Rend. Istit. Mat. Univ. Trieste
Suppl. Vol. XXVIII, 383–419 (1997)

Hopf's Lemma and Two Dimensional Domains with Corners

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Summary. - *Let $\Omega \subset I\!R^2$ be a bounded domain that is smooth except for a finite number of corners. The aim of this paper is to obtain conditions such that the solution of the Poisson problem $-\Delta u = f \geq 0$ in Ω ($f \neq 0$), with zero Dirichlet boundary condition, behaves similarly at the boundary as does the first eigenfunction ϕ_1. That is: there exist $c_1, c_2 > 0$ such that*

$$c_1 \phi_1 (x) \leq u (x) \leq c_2 \phi_1 (x) .$$

As a consequence we can improve some results for smooth domains to domains with corners, such as: the maximum principle for an equation with a potential which is unbounded near the boundary; the anti-maximum of Clément and Peletier; a Green function estimate of Zhao.

1. Introduction

If Ω is smooth enough and $f \geq 0$ sufficiently regular one finds pointwise estimates for the solution u of

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

by the maximum principle and Hopf's boundary point lemma. Indeed they imply the following. For a nonnegative nonzero right hand

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side $f \in C(\bar{\Omega})$ there is a constant $k > 0$ such that

$$k\ d(x,\Omega) \leq u(x) \qquad \text{for all } x \in \Omega. \tag{2}$$

For smooth Ω regularity theory shows that there is another constant $k' \in (0,\infty)$ such that

$$u(x) \leq k'\ d(x,\Omega) \qquad \text{for all } x \in \Omega, \tag{3}$$

where the distance function $d(\cdot,\Omega)$ is defined by

$$d(x,\Omega) = \inf\{|x-y|\,;y \notin \Omega\}. \tag{4}$$

If there can be no doubt we write $d(x) = d(x,\Omega)$.

The standard boundary point lemma of E. Hopf for the Laplacian is as follows. See Theorem 2.7. in [22].

- ***Hopf's Boundary Point Lemma***[1]***.*** *Let u satisfy $\Delta u \geq 0$ in D and $u \leq M$ in D, $u(P) = M$ for some $P \in \partial D$. Assume that P lies on the boundary of a ball $B \subset D$. If u is continuous on $D \cup P$ and if the outward directional derivative $\frac{\partial u}{\partial n}$ exists at P, then $u \equiv M$ or*

$$\frac{\partial u}{\partial n}(P) > 0.$$

If Ω is not smooth the estimates (2) and (3) are no longer true in general. Roughly speaking the estimate in (2) goes wrong for domains with corners pointing outside and (3) goes wrong for domains with inside pointing corners.

Instead of comparing with $d(x)$ in (2) and (3), one could try to compare with $\phi_1(x)$, the first eigenfunction of

$$\begin{cases} -\Delta\phi = \lambda\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

For smooth Ω there is not much difference since $c\ d(x) \leq \phi_1(x) \leq c'\ d(x)$ for some $c, c' > 0$.

[1] For the Laplacian in two dimensional domains the result goes back to C. Neumann in 1888. For general elliptic operators (in general dimensions) the result was published in 1952 independently by E. Hopf and by Oleinik. See page 156 of [22] for references and more bibliographical details.

For nonsmooth Ω the question is the following: do $k, k' \in (0,\infty)$ exist such that

$$k\ \phi_1(x) \leq\ u(x) \qquad \text{for all } x \in \Omega, \tag{6}$$

and

$$u(x) \leq k'\ \phi_1(x) \qquad \text{for all } x \in \Omega\,? \tag{7}$$

We will show that for domains with finitely many corners that satisfy a uniform interior cone condition with large angle the answer is yes. If such a domain has a corner with a small angle the answer is a conditional yes. The condition will be sharp. Related estimates for plane domains with corners are obtained by Oddson in [19]. His estimates are not sufficient for the results we are interested in. In several other papers related estimates can be found. One should mention Kondrat'ev [13], [14] and also [17], [18]. However, collecting the literature on elliptic boundary value problems on angular domains is *la mer à boire.*

The domains that we consider will have a finite number of corners. It would be interesting to see if similar results hold for Lipschitz domains. On such more general domains the conditions will depend on the Lipschitz coefficient.

