Rend. Istit. Mat. Univ. Trieste
Vol. XXIX, 1–17 (1997)

On the Tensor Product of Sections of Vector Bundles on an Algebraic Curve

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Summary. - *Here we study several properties of spanned vector bundles on an algebraic curve X. In particular we study the multiplication map $H^0(X,E) \otimes H^0(X,F) \to H^0(X,E\otimes F)$.*

1. Fix a smooth complete algebraic curve X and let E, F be vector bundles on X. Set $r := \operatorname{rank}(E)$, $s := \operatorname{rank}(S)$. Take linear subspaces $V \subseteq H^0(X,E)$, $W \subseteq H^0(X,F)$ and consider the multiplication map $\mathbf{m}(V,M) : V \otimes W \to H^0(X, E\otimes F)$. Set $m(V,W) := \dim(\mathbf{m}(V,W))$. Assume that V generically spans E and that W generically spans F. In [9] and [3] it was proved that

$$m(V,W) \geq r(\dim(W)) + s(\dim(V)) - rs. \qquad (1)$$

Here we continue the program started in [4] and [3] and study several geometric properties of the multiplication map. A preliminary version of part of this paper is contained in [2]. In Section 2 we will give an extension of most of [9] to the positive characteristic case. In Section 3 we study a geometric property (see Definition 1.2) of a vector space of sections of a rank $r > 1$ vector bundle.

We will use the following convention.

Definition 1.1. Fix a variety X, a vector bundle E over X and a finite dimensional vector space $V \subseteq H^0(X,E)$. The pair (E,V) is said to be a *spanned pair* (resp. a *generically spanned pair*) if V

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This research was partially supported by MURST and GNSAGA of CNR (Italy).

spans E (resp. if there is a Zariski dense open subset U of X such that V spans $E|U$).

Now we introduce the following definition which seems to have a geometric meaning and which is the key for the results of Section 3.

DEFINITION 1.2. Let X be a variety and (E, V) a generically spanned pair with $r := \mathrm{rank}(E) \geq 2$. Fix an integer t with $2 \leq t \leq r$. The pair (E, V) is called *t-generic* if every t-dimensional subspace of V spans a rank t subsheaf of E.

In Section 4 we improve with elementary methods the results on the multiplication map of [9] and [3]. However, most of the results hold only for line bundles. Here is the statement of our main result.

THEOREM 1.3. *Let X be a smooth curve of genus $g(X) \geq 7$. Let (L, V) and (M, W) be spanned pairs on X. Set $v := \dim(V)$ and $w := \dim(W)$. Assume $v \geq 2$, $w \geq 3$, $\deg(M) > 3v + 2w - 3$ and that the morphisms induced by $|V|$ and $|W|$ are birational and different. Then $m(V, W) \geq 2w + 2v - 4$.*

In Section 5 we study the multiplication map on a multiple cover $f : X \to C$ of smooth curves focusing on the cases "f unramified" and "f Galois". Then in Section 6 we use the theory of mock covers (see [6]) to study when $h^0(X, f^*(A)) > h^0(C, A)$ with A vector bundle on C. Here C is fixed and we take X and f general. In the last section we consider (with a few examples) when a given vector bundle E on X is of the form $u^*(E')$ with E' vector bundle on a lower genus smooth curve C and $u : X \to C$ finite map.

2. Recall the following results proved by R. Re ([9, Th. 1 and 2]).

THEOREM 2.1. ([9, Th. 1 and Th. 2]) *Assume characteristic* 0. *Let X be a smooth projective variety. Let (E, V) and (F, W) be generically spanned pairs on X. Then*
$m(V, W) \geq \dim(V)\mathrm{rank}(F) + \dim(W)\mathrm{rank}(E) - \mathrm{rank}(E)\mathrm{rank}(F)$.
Furthermore, if X is a smooth curve, (E, V) and (F, W) are spanned pairs and
$m(V, W)) = \dim(V)\mathrm{rank}(F) + \dim(W)\mathrm{rank}(E) - \mathrm{rank}(E)\mathrm{rank}(F)$,

then there exist a morphism $f : X \to \boldsymbol{P}^1$ and vector bundles E', F' on $\boldsymbol{P}^1$ with $E = f^*(E')$, $F = f^*(F')$, $V = f^*(H^0(\boldsymbol{P}^1, E'))$ and $W = f^*(H^0(\boldsymbol{P}^1, F'))$.

To make a more complete picture in this section we will show why the proofs of Theorem 1 and Theorem 2 (over a smooth curve) of [9] work in positive characteristic. We will show also that the proof of Theorem 1 in [9] can be extended easily to the case in which the variety is not locally Cohen - Macaulay.

