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Carleman Estimates and Exact Boundary Controllability for a System of Coupled Non-Conservative Schrödinger Equations

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In memory of my friend, Pierre Grisvard

Summary. - *We consider a system of 2 (or more) coupled Schrödinger equations in the difficult situation where the equations have first-order, lower-order terms, as well as first-order coupling in all space variables. By using a general differential multiplier we give a "friendly" proof of Carleman estimates. Under more restrictive intrinsic conditions mostly on the coupling operators, we obtain exact controllability results for the coupled system, under various combinations of boundary controls: Dirichlet/Dirichlet; Dirichlet/Neumann; Neumann/Neumann. The controls are active on a suitable portion of the boundary. These results* cannot *be obtained by standard multipliers.*

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Nota dell'Autore:
The results of this paper for a single Schrödinger equation (Section 2) were obtained during the Fall of 1992, when the author was visiting the Dipartimento di Matematica dell'Universitá degli Studi di Trento, Povo, Italy, whose hospitality is gratefully acknowledged. In particular, the author wishes to thank Prof. Mimmo Ianneli. Much of this paper was originally meant to be part of a book which the author is co-writing on control problems for partial differential equations. It is

1. Introduction

Problem formulation. Consider the following *coupled* system of two Schrödinger equations in the unknown $w(t,x)$ and $z(t,x)$:

$$iw_t = \Delta w + F_1(w) + P_1(z) \quad \text{in } (0,T] \times \Omega \equiv Q; \tag{1.1}$$

$$iz_t = \Delta z + F_2(z) + P_2(w) \quad \text{in } Q, \tag{1.2}$$

defined on a bounded domain $\Omega \subset R^n$ with smooth boundary Γ, say of class C^1. Here, F_1 and F_2 are (linear) differential operators of order one in the space variables $x_1, \dots, x_n$, with (possibly complex) $L_\infty(\bar{Q})$-coefficients, thus satisfying the pointwise bounds:

$$|F_1(w)|^2 \le C_T[|\nabla w|^2 + |w|^2]; \quad |F_2(z)|^2 \le C_T[|\nabla z|^2 + |z|^2], \quad \forall\, t, x \in \bar{Q}. \tag{1.3}$$

Assumption (1.3) on F_1 and F_2 will remain in force throughout the paper. For our first main result, the Carleman estimates of Theorem 1.1 below, we similarly assume that $P_1(z)$ and $P_2(w)$ are, like $F_1(w)$ and $F_2(z)$, first-order (linear) differential operators in the space variables $x_1, \dots, x_n$, with (possibly complex) $L_\infty(\bar{Q})$-coefficients, thus satisfying pointwise bounds as in (1.3):

$$|P_1(z)|^2 \le C_T[|\nabla z|^2 + |z|^2]; \quad |P_2(w)|^2 \le C_T[|\nabla w|^2 + |w|^2], \quad \forall\, t, x \in \bar{Q}. \tag{1.4}$$

Main results. No boundary conditions need to be imposed at this stage. For the purposes of this paper, the operator $(-\Delta)$ in (1.1) and (1.2) may be replaced by two, possibly different, uniformly elliptic operators of order two, with constant coefficients, without

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offered here in memory of Pierre Grisvard, on a topic which occupied the talent and energy of Pierre Grisvard in the latest years of his productive life: the problem of exact controllability for second-order hyperbolic equations, as well as of other evolution equations, in domains with corners and cracks. The author has fond memories of the numerous conversations (at the International Conference held at Delft University, The Netherlands, September 1989; at the International Workshop held at Han-sur-Lesse, Belgium,October 1991), as well as of the private correspondence, which he shared with Pierre Grisvard on these topics.

effecting the proofs and results below. Under the above assumptions we shall prove Carleman estimates for (1.1), (1.2): To state them, we let

$$E_w(t) \equiv \int_\Omega |\nabla w(t)|^2 d\Omega; \quad E_z(t) = \int_\Omega |\nabla z(t)|^2 d\Omega;$$
$$E(t) = E_w(t) + E_z(t), \tag{1.5}$$

and we let $\phi(x,t) = |x - x_0|^2 - c\left|t - \frac{T}{2}\right|^2$, $x_0 \in R^n$, be the pseudo-convex function discussed in (2.1.4) below in Section 2.

Theorem 1.1. (Carleman estimates) Assume (1.3) and (1.4). Let w, z be solutions of (1.1), (1.2) in the following class

$$\begin{cases} w, z \in C([0,T]; H^1(\Omega)) & (1.6) \\ w_t,\ \dfrac{\partial w}{\partial \nu},\ z_t,\ \dfrac{\partial z}{\partial \nu} \in L_2(0,T; L_2(\Gamma)), & (1.7) \end{cases}$$

ν being a unit outward vector on Γ. Let $T > 0$ be arbitrary. Then, for all τ sufficiently large, the following one-parameter family of estimates hold true:

$$\begin{aligned} &\left(2 - \frac{3C_T}{\tau} - \frac{1}{\tau}\right) \int_Q e^{\tau\phi}[|\nabla w|^2 + |\nabla z|^2] dQ \\ &- \frac{e^{-\delta\tau}}{\tau}[E(T) + E(0)] \ \leq \\ &\leq \ BT(w)|_\Sigma + BT(z)|_\Sigma \\ &\quad + C_{T,\phi,\tau}\left[\|w\|^2_{C([0,T];L_2(\Omega))} + \|z\|^2_{C([0,T];L_2(\Omega))}\right], \end{aligned} \tag{1.8}$$

where the boundary terms $BT(w)|_\Sigma$ are defined by

$$\begin{aligned} BT(w)|_\Sigma \ = \ & \mathrm{Re}\left(\int_\Sigma e^{\tau\phi}\frac{\partial w}{\partial \nu}\nabla\phi \cdot \nabla\bar{w}\, d\Sigma\right) \\ & - \frac{1}{2}\int_\Sigma e^{\tau\phi}|\nabla w|^2 \nabla\phi \cdot \nu\, d\Sigma \\ & + \left|\frac{1}{2}\int_\Sigma \frac{\partial w}{\partial \nu}\bar{w}\ \mathrm{div}\left(e^{\tau\phi}\nabla\phi\right) d\Sigma \right. \\ & \quad \left. - \frac{i}{2}\int_\Sigma \bar{w} w_t e^{\tau\phi}\nabla\phi \cdot \nu\, d\Sigma\right|, \end{aligned} \tag{1.9}$$

and similarly for $BT(z)|_\Sigma$. Finally, $\delta > 0$ is the constant associated with the pseudo-convex function $\phi(x,t)$ as in (2.1.4c). $\square$

As discussed in detail in the *Comments* below, a main source of difficulty in proving estimate (1.8) is due to the presence of arbitrary *first*-order terms F_1, F_2 in the original variable as well as P_1 and P_2 in the coupled variable, in each of the two equations (1.1) and (1.2). The proof of Theorem 1.1 will be given in Section 3, in Theorem 3.1, and will be critically based on Theorem 2.1.1 for a single equation. However, in order to deduce from estimate (1.8) further significant inequalities, we need to slightly specialize the operators F_1, F_2, and, more seriously, restrict the coupling operators $P_1(z)$ and $P_2(w)$ to be of zero order. Such latter restriction is *intrinsic* to the issue of describing E_w and E_z as a function of t; it rests on the fact that the multiplier $\bar{w}_t$, needed to obtain such description, does not (unlike the generalized wave equation case) yield $\int_\Omega |w_t|^2 d\Omega$ as an energy-term (see Section 2.3). More precisely, we henceforth require that the coefficients of F_1 and F_2 be *real* so that $F_1(w)$ and $F_2(z)$ are of the form

$$
\left\{
\begin{array}{rcll}
F_1(w) & = & \nabla r_1(t,x) \cdot \nabla w(t,x) + \rho_1(t,x) w(t,x) & \quad (1.10\text{a}) \\
F_2(z) & = & \nabla r_2(t,x) \cdot \nabla z(t,x) + \rho_2(t,x) z(t,x), & \quad (1.10\text{b}) \\
\multicolumn{3}{l}{\text{under the following assumptions}} & \\
r_i(t,x) & = & [r_i^{(1)}(t,x), \dots, r_i^{(n)}(t,x)] & \\
 & = & \text{real } n\text{-vector field, with} & \quad (1.10\text{c}) \\
 & & |\nabla r_i(t,x)| \in L_\infty(\bar{Q}),\ i = 1,2 & \\
\rho_i(t,x) & = & \text{real function in } L_\infty(\bar{Q}), & \\
 & & \ \text{with } \rho_{i,t}(t,x) \in L_\infty(\bar{Q}). & \quad (1.10\text{d})
\end{array}
\right.
$$

A fortiori, (1.10) implies (1.3). Moreover, as to $P_1(z)$ and $P_2(w)$, we require that they are of zero order

$$\begin{cases} P_1(z) = \alpha(t,x)z; \quad P_2(w) = \beta(t,x)w; & \text{(1.11a)} \\ \alpha(t,x),\ \beta(t,x) \in L_\infty(\bar{Q}); \\ \ |\nabla\alpha(t,x)|,\ |\nabla\beta(t,x)| \in L_\infty(\bar{Q}), & \text{(1.11b)} \\ \nabla = \text{gradient in } x, \text{ so that} \\ \qquad \text{the following estimates hold true:} \\ |P_1(z)| \le C_T|z|; \quad |P_2(w)| \le C_T|w|, \quad \forall\, t,x \in \bar{Q}; & \text{(1.11c)} \\ \|P_1(z)\|_{H^1(\Omega)} \le C_T\|z\|_{H^1(\Omega)}; \\ \|P_2(w)\|_{H^1(\Omega)} \le C_T\|w\|_{H^1(\Omega)}. & \text{(1.11d)} \end{cases}$$

Our goal in the present paper is twofold:

(i) To establish the energy estimate of our main Theorem 1.2 below, which reconstructs the energy from the boundary measurements modulo lower-order terms;

(ii) to provide a "friendly" and explicit proof of Theorem 1.2.

THEOREM 1.2. Assume (1.10) and (1.11). Let w, z be solutions of (1.1), (1.2) in the class (1.6), (1.7). Let $T > 0$ be arbitrary. Then,

(i) there exists a constant $k_T > 0$ such that the following inequality holds true for all $\epsilon_0 > 0$:

$$\begin{aligned} k_T[E(T) + E(0)] \ &\le \\ \le \ &\int_0^T \int_\Gamma \left[|w_t|^2 + |z_t|^2 + \left|\frac{\partial w}{\partial \nu}\right|^2 + \left|\frac{\partial z}{\partial \nu}\right|^2 \right] d\Gamma\, dt \\ &+ C_{T,\epsilon_0}\left\{ \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} + \|z\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} \right\} \end{aligned} \tag{1.12}$$

(ii) Assume further that

$$w|_{\Sigma_0} \equiv 0 \text{ and/or, respectively, } z|_{\Sigma_0} \equiv 0,\ \ \Sigma_0 = (0,T] \times \Gamma_0, \tag{1.13}$$

where Γ_0 is the portion of the boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, defined by Eqn. (2.1.16) in Section 2 in terms of $\nabla\phi$. Then, the right-hand side of (1.12) is refined, in the sense that integration over Γ of the w-terms, and/or, respectively, of the z-terms, is replaced by a corresponding integration over Γ_1 only, so that the right-hand side of (1.12) contains instead

$$\int_0^T \int_{\Gamma_1} \left[|w_t|^2 + \left| \frac{\partial w}{\partial \nu} \right|^2 \right] d\Gamma_1 dt, \quad \text{and/or, respectively,} \quad \int_0^T \int_{\Gamma_1} \left[|z_t|^2 + \left| \frac{\partial z}{\partial \nu} \right|^2 \right] d\Gamma_1 dt, \tag{1.14}$$

according to whichever of the conditions in (1.13) holds true (possibly both). □

Once (1.12) (or (1.13)) is established, one may further refine it by absorbing the interior lower-order terms in w and z by a compactness/uniqueness argument, and thus obtain the desired final estimate.

THEOREM 1.3. Assume (1.10) and (1.11). Let w, z be solutions of Eqns. (1.1) and (1.2) in the class (1.6), (1.7). Let both w and z satisfy the boundary conditions in (1.13) on Σ_0. Assume, further, the following uniqueness property: that the only solution of (1.1), (1.2) subject to the overdetermined homogeneous B.C.

$$\left\{ \begin{array}{rcl} w|_\Sigma & \equiv & 0; \\ \left. \dfrac{\partial w}{\partial \nu} \right|_{\Sigma_1} & \equiv & 0, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} z|_\Sigma & \equiv & 0; \\ \left. \dfrac{\partial z}{\partial \nu} \right|_{\Sigma_1} & \equiv & 0, \end{array} \right. \tag{1.15}$$

is the trivial solution $w \equiv z \equiv 0$. Then, there exists a positive constant $k_T > 0$ such that the following energy estimate holds true:

$$k_T[E(T) + E(0)] \leq \int_0^T \int_{\Gamma_1} \left[|w_t|^2 + |z_t|^2 + \left| \frac{\partial w}{\partial \nu} \right|^2 + \left| \frac{\partial z}{\partial \nu} \right|^2 \right] d\Gamma_1 \, dt. \tag{1.16}$$

□

Remark 1.1. The uniqueness result always holds true with sufficiently smooth coefficients [I.1].

□

Consequences on Exact Controllability. We now supplement Eqns. (1.1), (1.2) with boundary conditions. We need to consider essentially three cases, where $\Sigma_i = (0,T] \times \Gamma_i$, $i = 0, 1$, Γ_i defined by (2.1.16), (2.1.17) of Section 2.

Case 1. (Dirichlet/Dirichlet)

$$\begin{cases} w|_{\Sigma_0} \equiv 0; \\ w|_{\Sigma_1} \equiv u_1, \end{cases} \quad \text{and} \quad \begin{cases} z|_{\Sigma_0} \equiv 0 \\ z|_{\Sigma_1} \equiv u_2. \end{cases} \tag{1.17}$$

Case 2. (Neumann/Neumann)

$$\begin{cases} w|_{\Sigma_0} \equiv 0; \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} \equiv u_1, \end{cases} \quad \text{and} \quad \begin{cases} z|_{\Sigma_0} \equiv 0; \\ \left.\dfrac{\partial z}{\partial \nu}\right|_{\Sigma_1} \equiv u_2. \end{cases} \tag{1.18}$$

Case 1. (Dirichlet/Neumann)

$$\begin{cases} w|_{\Sigma_0} \equiv 0; \\ w|_{\Sigma_1} \equiv u_1, \end{cases} \quad \text{and} \quad \begin{cases} z|_{\Sigma_0} \equiv 0; \\ \left.\dfrac{\partial z}{\partial \nu}\right|_{\Sigma_1} \equiv u_2. \end{cases} \tag{1.19}$$

The well-posedness result in each of the three cases is standard. As a consequence of the basic energy estimate in (1.16), we obtain exact controllability at $t = T$ for problem (1.1), (1.2) in each of the three foregoing cases, within the class of $L_2(0,T; L_2(\Gamma_1))$-controls u_1 and u_2 in the (optimal) space (of regularity) $H^{-1}(\Omega)$ in the case of Dirichlet B.C., and in the space $H^1_{\Gamma_0}(\Omega)$ in the case of Neumann B.C.

Theorem 1.4. Let the hypotheses of Theorem 1.3 hold true.