In order to simplify notations we use:

Definition 1. *Set $u \preceq v$ on A if and only if there exists $k \in (0,\infty)$ such that*

$$u(x) \leq k\ v(x) \ \text{ for all } x \in A.$$

Set $u \not\preceq v$ on A if such a constant does not exist.

Definition 2. *Set $u \simeq v$ on A if and only if there exists $k \in (0,\infty)$ such that*

$$k^{-1}v(x) \leq u(x) \leq k\ v(x) \ \text{ for all } x \in A.$$

We will use results of Kadlec [12] and Grisvard [9]. For the pointwise results we are interested in, we will compare the solutions with appropriate sub- and supersolutions. In the last sections we will show how the estimates can be used to extend the results mentioned in the abstract. In the appendix one finds some of the results for conformal mappings that we use.

2. Some standard preliminaries

Let us recall some regularity results for solutions u of (1) on a bounded domain $\Omega \subset \mathbb{R}^n$. The function u will denote a solution of (1) in appropriate sense.

R1) If Ω satisfies an exterior cone condition then (see Theorem 8.30 of [8]) $f \in L^q(\Omega)$ with $q > \frac{1}{2}n$ implies $u \in C_0(\bar{\Omega})$.

R2) If $\partial\Omega$ is $C^{1,1}$ (see Theorem 9.15 of [8]) then $f \in L^q(\Omega)$, $q > 1$ implies $u \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$.

R3) If Ω is convex (see [12], [9]), then $f \in L^2(\Omega)$ implies $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$.

The following imbedding results hold:

I1) On general Ω (Theorem 7.10 of [8]) $u \in W_0^{1,q}(\Omega)$, $q > n$ implies $u \in C(\bar{\Omega})$.

I2) If Ω satisfies a uniform interior cone condition, then $u \in W^{2,q}(\Omega)$, $q > \frac{1}{2}n$ implies $u \in C(\Omega) \cap L^\infty(\Omega)$.

I3) If $\partial\Omega$ is $C^{0,1}$, then (Corollary 7.11 of [8]) $u \in W^{2,q}(\Omega)$, $q > \frac{1}{2}n$ implies $u \in C(\bar{\Omega})$, (and $q > n$ implies $u \in C^1(\bar{\Omega})$).

Notice that conditions on the outside of the domain, an exterior cone condition or convexity (which might be called exterior plane condition), imply the regularity of the solution. Conditions on the inside allow one to transfer integrability properties of first or second order derivatives to continuity of the function itself.

3. Is the first eigenfunction the lowest positive superharmonic function?

On smooth domains the results that we mentioned in the introduction imply that any positive superharmonic function, that is continuous, lies above a positive constant times the first eigenfunction. A proof of this result uses the smoothness of the boundary. However, it is not clear if this regularity is necessary. Let us state this open question as a conjecture.

CONJECTURE 3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and suppose that there is a first eigenfunction $\phi_1 \in C_0\left(\bar{\Omega}\right) \cap C^2(\Omega)$ (we fix $\max \phi_1 = 1$). Then for every $u \in W^{2,p}_{loc}(\Omega)$, $p > \frac{1}{2}n$, with $u \geq 0, -\Delta u \geq 0$ in Ω, either $u \equiv 0$ or there exists $c > 0$ such that $u \geq c\phi_1 \quad$ in Ω.*

This estimate from below by the first eigenfunction could take care of some of the results in this paper. But since there does not seem to be a proof in the general case we will (only) show it under some conditions on the boundary. By the way, the estimate from above will not be true in general.

The eigenvalue problem in general domains has recently been studied by Bañuelos in [3].