PROPOSITION 2.2. *The statement of Theorem 2.1 holds in arbitrary characteristic. The first assertion of Theorem 2.1 holds for any reduced and irreducible projective variety X.*

For the proof of the first assertion of Theorem 2.1 for curves it is sufficient to use the fact that, even in positive characteristic, the dimensional part of the classical Bertini theorem holds (see [8] or [7]). For the extension to the case $\dim(X) > 1$, X not locally Cohen - Macaulay, just note that in the proof in [9] it was used only the fact that if E is a vector bundle on X and H a very ample divisor on X, we have $H^t(X, E \otimes \boldsymbol{O}_X(-nH)) = 0$ for $t = 0$ and 1 and for very large n. This is equivalent to the condition that $\boldsymbol{O}_X$ has everywhere at least depth 1 (again by the original proof of Serre in F.A.C.). However, to apply the fact that the evaluation and multiplication maps commute with restriction, it is sufficient to have the injectivity at the level of global sections, i.e. Serre vanishing in the case $t = 0$. This is true if X has no embedded points. In particular it is true for reduced X. The general divisor in a very ample linear system will be again reduced and we conclude inductively for all integral X, as wanted. Now we consider the extension to positive characteristic of the last assertion of Theorem 2.1. Note that the proof of the classical case of line bundles on smooth curves ([9], first and second part of the proof of Prop. 1) holds in positive characteristic. Then note that over a smooth curve the induction on the rank of the vector bundles ([9], lemmata 1 and 2) works verbatim.

3. Here we show how to use Definition 1.2.

PROPOSITION 3.1. *Let X be a smooth curve and E a rank r vector bundle on X generically spanned by $V \subseteq H^0(X,E)$. Assume that (E,V) is r-generic. Then $h^0(X,\det(E)) \geq r(v-r)+1$.*

Proof. By definition every r-dimensional linear subspace of V induces an injection of $r\boldsymbol{O}_X$ into E. Such an injection drops rank on a divisor in the projective space $\boldsymbol{P} := \boldsymbol{P}(H^0(X,\det(E)))^*$. Hence we have an everywhere defined morphism $f : G(r,v) \to \boldsymbol{P}$. We may assume $v > r$. Fix a general $P \in X$. There are linearly independent $s(j) \in V$, $1 \leq j \leq r+1$, with $s(i)(P)$, $1 \leq i \leq r$, spanning the fiber of E at P, but with $s(r+1)(P)$ depending on $s(j)(P)$, $1 \leq j < r$. Thus f is not constant. Hence we have $f^*(\boldsymbol{O}_{\boldsymbol{P}}(1)) = \boldsymbol{O}_{G(r,v)}(a)$ with $a > 0$. Since $\boldsymbol{O}_{G(r,v)}(a)$ is ample, f does not contract any subvariety of $G(r,v)$ and in particular $\dim(\boldsymbol{P}) \geq r(v-r)$. □

COROLLARY 3.2. *Let X be a smooth curve and E a rank r vector bundle on X generically spanned by $V \subseteq H^0(X,E)$. Assume that (E,V) is r-generic and that* $\det(E)$ *is special. Then $r(v-r)+1 \leq g$ and $2r(v-r) \leq \deg(E)$. Furthermore, if $h^1(X,\det(E)) \geq 2$, then $\deg(E) - 2r(v-r) \geq Cliff(X)$, where $Cliff(X)$ is the Clifford index of X.*

Proof. By Proposition 3.1 the second inequality is Clifford theorem ([1]), while the "furthermore" part is just the definition of Clifford index of X. □

PROPOSITION 3.3. *Let X be a projective variety and E a rank r vector bundle on X generically spanned by $V \subseteq H^0(X,E)$. Assume that (E,V) is t-generic. Let A be a spanned vector bundle. Then:*

(a) for every t-dimensional linear subspace U of $H^0(X,A)$ the multiplication map $m(V,U)$ is injective.

(b) for every linear subspace U of $H^0(X,A)$ we have $m(V,U) \geq t(v+\dim(U)-t)$.

Proof. First we assume $\mathrm{rank}(A) = 1$. To prove part (a) note that every element of $\mathbf{m}(V,U)$ is of the form $a_1 \otimes b_1 + ... + a_t \otimes b_t$. By definition of t-generic if $a_1, \ldots, a_t$ are linearly independent they generate a rank t subsheaf of E. Hence part (a) follows. Part (b) follows from part (a) and [5, Prop.1.3], i.e. from the notion of t-generic vector space of matrices in the sense of [5]. Now assume $\mathrm{rank}(A) > 1$. Take a sufficiently positive quotient line bundle A' of A such that the induced map $W \to H^0(X, A')$ is injective. Apply parts (a) and (b) to A'. □

LEMMA 3.4. *Let X be a projective variety and E a rank r vector bundle on X generically spanned by $V \subseteq H^0(X, E)$. Assume the existence of a subbundle A of E such that $V' := V \cap H^0(X, A)$ generically spans A. Then for every generically spanned pair (F, W) on X we have $m(V, W) = m(V', W) + m(V/V', W)$.*

Proof. We have a commutative diagram of evaluation maps induced by the exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & V' \otimes W & \to & V \otimes W & \to & (V/V') \otimes W & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A \otimes F & \to & E \otimes F & \to & E/A \otimes F & \to & 0. \end{array} \tag{2}$$

Although the global section functor is only left exact, we still have the equality and not only the inequality $m(V, W) \leq m(V', W) + m(V/V', W)$ because by 2 the image $\mathbf{m}(V/V', W)$ of the evaluation map of $(V/V') \otimes W$ is contained in the image of $H^0(X, E \otimes F)$. □

The same proof by induction on r gives the following result.