(a) (Continuous observability inequalities) The following inequalities hold true for (1.1), (1.2); supplemented by the following B.C.:

Case 1. Problem (1.1), (1.2), (1.17) with $u_1 \equiv u_2 \equiv 0$:

$$k_T E(0) \le \int_0^T \int_{\Gamma_1} \left[\left| \frac{\partial w}{\partial \nu} \right|^2 + \left| \frac{\partial z}{\partial \nu} \right|^2 \right] d\Gamma_1 dt. \tag{1.20}$$

Case 2. Problem (1.1), (1.2), (1.18) with Eqns. (1.18b) replaced by

$$\left[\frac{\partial w}{\partial \nu} - \alpha w \right]_{\Sigma_1} \equiv 0; \quad \left[\frac{\partial z}{\partial \nu} - \beta z \right]_{\Sigma_1} \equiv 0, \quad \alpha, \beta \in L_\infty(\Sigma_1):$$

$$k_T E(0) \le \int_0^T \int_{\Gamma_1} \left[|w_t|^2 + |z_t|^2 \right] d\Gamma_1 dt. \tag{1.21}$$

Case 3. Problem (1.1), (1.2), (1.19) with $u_1 \equiv 0$ and Equation (1.18b) for z replaced by $\left[\frac{\partial z}{\partial \nu} - \beta z \right]_{\Sigma_1} \equiv 0$:

$$k_T E(0) \le \int_0^T \int_{\Gamma_1} \left[\left| \frac{\partial w}{\partial \nu} \right|^2 + |z_t|^2 \right] d\Gamma_1 dt. \tag{1.22}$$

(b) (Exact controllability at $t = T$). By duality, e.g., [L–T.2] and Appendix, problem (1.1), (1.2) with $\Delta r \in L_\infty(\bar{Q})$ is exactly controllable as specified in each of the following cases:

Case 1. Assume the following initial conditions,

$$w_0 \in H^{-1}(\Omega); \qquad z_0 \in H^{-1}(\Omega). \tag{1.23}$$

Then, there exist controls

$$\{u_1, u_2\} \in L_2(0, T; L_2(\Gamma_1)) \times L_2(0, T; L_2(\Gamma_1)), \tag{1.24}$$

such that the corresponding solution of (1.1), (1.2), (1.17) satisfies

$$w(T) = z(T) = 0. \tag{1.25}$$

Case 2. Assume the following initial conditions,

$$w_0 \in H^1_{\Gamma_0}(\Omega); \qquad z_0 \in H^1_{\Gamma_0}(\Omega). \tag{1.26}$$

Then, there exist controls $\{u_1, u_2\}$ as in (1.24) such that the corresponding solution of (1.1), (1.2), (1.18) satisfies (1.25).

Case 3. Assume the following initial conditions,

$$w_0 \in H^{-1}(\Omega); \qquad z_0 \in H^1_{\Gamma_0}(\Omega). \tag{1.27}$$

Then, there exist controls $\{u_1, u_2\}$ as in (1.24) such that the corresponding solution of (1.1), (1.2), (1.19) satisfies (1.25).

Comments, literature. To put the above estimate (1.8) for the coupled problem (1.1), (1.2) in perspective, let us consider at first only the w-equation (1.1) with no coupling; i.e., the equation

$$iw_t = \Delta w + F_1(w) \quad \text{on } Q, \tag{1.28}$$

with F_1 a first-order differential operator in $x_1, \dots, x_n$ satisfying (1.3). The energy (multiplier) method, based on the principal multiplier $h(x) \cdot \nabla \bar{w}(x)$, $h(x)$ a suitable coercive vector field over $\bar{\Omega}$, permits to establish a number of key inequalities:

(i) the "regularity inequality" in the Dirichlet homogeneous case $w|_\Sigma \equiv 0$ (the $L_2(\Sigma_T)$-norm of $\frac{\partial w}{\partial \nu}$ is bounded above by $E_w(0)$, for all T) [L–T.2, Thm. 1.1], indeed, even in the case of a (symmetric) principal part with variable coefficients;

(ii) the reverse "continuous observability inequality," when coupled with the second multiplier $\bar{w}$ div h, however, *only when* F_1 is actually a *zero*-order operator [L–T.2]. If F_1 is a bonafide first-order operator, the method fails. To obtain "continuous observability" reverse inequalities, more sophisticated methods were subsequently introduced:

(a) methods of microlocal analysis, after a rescaling of time, depending on the frequency [L]: the final statement, which assumes analytic boundary and delivers a control acting on a pair $(\bar{\Gamma}, T)$, which geometrically controls Ω, refers, however, to the pure Schrödinger equation (1.28) with $F_1 \equiv 0$;

(b) pseudo-differential methods to extend Carleman estimates—which were available in the literature for solutions with compact support and, generally, isotropic operators—to the case of domains with boundary and to anisotropic operators, as carried out in the general and unifying work of [Ta.1–3], with constant coefficient principal part (to assert the existence of a pseudo-convex function).

Both works have "unfriendly" proofs, not readily accessible outside specialized circles. It would be a problem to just quote or dig out the required estimates from either [L] or from [Ta.1–3] to prove Theorem 1.1 for the coupled system (1.1), (1.2)—or how to dispense with geometric conditions by appealing to the methods behind these proofs. Moreover, [Ta.1–3], at least in this first effort, takes the control over the entire boundary.

Finally, we quote references [Li-Ta], [H–L] where an altogether different approach is pursued, which aims at obtaining steering controls directly through the principle *local smoothing + reversibility + uniqueness → exact controllability*. This method allows for variable C^∞-coefficients of the (strongly elliptic and self-adjoint) principal part, but delivers only controls which belong to $C^\infty(\partial\Omega)$ for $t > 0$. For many purposes, we would instead need a precise relationship, in terms of Sobolev spaces, between the space $L_2(0, T; L_2(\Gamma))$ of controls on $[0, T]$, and the space Y of exact controllability at $t = T$; i.e., $Y = H^{-1}(\Omega)$ (Dirichlet case), and $Y = H^1(\Omega)$ (Neumann case).

In this paper, we pursue the Carleman estimate approach proposed by [Ta]. We thus provide—first, in Section 2, for the single equation (1.28); next, in Section 3, for the coupled system (1.1), (1.2)—explicit, direct, friendly-to-follow energy computations and estimates, conducted throughout at the differential level with differential multipliers (rather than at the pseudo-differential level with pseudo-differential multipliers as in the general work of [Ta]). In the process, we dispense with geometric conditions (by virtue of our result Theorem 2.1.4) and, moreover, allow the control action to be active only on a portion of the boundary (unlike [Ta.1–3]). Thus, the method given explicitly here reveals itself as a differential multiplier method, with multipliers which generalize directly but in a non-trivial way the original multipliers $h \cdot \nabla\bar{w}$ and $\bar{w}$ div h mentioned above, and used for continuous observability inequalities and in uniform stabilization inequalities, only for *canonical* models [L–T.2]. The method provides a one-parameter family of (Carleman) estimates in terms of a parameter $\tau > 0$, and its virtue in absorbing first-order terms is clearly displayed by using the additional flexibility of the parameter τ, once taken large enough (see Remark 2.2.1, and proof of Theorem 2.2.4). The estimates refer to solutions of the Schrödinger equation (1.28) above (Section 2) and respectively, of the coupled system (1.1) and (1.2) above (Section 3), with no boundary conditions. They are expressed explicitly in terms of boundary traces as well. As a consequence, we derive exact controllability results in the Dirichlet and Neumann cases, i.e., in cases which rely on $H^1(\Omega)$-energy level estimates.

Finally, one can obtain results for the Euler-Bernoulli-type equation by factoring it as the product of two Schrödinger-type equations. This will appear elsewhere.

2. A-priori P.D.E.'s Estimates for Schrödinger Equations

2.1 Dynamical Model and Statement of Main Results

Dynamical model. Let Ω be an open bounded domain in R^n with sufficiently smooth boundary Γ, say of class C^1. Throughout this section we shall consider the following Schrödinger equation in the unknown $w(t,x)$:

$$i\, w_t = \Delta w + F(w) + f \quad \text{in } (0,T] \times \Omega \equiv Q, \tag{2.1.1}$$

where (at least) $f \in L_2(Q)$ is a forcing term and where $F(w)$ is a linear first-order differential operator in the space variables $\{x_1, \dots, x_n\}$ on w with $L_\infty(Q)$-(possibly complex) coefficients, thus satisfying the following pointwise estimate: There exists a constant $C_T > 0$ such that

$$|F(w)|^2 \le C_T[|\nabla w|^2 + |w|^2], \quad \forall\; t, x \in Q. \tag{2.1.2}$$

In this section, we introduce (no confusion is likely to arise with (1.5))

$$E(t) \equiv \int_\Omega |\nabla w(t)|^2 d\Omega. \tag{2.1.3}$$

REMARK 2.1.1. In the analysis below the operator $(-\Delta)$ in (2.1.1) [Laplacian in the space variables] could be replaced by a second-order uniformly elliptic operator with constant coefficients. $\square$

Pseudo-convex function $\phi(x,t)$. Let $\phi : \Omega \times \mathbf{R} \to \mathbf{R}$ be the (pseudo-convex) function defined by

$$\phi(x,t) \equiv |x-x_0|^2 - c\left|t - \frac{T}{2}\right|^2, \tag{2.1.4a}$$

where $x_0 \in R^n$, whereby there exists a subinterval $[t_0, t_1] \subset (0,T)$ such that

$$\phi(x,t) > 1 \text{ for } t \in [t_0,t_1];\ x \in \Omega; \tag{2.1.4b}$$

$$\phi(x,0) < -\delta < 0;\ \phi(x,T) < -\delta < 0, \text{ uniformly in } x \in \Omega, \tag{2.1.4c}$$

for a suitable constant $\delta > 0$. We set, for future use

$$\nabla\phi(x,t) = 2(x-x_0) \equiv h(x); \quad \nabla(e^{\tau\phi}) = \tau e^{\tau\phi}\nabla\phi. \tag{2.1.5}$$

REMARK 2.1.2. (Optimal choice of T) By choosing c large enough we may obtain any $T > 0$ small. Henceforth, in all results to follow, $T > 0$ may be taken arbitrarily small, since the proofs put no further constraint on c. $\square$

Main results. The main results of the present section are as follow. They are listed in the order in which they are proved.

THEOREM 2.1.1. (Carleman estimates) Assume (2.1.2) and $f \in L_2(Q)$. Let w be a solution of Eqn. (2.1.1) in the following class:

$$\begin{cases} w \in C([0,T]; H^1(\Omega)); & (2.1.6) \\ w_t,\ \dfrac{\partial w}{\partial \nu} \in L_2(0,T; L_2(\Gamma)). & (2.1.7) \end{cases}$$

Let $\phi(x,t)$ be the pseudo-convex function defined by (2.1.4). Then, for $\tau > 0$ sufficiently large, the following one-parameter family of estimates holds true, with $E(t)$ as in (2.1.3):

(i)

$$\left(2-\frac{C_T}{\tau}-\frac{1}{\tau}\right)\int_Q e^{\tau\phi}|\nabla w|^2 dQ-\frac{e^{-\delta\tau}}{\tau}[E(T)+E(0)] \ \leq$$
$$\leq \ (BT)|_\Sigma+\frac{2}{\tau}\int_Q e^{\tau\phi}|f|^2 dQ+C_{T,\phi,\tau}\|w\|^2_{C([0,T];L_2(\Omega))} \tag{2.1.8}$$

(ii)

$$\left(2-\frac{C_T}{\tau}-\frac{1}{\tau}\right)e^{\tau}\int_{t_0}^{t_1} E(t)dt-\frac{e^{-\delta\tau}}{\tau}[E(T)+E(0)]$$
$$\leq \ (BT)|_\Sigma+\frac{2}{\tau}\int_Q e^{\tau\phi}|f|^2 dQ+C_{T,\phi,\tau}\|w\|^2_{C([0,T];L_2(\Omega))} \tag{2.1.9}$$

where the boundary terms $(BT)|_\Sigma$ over $\Sigma=[0,T]\times\Gamma$ are given by (see (2.2.23) below)

$$(BT)|_\Sigma \equiv \mathrm{Re}\left(\int_\Sigma e^{\tau\phi}\frac{\partial w}{\partial\nu}\nabla\phi\cdot\nabla\bar w\,d\Sigma\right)-\frac{1}{2}\int_\Sigma e^{\tau\phi}|\nabla w|^2 h\cdot\nu\,d\Sigma$$
$$+\left|\frac{1}{2}\int_\Sigma\frac{\partial w}{\partial\nu}\bar w\ \mathrm{div}(e^{\tau\phi}h)d\Sigma-\frac{i}{2}\int_\Sigma\bar w w_t e^{\tau\phi}h\cdot\nu\cdot d\Sigma\right|. \tag{2.1.10}$$

$\nabla\phi=h$ by (2.1.5). The constant δ is defined by (2.1.4c). □

Remark 2.1.3. The presence of the factor $\frac{1}{\tau}$ in front of the integral term containing f in (2.1.8) is critical to extend Theorem 2.1.1 to the coupled case as in Theorem 1.1 of Section 1. □

The proof of Theorem 2.1.1 will be given in Section 2.2, where additional interesting results are contained. Henceforth, we specialized the first-order differential operator $F(w)$ to have *real* (still, possibly time dependent) coefficients, so that $F(w)$ is of the form

$$\left\{\begin{array}{l} F(w)=\nabla r(t,x)\cdot\nabla w(t,x)+\rho(t,x)w(t,x) \\ \\ \text{under the following assumptions} \\ \\ \begin{array}{rcl} r &=& r(t,x)=[r_1(t,x),\dots,r_n(t,x)] \\ &=& \text{real vector field with } |\nabla r(t,x)|\in L_\infty(\bar Q); \\ \\ \rho(t,x) &=& \text{real function in } L_\infty(\bar Q) \\ && \quad\text{with } \rho_t(t,x)\in L_\infty(\bar Q). \end{array}\end{array}\right. \tag{2.1.11}$$

THEOREM 2.1.2.

(i) Let $f \in L_2(0,T;H^1(\Omega))$, and let w be a solution of Eqn. (2.1.1) in the class (2.1.6), (2.1.7). Finally, let the coefficients ∇r and ρ of F be as in (2.1.11). Then, for $\tau > 0$ sufficiently large, there exists a constant $c_{\phi,\tau} > 0$ such that the following estimates hold true:

(i$_1$)
$$
\begin{aligned}
c_{\phi,\tau} E(0) \;\leq\; & C_T (BT_1)|_\Sigma + \\
& + \frac{2}{\tau} \int_Q |f|^2 e^{\tau\phi} dQ + C \|f\|^2_{L_2(0,T;H^1(\Omega))} \\
& + C_{\phi,T,\tau} \|w\|^2_{C([0,T];L_2(\Omega))}.
\end{aligned}
\tag{2.1.12}
$$

(i$_2$) With $k > \|R\|_{L_\infty(Q)}$, R the $n \times n$ matrix $R = \left[\frac{\partial r_i}{\partial x_j}\right]$, see below (2.3.17), we have

$$
\begin{aligned}
& \frac{e^{-kT}}{2} c_{\phi,\tau} [E(0) + E(T)] \;\leq \\
& \leq\; C_T (BT_1)|_\Sigma + \frac{2}{\tau} \int_Q |f|^2 e^{\tau\phi} dQ \\
& \qquad + C \|f\|^2_{L_2(0,T;H^1(\Omega))} + C_{\phi,T,\tau} \|w\|^2_{C([0,T];L_2(\Omega))},
\end{aligned}
\tag{2.1.13}
$$

where the boundary terms $(BT_1)|_\Sigma$ over Σ are given by

$$
\begin{aligned}
(BT_1)|_\Sigma \;=\; & (BT)|_\Sigma + \int_0^T \int_\Gamma \left| \frac{\partial w}{\partial \mu} \, \frac{\partial \bar{w}}{\partial \nu} \right| d\Gamma \, dt \\
& + \int_0^T \int_\Gamma \left| f \frac{\partial \bar{w}}{\partial \nu} \right| d\Gamma \, dt \\
& + \int_0^T \int_\Gamma \left| \frac{\partial w}{\partial \nu} \bar{w}_t \right| d\Gamma \, dt,
\end{aligned}
\tag{2.1.14}
$$

μ being a unit tangential vector on Γ, hence $\frac{\partial}{\partial \mu}$ being a tangential derivative, with $(BT)|_\Sigma$ defined by (2.1.10).

(ii) Assume, further, that $f \in L_2(0,T;H^1_{\Gamma_0}(\Omega))$, $H^1_{\Gamma_0}(\Omega) = \{f \in H^1(\Omega) : f|_{\Gamma_0} = 0\}$, and that w satisfies the boundary condition

$$
w|_{\Sigma_0} \equiv 0, \quad \Sigma_0 = (0,T] \times \Gamma_0,
\tag{2.1.15}
$$

where we divide the boundary Γ as $\Gamma = \Gamma_0 \cup \Gamma_1$, with

$$\Gamma_0 = \{x \in \Gamma : \nabla\phi \cdot \nu(x) \leq 0\}; \tag{2.1.16}$$

$$\Gamma_1 = \{x \in \Gamma : \nabla\phi \cdot \nu(x) > 0\}, \tag{2.1.17}$$

with $\nu(x)$ = unit outward normal vector at $x \in \Gamma$, and $x_0 \in R^n$ the point entering into the definition of ϕ in (2.1.4) so that $2(x - x_0) = h = \nabla\phi$.

