cambridge University Press, New York, 1995.

[4] GARDNER R.J. and MCMULLEN P., *on Hammer's X-ray problem*, J. London Math. Soc. **21** (1980), 171–175.

[5] LONGINETTI M., *An isoperimetric inequality for convex polygons and convex sets with the same symmetrals*, Geom. Dedicata **20** (1986), 27–41.

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