PROPOSITION 3.5. *Let E be a rank r vector bundle generically spanned by $V \subseteq H^0(X, E)$. Let $\{E_i\}$, $0 \leq i \leq r$, be an increasing filtration of E by subbundles with $L_i := E_{i+1}/E_i$ of rank 1, $E_0 = 0$. Set $V_i := V \cap H^0(X, E_i)$ and $v_i := \dim(V_i)$. Assume $v_i < v_{i+1}$ for every $i \geq 0$. Then for every generically spanned pair (F, W) we have $m(V, W) = \sum_i m(V_{i+1}/V_i, W)$.*

Motivated by Proposition 3.5 we introduce the following definition.

DEFINITION 3.6. Let X be a projective variety and E a rank r vector bundle generically spanned by $V \subseteq H^0(X, E)$. Let $\{Ei\}$, $0 \leq i \leq r$, be an increasing filtration of E by subbundles with $L_i := E_{i+1}/E_i$ of rank 1, $E_0 = 0$. Set $V_i := V \cap H^0(X, E_i)$ and $v_i := \dim(V_i)$. Assume $v_i < v_{i+1}$ for every $i \geq 0$. Then the filtration $\{E_i\}$ of E will be called a *spanning filtration.*

PROPOSITION 3.7. *Let X be a projective variety and (E, V) and (F, W) be generically spanned pairs on X admitting a spanning filtration. Set $r := \mathrm{rank}(E)$, $s := \mathrm{rank}(F)$. Assume $m(V, W) > rw + sv - rs$. Then $m(V, W) \geq rw + sv - rs + \min(r, s)$.*

Proof. Let E_i, V_i, v_i (resp. F_j, W_j, w_j) be the data associated to a spanning filtration of (E, V) (resp. (F, W)). By 3.2 and the assumption on $m(V, W)$ there is at least a pair (i, j) with $m(V_{i+1}/V_i, W_{i+1}, W_i) \geq \dim(V_{i+1}/V_i) + \dim(W_{j+1}/W_j) - 1$. By 3.6 it is sufficient to prove that there are at least $\min(r, s)$ such pairs (i, j). By [9, Th. 2], if this inequality fails for a pair (a, b) there is a map $f_{ab} : X \to \boldsymbol{P}^1$ inducing the corresponding rank 1 spanned pairs. Note that if we have such maps f_{12}, f_{21} and f_{22}, we have also the map f_{11}. Hence we conclude easily. □

4. In the first part of this section we prove Theorem 1.3 and a few related results, i.e. we improve with elementary methods the results on the multiplication map in the case of line bundles over a smooth projective curve X of genus $g(X) \geq 2$. In the second part of this section we will extend the methods and proofs to cover the higher rank case. In this section we assume characteristic 0. We fix two line bundles L, M on X and vector spaces $V \subseteq H^0(X, L)$, $W \subseteq H^0(X, M)$ without base points. Set $v := \dim(V)$, $w := \dim(W)$ and let $|V|$, $|W|$ be the associated projective spaces. Let $\mathbf{m}|V, W| = |\mathbf{m}(V, W)|$ be the span of the image of the multiplication map in the complete linear system $|L \otimes M|$ associated to $L \otimes M$. Note that $m(V, W) \leq vw$. For any linear system $|U|$ on X and any $P \in X$, set $|U - P| := |U(-P)| := \{D \in |U| : P \in D_{red}\}$.

Proof of Theorem 1.3. The proof is divided into 4 steps labelled 4.1, 4.2, 4.3 and 4.4. The bounds on $\deg(M)$ and $g(X)$ assumed in the statement of 1.3 are very rough. For better bounds, see 4.2, 4.3, 4.4 and the proof of 4.3.

PART 4.1. If $v = w = 2$ and $|V|$ and $|W|$ are not composed with the same pencil, then by [9, Prop. 1], we have $m(V, W) = 4$.

PART 4.2. Assume $v = 2$, $w = 3$ and $|W|$ not composed with a rational involution. We claim that $m(V, W) = 6$. Assume by contradiction $m(V, W) = 5$. Let $t : X \to C \subset \boldsymbol{P}^2$ be the morphism induced by $|W|$, Y the normalization of C and $t' : X \to Y$ the map induced by t. For a general $P \in X$ we have $3 \le m(V, W - P) \le 4$. First assume that for a general P we have $m(V, W - P) = 3$. By [9, Prop. 1], $|V|$ and $|W - P|$ are composed with the same rational pencil. In particular $M(-P)$ and $M(-P')$ are linearly equivalent for general P, P' in X, contradiction. Hence we may assume $m(V, W - P) = 4$ for general P. Thus $\mathbf{m}|V, W - P|$ is the hyperplane $|\mathbf{m}(V, W) - P|$ of $|\mathbf{m}(V, W)|$. Since $|W|$ is not composed with a rational pencil, we see that for a general point P there is $P' \in |V - P|$ with $|W - P - P'| \neq |W - P|$. Note that the image $Z \subset |\mathbf{m}(V, W)| \cong \boldsymbol{P}^4$ of $|V| \times |W|$ is a hypersurface with a 1-dimensional ruling by planes $\{D + |W|$ with $D \in |V|\}$ and a 2-dimensional family of lines $\{|V| + D$ with $D \in W\}$. Thus the line bundle associated to the morphism $|V| \times |W| \to |\mathbf{m}(V, W)|$ has bidegree $(1, 1)$. Thus an easy computation gives $\deg(Z) = 3$. Note that the plane $|\mathbf{m}(V, W)|(-P - P')$ intersects Z at least in the family of lines $\{|V| + P''$ with $P'' \in |W(-P - P')|\}$. Thus $\deg(M) \le 5$. Assume $\deg(M) = 5$. Since 5 is prime, we see that the map induced by $|W|$ is birational. Hence $g(X) \le 6$. Now assume $\deg(M) = 4$. If the map induced by $|W|$ is birational, then $g(X) \le 3$. If the map induced by M is not birational, then X is hyperelliptic and $|W| = 2g^1_2$.