Then, estimates (2.1.12), (2.1.13) hold true for $\tau > 0$ sufficiently large, with the boundary terms $(BT_1)|_\Sigma$ replaced by $(BT_1)|_{\Sigma_1}$, i.e., evaluated only on $\Sigma_1 = [0, T] \times \Gamma_1$. □

The proof of Theorem 2.1.2 will be given in Section 2.3. As a consequence of Theorem 2.1.2 and of a uniqueness theorem, we then obtain the desired "continuous observability inequality" for the corresponding problem with homogeneous Dirichlet B.C.,

$$\begin{cases} i\psi_t = \Delta\psi + F(\psi) & \text{in } (0, T] \times \Omega = Q; & (2.1.18\text{a}) \\ \psi(0, \cdot\,) = \psi_0 & \text{in } \Omega; & (2.1.18\text{b}) \\ \psi|_\Sigma \equiv 0 & \text{in } (0, T] \times \Gamma = \Sigma, & (2.1.18\text{c}) \end{cases}$$

where F is as in (2.1.11). The reversed "trace regularity inequality" was proved in [L–T.2] for any $T > 0$ (the presence of a first-order operator F in the space variables does not affect the trace regularity inequality).

Theorem 2.1.3. (Continuous observability, Dirichlet case) Let $T > 0$ be arbitrary. Let F be as in (2.1.11). Let the homogeneous, over-determined problem defined by (2.1.18a-b-c), as well as $\left.\frac{\partial\psi}{\partial\nu}\right|_{\Sigma_1} \equiv 0$ on $(0, T] \times \Gamma_1 \equiv \Sigma_1$ admit the unique solution $\psi \equiv 0$. Then, with reference to problem (2.1.18), there exists a positive constant $\text{const}_T > 0$ such that

$$\int_0^T \int_{\Gamma_1} \left|\frac{\partial\psi}{\partial\nu}\right|^2 d\Gamma_1 dt \geq \text{const}_T \|\psi_0\|^2_{H_0^1(\Omega)}. \tag{2.1.19}$$

□

REMARK 2.1.4. The uniqueness result invoked in Theorem 2.1.3, as well as in Theorem 2.1.7 below, always holds true with sufficiently smooth coefficients [I.1]. □

The proof of Theorem 2.1.3 will be given in Section 2.4. The next result 'absorbs' the tangential derivatives $\frac{\partial w}{\partial s} = \nabla_s w$ (tangential gradient) by the normal derivative $\frac{\partial w}{\partial \nu}$ and w_t.

THEOREM 2.1.4. Let w be a solution of Eqn. (2.1.1) in the class (2.1.6), (2.1.7).

(i) Given $\epsilon > 0$ and $\epsilon_0 > 0$ arbitrarily small, and given $T > 0$, there exists a constant $C_{\epsilon,\epsilon_0,T} > 0$ such that

$$\int_{\epsilon}^{T-\epsilon} \int_{\Gamma} |\nabla_s w|^2 d\Gamma\, dt \ \leq\ C_{\epsilon,\epsilon_0,T} \left\{ \int_0^T \int_{\Gamma} \left[\left| \frac{\partial w}{\partial \nu} \right|^2 + |w_t|^2 \right] d\Sigma \right.$$
$$\left. + \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} + \|f\|^2_{H^{-\frac{1}{2}+\epsilon_0}(Q_T)} \right\}. \tag{2.1.20}$$

(ii) If, moreover, w satisfies the boundary condition (2.1.15), then (2.1.20) holds true with Γ replaced by Γ_1. □

The proof of Theorem 2.1.4 follows from the proof of a more demanding, corresponding result for second-order hyperbolic equations, as given in [L–T.3, Section 7.2], and is sketched in Section 2.5.

The final estimate—the main result of the present section—is given next.

THEOREM 2.1.5. Let $f \in L_2(0,T;H^1(\Omega))$, left F be as in (2.1.11), and let w be a solution of Eqn. (2.1.1) in the class (2.1.6) and (2.1.7). Assume, moreover, that w satisfies the boundary condition (2.1.15). Then, the following estimate holds true. There exists a constand $k_{\phi,\tau} > 0$, ϕ the pseudo-convex function in (2.1.4) and τ a sufficiently large parameter, such that

$$\text{(i}_1\text{)}\quad k_{\phi,\tau}E(0) \le \int_0^T \int_{\Gamma_1} \left[\left| \frac{\partial w}{\partial \nu} \right|^2 + |w_t|^2 \right] d\Gamma_1\, dt + \text{const}_{\phi,\tau} \|f\|^2_{L_2(0,T;H^1(\Omega))} + C_{\phi,T,\tau,\epsilon_0} \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))}; \qquad (2.1.21)$$

(i$_2$) or, equivalently,

$$k_{\phi,\tau}[E(0)+E(T)] \le \int_0^T \int_{\Gamma_1} \left[\left| \frac{\partial w}{\partial \nu} \right|^2 + |w_t|^2 \right] d\Gamma_1\, dt + \text{const}_{\phi,\tau} \|f\|^2_{L_2(0,T;H^1(\Omega))} + C_{\phi,T,\tau,\epsilon_0} \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))}. \qquad (2.1.22)$$

□

The proof of Theorem 2.1.5 combines Theorem 2.1.2 and 2.1.4, and is similar to that given in Section 2.5 [L–T.3, p. 221]. A similar proof for the coupled problem (1.1), (1.2) is given in Proposition 3.5 below.

Consequences on exact controllability. By the standard duality between continuous observability and exact controllability [the input-solution operator is surjective $L_2(0,T;U)$ onto Y at time T if and only if its adjoint is bounded below], see Appendix, we obtain exact controllability of Eqn. (2.1.1) with $f \equiv 0$ and $L_2(0,T;L_2(\Gamma_1))$-boundary control u either in the Dirichlet B.C. and on the space (of optimal regularity) $Y = H^{-1}(\Omega)$ [L–T.2]; or else in the Neumann B.C. and on the space $Y = H^1_{\Gamma_0}(\Omega)$. Details are omitted.

THEOREM 2.1.6. (Exact controllability, Dirichlet case) Assume the hypotheses of Theorem 2.1.3, so that inequality (2.1.19) holds true for the homogeneous problem (2.1.18). Equivalently, the mixed problem,

$$\begin{cases} iw_t = \Delta w + F(w) & \text{in } Q; & (2.1.23a) \\ w(0,\,\cdot\,) = w_0 & \text{in } \Omega; & (2.1.23b) \\ w|_{\Sigma_0} \equiv 0 & \text{in } \Sigma_0; & (2.1.23c) \\ w|_{\Sigma_1} = u & \text{in } \Sigma_1, & (2.1.23d) \end{cases}$$

with F as in (2.1.11) and Γ_1 as in (2.1.17) for a fixed $x_0 \in R^n$, is exactly controllable (to, or from, the origin) over the space $Y = H^{-1}(\Omega)$, within the class of $L_2(0,T;L_2(\Gamma_1))$-controls, with $T > 0$ arbitrary. Specifically, given $w_0 \in Y$ and $v_0 \in Y$, there exists $u \in L_2(0,T;L_2(\Gamma_1))$ such that the corresponding solution of (2.1.23) satisfies $w \in C([0,T];Y)$ and $w(T;\,\cdot\,) = v_0$. □

Turning to the Neumann case, we have that exact controllability of the non-homogeneous problem,

$$\begin{cases} iw_t = \Delta w + F(w) & \text{in } (0,T] \times \Omega = Q; & (2.1.24\text{a}) \\ w(0, \quad) = w_0 & \text{in } \Omega; & (2.1.24\text{b}) \\ w|_{\Sigma_0} \equiv 0 & \text{in } (0,T] \times \Gamma_0 \equiv \Sigma_0; & (2.1.24\text{c}) \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} \equiv u & \text{in } (0,T] \times \Gamma_1 \equiv \Sigma_1, & (2.1.24\text{d}) \end{cases}$$

on the space $Y = H^1_{\Gamma_0}(\Omega)$, with $u \in L_2(\Sigma_1)$, for any $T > 0$, is equivalent (see Appendix) to the continuous observability inequality, at any $T > 0$, for the following homogeneous problem

$$\begin{cases} i\psi_t = \Delta\psi + \tilde{F}(\psi) & \text{in } (0,T] \times \Omega = Q; & (2.1.25\text{a}) \\ \psi(0,\,\cdot\,) = \psi_0 & \text{in } \Omega; & (2.1.25\text{b}) \\ \psi|_{\Sigma_0} \equiv 0 & \text{in } (0,T] \times \Gamma_0 = \Sigma_0; & (2.1.25\text{c}) \\ \left[\dfrac{\partial \psi}{\partial \nu} - \psi \dfrac{\partial r}{\partial \nu}\right]_{\Sigma_1} \equiv 0 & \text{in } (0,T] \times \Gamma_1 = \Sigma_1, & (2.1.25\text{d}) \end{cases}$$

with $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$, and

$$\tilde{F}(\psi) = -\text{div}(\psi \nabla r) + \rho\psi = -\nabla\psi \cdot \nabla r + \{\rho - \Delta r\}\psi, \qquad (2.1.26)$$

which is a first-order operator of the same type as the original $F(\psi)$ when $\Delta r \in L_\infty(Q)$. In this case, Theorem 2.1.5 proves the major estimate responsible for the continuous observability inequality of (2.1.25). After absorption of the lower order terms by compactness/uniqess, we arrive at the following result.

THEOREM 2.1.7. (Continuous observability, exact controllability, Neumann case) Let $T > 0$. Let F be as in (2.1.11) with $\Delta r \in L_\infty(Q)$. Let the homogeneous, over-determined problem (2.1.25a-b-c), along with $\psi|_\Sigma \equiv 0$ admit the unique solution $\psi \equiv 0$. Then, with reference to problem (2.1.25), there exists a positive constant $C_T > 0$ such that the following continuous observability inequality holds true

$$\int_0^T \int_{\Gamma_1} |\psi_t|^2 d\Gamma_1 dt \geq C_T E(0). \tag{2.1.27}$$

Equivalently, problem (2.1.24) is exactly controllable (to, from) the origin on the space $Y = H^1_{\Gamma_0}(\Omega)$ within the class of $L_2(0,T; L_2(\Gamma_1))$-controls, with $T > 0$ arbitrary. Specifically, given $w_0 \in Y$ and $v_0 \in Y$ there exists $u \in L_2(0,T; L_2(\Gamma_1))$ such that the corresponding solution of (2.1.24) satisfies $w(T, \,\cdot\,) = v_0$.

□

2.2 Proof of Theorem 2.2.1: Carleman Estimates

In order to handle a general *first-order* operator F in Eqn. (2.1.1) [i.e., at the "energy level"], a major conceptual and technical jump is called for over the energy method, which is used either for the reverse regularity inequality [L–T.2], or else for the continuous observability inequality with F of *order zero* [L–T.2]. Such definitely non-trivial extension is based on the more sophisticated main multiplier

$$e^{\tau\phi(x,t)} \nabla\phi(x,t) \cdot \nabla\bar{w}(t,x), \tag{2.2.1}$$

where ϕ is the pseudo-convex function introduced in (2.1.4), and τ is a positive free parameter of adjustment. Such parameter τ is eventually chosen sufficiently large, as to absorb an energy level term with a large negative constant $\left(-\frac{1}{\epsilon}\right)$ in front, which arises due to the fact that $F(w)$ is first order: see the key step described in Remark 2.2.1 below. A second multiplier is $[\bar{w}\ \mathrm{div}(e^{\tau\phi}\nabla\phi)]$, see (2.2.16). Henceforth, we shall write freely and interchangeably $h(x) = \nabla\phi$ as in (2.1.5).

Step 1. Theorem 2.2.1. Assume (2.1.2). Let w be a solution of Eqn. (2.1.1) in the class (2.1.6), (2.1.7). Then, the following identity holds true, where $\Sigma = [0,T] \times \Gamma$; $Q = [0,T] \times \Omega$; $h = \nabla\phi$; and $\nu =$ an outward unit vector on Γ:

$$\begin{aligned}
&\mathrm{Re}\left(\int_\Sigma e^{\tau\phi}\frac{\partial w}{\partial\nu} h\cdot\nabla\bar{w}\, d\Sigma\right) - \frac{1}{2}\int_\Sigma e^{\tau\phi}|\nabla w|^2 h\cdot\nu\, d\Sigma \\
&\quad + \frac{1}{2}\int_\Sigma \frac{\partial w}{\partial\nu}\bar{w}\ \mathrm{div}(e^{\tau\phi}h)d\Sigma - \frac{i}{2}\int_\Sigma \bar{w}w_t e^{\tau\phi} h\cdot\nu\, d\Sigma \\
&= \quad 2\int_Q e^{\tau\phi}|\nabla w|^2 dQ + \tau\int_Q e^{\tau\phi}|h\cdot\nabla w|^2 dQ \\
&\quad -\mathrm{Re}\Big(\int_Q [F(w)+f]e^{\tau\phi}h\cdot\nabla\bar{w}\Big)\, dQ + \frac{1}{2}\int_Q \bar{w}\nabla w\cdot\nabla(\mathrm{div}(e^{\tau\phi}h))dQ \\
&\quad -\frac{1}{2}\int_Q [F(w)+f]\bar{w}\ \mathrm{div}(e^{\tau\phi}h)dQ \\
&\quad +\frac{i}{2}\int_Q \bar{w}\frac{d(e^{\tau\phi})}{dt}h\cdot\nabla w\, dQ - \frac{i}{2}\left[\int_\Omega \bar{w}\, e^{\tau\phi}h\cdot\nabla w\, d\Omega\right]_0^T. \qquad (2.2.2)
\end{aligned}$$

Proof. (a) We multiply both sides of Eqn. (2.1.1) by the multiplier $e^{\tau\phi}\nabla\phi\cdot\nabla\bar{w}$ in (2.2.1). We shall show that

(i)

$$\begin{aligned}
ia \ &= \int_\Sigma e^{\tau\phi}\frac{\partial w}{\partial\nu}h\cdot\nabla\bar{w}\, d\Sigma - \frac{1}{2}\int_\Sigma e^{\tau\phi}|\nabla w|^2 h\cdot\nu\, d\Sigma \\
&\quad -2\int_Q e^{\tau\phi}|\nabla w|^2 dQ + \frac{1}{2}\int_Q |\nabla w|^2\ \mathrm{div}(e^{\tau\phi}h)dQ \\
&\quad -\tau\int_Q e^{\tau\phi}|h\cdot\nabla w|^2 dQ + \int_Q [F(w)+f]e^{\tau\phi}h\cdot\nabla\bar{w}\, dQ, \quad (2.2.3)
\end{aligned}$$

where we have set

$$a = \int_0^T\int_\Omega w_t e^{\tau\phi}h\cdot\nabla\bar{w}\, dQ; \qquad (2.2.4)$$

(ii)

$$
\begin{aligned}
ia - i\bar{a} = -2(\mathrm{Im}\ a) \quad &= \quad i\int_\Sigma \bar{w}\, w_t e^{\tau\phi} h\cdot\nu\, d\Sigma \\
&\quad + i\int_Q \bar{w}\frac{d(e^{\tau\phi})}{dt} h\cdot\nabla w\, dQ \\
&\quad - i\int_Q \bar{w}\, w_t\ \mathrm{div}(e^{\tau\phi}h) dQ \\
&\quad - i\left[\int_\Omega \bar{w}\, e^{\tau\phi} h\cdot\nabla w\, d\Omega\right]_0^T. \qquad (2.2.5)
\end{aligned}
$$

Proof of (i). On the right-hand side of Eqn. (2.1.1) we compute by Green first theorem

$$
\begin{aligned}
\mathrm{R.H.S.}_1 \quad &= \quad \int_0^T \int_\Omega \Delta w\, e^{\tau\phi}\nabla\phi\cdot\nabla\bar{w}\, d\Omega\, dt \\
&= \quad \int_0^T \int_\Gamma e^{\tau\phi}\frac{\partial w}{\partial\nu}(\nabla\phi\cdot\nabla\bar{w}) d\Sigma \\
&\quad - \int_0^T \int_\Omega e^{\tau\phi}\nabla w\cdot\nabla(\nabla\phi\cdot\nabla\bar{w}) dQ \\
&\quad - \int_0^T \int_\Omega \nabla w\cdot\nabla(e^{\tau\phi})(\nabla\phi\cdot\nabla\bar{w}) dQ, \qquad (2.2.6)
\end{aligned}
$$

where we recall from [L–T.1, Eqn. (2.2.16)] that

$$
\begin{aligned}
&- \int_0^T \int_\Omega e^{\tau\phi}\nabla w\cdot\nabla(\nabla\phi\cdot\nabla\bar{w}) d\Omega\, dt \\
&= \quad -2\int_Q e^{\tau\phi}|\nabla w|^2 dQ - \frac{1}{2}\int_\Sigma e^{\tau\phi}|\nabla w|^2 h\cdot\nu\, d\Sigma \\
&\quad + \frac{1}{2}\int_Q |\nabla w|^2\ \mathrm{div}(e^{\tau\phi}h) dQ. \qquad (2.2.7)
\end{aligned}
$$

Indeed, with $\nabla\phi = h$, we recall the identity

$$
\nabla w\cdot\nabla(h\cdot\nabla\bar{w}) = H\nabla w\cdot\nabla\bar{w} + \frac{1}{2}h\cdot\nabla(|\nabla w|^2) \qquad (2.2.8)
$$

from [L–T.1, Appendix A], where in our present case the matrix $H = 2$ (Identity) by $h(x) = 2(x - x_0)$ in (2.1.5). Once (2.2.8) is inserted into the left-hand side of (2.2.7), we invoke the standard identity

$$\int_\Omega k \cdot \nabla\psi \, d\Omega = \int_\Gamma \psi k \cdot \nu \, d\Gamma - \int_\Omega \psi \text{ div } k \, d\Omega, \tag{2.2.9}$$

with $\psi = |\nabla w|^2$, $k = h$ and we thus arrive at the expression on the right-hand side of (2.2.7). Substituting (2.2.7) for the term before the last in (2.2.6) yields

$$\begin{aligned} \text{R.H.S.}_1 \quad = \quad & \int_\Sigma e^{\tau\phi} \frac{\partial w}{\partial \nu} \nabla\phi \cdot \nabla\bar{w} \, d\Sigma - \frac{1}{2} \int_\Sigma e^{\tau\phi} |\nabla w|^2 h \cdot \nu \, d\Sigma \\ & -2 \int_Q e^{\tau\phi} |\nabla w|^2 dQ + \frac{1}{2} \int_Q |\nabla w|^2 \text{ div}(e^{\tau\phi} h) dQ \\ & -\tau \int_Q e^{\tau\phi} |\nabla\phi \cdot \nabla w|^2 dQ. \end{aligned} \tag{2.2.10}$$

Setting

$$\text{R.H.S.}_2 = \int_Q [F(w) + f] e^{\tau\phi} \nabla\phi \cdot \nabla\bar{w} \, dQ, \tag{2.2.11}$$

we then see that multiplication by $e^{\tau\phi}\nabla\phi \cdot \nabla\bar{w}$ of both sides of Eqn. (2.1.1) and integration over Q has resulted into the identity: L.H.S. = R.H.S.$_1$ + R.H.S.$_2$, where L.H.S. $= ia$, a defined by (2.2.4), and R.H.S.$_1$ and R.H.S.$_2$ given by (2.2.10) and (2.2.11), respectively. This identity becomes, explicitly, (2.2.3), as desired.

Proof of (ii). Using identity (2.2.9) with $k = [w_t e^{\tau\phi} \nabla\phi]$, $\nabla\phi = h$, and $\psi = \bar{w}$, yields

$$\begin{aligned} \text{L.H.S.} = ia \quad = \quad & i \int_0^T \int_\Omega w_t e^{\tau\phi} \nabla\phi \cdot \nabla\bar{w} \, d\Omega \, dt \\ = \quad & i \int_0^T \int_\Gamma \bar{w} \, w_t e^{\tau\phi} h \cdot \nu \, d\Sigma - i \int_\Omega \int_0^T \bar{w} e^{\tau\phi} h \cdot \nabla w_t dQ \\ & -i \int_0^T \int_\Omega \bar{w} \, w_t \text{ div}(e^{\tau\phi} h) dQ, \end{aligned} \tag{2.2.12}$$

since $\operatorname{div}(w_t e^{\tau\phi} h) = e^{\tau\phi} h \cdot \nabla w_t + w_t \operatorname{div}(e^{\tau\phi} h)$. Integrating by parts in t in the second integral on the right-hand side of (2.2.12), we obtain

$$\begin{aligned}
\text{L.H.S.} = ia &= i \int_Q w_t e^{\tau\phi} h \cdot \nabla \bar{w}\, dQ \\
&= i \int_\Sigma \bar{w}\, w_t e^{\tau\phi} h \cdot \nu\, d\Sigma - i \left[\int_\Omega \bar{w} e^{\tau\phi} h \cdot \nabla w\, d\Omega \right]_0^T \\
&\quad + i\bar{a} + i \int_Q \bar{w} \frac{d(e^{\tau\phi})}{dt} h \cdot \nabla w\, dQ \\
&\quad - i \int_Q \bar{w}\, w_t \operatorname{div}(e^{\tau\phi} h) dQ. \qquad (2.2.13)
\end{aligned}$$

Moving $i\bar{a}$ on the left-hand side of identity (2.2.13) yields (2.2.5), as desired. Thus (i) and (ii) are proved.

(b) We shall now refine identity (2.2.5) in (ii) above and obtain (iii)

$$\begin{aligned}
-\operatorname{Im} a &= -\frac{1}{2} \int_\Sigma \frac{\partial w}{\partial \nu} \bar{w} \operatorname{div}(e^{\tau\phi} h) d\Sigma + \frac{i}{2} \int_\Sigma \bar{w}\, w_t e^{\tau\phi} h \cdot \nu\, d\Sigma \\
&\quad + \frac{1}{2} \int_Q |\nabla w|^2 \operatorname{div}(e^{\tau\phi} h) dQ + \frac{i}{2} \int_Q \bar{w} \frac{d(e^{\tau\phi})}{dt} h \cdot \nabla w\, dQ \\
&\quad + \frac{1}{2} \int_Q \bar{w}\, \nabla w \cdot \nabla(\operatorname{div}(e^{\tau\phi} h)) dQ \\
&\quad - \frac{1}{2} \int_Q [F(w) + f] \bar{w} \operatorname{div}(e^{\tau\phi} h) dQ \\
&\quad - \frac{i}{2} \left[\int_\Omega \bar{w}\, e^{\tau\phi} h \cdot \nabla w\, d\Omega \right]_0^T. \qquad (2.2.14)
\end{aligned}$$

Proof of (iii). We return to (2.2.5) already proved and rewrite its last integral term over Q. If $m = m(x,t)$ is a real function in $C^1(Q)$, we may verify the identity

$$i \int_Q w_t \bar{w}\, m\, dQ \;=\; \int_\Sigma \frac{\partial w}{\partial \nu} \bar{w}\, m\, d\Sigma + \int_Q [F(w) + f] \bar{w}\, m\, dQ$$

$$- \int_Q |\nabla w|^2 m\, dQ - \int_Q \bar{w} \nabla w \cdot \nabla m\, dQ. \tag{2.2.15}$$

Indeed, we multiply both sides of Eqn. (2.1.1) by $\bar{w}m$, we use on the right-hand side the Green's first identity, and we obtain (2.2.15). Specializing (2.2.15) to the choice $m = \operatorname{div}(e^{\tau\phi}h)$, we obtain

$$i \int_Q \bar{w}\, w_t \operatorname{div}(e^{\tau\phi}h) dQ \;=\; \int_\Sigma \frac{\partial w}{\partial \nu} \bar{w} \operatorname{div}(e^{\tau\phi}h) d\Sigma$$

$$+ \int_Q [F(w) + f] \bar{w} \operatorname{div}(e^{\tau\phi}h) dQ$$

$$- \int_Q |\nabla w|^2 \operatorname{div}(e^{\tau\phi}h) dQ$$

$$- \int_Q \bar{w} \nabla w \cdot \nabla(\operatorname{div}(e^{\tau\phi}h)) dQ. \tag{2.2.16}$$

Substituting (2.2.16) for the last integral term over Q on the right-hand side of (2.2.5) yields (2.2.14), as desired.

(c) We return to identity (2.2.3): we rewrite its left-hand side as $ia = i\{\text{Re } a + i \text{ Im } a\} = (-\text{Im } a) + i(\text{Re } a)$. Thus, equating $(-\text{Im } a)$, as given explicitly by (2.2.14), to the *real part* of Eqn. (2.2.3), yields the desired identity (2.2.2), after a cancellation of the term

$$\frac{1}{2} \int_Q |\nabla w|^2 \operatorname{div}(e^{\tau\phi}h) dQ.$$

The proof of Theorem 2.2.1 is complete. □

Step 2. Lemma 2.2.2. Let w be a solution of Eqn. (2.1.1) in the class (2.1.6), (2.1.7).

(i) With reference to the first three $\int_Q$-terms on the right-hand side of Eqn. (2.2.2), we have for any $\epsilon > 0$:

$$
\begin{aligned}
&\Big| 2\int_Q e^{\tau\phi}|\nabla w|^2 dQ + \tau \int_Q e^{\tau\phi}|h\cdot\nabla w|^2 dQ \\
&\qquad - \mathrm{Re}\Big(\int_Q [F(w)+f] e^{\tau\phi} h\cdot\nabla\bar{w}\, dQ\Big)\Big| \\
\geq\ & (2-\epsilon C_T)\int_Q e^{\tau\phi}|\nabla w|^2 dQ + \Big(\tau - \frac{1}{2\epsilon}\Big)\int_Q e^{\tau\phi}|h\cdot\nabla w|^2 dQ \\
-\ & \epsilon C_{\phi,T}\|w\|^2_{C([0,T];L_2(\Omega))} - \epsilon\int_Q e^{\tau\phi}|f|^2 dQ. \qquad (2.2.17)
\end{aligned}
$$

(ii) Regarding the last three $\int_Q$-terms in (2.2.2), we have for any $\epsilon > 0$:

$$
\begin{aligned}
&\Big|\frac{1}{2}\int_Q \bar{w}\,\nabla w\cdot\nabla(\mathrm{div}(e^{\tau\phi}h))dQ \\
&-\frac{1}{2}\int_Q [F(w)+f]\bar{w}\ \mathrm{div}(e^{\tau\phi}h)dQ + \frac{i}{2}\int_Q \bar{w}\frac{d(e^{\tau\phi})}{dt}h\cdot\nabla w\, dQ\Big| \\
&\geq\ -\epsilon\int_Q |\nabla w|^2 e^{\tau\phi} dQ \\
&\qquad - \frac{C_{\phi,T}}{\epsilon}\int_Q |w|^2 e^{\tau\phi} dQ - \epsilon\int_Q |f|^2 e^{\tau\phi} dQ. \qquad (2.2.18)
\end{aligned}
$$

Proof. (i) We compute, recalling (2.1.2),

$$
\begin{aligned}
|[F(w)+f]e^{\tau\phi}h\cdot\nabla\bar{w}| &\geq -\frac{\epsilon}{2}|F(w)+f|^2 e^{\tau\phi} - \frac{1}{2\epsilon}e^{\tau\phi}|h\cdot\nabla w|^2 \\
\text{(by (2.1.2))} \qquad &\geq -\epsilon C_T|\nabla w|^2 e^{\tau\phi} - \epsilon C_T|w|^2 e^{\tau\phi} \\
&\quad -\epsilon|f|^2 e^{\tau\phi} - \frac{1}{2\epsilon}e^{\tau\phi}|h\cdot\nabla w|^2. \qquad (2.2.19)
\end{aligned}
$$

Using (2.2.19) on the left-hand side of (2.2.17) yields the right-hand side of (2.2.17).

(ii) Similarly, we use the inequality $2ab \leq \epsilon a^2 + \frac{1}{\epsilon}b^2$, as well as (2.1.2), where a denotes "energy level" terms: ∇w, $F(w)$, as well as f, and b denotes lower-order terms, i.e., w. $\square$

REMARK 2.2.1. In the third integral over Q on the left-hand side of (2.2.17), both factors $F(w)$ and $h \cdot \nabla \bar{w}$ are energy level, with F a general first-order operator. The virtue of the free parameter τ is seen in the second term on the right-hand side of (2.2.17), in making the coefficient $\tau - \frac{1}{2\epsilon} > 0$, after ϵ has been fixed, and dropping that term, see next Theorem 2.2.4. In part (ii), Eqn. (2.2.23) and (iii) Eqn. (2.2.24) of Theorem 2.2.4 below, we obtain the desired estimates (2.1.8) and (2.1.9) of Theorem 2.1.1. □

LEMMA 2.2.3. Let w be a solution of Eqn. (2.1.1) in the class (2.1.6), (2.1.7). With reference to the last term on the right-hand side of (2.2.2),

$$\beta_{0,T} = -\frac{i}{2}\left[\int_\Omega \bar{w}\, e^{\tau\phi} h \cdot \nabla w \, d\Omega\right]_0^T, \tag{2.2.20}$$

we have for any $\epsilon > 0$,

$$\begin{aligned} |\beta_{0,T}| \quad \geq \quad & -\epsilon e^{-\delta\tau} \int_\Omega \left[|\nabla w(T)|^2 + |\nabla w(0)|^2\right] d\Omega \\ & - \frac{C_\phi}{\epsilon} e^{-\delta\tau} \int_\Omega \left[|w(T)|^2 + |w(0)|^2\right] d\Omega. \end{aligned} \tag{2.2.21}$$

Proof. We use property (2.1.4c) for $\phi(x,0)$ and $\phi(x,T)$. □

Step 4. Theorem 2.2.4. Assume (2.1.2). Let w be a solution of Eqn. (2.1.1) in the class (2.1.6), (2.1.7). Then

(i) the following inequality holds true, with $\tau = \frac{1}{\epsilon}$, $\epsilon > 0$ as in Lemma 2.2.2,

$$\begin{aligned} (BT)|_\Sigma + \frac{2}{\tau}\int_Q |f|^2 e^{\tau\phi} dQ + TC_{\phi,\tau,T}\|w\|^2_{C([0,T];L_2(\Omega))} \\ \geq \quad \left(2 - \frac{C_T}{\tau} - \frac{1}{\tau}\right)\int_Q e^{\tau\phi}|\nabla w|^2 dQ \\ + \frac{\tau}{2}\int_Q e^{\tau\phi}|h \cdot \nabla w|^2 dQ + \beta_{0,T}, \end{aligned} \tag{2.2.22}$$

where C_T is the constant in (2.1.2), and where we have set (in agreement with (2.1.10)),

$$
\begin{aligned}
(BT)|_\Sigma &= \operatorname{Re}\left(\int_\Sigma e^{\tau\phi}\frac{\partial w}{\partial\nu}\nabla\phi\cdot\nabla\bar{w}\,d\Sigma\right) \\
&\quad -\frac{1}{2}\int_\Sigma e^{\tau\phi}|\nabla w|^2 h\cdot\nu\,d\Sigma \\
&\quad +\left|\frac{1}{2}\int_\Sigma\frac{\partial w}{\partial\nu}\bar{w}\ \operatorname{div}(e^{\tau\phi}h)d\Sigma\right. \\
&\qquad\left.-\frac{i}{2}\int_\Sigma\bar{w}\,w_t e^{\tau\phi}h\cdot\nu\,d\Sigma\right|. \qquad (2.2.23)
\end{aligned}
$$

(ii) Inequality (2.2.22) may be made more explicit in the following form: for τ sufficiently large,

$$
\begin{aligned}
&(BT)|_\Sigma+\frac{2}{\tau}\int_Q|f|^2e^{\tau\phi}dQ+TC_{\phi,T,\tau}\|w\|^2_{C([0,T];L_2(\Omega))} \\
&\quad\geq\left(2-\frac{C_T}{\tau}-\frac{1}{\tau}\right)\int_Q e^{\tau\phi}|\nabla w|^2dQ \\
&\qquad-\frac{e^{-\delta\tau}}{\tau}[E(T)+E(0)], \qquad (2.2.24)
\end{aligned}
$$

with $E(t)$ as in (2.1.3).