PART 4.3. Now assume $v = 2$, $w \ge 4$, $\deg(M) \ge (3w/2)$ and that the morphism induced by $|W|$ is birational. Then $m(V, W) \ge 2w$. Assume by contradiction $m(V, W) \le 2w - 1$. By induction on w we may assume $m(V, W) = 2w - 1$ and that for a general $P \in X$ $m(V, W - P) = 2w - 2$. Let $Z \subset \boldsymbol{P}^{2w-2}$ be the image of $|V| \times |W|$. As in 4.2 we see that $\deg(Z) = w$. By assumption for general points $P_j \in X$, $1 \le j \le w - 1$, the space $|\mathbf{m}(V, W)|(-P1 - \ldots - P_{w-1})$ has dimension $w - 1$ and intersects Z in at least $\deg(M) - w + 1$ lines D_j. By the birationality assumption and the fact that in characteristic 0 the monodromy group of a general hyperplane section of an integral projective curve is the full symmetric group ([1, Ch. II]), there are integers a_2 and a_1 such that the linear span of any t of the lines D_j's has dimension $2t-1$ if $2 \leq t \leq a_2$, dimension $2a_2 + (t - a_1) = a_2 + t$ if $a_2 < t \leq a_1 + a_2$, dimension $2w - 2$ if $t \geq a_1 + a_2$. Hence $2a_2 + a_1 = 2w - 2$. Furthermore, since any line is contained in the span of any two of its points, two different sets of a_2 lines D_j's span $|\mathbf{m}(V, W)|$. Hence $4a_2 - 1 \geq 2w - 2$, i.e. $2a_2 \geq w$. Thus if $\deg(M) \geq w + (w/2)$, $|\mathbf{m}(V, W)|(-P1 - \ldots - P_{w-1})| \cap Z$ contains at least $1 + (w/2)$ lines D_j's, contradiction.

PART 4.4. Now assume $3 \leq v \leq w$ and that the morphisms induced by $|V|$ and $|W|$ are birational and different. We want to prove Theorem 1.3. Assume by contradiction $m(V, W) \leq 2w + 2v - 4$. By 4.3 and induction on v, we may assume $m(V, W) = 2w + 2v - 4$ and that for a general $P \in X$ we have $m(V - P, W) = 2w + 2v - 5$. We follow the proof of 4.3 reversing the roles of $|V|$ and $|W|$. Now we have integers a_j, $1 \leq j \leq w$, such that $a_j \geq 0$, $a_v > 0$, $\sum_{1 \leq j \leq w} ja_j = 2w + 2v - 3$ for general P_j, $1 \leq j \leq t$ with, say, $a_w + a_{w-1} + \ldots + a_{i+1} < t \leq a_w + a_{w-1} + \ldots + a_i$, the linear span of the subspaces $\{P_j\} \times |W|$ of $|\mathbf{m}(V, W)|$ has dimension $\sum_{i+1 \leq j \leq w} ja_j - 1 + i(t - a_w + a_{w-1} + \ldots + a_{i+1})$. Now $\deg(Z) = (v + w - 2)!/((v-1)!(w-1)!)$. Taking $v - 1$ general points P_j's we find a contradiction as in 4.4 when $\deg(M) - v + 1$ is large, e.g. if $\deg(M) \geq 2w + 3v - 3$. □

REMARK 4.5. If we do not assume that the morphism induced by $|V|$ is birational, the proof of Theorem 1.3 (see in particular 4.3 and 4.4) gives $m(V, W) \geq 2w + v - 2$ if $\deg(M) > 2v + 2w - 3$.

REMARK 4.6. Let (E, V) and (F, W) generically spanned pairs with $\mathrm{rank}(E) = r$ and $\mathrm{rank}(F) = s$. Assume that these pairs have a spanning filtrations such that the associated rank 1 graded pieces induce birational morphisms. The proof of the inequality (1) given in [3] (i.e. in particular the proof of [3, Prop. 1.2]) and the proof of Theorem 1.3 and Remark 4.5 give a better bound on $m(V, W)$ (roughly twice the naive one given by (1)).

THEOREM 4.7. *Let (E, V) be a generically spanned pair and (M, W) a spanned line bundle with $|M|$ birational, $w \geq 3$ and $\deg(M) \geq (3w/2)$. Then $m(V, W) \geq (r+1)w + v - r + 3$.*

Proof. By [3, Prop. 1.2] we reduce to the case $E \cong (r-1)\boldsymbol{O}_X \oplus \det(E)$, $V = (r-1)\mathbf{K} \oplus V'$ with the direct sum of V corresponding to the direct sum of E, $\dim(V') = v - r + 3$. Hence $m(V,W) \geq (r-1)w + m(V',W)$. Now apply 4.3. □

REMARK 4.8. Note that we have $m(V' \oplus V'', W) = m(V',W) + m(V'',W)$ for all (generically) spanned pairs (E',V'), (E'',V'') and (F,W). Fix a generically spanned pair (E,V) with invariants r, v on the smooth curve X. In the proof of [3, Prop. 1.2], we gave a degeneration of the pair (E,V) to the pair $((r-1)\boldsymbol{O}_X \oplus \det(E), (r-1)\mathbf{K} \oplus V')$ for some V'. Let $S^n(V)^=$ be the image of $S^n(V)$ into $S^n(E)$. Hence by semicontinuity, with obvious notations for any other generically spanned pair (F,W), we have $m(S^n(V)^=, W) \geq \sum_{1 \leq i \leq n} m(S^i(V')^=, W) + ((n+r-1)!/(n!(r-1)!)w$. A similar result holds for every other Schur functor in any number of variables (e.g. tensor products).