(iii) Recalling (2.1.4b) for ϕ, estimate (2.2.24) implies: for τ sufficiently large

$$
\begin{aligned}
&(BT)|_\Sigma+\frac{2}{\tau}\int_Q|f|^2e^{\tau\phi}dQ+TC_{\phi,T,\tau}\|w\|^2_{C([0,T];L_2(\Omega))} \\
&\quad\geq\left(2-\frac{C_T}{\tau}-\frac{1}{\tau}\right)e^{\tau}\int_{t_0}^{t_1}E(t)dt \\
&\qquad-\frac{e^{-\delta\tau}}{\tau}[E(T)+E(0)]. \qquad (2.2.25)
\end{aligned}
$$

Proof. (i) On the right-hand side of the fundamental identity (2.2.2), we use (2.2.17), (2.2.18), and thus obtain inequality (2.2.22) at once, with τ chosen as $\tau = \frac{1}{\epsilon}$, $\epsilon > 0$ as in Lemma 2.2.2, and $BT|_\Sigma$ as in (2.2.23).

(ii), (iii) On the right-hand side of (2.2.22) we drop the last (positive) $\int_Q$-term [the benefit of having chosen τ large enough]; we invoke estimate (2.2.21) for $|\beta_{0,T}|$, and thus get (2.2.24). Moreover, to obtain (2.2.25) from (2.2.24), we recall property (2.1.4b) for ϕ.

$\square$

2.3 Proof of Theorem 2.1.2

Here we shall refine the basic estimate (2.1.8) = (2.2.24) of Theorem 2.1.1 in terms of only $E(T)$, or only $E(0)$, by examining the behavior of $E(t)$ in (2.1.3). To this end, we shall need the additional assumption that the coefficients (in $L_\infty(\bar{Q})$) of $F(w)$ be *real*, and as assumed in (2.1.11). Moreover, we shall require that $f \in L_2(0,T;H^1(\Omega))$.

Step 1. Lemma 2.3.1. Assume (2.1.11). Let w be a solution of Eqn. (2.1.1) in the class (2.1.6), (2.1.7), and let $f \in L_2(0,T;H^1(\Omega))$. Then, with reference to $E(t)$ defined by (2.1.3), we have

(i) for all t, s:

$$\begin{aligned} E(t) \quad = \quad & E(s) + 2\mathrm{Re}\left(\int_s^t \int_\Gamma \frac{\partial w}{\partial \nu} \bar{w}_t \, d\Gamma \, d\sigma\right) \\ & + 2\mathrm{Re}\left(\int_s^t \int_\Omega [F(w) + f]\bar{w}_t \, d\Omega \, d\sigma\right); \quad (2.3.1) \end{aligned}$$

(ii) with μ a unit tangential vector on Γ:

$$\begin{aligned} \mathrm{Re}\left(\int_s^t \int_\Omega F(w)\bar{w}_t \, d\Omega \, d\sigma\right) \; = \; & \mathrm{Re}\left(i\int_s^t \int_\Gamma (\nabla r \cdot \mu) \frac{\partial w}{\partial \mu} \, \frac{\partial \bar{w}}{\partial \nu} d\Gamma \, d\sigma\right) \\ & - \mathrm{Re}\left(i\int_s^t \int_\Omega R\nabla \bar{w} \cdot \nabla w \, d\Omega \, d\sigma\right) \\ & + \mathrm{Re}\left(i\int_s^t \int_\Omega (\nabla r \cdot \nabla w)(\rho\bar{w} + \bar{f}) d\Omega \, d\sigma\right) \\ & + \frac{1}{2}\left[\int_\Omega \rho|w|^2 d\Omega\right]_s^t - \frac{1}{2}\int_\Omega \int_s^t \rho_t |w|^2 dQ; \qquad (2.3.2) \end{aligned}$$

$$R \equiv \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial r_n}{\partial x_1} & \cdots & \frac{\partial r_n}{\partial x_n} \end{bmatrix}. \tag{2.3.3}$$

(iii)

$$\begin{aligned} \mathrm{Re}\left(\int_s^t \int_\Omega f \bar{w}_t \, d\Omega \, d\sigma\right) &= \\ &= \mathrm{Re}\left(i \int_s^t \int_\Gamma f \, \frac{\partial \bar{w}}{\partial \nu} d\Gamma \, d\sigma\right) \\ &\quad - \mathrm{Re}\left(i \int_s^t \int_\Omega \nabla f \cdot \nabla \bar{w} \, d\Omega \, d\sigma\right) \\ &\quad + \mathrm{Re}\left(i \int_s^t \int_\Omega f [\nabla r \cdot \nabla \bar{w} + \rho \bar{w}] d\Omega \, d\sigma\right) \end{aligned} \tag{2.3.4}$$

Proof. (i) We multiply Eqn. (2.1.1) by $\bar{w}_t$ and integrate over $(s,t) \times \Omega$, obtaining by virtue of Green's first identity

$$\begin{aligned} 0 = \mathrm{Re}\left(i \int_s^t \int_\Omega |w_t|^2 \, d\Omega \, d\sigma\right) &= \\ &= \mathrm{Re}\left(\int_s^t \int_\Gamma \frac{\partial w}{\partial \nu} \bar{w}_t d\Gamma \, d\sigma\right) \\ &\quad - \mathrm{Re}\left(\int_s^t \int_\Omega \nabla w \cdot \nabla \bar{w}_t \, d\Omega \, d\sigma\right) \\ &\quad + \mathrm{Re}\left(\int_s^t \int_\Omega [F(w) + f] \bar{w}_t d\Omega \, d\sigma\right). \end{aligned} \tag{2.3.5}$$

By (2.1.3),

$$\begin{aligned} E(t) - E(s) &= \int_\Omega \int_s^t \frac{\partial}{\partial \sigma}(|\nabla w|^2) d\sigma \, d\Omega \\ &= 2 \, \mathrm{Re}\left(\int_s^t \int_\Omega \nabla w \cdot \nabla \bar{w}_t \, d\Omega \, d\sigma\right). \end{aligned} \tag{2.3.6}$$

Inserting (2.3.6) for the second term on the right-hand side of (2.3.5) yields (2.3.1), as desired.

(ii) Recalling $F(w) = \nabla r \cdot \nabla w + \rho w,\ r, \rho$ real, from (2.1.11), we have

$$\int_s^t \int_\Omega F(w)\bar{w}_t d\Omega\, d\sigma = \int_s^t \int_\Omega \rho w \bar{w}_t\, d\Omega\, d\sigma + \int_s^t \int_\Omega (\nabla r \cdot \nabla w)\bar{w}_t d\Omega\, d\sigma. \tag{2.3.7}$$

As to the first term on the right-hand side of (2.3.7), we compute after integration by parts in σ

$$\mathrm{Re}\left(\int_s^t \int_\Omega \rho w \bar{w}_t d\Omega\, d\sigma\right) = \frac{1}{2}\left[\int_\Omega \rho |w|^2 d\Omega\right]_s^t - \frac{1}{2}\int_s^t \int_\Omega \rho_t |w|^2 d\Omega\, d\sigma. \tag{2.3.8}$$

As to the second term on the right-hand side of (2.3.7), we compute with $\bar{w}_t = i\Delta\bar{w} + i\bar{F}(w) + i\bar{f}$, by use of Green's first identity:

$$\begin{aligned}
\int_s^t \int_\Omega (\nabla r \cdot \nabla w)\bar{w}_t\, d\Omega\, d\sigma \ &= \\
= \ & \int_s^t \int_\Omega (\nabla r \cdot \nabla w)[i\Delta\bar{w} + i\bar{F}(w) + i\bar{f}]d\Omega\, d\sigma \\
= \ & i\int_s^t \int_\Gamma (\nabla r \cdot \nabla w)\frac{\partial \bar{w}}{\partial \nu}\, d\Gamma\, d\sigma - i\int_s^t \int_\Omega \nabla(\nabla r \cdot \nabla w)\cdot \nabla\bar{w}\, d\Omega\, d\sigma \\
& + i\int_s^t \int_\Omega (\nabla r \cdot \nabla w)[\nabla r \cdot \nabla \bar{w} + \rho\bar{w} + \bar{f}]d\Omega\, d\sigma,
\end{aligned} \tag{2.3.9}$$

recalling (2.1.11) in the last step. But, invoking (2.2.8), we have

$$\nabla(\nabla r \cdot \nabla w)\cdot\nabla\bar{w} = R\nabla\bar{w}\cdot\nabla w + \frac{1}{2}r \cdot \nabla(|\nabla w|^2), \tag{2.3.10}$$

where R is the $n \times n$ matrix defined by (2.3.3). Proceeding as in the argument from (2.2.8) to (2.2.10), i.e., using (2.2.9) with $\psi = |\nabla w|^2$, we obtain via (2.3.10),

$$\begin{aligned}
-i\int_s^t \int_\Omega \nabla(\nabla r \cdot \nabla w)\cdot\nabla\bar{w}\, d\Omega\, d\sigma \ &= \\
&= \ -i\int_s^t \int_\Omega \left[R\nabla\bar{w}\cdot\nabla w + \frac{1}{2}r\cdot\nabla(|\nabla w|^2)\right] d\Omega\, d\sigma \\
\text{(by (2.2.9))} \quad &= \ -i\int_s^t \int_\Omega R\nabla\bar{w}\cdot\nabla w\, d\Omega\, d\sigma
\end{aligned}$$

$$- \frac{i}{2}\int_s^t\int_\Gamma |\nabla w|^2 r\cdot\nu\, d\Gamma\, d\sigma$$
$$+ \frac{i}{2}\int_s^t\int_\Omega |\nabla w|^2 \text{ div } r\, d\Omega\, d\sigma. \tag{2.3.11}$$

Substituting (2.3.11) for the second term on the right-hand side of (2.3.9), we obtain, since the last two terms in (2.3.11), as well as one term in (2.3.9), have Re part equal to zero:

$$\begin{aligned}
\text{Re}\left(\int_s^t\int_\Omega (\nabla r\cdot\nabla w)\bar{w}_t d\Omega\, d\sigma\right) &= \\
&= \text{Re}\left(i\int_s^t\int_\Gamma (\nabla r\cdot\nabla w)\frac{\partial \bar{w}}{\partial\nu} d\Gamma\, d\sigma\right) \\
&\quad - \text{Re}\left(i\int_s^t\int_\Omega R\nabla\bar{w}\cdot\nabla w\, d\Omega\, d\sigma\right) \\
&\quad + \text{Re}\left(i\int_s^t\int_\Omega (\nabla r\cdot\nabla w)(\rho\bar{w}+\bar{f})\, d\Omega\, d\sigma\right).
\end{aligned} \tag{2.3.12}$$

Finally, with reference to the first (boundary) term on the right-hand side of (2.3.12), we have, with μ a tangential unit vector on Γ,

$$\nabla r = (\nabla r\cdot\nu)\nu + (\nabla r\cdot\mu)\mu,\ \nabla r\cdot\nabla w = (\nabla r\cdot\nu)\frac{\partial w}{\partial\nu} + (\nabla r\cdot\mu)\frac{\partial w}{\partial\mu}, \tag{2.3.13}$$

and hence

$$\begin{aligned}
\text{Re}\left(i\int_s^t\int_\Gamma (\nabla r\cdot\nabla w)\frac{\partial \bar{w}}{\partial\nu}\, d\Gamma\, d\sigma\right) &= \\
&= \text{Re}\left(i\int_s^t\int_\Gamma \left|\frac{\partial w}{\partial\nu}\right|^2 \nabla r\cdot\nu\, d\Gamma\, d\sigma\right) \\
&\quad + \text{Re}\left(i\int_s^t\int_\Gamma (\nabla r\cdot\mu)\frac{\partial w}{\partial\mu}\ \frac{\partial\bar{w}}{\partial\nu}\, d\Gamma\, d\sigma\right) \\
&= \text{Re}\left(i\int_s^t\int_\Gamma (\nabla r\cdot\mu)\frac{\partial w}{\partial\mu}\ \frac{\partial\bar{w}}{\partial\nu}\, d\Gamma\, d\sigma\right),
\end{aligned} \tag{2.3.14}$$

since the first term has real part zero. Substituting (2.3.14) for the first term on the right-hand side of (2.3.12), we obtain

$$\begin{aligned}
\operatorname{Re}\left(\int_s^t \int_\Omega (\nabla r \cdot \nabla w)\bar{w}_t \, d\Omega \, d\sigma\right) &= \\
= \ & \operatorname{Re}\left(i \int_s^t \int_\Gamma (\nabla r \cdot \mu) \frac{\partial w}{\partial \mu} \frac{\partial \bar{w}}{\partial \nu} \, d\Gamma \, d\sigma\right) \\
& - \operatorname{Re}\left(i \int_s^t \int_\Omega R \nabla \bar{w} \cdot \nabla w \, d\Omega \, d\sigma\right) \\
& + \operatorname{Re}\left(i \int_s^t \int_\Omega (\nabla r \cdot \nabla w)(\rho \bar{w} + \bar{f}) \, d\Omega \, d\sigma\right). \quad (2.3.15)
\end{aligned}$$

In conclusion, taking the Real part of (2.3.7) and invoking (2.3.8) and (2.3.15) yields (2.3.2), as desired.

(iii) As in (ii), we compute via Green's first identity and (2.1.11):

$$\begin{aligned}
\int_s^t \int_\Omega f \bar{w}_t \, d\Omega \, d\sigma &= \int_s^t \int_\Omega f [i \Delta \bar{w} + i \bar{F}(w) + i \bar{f}] \, d\Omega \, d\sigma \\
&= i \int_s^t \int_\Gamma f \frac{\partial \bar{w}}{\partial \nu} \, d\Gamma \, d\sigma - i \int_s^t \int_\Omega \nabla f \cdot \nabla \bar{w} \, d\Omega \, d\sigma \\
&\quad + i \int_s^t \int_\Omega f [\nabla r \cdot \nabla \bar{w} + \rho \bar{w}] d\Omega \, d\sigma \\
&\quad + i \int_s^t \int_\Omega |f|^2 d\Omega \, d\sigma. \qquad (2.3.16)
\end{aligned}$$

Taking the Real part of (2.3.16) yields (2.3.4). □

Step 2. Lemma 2.3.2. Let $f \in L_2(0, T; H^1(\Omega))$ and assume (2.1.11). Let w be a solution of Eqn. (2.1.1) in the class (2.1.6), (2.1.7). Then

(i) for any $\epsilon > 0$ and $t \geq s \geq 0$,

$$\begin{aligned}
&\left| \operatorname{Re}\left(\int_s^t \int_\Omega [F(w) + f] \bar{w}_t d\Omega \, d\sigma \right| \right. \\
&\leq \ (\|R\|_\infty + \epsilon) \int_s^t E(\sigma) d\sigma \\
&\quad + C \int_0^T \int_\Gamma \left| \frac{\partial w}{\partial \mu} \ \frac{\partial \bar{w}}{\partial \nu} \right| d\Gamma \, dt + \int_0^T \int_\Gamma \left| f \frac{\partial \bar{w}}{\partial \nu} \right| d\Gamma \, d\sigma
\end{aligned}$$

$$+ C_{T,\epsilon,\rho,r}\|w\|^2_{C([s,t];L_2(\Omega))} + \frac{C_{r,\rho}}{\epsilon}\|f\|^2_{L_2(s,t;H^1(\Omega))} \quad (2.3.17)$$

where $\|R\|_\infty$ is the L_∞-norm over Q of R in (2.3.3).

(ii) With reference to $E(t)$ in (2.1.3), the following inequalities hold true for $T \geq t > 0$:

$$e^{-kt}E(0) - \Lambda(T) \leq E(t) \leq [E(0) + \Lambda(T)]e^{kt}, \quad (2.3.18)$$

where $k = 2(\|R\|_\infty + \epsilon)$ and

$$\begin{aligned}\Lambda(T) \;=\; & 2C\int_0^T \int_\Gamma \left|\frac{\partial w}{\partial \mu}\;\frac{\partial \bar{w}}{\partial \nu}\right| d\Gamma\, dt + 2\int_0^T \int_\Gamma \left| f\frac{\partial \bar{w}}{\partial \nu}\right| d\Gamma\, d\sigma \\ & +2\int_0^T \int_\Gamma \left|\frac{\partial w}{\partial \nu}\,\bar{w}_t\right| d\Gamma\, d\sigma + C_{T,\epsilon,\rho,r}\|w\|^2_{C([0,T];L_2(\Omega))} \\ & +\frac{C_{r,\rho}}{\epsilon}\|f\|^2_{L_2(0,T;H^1(\Omega))}. \end{aligned} \quad (2.3.19)$$

Proof. (i) Identities (2.3.2) and (2.3.4) are estimated by $2|ab| \leq \epsilon|a|^2 + \frac{1}{\epsilon}|b|^2$, with $|a|$ being an energy term, i.e., $|\nabla w|$. One readily obtains (2.3.17).

(ii) We return to identity (2.3.1) and use estimate (2.3.17) to obtain for $t \geq s \geq 0$ via (2.3.19):

$$E(t) \;\leq\; [E(s) + \Lambda(T)] + k\int_s^t E(\sigma)d\sigma; \quad (2.3.20)$$

$$E(s) \;\leq\; [E(t) + \Lambda(T)] + k\int_s^t E(\sigma)d\sigma. \quad (2.3.21)$$

We next apply the classical argument of the Gronwall's inequality to (2.3.20) and (2.3.21), where we note that the terms in the brackets are independent of t in (2.3.20), and independent of s in (2.3.21). We thus obtain for $t \geq s \geq 0$:

$$E(t) \leq [E(s)+\Lambda(T)]e^{k(t-s)}; \quad E(s) \leq [E(t)+\Lambda(T)]e^{k(t-s)}. \quad (2.3.22)$$

Setting $s = 0$ and thus taking $t > 0$ in (2.3.22) yields (2.3.18). □

Step 3. Theorem 2.3.3. Let $f \in L_2(0,T;H^1(\Omega))$, and let w be a solution of Eqn. (2.1.1) in the class (2.1.6), (2.1.7). Finally, let the coefficients ∇r and ρ of F be as in (2.1.11). Then, the following inequality holds true, with τ sufficiently large:

$$\left\{\left(2-\frac{C_T}{\tau}-\frac{1}{\tau}\right)C_{t_0,t_1}e^{\tau}-\frac{1}{\tau}e^{-\delta\tau}(1+e^{kT})\right\}E(0)$$

$$\leq \; c_T(BT_1)|_\Sigma+\frac{2}{\tau}\int_Q|f|^2e^{\tau\phi}dQ+\frac{C}{\epsilon}\|f\|^2_{L_2(0,T;H^1(\Omega))}$$

$$+TC_{\phi,T,\tau,\epsilon}\|w\|^2_{C([0,T];L_2(\Omega))}, \tag{2.3.23}$$

where the coefficient $\{\quad\}$ of $E(0)$ may be made positive for τ large,

$$\begin{aligned}(BT_1)|_\Sigma \;&=\; (BT)|_\Sigma+\int_0^T\int_\Gamma\left|\frac{\partial w}{\partial\mu}\;\frac{\partial\bar w}{\partial\nu}\right|d\Gamma\,dt\\ &\quad+\int_0^T\int_\Gamma\left|f\frac{\partial\bar w}{\partial\nu}\right|d\Gamma\,dt+\int_0^T\int_\Gamma\left|\frac{\partial w}{\partial\nu}\,\bar w_t\right|d\Gamma\,dt\end{aligned} \tag{2.3.24a}$$

$$\begin{aligned}(\text{by (2.2.23)}) \;&=\; \mathrm{Re}\left(\int_\Sigma e^{\tau\phi}\frac{\partial w}{\partial\nu}h\cdot\nabla\bar w\,d\Sigma\right)-\frac{1}{2}\int_\Sigma e^{\tau\phi}|\nabla w|^2h\cdot\nu\,d\Sigma\\ &\quad+\left|\frac{1}{2}\int_\Sigma\frac{\partial w}{\partial\nu}\bar w\;\mathrm{div}(e^{\tau\phi}h)d\Sigma-\frac{i}{2}\int_\Sigma\bar w\,w_te^{\tau\phi}h\cdot\nu\,d\Sigma\right|\\ &\quad+\int_0^T\int_\Gamma\left|\frac{\partial w}{\partial\mu}\;\frac{\partial\bar w}{\partial\nu}\right|d\Gamma\,dt+\int_0^T\int_\Gamma\left|\frac{\partial w}{\partial\nu}\bar w_t\right|d\Gamma\,d\sigma\\ &\quad+\int_0^T\int_\Gamma\left|f\frac{\partial\bar w}{\partial\nu}\right|d\Gamma\,d\sigma.\end{aligned} \tag{2.3.24b}$$

Proof. We return to inequality (2.2.25): on the right-hand side we use the estimates of $E(\cdot)$ in (2.3.18). We thus obtain by the left-hand inequality in (2.3.18),

$$\int_{t_0}^{t_1}E(t)dt\geq\int_{t_0}^{t_1}\left[e^{-kt}E(0)-\Lambda(T)\right]dt=C_{t_0,t_1}E(0)-(t_1-t_0)\Lambda(T). \tag{2.3.25}$$

Moreover, by the right-hand inequality in (2.3.18) we have

$$E(T)+E(0)\leq[1+e^{kT}]E(0)+e^{kT}\Lambda(T). \tag{2.3.26}$$

Inserting (2.3.25) and (2.3.26) on the right-hand side of (2.2.25) yields (2.3.23) by virtue of (2.3.19) on $\Lambda(T)$ via (2.3.24). $\square$

Estimate (2.3.23) of Theorem 2.3.3 proves Theorem 2.1.2, Equation (2.1.12). Using $E(0) \geq e^{-kT}E(T) - \Lambda(T)$ from (2.3.18) once more, yields

$$E(0) = \frac{E(0)}{2} + \frac{E(0)}{2} \geq \frac{e^{-kT}}{2}[E(0) + E(T)] - \frac{\Lambda(T)}{2}, \qquad (2.3.27)$$

which inserted on the left-hand side of (2.3.23) produces estimate (2.1.13) in Theorem 2.1.2. Thus, Theorem 2.1.2, part (i), is fully proved. $\square$

COROLLARY 2.3.4. Let $f \in L_2(0,T; H^1_{\Gamma_0}(\Omega))$. Let w be a solution of Eqn. (2.1.1) in the class (2.1.6), (2.1.7), which, moreover, satisfies the boundary condition

$$w|_{\Sigma_0} \equiv 0, \text{ where } \Gamma_0 = \{x \in \Gamma : \nabla\phi \cdot \nu \leq 0\}, \qquad (2.3.28)$$

see (2.1.16). Let the coefficients ∇r and ρ of F be as in (2.1.11). Then the following inequality holds true for τ sufficiently large: there is a positive constant $k_{\phi,\tau} > 0$ such that, if $(BT_1)|_{\Sigma_1}$ are the boundary terms (BT_1) in (2.3.24) evaluated, however, on $\Sigma_1 = (0,T] \times \Gamma_1$, $\Gamma_1 = \Gamma \setminus \Gamma_0$, then

$$\begin{aligned} k_{\phi,\tau}E(0) \;\leq\; & c_T(BT_1)|_{\Sigma_1} + \frac{2}{\tau}\int_Q |f|^2 e^{\tau\phi} dQ \\ & + \frac{C}{\epsilon}\|f\|^2_{L_2(0,T;H^1(\Omega))} + TC_{\phi,T,\tau}\|w\|^2_{C([0,T];L_2(\Omega))} \end{aligned} \qquad (2.3.29)$$

Proof. We split the boundary terms (BT_1) in (2.3.24b) on Σ_0 and on Σ_1. On Σ_0, the boundary condition (2.3.28) and the assumption $f|_{\Gamma_0} = 0$ make all terms vanish, except the first two of them. ($w|_{\Sigma_0} \equiv 0 \Rightarrow \frac{\partial w}{\partial \mu}|_{\Sigma_0} \equiv 0$ since μ is tangential). Moreover, we have on Σ_0 : $h \cdot \nabla \bar{w} = \frac{\partial \bar{w}}{\partial \nu} h \cdot \nu$; $|\nabla w| = \left|\frac{\partial w}{\partial \nu}\right|$, where $h = \nabla\phi$ by (2.1.5), and thus

$$(BT_1)|_{\Sigma} \;=\; (BT_1)|_{\Sigma_0} + (BT_1)|_{\Sigma_1}$$

$$
\begin{aligned}
&= \int_{\Sigma_0} e^{\tau\phi} \frac{\partial w}{\partial \nu} \frac{\partial \bar{w}}{\partial \nu} h \cdot \nu \, d\Sigma_0 \\
&\quad - \frac{1}{2} \int_{\Sigma_0} e^{\tau\phi} |\nabla w|^2 h \cdot \nu \, d\Sigma_0 + (BT_1)|_{\Sigma_1} \\
&= \frac{1}{2} \int_{\Sigma_0} e^{\tau\phi} \left| \frac{\partial w}{\partial \nu} \right|^2 h \cdot \nu \, d\Sigma_0 + (BT_1)|_{\Sigma_1} \qquad (2.3.30) \\
&\leq (BT_1)|_{\Sigma_1}, \qquad (2.3.31)
\end{aligned}
$$

recalling $h \cdot \nu \leq 0$ on Γ_0 via (2.1.16) = (2.3.28). Inserting (2.3.31) into the right-hand side of (2.1.12) = (2.3.23) yields (2.3.29), as desired. □

Theorem 2.3.3 and Corollary 2.3.4 prove part (i) and part (ii), respectively, of Theorem 2.1.2.

2.4 Proof of Theorem 2.1.3: Continuous Observability Inequality, Dirichlet case

Step 1. Proposition 2.4.1. For the solution of the problem

$$
\begin{cases}
i\psi_t = \Delta\psi + F(\psi) & \text{in } (0, T] \times \Omega; \qquad (2.4.1a) \\
\psi(0, \cdot\,) = \psi_0 & \text{in } \Omega; \qquad (2.4.1b) \\
\psi|_\Sigma \equiv 0 & \text{in } (0, T] \times \Gamma, \qquad (2.4.1c)
\end{cases}
$$

where F is the first-order differential operator as in (2.1.11), and $T > 0$ arbitrary, the following estimate holds true: for τ sufficiently large, there is a positive constant $C_{\phi,\tau} > 0$ such that

$$
C_{\phi,\tau} E(0) \leq \int_0^T \int_{\Gamma_1} \left| \frac{\partial \psi}{\partial \nu} \right|^2 d\Gamma_1 dt + \text{const}_\tau \|\psi\|^2_{C([0,T];L_2(\Omega))}. \qquad (2.4.2)
$$

Proof. As in the proof of Corollary 2.3.4, Eqn. (2.3.30), we obtain by the B.C. (2.4.1c) used in (2.3.24b),

$$
(BT_1)|_\Sigma = \text{Re} \left(\int_\Sigma e^{\tau\phi} \frac{\partial \psi}{\partial \nu} h \cdot \nabla \bar{\psi} \, d\Sigma \right) - \frac{1}{2} \int_\Sigma e^{\tau\phi} |\nabla \psi|^2 h \cdot \nu \, d\Sigma
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\Sigma} e^{\tau\phi}\left|\frac{\partial\psi}{\partial\nu}\right|^2 h\cdot\nu\, d\Sigma \\
&\le C_{\phi,\tau}\int_0^T\int_{\Gamma_1}\left|\frac{\partial\psi}{\partial\nu}\right|^2 d\Gamma_1 dt, \qquad (2.4.3)
\end{aligned}
$$

where in the last step we have recalled the definition (2.1.16) = (2.3.28) of Γ_0. Setting then $f \equiv 0$ in (2.3.23) yields (2.4.2) by (2.4.3). □

Step 2. (Absorption of the lower-order term)

LEMMA 2.4.2. Let ψ be a solution of problem (2.4.1) with F as in (2.1.11) and with $\psi_0 \in H_0^1(\Omega)$, so that inequality (2.4.2) holds true. Let the homogeneous, over-determined problem defined by (2.4.1a–c), as well as $\left.\frac{\partial\psi}{\partial\nu}\right|_{\Sigma_1} \equiv 0$ on $(0,T]\times\Gamma_1 \equiv \Sigma_1$ admits the unique solution $\psi \equiv 0$. Then, there exists a constant $c_T > 0$ such that

$$
\|\psi\|^2_{C([0,T];L_2(\Omega))} \le c_T\int_0^T\int_{\Gamma_1}\left|\frac{\partial\psi}{\partial\nu}\right|^2 d\Gamma_1 dt.
$$

Proof. This follows by a compactness/uniqueness argument, e.g., [L–T.2, Lemma 3.3 or Lemma 4.1]. □

2.5 Proof of Theorem 2.1.4: Absorption of Tangential Derivatives

The proof of Theorem 2.1.4 for the Schrödinger equation (2.1.1) is a minor modification of the proof of a similar result for second-order hyperbolic equations [L–T.3, Section 7.2]. Assuming [L–T.3] at hand, we shall limit ourselves here to note the necessary modifications. The proof is given for the corresponding half-space problem $\Omega = R^1_{x+}\times R^1_y$ with $\Gamma = R_y^{n-1} = \Omega|_{x=0}$ its boundary. Equation (2.1.11) is then rewritten as

$$
Pw = f \qquad (0,\infty)\times\Omega = Q_\infty \qquad (2.5.1)
$$

via partition of unity, where the characteristic polynomial of P is

$$
\begin{aligned}
p(x,y;\tau,\xi,\eta) &= -a\tau + \sum_{i,j=1}^{n-1} a_{ij}\eta_i\eta_j + 2\xi \sum_{j=1}^{n-1} a_{nj}\eta_j + \xi^2 \\
&= -a\tau + \left(\xi + \sum_{j=1}^{n-1} a_{nj}\eta_j\right)^2 \\
&\quad + \sum_{i,j=1}^{n-1} a_{ij}\eta_i\eta_j - \left(\sum_{j=1}^{n-1} a_{nj}\eta_j\right)^2 \\
&= -a\tau + \tilde{\xi}^2 + \frac{d(x,y;\eta)}{a^2(x,y)}, \qquad (2.5.2)
\end{aligned}
$$

in the notation of [L–T.3, Section 7.2].

Here the quadratic form in η, as in [L–T.3, Eqn. (5.5)],

$$
\begin{aligned}
d(x,y;\eta) &= a^2(x,y)\left\{\sum_{i,j=1}^{n-1} a_{ij}(x,y)\eta_i\eta_j - \left(\sum_{j=1}^{n-1} a_{nj}(x,y)\eta_j\right)^2\right\} \\
&\geq c|\eta|^2, \qquad c>0, \qquad (2.5.3)
\end{aligned}
$$

is positive definite uniformly in $(x,y) \in \Omega$. With $\tau = \sigma - i\gamma$, $\gamma > 0$, $\sigma \in R^1$, the Laplace variable corresponding to $t: \ D_t \to \tau$, and $\eta \in R^{n-1}$ the Fourier variable corresponding to $y: \ D_y \to \eta$, we define the cones [L–T.3, Eqn. (5.15)–(5.17)],

$$\mathcal{R}_1 = \left\{(x,y;\sigma;\eta) \in R^{2n}(+) : \frac{3}{4} m|\eta| < \sigma\right\}, \qquad (2.5.4)$$

$$\mathcal{R}_{\mathrm{tr}} = \left\{(x,y;\sigma;\eta) \in R^{2n}(+) : \frac{m}{2}|\eta| \leq \sigma \leq \frac{3}{4} m|\eta|\right\}, \qquad (2.5.5)$$

$$\mathcal{R}_2 = \left\{(x,y;\sigma;\eta) \in R^{2n}(+) : \sigma < \frac{m}{2}|\eta|\right\}. \qquad (2.5.6)$$

Then, *a fortiori* over the hyperbolic case in [L–T.3, Eqn. (5.18), (6.17)],

$$\mathcal{R}_{\mathrm{tr}} \cup \mathcal{R}_2 \subset \text{ elliptic cone } = \{(x,y;\sigma,\eta) \in R^{2n}(+) : \ \sigma < m|\eta|\}, \tag{2.5.7}$$

and

$$|p(x,y;\tau;\xi,\eta)| \geq c[\tilde{\xi}^2 + |\eta|^2 + \sigma^2] \text{ in } \mathcal{R}_{\mathrm{tr}} \cup \mathcal{R}_2, \tag{2.5.8}$$

so that p is elliptic of order 2 in all variables in $\mathcal{R}_{\mathrm{tr}} \cup \mathcal{R}_2$. Defining $w_c(t,\,\cdot\,) = \psi(t)w(t,\,\cdot\,)$ where $\psi \in C_0^\infty(R)$ is identically equal to 1 on $[\epsilon, T-\epsilon]$ and vanishes outside $(0,T)$, we obtain from (2.5.1),

$$Pw_c = [P,\psi]w + \psi f, \quad [P,\psi]w = ia\psi' w \tag{2.5.9}$$

counterpart of [L–T.3, Eqn. (7.8)]. We then obtain the counterpart of [L–T.3, Lemma 7.2].