REMARK 4.9. We may combine Remark 4.8 with the proof of Theorem 1.3 (in particular with part 4.3) and with [3, Prop. 1.2]. Assume that X has no map of degree > 1 to a curve of genus > 0. Fix $R \in \mathrm{Pic}(X)$ and $U \subseteq H^0(X,R)$; let B be the base locus of U and set $U' := U(-B) \cong U$. Note that $|U'|$ induces a birational morphism if $U' = H^0(X, R(-B))$ and $h^0(X, R(-B))$ does not divide $\deg(R(-B))$ because a rational normal curve in $\boldsymbol{P}^m$ has degree m. For the same reason, if we assume that $|U'|$ factors through a curve of geometric genus 1, then either $h^0(X, R(-B))$ or $h^0(X, R(-B)) + 1$ divides $\deg((R(-B))$. And so on. If there is no assumption on U' but X does not map to a curve of genus > 0, at least we know that the morphism induced by U' is birational if either $\deg(R(-B))$ is prime or $\deg(R(-B)) < k^2$, with k the gonality of X.

The following lemma may be used to give weak extensions of Theorem 4.7 to the case $r \geq 2$.

LEMMA 4.10. *Let (E,V), (F,W) be generically spanned pairs with invariants r,s,v,w on the smooth curve X. Fix $v-r-1$ general points $Q(1),\dots,Q(v-r-1)$ of X and $v-r+1$ sections $s(1),\dots,s(v-r+1) \in V$ such that for all i,j we have $s(j)(Q(i)) = 0$ if $i \neq j$,*

$s(i)(Q(i)) \neq 0$. Fix a vector subspace V', $\dim(V') = r+1$, which is a complement of the subspace V'' of V spanned by the sections $s(j)'s$ and which generically spans E. Then $m(V,W) \geq m(V',W) + s(v-r-1)$.

Proof. Take s sections $a(1), \dots, a(s) \in W$ such that their evaluation on each fiber $F|A(i)$ induces a base of the vector space $F|A(i)$. Then evaluate at each point $A(i)$ every linear relation between $\mathbf{m}(V',W)$ and $\mathbf{m}(V'',W)$ to show that their linear span in $H^0(X, E\otimes F)$ contains $\mathbf{m}(V',W)$ as a subspace of codimension $\geq s(v-r-1)$. $\square$

5. In this section we study the multiplication map on a curve X which is a multiple cover. Let $f : X \to C$ be a finite separable map with X smooth genus g curve and C smooth curve. Let $Y \to X \to C$ be the Galois extension determined by f; let G be its Galois group, $H \subset G$ the subgroup corresponding to $Y \to X$; we will always assume either characteristic zero or that $\mathrm{card}(G)$ is not divisible by the characteristic of the algebraically closed base field. Set $t := \deg(f)$ and $q := p_a(C) \geq 0$. Fix a spanned pair (E,V) on X. Set $r := \mathrm{rank}(E)$, $v := \dim(V)$.

PROPOSITION 5.1. *Assume f unramified. Let (E,V) be a spanned pair with invariants (r,v). Then there is a spanned pair (E',V') on C with invariants $r' := r(\mathrm{card}(G))$, $v' := v(\mathrm{card}(G))$ such that for every spanned pair (F',W') on C with invariants (s,w) we have $m(V,W) = m(V',W')/\mathrm{card}(G)$, where $F := f^*(F')$, $W := f^*(W')$.*

Proof. (a) First, we check that $Y \to C$ is unramified. Let H' be the subgroup of G generated by all stabilizers of the points of Y. Since f is unramified, we have $H' \subseteq H$. Since H' is normal and $Y \to C$ is the minimal Galois extension of f, H' is trivial, as wanted.
(b) Let (E'',V'') be the pull-back on Y of the pair (E,V). Set $A := \bigoplus_{g\in G} g^*(E'')$ and $B := \bigoplus_{g\in G} g^*(V'')$. The spanned pair (A,B) is G-invariant. Since $Y \to C$ is étale, by descent theory (A,B) is the pull-back of a spanned pair (E',V') on C. Let (F',W') be a spanned pair on C with invariants (s,w) and let (F,W) (resp. (F'',W'')) its pull-back to X (resp. Y). By construction we have $m(V,W) =$

$m(V'', W'') = m(A, W'')/\mathrm{card}(G) = m(V', W')/\mathrm{card}(G)$, as wanted. $\square$