THEOREM 2.5.1. Let w be a solution of (2.5.1). Let $\frac{\partial}{\partial y}$ be the tangential derivative on $\Gamma = \Omega|_{x=0}$. Then, the following estimate holds true: given $\epsilon > 0$ and $\epsilon_0 > 0$ arbitrarily small, and given $T > 0$, there exists a constant $C_{\epsilon,\epsilon_0,T} > 0$ such that

$$\begin{aligned} \left\| \frac{\partial w}{\partial y} \right\|_{L_2([\epsilon,T-\epsilon]\times\Gamma)} \leq \ & C_{\epsilon,\epsilon_0,T} \Big\{ \|\tilde{D}_x w\|_{L_2(\Sigma_T)} + \|w_t\|_{L_2(\Sigma_T)} \\ & + \|w\|_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} + \|f\|_{H^{-\frac{1}{2}+\epsilon_0}(Q_T)} \Big\}, \end{aligned} \tag{2.5.10}$$

where $\tilde{D}_x$ is the operator corresponding to the symbol $\tilde{\xi}_x$, so that $\tilde{D}_x$, restricted on Γ, coincides with the co-normal operator. □

Proof. The proof follows as in [L–T.3, Section 7.2]. In synthesis: if $\chi = \chi(x,y;\sigma,\eta)$ is the symbol of order zero of localizaiton on $\mathcal{R}_1$ and $\mathcal{X}$ the corresponding pseudo-differential operator, then $\mathcal{X}w_c$ obeys elliptic estimates in all variables. Instead, $(1-\mathcal{X})w_c$ has its tangential derivative $\frac{\partial}{\partial y}(1-\mathcal{X})w_c$ dominated in the $L_2(\Sigma_\infty)$-norm by its time derivative $(1-\mathcal{X})(w_c)_t$, since in the region $\mathcal{R}_1 \cup \mathcal{R}_{\mathrm{tr}}$, where $\mathrm{supp}(1-\chi)$ lies, we have $\sigma \geq \frac{m}{2}|\eta|$. Details are as in [L–T.3, Section 7.2.]. □

Theorem 2.1.4 is the version of Theorem 2.5.1 just proved corresponding to a bounded smooth domain.

3. Proof of Theorems 1.1 and 1.2

We shall heavily rely on the results in the preceding Section 2 for the simple Eqn. (2.1.1). Theorem 1.1 is here restated as Theorem 3.1. The counterpart of Theorem 2.1.1 is:

Step 1. Theorem 3.1. (Carleman estimates) Assume (1.3) and (1.4). Let w, z be solutions of Eqns. (1.1) and (1.2) in the class (1.6), (1.7). Let $\phi(x,t)$ be the pseudo-convex function defined by (2.1.4). Then, for all τ sufficiently large, the following one-parameter family of estimates hold true:

$$\begin{aligned}\left(2-\frac{3C_T}{\tau}-\frac{1}{\tau}\right)\int_Q e^{\tau\phi}[|\nabla w|^2+|\nabla z|^2]dQ-\frac{e^{-\delta\tau}}{\tau}[E(T)+E(0)]\\ \leq\ BT(w)|_\Sigma+BT(z)|_\Sigma\\ +C_{T,\phi,\tau}\left[\|w\|^2_{C([0,T];L_2(\Omega))}+\|z\|^2_{C([0,T];L_2(\Omega))}\right],\end{aligned}\tag{3.1}$$

where the boundary terms $BT(w)|_\Sigma$ are defined by (1.9) [or (2.1.10)], and similarly for $BT(z)|_\Sigma$ with respect to z. Finally, $E(t)$ is defined by (1.5).

Proof. We apply the Carleman estimates Eqn. (2.1.8) of Theorem 2.1.1 to the w-equation (1.1) with $f=P_1(z)$ and, respectively, to the z-equation (1.2) with $f=P_2(w)$. We thus obtain, respectively,

$$\begin{aligned}\left(2-\frac{C_T}{\tau}-\frac{1}{\tau}\right)\int_Q e^{\tau\phi}|\nabla w|^2dQ-\frac{e^{-\delta\tau}}{\tau}[E_w(T)+E_w(0)]\\ \leq\ BT(w)|_\Sigma+\frac{2}{\tau}\int_Q e^{\tau\phi}|P_1(z)|^2dQ\\ +C_{T,\phi,\tau}\|w\|^2_{C([0,T];L_2(\Omega))};\end{aligned}\tag{3.2}$$

$$\left(2-\frac{C_T}{\tau}-\frac{1}{\tau}\right)\int_Q e^{\tau\phi}|\nabla z|^2dQ-\frac{e^{-\delta\tau}}{\tau}[E_z(T)+E_z(0)]$$

$$\leq \quad BT(z)|_\Sigma + \frac{2}{\tau}\int_Q e^{\tau\phi}|P_2(w)|^2 dQ$$

$$+ C_{T,\phi,\tau}\|z\|^2_{C([0,T];L_2(\Omega))}, \tag{3.3}$$

for τ sufficiently large. We next recall the pointwise bounds (1.4) for $P_1(z)$ and $P_2(w)$ in, respectively, (3.2) and (3.3), to obtain

$$\left(2 - \frac{C_T}{\tau} - \frac{1}{\tau}\right)\int_Q e^{\tau\phi}|\nabla w|^2 dQ - \frac{e^{-\delta\tau}}{\tau}[E_w(T) + E_w(0)]$$

$$\leq \quad BT(w)|_\Sigma + \frac{2C_T}{\tau}\int_Q e^{\tau\phi}[|\nabla z|^2 + |z|^2]dQ$$

$$+C_{T,\phi,\tau}\left\{\|w\|^2_{C([0,T];L_2(\Omega))} + \|z\|^2_{C([0,T];L_2(\Omega))}\right\}; \tag{3.4}$$

$$\left(2 - \frac{C_T}{\tau} - \frac{1}{\tau}\right)\int_Q e^{\tau\phi}|\nabla z|^2 dQ - \frac{e^{-\delta\tau}}{\tau}[E_z(T) + E_z(0)]$$

$$\leq \quad BT(z)|_\Sigma + \frac{2C_T}{\tau}\int_Q e^{\tau\phi}[|\nabla w|^2 + |w|^2]dQ$$

$$+C_{T,\phi,\tau}\left\{\|z\|^2_{C([0,T];L_2(\Omega))} + \|w\|^2_{C([0,T];L_2(\Omega))}\right\}. \tag{3.5}$$

Next, we sum up (3.4) and (3.5) and move from the right to the left-hand side the energy terms involving $|\nabla z|^2$ and $|\nabla w|^2$, with coefficient $\frac{2C_T}{\tau}$, and we readily obtain (3.1), as desired. □

We next specialize F_1, F_2, P_1, P_2 as in (1.10), (1.11). The counterpart of Lemma 2.3.2(ii) is:

Step 2. Proposition 3.2. Assume (1.10) and (1.11). Let w, z be solutions of Eqns. (1.1) and (1.2) in the class (1.6), (1.7). Then, with reference to (1.5), the following inequalities hold true for $T \geq t > 0$:

$$e^{-\tilde{k}t}E(0) - \tilde{\Lambda}(T) \leq E(t) \leq [E(0) + \tilde{\Lambda}(T)]e^{\tilde{k}t}, \tag{3.6}$$

where $\tilde{k} = 2\max\left\{\|R_1\|_\infty + \epsilon, \|R_2\|_\infty + \epsilon, \frac{CC_T}{\epsilon}\right\}$, see below, and

$$\begin{aligned}
\tilde{\Lambda}(T) &= 2\int_0^T \int_\Gamma \left|\frac{\partial w}{\partial \nu}\,\bar{w}_t\right| d\Gamma\,dt + 2C\int_0^T \int_\Gamma \left|\frac{\partial z}{\partial \nu}\,\bar{z}_t\right| d\Gamma\,dt \\
&\quad +2C\int_0^T \int_\Gamma \left|\frac{\partial w}{\partial \mu}\,\frac{\partial \bar{w}}{\partial \nu}\right| d\Gamma\,dt + 2C\int_0^T \int_\Gamma \left|\frac{\partial z}{\partial \mu}\,\frac{\partial \bar{z}}{\partial \nu}\right| d\Gamma\,dt \\
&\quad +C_T\int_0^T \int_\Gamma \left|z\,\frac{\partial \bar{w}}{\partial \nu}\right| d\Gamma\,dt + C_T\int_0^T \int_\Gamma \left|w\,\frac{\partial \bar{z}}{\partial \nu}\right| d\Gamma\,dt \\
&\quad +C_{T,\epsilon}\left[\|w\|^2_{C([0,T];L_2(\Omega))} + \|z\|^2_{C([0,T];L_2(\Omega))}\right]. \qquad (3.7)
\end{aligned}$$

Proof. We return to identity (2.3.1) which we write for the w-equation (1.1) with $f = P_1(z)$ and, respectively, for the z-equation (1.2) with $f = P_2(w)$. We obtain, respectively, for all s, t:

$$\begin{aligned}
E_w(t) &= E_w(s) + 2\ \mathrm{Re}\left(\int_s^t \int_\Gamma \frac{\partial w}{\partial \nu}\,\bar{w}_t\,d\Gamma\,d\sigma\right) \\
&\quad +2\ \mathrm{Re}\left(\int_s^t \int_\Omega [F_1(w) + P_1(z)]\bar{w}_t\,d\Omega\,d\sigma\right); \qquad (3.8)
\end{aligned}$$

$$\begin{aligned}
E_z(t) &= E_z(s) + 2\ \mathrm{Re}\left(\int_s^t \int_\Gamma \frac{\partial z}{\partial \nu}\,\bar{z}_t\,d\Gamma\,d\sigma\right) \\
&\quad +2\ \mathrm{Re}\left(\int_s^t \int_\Omega [F_2(z) + P_2(w)]\bar{z}_t\,d\Omega\,d\sigma\right). \qquad (3.9)
\end{aligned}$$

Summing up (3.8) and (3.9) yields by (1.5),

$$\begin{aligned}
E(t) &= E(s) + 2\ \mathrm{Re}\left(\int_s^t \int_\Gamma \frac{\partial w}{\partial \nu}\,\bar{w}_t\,d\Gamma\,d\sigma\right) \\
&\quad + 2\ \mathrm{Re}\left(\int_s^t \int_\Gamma \frac{\partial z}{\partial \nu}\,\bar{z}_t\,d\Gamma\,d\sigma\right)
\end{aligned}$$

$$+ 2\ \mathrm{Re}\left(\int_s^t \int_\Omega [F_1(w) + P_1(z)]\bar{w}_t \, d\Omega \, d\sigma\right)$$

$$+ 2\ \mathrm{Re}\left(\int_s^t \int_\Omega [F_2(z) + P_2(w)]\bar{z}_t \, d\Omega \, d\sigma\right). \quad (3.10)$$

On the other hand, recalling (2.3.17) as applied to the w-equation (1.1), we have for any $\epsilon > 0$, and $t \geq s > 0$:

$$\left|\mathrm{Re}\left(\int_s^t \int_\Omega [F_1(w) + P_1(z)]\bar{w}_t \, d\Omega \, d\sigma\right| \leq (\|R_1\|_\infty + \epsilon) \int_s^t E_w(\sigma) d\sigma\right.$$

$$+ \quad C \int_0^T \int_\Gamma \left|\frac{\partial w}{\partial \mu}\,\frac{\partial \bar{w}}{\partial \nu}\right| d\Gamma \, dt + \int_0^T \int_\Gamma \left|P_1(z)\,\frac{\partial \bar{w}}{\partial \nu}\right| d\Gamma \, d\sigma$$

$$+ \quad C_{T,\epsilon}\|w\|^2_{C([0,T];L_2(\Omega))} + \frac{C}{\epsilon}\|P_1(z)\|^2_{L_2(s,t;H^1(\Omega))}, \quad (3.11)$$

where R_1 is defined by (2.3.3) with r replaced by r_1. Similarly, (2.3.17), as applied to the z-equation (1.2), gives for $\epsilon > 0$ and $t \geq s > 0$,

$$\left|\mathrm{Re}\left(\int_s^t \int_\Omega [F_2(z) + P_2(w)]\bar{z}_t d\Omega \, d\sigma\right) \leq (\|R_2\|_\infty + \epsilon) \int_s^t E_z(\sigma) d\sigma\right.$$

$$+ \quad C \int_0^T \int_\Gamma \left|\frac{\partial z}{\partial \mu}\,\frac{\partial \bar{z}}{\partial \nu}\right| d\Gamma \, dt + \int_0^T \int_\Gamma \left|P_2(w)\,\frac{\partial \bar{z}}{\partial \nu}\right| d\Gamma \, d\sigma$$

$$+ \quad C_{T,\epsilon}\|z\|^2_{C([0,T];L_2(\Omega))} + \frac{C}{\epsilon}\|P_2(w)\|^2_{L_2(s,T;H^1(\Omega))}, \quad (3.12)$$

where μ is a unit tangential vector on Γ.

Using the assumptions (1.11d) on $P_1(z)$ and $P_2(w)$, we obtain

$$\|P_1(z)\|^2_{L_2(s,t;H^1(\Omega))} + \|P_2(w)\|^2_{L_2(s,t;H^1(\Omega))} \leq$$

$$\leq \quad C_T \int_s^t \int_\Omega \left[|\nabla z|^2 + |\nabla w|^2 + |z|^2 + |w|^2\right] d\Omega \, d\sigma$$

$$\text{(by (1.5))} \quad \leq \quad C_T \int_s^t E(\sigma) d\sigma + c_T \Big[\|z\|^2_{C([0,T];L_2(\Omega))} + \|w\|^2_{C([0,T];L_2(\Omega))} \Big]. \tag{3.13}$$

Summing up (3.11) and (3.12) and using (3.13) results in

$$\begin{aligned} & \left| \mathrm{Re}\left(\int_s^t \int_\Omega [F_1(w) + P_1(z)] \bar{w}_t d\Omega \, d\sigma \right) \right| \\ & \quad + \left| \mathrm{Re}\left(\int_s^t \int_\Omega [F_2(z) + P_1(w)] \bar{z}_t d\Omega \, d\sigma \right) \right| \leq \\ & \qquad \leq \frac{\tilde{k}}{2} \int_s^t E(\sigma) d\sigma \\ & \qquad +C \int_0^T \int_\Gamma \left| \frac{\partial w}{\partial \mu} \frac{\partial \bar{w}}{\partial \nu} \right| d\Gamma \, dt + \int_0^T \int_\Gamma \left| P_1(z) \frac{\partial \bar{w}}{\partial \nu} \right| d\Gamma \, d\sigma \\ & \qquad +C \int_0^T \int_\Gamma \left| \frac{\partial z}{\partial \mu} \frac{\partial \bar{z}}{\partial \nu} \right| d\Gamma \, dt + \int_0^T \int_\Gamma \left| P_2(w) \frac{\partial \bar{z}}{\partial \nu} \right| d\Gamma \, d\sigma \\ & \qquad +C_{T,\epsilon} \left[\|w\|^2_{C([0,T];L_2(\Omega))} + \|z\|^2_{C([0,T];L_2(\Omega))} \right], \end{aligned} \tag{3.14}$$

where $\tilde{k}$ is a suitable constant defined below (3.6). Since inequalities (1.11c) hold true also on Σ, we then obtain by substituting (3.14) into (3.10),

$$E(t) \leq [E(s) + \tilde{\Lambda}(T)] + \tilde{k} \int_s^t E(\sigma) d\sigma, \quad t \geq s \geq 0, \tag{3.15}$$

where $\tilde{\Lambda}(T)$ is defined by (3.7). Similarly, from (3.10),

$$E(s) \leq [E(t) + \tilde{\Lambda}(T)] + \tilde{k} \int_s^t E(\sigma) d\sigma, \quad t \geq s \geq 0. \tag{3.16}$$

Thus, we obtain the counterpart of Eqns. (2.3.20) and (2.3.21) in the case of one equation (2.1.1). We then obtain for $t \geq s > 0$,

$$E(t) \leq [E(s) + \tilde{\Lambda}(T)] e^{\tilde{k}(t-s)}; \quad E(s) \leq [E(t) + \tilde{\Lambda}(T)] e^{\tilde{k}(t-s)}, \tag{3.17}$$

i.e., the counterpart of (2.3.22). Setting $s = 0$ in (3.17) yields (3.6), as desired. □

The counterpart of Theorem 2.3.3 is:

Step 3. Theorem 3.3. Assume (1.10) and (1.11). Let w, z be solutions of Eqns. (1.1) and (1.2) in the class (1.6), (1.7). Then, with reference to (1.5), the following inequality holds true, with τ sufficiently large, and $\epsilon_0 > 0$ arbitrarily small > 0:

$$\left\{ \left(2 - \frac{3C_T}{\tau} - \frac{1}{\tau} \right) C_{t_0,t_1} e^{\tau} - \frac{e^{-\delta\tau}}{\tau} \left(1 + e^{\tilde{k}T} \right) \right\} E(0)$$

$$\leq \; (BT(w,z))|_{\Sigma} + \text{const}_{T,\tau} \Big\{ \|w\|^2_{C([0,T];L_2(\Omega))} + \\ + \|z\|^2_{C([0,T];L_2(\Omega))} \Big\}, \qquad (3.18)$$

where taking τ sufficiently large makes the coefficient $\{ \quad \}$ in front of $E(0)$ positive in (3.18). Moreover, we have set

$$BT(w,z)|_{\Sigma} \equiv BT(w)|_{\Sigma} + BT(z)|_{\Sigma} + \left[(t_1 - t_0) + \frac{e^{-\delta\tau}}{\tau} e^{\tilde{k}T} \right] \tilde{\Lambda}(T), \qquad (3.19)$$

where $BT(w)$ is defined by (1.9) (or (2.1.10)), and similarly for $BT(z)$, while $\tilde{\Lambda}(T)$ is defined by (3.7); [Estimate (3.18) is the counterpart for system (1.1), (1.2) of Eqn. (2.3.23) of Theorem 2.3.3 for the single equation (2.1.1).]

Proof. We proceed as in the proof of Theorem 2.3.3. Using the left-hand side inequality in (3.6), we compute

$$\int_{t_0}^{t_1} E(t)dt \geq \int_{t_0}^{t_1} \left[e^{-\tilde{k}t} E(0) - \tilde{\Lambda}(T) \right] dt = C_{t_0,t_1} E(0) - (t_1 - t_0)\tilde{\Lambda}(T). \qquad (3.20)$$

Moreover, recalling property (2.1.4c) for ϕ and (1.5), we estimate with reference to (3.1)

$$\int_Q e^{\tau\phi} [|\nabla w|^2 + |\nabla z|^2] dQ \geq e^{\tau} \int_{t_0}^{t_1} E(t)dt. \qquad (3.21)$$

Using now the right-hand side inequality in (3.6), we estimate as in (2.3.26),

$$E(T)+E(0)\leq\left[1+e^{\tilde{k}T}\right]E(0)+e^{\tilde{k}T}\tilde{\Lambda}(T). \tag{3.22}$$

We now use (3.20) in (3.21) and substitute the result, along with (3.22), into the left-hand side of (3.1). This way we obtain (3.18). □

Step 4. We now absorb the tangential traces from the boundary terms $BT(w,z)$.

PROPOSITION 3.4. Assume (1.3) and (1.4). Let w, z be solutions of Eqns. (1.1) and (1.2) in the class (1.6), (1.7). Then

(i) given $\epsilon > 0$ and $\epsilon_0 > 0$ arbitrary small, and given $T > 0$, there exists a constant $C_{\epsilon,\epsilon_0,T} > 0$ such that

$$\begin{aligned}&\int_{\epsilon}^{T-\epsilon}\int_{\Gamma}\left[\left|\frac{\partial w}{\partial\mu}\right|^2+\left|\frac{\partial z}{\partial\mu}\right|^2\right]d\Gamma\,dt\\ &\quad\leq\; C_{\epsilon,\epsilon_0,T}\left\{\int_0^T\int_{\Gamma}\left[\left|\frac{\partial w}{\partial\nu}\right|^2+|w_t|^2+\left|\frac{\partial z}{\partial\nu}\right|^2+|z_t|^2\right]d\Gamma\,dt\right.\\ &\qquad\left.+\|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))}+\|z\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))}\right.,\end{aligned} \tag{3.23}$$

where $|\frac{\partial}{\partial\mu}|=|\nabla_{\mu}|=|\text{tangential gradient}|$.

(ii) If, moreover, w and/or z satisfy the boundary condition (1.13) on Σ_0, then the corresponding integral term in (3.23) for w and/or for z replaces Γ with Γ_1. [This result is the counterpart for system (1.1), (1.2) of Eqn. (2.1.20) of Theorem 2.1.4 for the single equation (2.1.1).]

Proof. (i) We apply (2.1.20) of Theorem 2.1.4 to the w-equation (1.1) with $f = P_1(z)$ and, respectively, to the z-equation (1.2) with

$f = P_2(w)$. We obtain

$$\int_{\epsilon}^{T-\epsilon} \int_{\Gamma} \left|\frac{\partial w}{\partial \mu}\right|^2 d\Gamma\, dt \;\le\; C_{\epsilon,\epsilon_0,T}\left\{ \int_0^T \int_{\Gamma} \left[\left|\frac{\partial w}{\partial \nu}\right|^2 + |w_t|^2 \right] d\Sigma \right.$$

$$\left. + \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} + \|P_1(z)\|^2_{H^{-\frac{1}{2}+\epsilon_0}(Q_T)} \right\}; \quad (3.24)$$

$$\int_{\epsilon}^{T-\epsilon} \int_{\Gamma} \left|\frac{\partial z}{\partial \mu}\right|^2 d\Gamma\, dt \;\le\; C_{\epsilon,\epsilon_0,T}\left\{ \int_0^T \int_{\Gamma} \left[\left|\frac{\partial z}{\partial \nu}\right|^2 + |z_t|^2 \right] d\Sigma \right.$$

$$\left. + \|z\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} + \|P_2(w)\|^2_{H^{-\frac{1}{2}+\epsilon_0}(Q_T)} \right\}. \quad (3.25)$$

Next, we sum up (3.24) and (3.25) and use

$$\|P_1(z)\|_{H^{-\frac{1}{2}+\epsilon_0}(Q_T)} \;\le\; C_T\|z\|_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))}; \qquad (3.26)$$

$$\|P_2(w)\|_{H^{-\frac{1}{2}+\epsilon_0}(Q_T)} \;\le\; C_T\|w\|_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))}, \qquad (3.27)$$

since P_1 and P_2 are, by (1.4), first-order operator in the space variables $x_1, \ldots, x_n$, with L_∞-coefficients in time. This way, we obtain (3.23).

(ii) The proof of part (ii) is the same, recalling part (ii) of Theorem 2.1.4.

□

We now complete the proof of Theorem 1.2.

Step 5. Proposition 3.5. Assume (1.10), (1.11). Let w, z be solutions of Eqns. (1.1), (1.2) within the class (1.6), (1.7). Then

(i) the following inequality holds true for τ sufficiently large: there exists a constant $\text{const}_{\phi,\tau} > 0$ such that, with reference to (1.5),

$$\text{const}_{\phi,\tau} E(0) \le \int_0^T \int_{\Gamma} \left[\left|\frac{\partial w}{\partial \nu}\right|^2 + |w_t|^2 + \left|\frac{\partial z}{\partial \nu}\right|^2 + |z_t|^2 \right] d\Gamma\, dt$$

$$+ \text{const}_{T,\tau,\epsilon_0}\left[\|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} + \|z\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} \right] \quad (3.28)$$

or equivalently,

$$\begin{aligned}
\text{const}_{\phi,\tau}[E(0)+E(T)] \ &\leq \\
&\leq \ \int_0^T \int_\Gamma \left[\left| \frac{\partial w}{\partial \nu} \right|^2 + |w_t|^2 + \left| \frac{\partial z}{\partial \nu} \right|^2 + |z_t|^2 \right] d\Gamma \, dt \\
&\quad + \text{const}_{T,\tau,\epsilon_0} \Big[\|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} \\
&\qquad + \|z\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} \Big]. \qquad (3.29)
\end{aligned}$$

(ii) If, moreover, w and/or z satisfy the boundary condition (1.13) on Σ_0, then the corresponding integral term for w and/or z replaces Γ with Γ_1 defined by (2.1.17). [This result is the counterpart for system (1.1), (1.2) of Eqn. (2.1.21) and (2.1.22) of Theorem 2.1.5 for the single equation (2.1.1).]

Proof. (i) We proceed as in the proof of Theorem 2.1.5 or as in [L–T.3, p. 221]. For fixed $\epsilon > 0$ small we apply estimate (3.18) of Theorem 3.3 over the interval $[\epsilon, T-\epsilon]$, rather than over $[0,T]$. We obtain for $k_{\phi,\tau,\epsilon} > 0$,

$$\begin{aligned}
k_{\phi,\tau,\epsilon} E(\epsilon) \ &\leq \ BT(w,z)|_{[\epsilon,T-\epsilon]\times\Gamma} \\
&\quad + \text{const}_{T,\tau,\epsilon} \Big\{ \|w\|^2_{C([0,T];L_2(\Omega))} \\
&\qquad + \|z\|^2_{C([0,T];L_2(\Omega))} \Big\}. \qquad (3.30)
\end{aligned}$$

Using the left-hand side inequality in (3.6) with $t = \epsilon$ in the left-hand side of (3.30), we obtain

$$\begin{aligned}
k_{\phi,\tau,\epsilon} e^{-\tilde{k}\epsilon} E(0) \ &\leq \ BT(w,z)|_{[\epsilon,T-\epsilon]\times\Gamma} + \tilde{\Lambda}(T) \\
&\quad + \text{const}_{T,\tau,\epsilon} \Big\{ \|w\|^2_{C([0,T];L_2(\Omega))} \\
&\qquad + \|z\|^2_{C([0,T];L_2(\Omega))} \Big\}. \qquad (3.31)
\end{aligned}$$

But by virtue of estimate (3.23) of Proposition 3.4, we readily see via the definition $BT(w,z)$ over the domain $[\epsilon, T-\epsilon] \times \Gamma$ in (3.19) [and the definition of $BT(w)$ or $BT(z)$ in (1.9) (or (2.1.10)], except on $[\epsilon, T-\epsilon] \times \Gamma]$ and via the definition of $\tilde{\Lambda}(T)$ in (3.7) along with trace theory and $|\nabla w|^2 = \left|\frac{\partial w}{\partial \nu}\right|^2 + \left|\frac{\partial w}{\partial \mu}\right|^2$ that:

$$\begin{aligned} BT(w,z)|_{[\epsilon,T-\epsilon]\times\Gamma} + \tilde{\Lambda}(T) \ &\leq \\ &\leq \ C_{\epsilon,\epsilon_0,T}\left\{ \int_0^T \int_\Gamma \left[\left|\frac{\partial w}{\partial \nu}\right|^2 + |w_t|^2 + \left|\frac{\partial z}{\partial \nu}\right|^2 + |z_t|^2 \right] d\Gamma\, dt \right. \\ &\qquad \left. + \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} + \|z\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} \right. . \end{aligned} \tag{3.32}$$

Inserting (3.32) into the right-hand side of (3.31) yields (3.28), as desired.

To prove (3.29), we write as in (2.3.27),

$$E(0) = \frac{E(0)}{2} + \frac{E(0)}{2} \geq \frac{e^{-\tilde{k}T}}{2}[E(0) + E(T)] - \frac{\tilde{\Lambda}(T)}{2}, \tag{3.33}$$

recalling $E(0) \geq e^{-\tilde{k}T}E(T) - \tilde{\Lambda}(T)$ from the right-hand side inequality (3.6). We then insert (3.33) into (3.28) already proved, and we then establish (3.29) again via (3.7) for $\tilde{\Lambda}(T)$ as above. Part (i) of Proposition 3.5 is proved.

For part (ii), we use part (ii) of Proposition 3.4 instead. □

Appendix: Duality maps

The appendix sketches the duality between continuous observability inequalities and corresponding exact controllability results for a single equation.

Dirichlet case. The following two problems

$$\begin{cases} iw_t &=& \Delta w + F(w), \\ w(0,\,\cdot\,) &=& w_0, \\ w|_{\Sigma_0} &\equiv& 0, \\ w|_{\Sigma_1} &=& u, \end{cases} \tag{A.1}$$

and

$$\begin{cases} i\psi_t &=& \Delta\psi + \tilde{F}(\psi) & \text{in } Q, \\ \psi(T,\,\cdot\,) &=& \psi_0 & \text{in } \Omega, \\ \psi|_{\Sigma} &\equiv& 0 & \text{in } \Sigma, \end{cases}$$

where $F(w)$ is given by (2.1.11), and where

$$\tilde{F}(\psi) \equiv -\text{div}(\psi\nabla r) + \rho\psi = -\nabla\psi\cdot\nabla r + (\rho - \Delta r)\psi, \;\; \Delta r \in L_\infty(\bar{Q}), \tag{A.2}$$

are dual of each other in the following sense: the map

$$\begin{cases} u \in L_2(0,T;L_2(\Gamma_1)) \\ w_0 = 0 \end{cases} \Rightarrow \mathcal{L}_T u = w(T,\,\cdot\,) \in H^{-1}(\Omega), \tag{A.3}$$

whose regularity always holds true [L–T.2, Thm. 1.1], is dual to the map

$$\psi_0 \in H_0^1(\Omega) \Rightarrow \mathcal{L}_T^*\psi_0 = -i\frac{\partial\psi}{\partial\nu} \in L_2(0,T;L_2(\Gamma_1)). \tag{A.4}$$

This is shown by multiplying the w-equation in (A.1) by $\bar{\psi}$ and integrating in Q by parts, thus arriving at the identity

$$i\int_{\Sigma_1} u\,\frac{\partial\bar{\psi}}{\partial\nu}\,d\Sigma_1 = \int_\Omega w(T)\bar{\psi}(T)d\Omega, \tag{A.5}$$

which proves the assertion in (A.3), (A.4).

Neumann case. The two following problems,

$$\begin{cases} iw_t &= \Delta w + F(w), \\ w(0,\,\cdot\,) &= w_0, \\ w|_{\Sigma_0} &\equiv 0, \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} &= u, \end{cases}$$

and (A.6)

$$\begin{cases} i\psi_t &= \Delta\psi + \tilde{F}(\psi) & \text{in } Q, \\ \psi(T,\,\cdot\,) &= \psi_0 & \text{in } \Omega, \\ \psi|_{\Sigma_0} &\equiv 0 & \text{in } \Sigma, \\ \left[\dfrac{\partial \psi}{\partial \nu} - \psi \dfrac{\partial r}{\partial \nu}\right]_{\Sigma_1} &\equiv 0 & \text{in } \Sigma_1, \end{cases}$$

$\tilde{F}$ as in (A.2), are dual of each other in the following sense:

$$\begin{cases} w_0 = 0 \\ u \in L_2(0,T;L_2(\Gamma_1)) \end{cases} \quad \text{such that } \mathcal{L}_T u = w(T,\cdot) \in H^1_{\Gamma_0}(\Omega), \tag{A.7}$$

is dual to the map

$$\psi_0 \in H^{-1}_{\Gamma_0}(\Omega) \Rightarrow \mathcal{L}^*_T \psi_0 = i\psi|_{\Sigma_1} \in L_2(0,T;L_2(\Gamma_1)), \tag{A.8}$$

equivalently,

$$\psi_0 \in H^1_{\Gamma_0}(\Omega) \Rightarrow \psi_t|_{\Sigma_1} \in L_2(0,T;L_2(\Gamma_1)). \tag{A.9}$$

Indeed, multiplying the w-equation in (A.6) by $\bar{\psi}$ and integrating over Q by parts yields the identity

$$-i\int_{\Sigma_1} u\bar{\psi}\, d\Sigma_1 = \int_\Omega w(T)\bar{\psi}(T) d\Omega, \tag{A.10}$$

which proves the assertion in (A.7)–(A.9).

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