Proposition 5.2. *Assume f unramified. Let (E, V) be a spanned pair with invariants (r, v). Then there is a spanned pair (E', V') on C with invariants $r' := r(\mathrm{card}(G))$, $v' := v(\mathrm{card}(G))$ such that for every spanned pair (F, W) on X with invariants (s, w) there is a spanned pair (M, N) on C with invariants $(s(\mathrm{card}(G), w(\mathrm{card}(G))$ such that $m(V, W) = m(V', N)/\mathrm{card}(G)^2$, where $F := f^*(F')$, $W := f^*(W')$.*

Proof. Apply to the pair (F, W) the construction in the proof of 5.1 (pull-back to Y and then averaging with respect to the action of G). $\square$

5.3. In this subsection we assume that $f : X \to C$ is a Galois extension. For every ramification point $P \in X$ of f let $G_P \subseteq G$ be the stabilizer subgroup and $A|\{P\}$ the fiber of any vector bundle on X. Let B be the set of ramification points. Note that if $h^*(A) \cong A$ for every $h \in G$, then G_P acts on $A|\{P\}$ $(P \in B)$. For this action of G_P the restriction map $A \to A|\{P\}$ is G_P-equivariant. Let $A|\{P\} \to A(P)$ the G_P-equivariant projection into the subspace of $A|\{P\}$ on which G_P does not act trivially. Let $A(G)$ be the subsheaf of A which is the intersection of all the kernels of the surjections $A \to A(P)$, $P \in B$. Since X is a smooth curve, $A(G)$ is a locally free subsheaf of A with $\mathrm{rank}(A(G)) = \mathrm{rank}(A)$. By construction $h^*(A(G)) \cong A(G)$ for every $h \in G$ and G_P acts trivially on $A(G)|\{P\}$ for every $P \in B$. Hence by descent theory there is a vector bundle A' on C such that $f^*(A') \cong A(G)$. If W spans A and G acts on W, set $W(G) := W \cap H^0(X, A(G))$ and $G(W) := \dim(W) - \dim(W(G))$; we will say that (A, W) is a G-spanned pair. Note that there is $W' \subseteq H^0(C, A')$ with $f^*(W') = W(G)$ (but of course we may even have $W(G) = \{0\}$).

Proposition 5.3. *Let (E, V) be a G-spanned pair. Then for every spanned pair (M, N) on C, we have*

$$m(V, f^*(N)) = m(V(G), f^*(N)) + G(V)\dim(N).$$

Proof. Note that for every $P \in B$, G acts on $E \otimes f^*(M)|\{P\}$ and for this action $V(G) \otimes f^*(N)$ is the part on which G acts trivially. $\square$

REMARK 5.4. If we have another G-spanned pair, say (M'', N'') instead of $(f^*(M), f^*(N))$ as in 5.3 we may still have a decomposition of the fibers of $E \otimes M''|\{P\}$, $P \in G$, in terms of the action of G_P on $E|\{P\}$ and $M''|\{P\}$ and still apply the method of 5.3 (if the decomposition is known). If (F, W) is any spanned pair, we may apply the averaging method of 5.3 to obtain a G-spanned pair.

6. Fix a smooth genus q curve C, $q \geq 0$ and a vector bundle M on C. Let $f : X \to C$ be a finite cover with X smooth genus g curve. Set $k := \deg(f)$. The general question is when $h^0(X, f^*(M)) > h^0(C, M)$, i.e. although the vector bundle $f^*(M)$ is induced by the cover f, it has sections which are not induced by the cover. This question may have a meaningful partial answer in term of q, g, k and numerical invariants of M, plus if necessary assumptions on C (if $q > 0$), and on f (e.g. f general with C fixed, or f general with C general or f cyclic or f Galois). Here we consider the case in which C is fixed, f is a Galois extension and f is general. The main tool is the theory of mock covers introduced in [6]. In this section we assume characteristic 0. Fix C and a finite group G. Fix a finite set $B \subset C$. There is a generalized Hurwitz scheme $\mathrm{Hurw}(C, G, B)$ parameterizing Galois covers of C with group G and which are unramified outside B. On the boundary of $\mathrm{Hurw}(C, G, B)$ there are the mock covers introduced and studied in [6, Section 2 and Section 3]. Recall that a mock cover with respect to C, B and G is a reduced curve Y with a finite map $\pi : Y \to C$ with the following properties. Y has $\mathrm{card}(G)$ irreducible components $\{C(g)\}_{g \in G}$, each of them mapped isomorphically by π onto C. G permutes these components (i.e. $h \in G$ sends $C(g)$ into $C(gh)$), Y is seminormal, i.e. uniquely determined by the gluing data of the components $\{C(g)\}$. The gluing data of these components are given in terms of generators for the stabilizer subgroup H_P of each ramification point $P \in B$ (i.e. the sheets $C(h)$, with $h \in H_P$ are glued together at the point P).

PROPOSITION 6.1. *Let $\pi : Y \to C$ a mock cover with Galois group G and ramification set B. Fix a spanned pair (E, V) on C. Set*

$r := \mathrm{rank}(E)$, $v := \dim(V)$, $t := v/r$. Assume that t is an integer and that for k general points $P_1, \dots, P_k$ of C we have $V(-P_1 - \dots - P_k) := V \cap H^0(C, E(-P_1 - \dots - P_k)) = 0$. Assume that there is an ordering $C\{1\}, \dots, C\{\mathrm{card}(G)\}$ of the components of Y such that for every $i > 1$ $C\{i\} \cap (\bigcup_{1 \le j < i} C\{j\})$ contains at least t general points of $C\{i\}$. Then for a general cover $f : X \to C$ in $\mathrm{Hurw}(C, G, B)$ we have $h^0(X, f^*(E)) = h^0(C, E)$.

Proof. Let $\pi(i)$ be the restriction of π to $\bigcup_{1 \le j < i} C\{j\}$. First, using induction on i we see that for every $i \ge 1$ we have

$$h^0(\bigcup_{1 \le j < i} C\{j\}, \pi(i)^*(E)) = tr = h^0(C, E).$$

Then we apply [6] to show that this mock cover is the flat limit of covers from smooth irreducible Riemann Surfaces. □

Consider the following Remarks 6.2 and 6.3 related to the assumptions of Proposition 6.1 for $r = 2$ and $v = 4$.

REMARK 6.2. Assume that G is generated by the stabilizers H_P with $P \in B$. Then for every mock cover we may find an order of the sheets with $\mathrm{card}(C\{i\} \cap (\bigcup_{1 \le j < i} C\{j\})) \ge 2$ for every i and we may find such a mock cover such that (for fixed E) $C\{i\} \cap (\bigcup_{1 \le j < i} C\{j\})$ contains two sufficiently general points of $C(i)$. A smooth Galois cover $f : X \to C$ satisfies the condition on the stabilizers H_P, $P \in B$, if and only if f does not factor through an étale cover of degree > 1.

LEMMA 6.3. *Fix a spanned pair (E, V) on the smooth curve C with $\mathrm{rank}(E) = 2$, $\dim(V) = 4$. Either $E \cong \boldsymbol{O}_C \oplus J$ for a line bundle J with $\dim(V \cap H^0(C, J)) = 3$ or for general points P, Q of C we have $V(-P - Q) := H^0(C, E(-P - Q)) \cap V = 0$.*

Proof. Since V spans E, for every P we have $\dim(V(-P)) = 2$. If the thesis of the lemma fails, we see that $V(-P)$ spans a rank 1 subsheaf of $E(-P)$; let L_P be its saturation and $M_P := E(-P)/L$. By assumption the image of $V(-P)$ into L vanishes. If $\dim(V \cap H^0(C, L_P(P))) = 3$ for some P, then $M_P(P)$ is spanned by one section. Hence $M_P(P)$ is trivial. Since, up to a sign, the image of the constant section 1 of $H^0(C, \boldsymbol{O}_C)$ into $H^1(C, L_P(P)^*)$ is the extension class of the extension induced by $L_P(P)$ and $M_P(P)$ is spanned by V, this extension splits and $E \cong \boldsymbol{O}_C \oplus J$ for some J. If for general P, Q we have $L_P(P) = L_Q(Q)$ as subsheaves of E, then $V(-Q) \cap H^0(C, L_P(P))) = 2$, $\dim(V \cap H^0(C, L_P(P))) = 3$ and we conclude. If for general P, Q we have $L_P(P) \neq L_Q(Q)$, then the image of $L_P(P)$ into $M_Q(Q)$ vanishes and we see that too many sections of V have 0 as images into $M_Q(Q)$. □

7. In this section we consider the following problem which is classical and important in the rank 1 case. Fix a smooth genus g curve X, a vector bundle E on X and a vector space $V \subseteq H^0(X, E)$. When there exist a smooth curve C, a finite map $u : X \to C$, $\deg(u) > 1$, and a pair (E', V') on C inducing (E, V), i.e. with $E \cong u^*(E')$ and $V = u^*(V')$ (with the identification given by the previous isomorphism) ? From our point of view this problem seems to be reasonable only if V spans E (as it was in the classical rank 1 case). The following observation shows that the problem for the higher rank case has some specific flavour.

Remark 7.1. Assume that X is not a covering of a curve of genus > 0 and let E be a spanned rank r vector bundle on X. If the map $f : X \to G(r, h^0(X, E))$ induced by $H^0(X, E)$ is not birational, the image must be a rational (perhaps singular) curve Y and f factors through the normalization $\boldsymbol{P}^1$ of Y. Hence E is a direct sum of r line bundles. Similarly, if X covers non trivially only curves of genus ≤ 1, E is indecomposable and f is not birational, then the normalization of the image is an elliptic curve. Hence by Atiyah's classification of vector bundles on elliptic curves, E is semi-stable and if r and $\deg(E)$ are not coprime, E cannot be stable. Thus on "most" curves, spanned stable vector bundles do not factor in a non trivial way.

We need the following well-known result.

Lemma 7.2. *If a spanned vector bundle F on an integral complete variety Y has $\boldsymbol{O}_Y$ as quotient, then it has $\boldsymbol{O}_Y$ as direct factor.*

Proof. Since F is spanned we have a surjection $\boldsymbol{O}_Y^{\oplus t} \to E$ which induces a surjection $u : \boldsymbol{O}_Y^{\oplus t} \to \boldsymbol{O}_Y$. Since Y has only the constants as global sections, the surjection u splits and induces a splitting of the surjection $E \to \boldsymbol{O}_Y$. □

We will consider here only the case $V := H^0(X, E)$ and set $v := h^0(X, E)$, $r := \mathrm{rank}(E)$, $L := \det(E)$ and $d := \deg(L) = \deg(E)$. For all integers n, y with $n \geq 2$, $y \geq n$, we define the integer $p_a(n, y)$ in the following way. Set $y - 1 = m(n-1) + \varepsilon$ with $0 \leq \varepsilon \leq n-2$. Then $p_a(n, y) := (n-1)m(m-1)/2 + m\varepsilon$. By classical Castelnuovo theory (see e.g. [1, ch. II]) $p_a(n, y)$ is the maximal arithmetic genus of an integral non degenerate degree y curve in $\boldsymbol{P}^n$.

PROPOSITION 7.3. *Set $x := h^0(X, L)$. Assume E spanned and $x \geq 3$ (e.g. assume E non trivial, $r \geq 2$ and $g \geq 2$). Assume $g > p_a(x-1, d)$. Then there is a smooth curve C, a vector bundle E' on C and a degree > 1 finite map $u : X \to C$ with $E \cong u^*(E')$ and $h^0(X, E) = h^0(C, E')$.*

Proof. Note that we may split off from E any trivial factor $\boldsymbol{O}$ without changing x and d. Thus we may assume $h^0(X, E^*) = 0$. Take a general $V \subset H^0(X, E)$ with $\dim(V) = r + 1$. Since E is spanned by V, we obtain an exact sequence

$$0 \to L^* \to V \otimes \boldsymbol{O} \to E \to 0. \tag{3}$$

Since $h^0(X, E^*) = 0$, we have $V^* \subseteq H^0(X, L)$. Thus (L, V^*) induces a morphism $u(V) : X \to \boldsymbol{P}(V^*)$ which is a projection of the morphism $u(L)$ induced by $(L, H^0(X, L))$. Since $g > p_a(x-1, d)$, $u(L)$ is not birational and factors in a non trivial way with a degree > 1 finite map $u : X \to C$. Thus $u(V)$ factors through u. By 3 the pair (E, V) factors through u and we obtain (E', V') on C inducing (E, V). Since u is the same for all general V, we obtain that $(E, H^0(X, E))$ is induced by $(E', H^0(C, E'))$, as wanted. □

Since as datum for E it is much more natural to take $v := h^0(X, E)$, rather than $x := h^0(X, \det(L))$, we would like to obtain good lower bounds of x in terms of v. For a related result, see Proposition 3.1. We have the following results.

LEMMA 7.4. *If E is spanned, then $x \geq v - r + 1$.*

Proof. Since E is spanned, taking $r-1$ general sections of E we obtain

$$0 \to (r-1)\boldsymbol{O} \to E \to L \to 0, \tag{4}$$

and we conclude. □

REMARK 7.5. In general the bound $x \geq v - r + 1$ just given is optimal as shown by all spanned vector bundles on $\boldsymbol{P}^1$ and hence by the bundles coming from $\boldsymbol{P}^1$, at least in suitable ranges.

LEMMA 7.6. *If E has no trivial factor, then $x \geq r+1$.*

Proof. Since E has no trivial factor, $h^0(X, E^*) = 0$ and hence by 3 $V^* \subseteq H^0(X, L)$ with $\dim(V^*) = r+1$. □

Motivated by Proposition 7.3 we will give a few cases in which better lower bounds for x are available (under the assumption that E has no trivial factor). By Remark 7.1 the case of curves with low genus is particularly interesting, even assuming that on the low genus curve the bundle induces a birational map.

REMARK 7.7. If $h^1(E) = 0$, by Riemann-Roch we have $x \geq v + (r-1)(g-1)$.

REMARK 7.8. If X has genus 1 and E has no trivial factor by Atiyah's classification of bundles on elliptic curves we have $v = x$.

REMARK 7.9. Assume that X has genus 2, $h^1(E) \neq 0$ and that E has no trivial factor. Hence we may find a non zero section vanishing at a general $P \in X$. Let $A := \boldsymbol{O}(D)$ be the saturation of the corresponding subsheaf of E. If $r = 2$, we have $E/A \cong L(-D)$. Since $D \neq \emptyset$ and L is spanned, we find $x \geq v$ if $h^0(A) = 1$. This is the case if $D = \{P\}$, i.e. if the morphism induced by E is birational. Assume $h^0(X, A) \geq 2$. Since X has genus 2 we have either $A \cong K_X$ or $\deg(A) \geq 3$. In both cases we find $h^0(X, L(-D)) \leq h^0(L) - 2$. Hence we get the same bound $x \geq v$ unless $h^0(X, A) \geq 3$. If $h^0(X, A) \geq 3$, we have $h^1(X, A) = 0$. Since E has not $\boldsymbol{O}$ as a factor by Lemma 7.2, we have $\deg(L(-D)) > 0$. Since $h^1(X, E) \neq 0$ we have $h^1(X, L(-D)) \neq 0$. Since $L(-D)$ is spanned we have $L(-D) \cong K_X$. In particular E is not semi-stable. If E is indecomposable, then either $\deg(A) = 3$ or $\deg(A) = 4$ and $A \cong K_X^{\otimes 2}$. Now assume $r > 2$. If E/A has not $\boldsymbol{O}$ as a factor, we may obtain by induction on r the inequality $x \geq v$. If E/A has $\boldsymbol{O}$ as factor, $\boldsymbol{O}$ is a quotient of E. Since E is spanned, this implies that $\mathbf{O}$ is a factor of E by Lemma 7.2.

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Received May 14, 1997.