4. Elementary domains

4.1 Notations

Since results related with the Riemann Mapping Theorem are more easily stated using $\mathbb{C}$ instead of $\mathbb{R}^2$ we will use boldface for the complex alternative:

$$\begin{array}{lll} \text{for } x \in \mathbb{R}^2 & \text{set} & \boldsymbol{x} = x_1 + ix_2 \in \mathbb{C}, \\ \text{for } A \subset \mathbb{R}^2 & \text{set} & \boldsymbol{A} = \{x_1 + ix_2 \in \mathbb{C}; x \in A\}, \\ \text{for } h : \mathbb{R}^2 \to \mathbb{R}^2 & \text{set} & \boldsymbol{h}(\boldsymbol{x}) = h_1(x) + ih_2(x). \end{array}$$

We will start by considering the following domains in $\mathbb{R}^2$

$$D(\psi) = \left\{x \in \mathbb{R}^2; |\boldsymbol{x}| < 1, |\arg \boldsymbol{x}| < \tfrac{1}{2}\psi\right\}.$$

The angle of the domain $D(\psi)$ at zero (we will always measure from inside) equals ψ. Related are

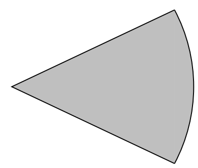
$$\begin{array}{lcl} \text{the circular boundary:} & \Gamma_\psi = & \left\{x \in \overline{D(\psi)}; |x| = 1\right\}, \\ \text{the conical boundary:} & S_\psi = & \left\{x \in \overline{D(\psi)}; |x| < 1\right\}, \\ \text{the related growth rate:} & \beta_\psi = & \dfrac{\pi}{\psi}. \end{array}$$

Notice that $\partial D(\psi) = \Gamma_\psi \cup S_\psi$. We will also use

$$\frac{1}{2}D(\psi) = \left\{x \in \mathbb{R}^2; 2x \in D(\psi)\right\}.$$

4.2 Cones that are convex near 0

We start with domains $\Omega = D(\psi)$ when $\psi < \pi$ and will be interested in the behavior near $(0,0)$. Note that the corner at $(0,0)$ is convex and hence there is no regularity problem (see Grisvard [9], [10]).



$D(\psi)$ for some $\psi \in (0,\pi)$.

We will compare the solutions of

$$\begin{cases} -\Delta v = 0 & \text{in } D(\psi), \\ v = 0 & \text{on } S_\psi, \\ v = \cos\beta_\psi\varphi & \text{on } \Gamma_\psi, \end{cases} \tag{8}$$

$$\begin{cases} -\Delta\phi_1 = \lambda_1\phi_1 & \text{in } D(\psi), \\ \phi_1 = 0 & \text{on } \partial D(\psi), \end{cases} \tag{9}$$

with λ_1 the first eigenvalue (take $\phi_1 > 0$), and for $\alpha > -1$

$$\begin{cases} -\Delta u_\alpha = r^\alpha & \text{in } D(\psi), \\ u = 0 & \text{on } \partial D(\psi). \end{cases} \tag{10}$$

Note that (1) has a solution $u \in C_0(\bar{\Omega})$ if $f \in L^q(\Omega)$ with $q > 1$ if Ω ($\subset \mathbb{R}^2$) satisfies an exterior cone condition at every boundary point.

LEMMA 4. *Let $D(\psi), v, \phi_1$ be as above. Then for all $\psi \in (0,\pi)$*

$$v \simeq \phi_1 \quad \text{on } \tfrac{1}{2}D(\psi).$$

Proof. We can write both functions explicitly. We have $v(r,\varphi) = r^{\beta_\psi}\cos(\beta_\psi\varphi)$ and by using a Bessel function of order β_ψ, namely $\phi_1(r,\varphi) = J_{\beta_\psi}\left(\rho_{\beta_\psi,1}r\right)\cos(\beta_\psi\varphi)$ with corresponding eigenvalue

$\lambda_1 = \left(\rho_{\beta_\psi,1}\right)^2$ (ϕ_1 is unique up to multiplication by a constant) where

$$J_\beta(s) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}s\right)^{\beta+2m}}{m!\,\Gamma(\beta+m+1)}$$

and $\rho_{\beta,1}$ the first positive zero of $J_\beta(\cdot)$.

Since $r^\beta \preceq J_\beta\left(\rho_{\beta,1}r\right) \preceq r^\beta$ on $\left[0,\frac{1}{2}\right]$ the estimate in i) holds. Remember that $J_\beta(s)$ solves $s^2J'' + sJ' + \left(s^2-\beta^2\right)J = 0$. □

THEOREM 5. *Suppose $\alpha > -1$ and fix $\psi \in (0,\pi)$. Let $D(\psi), \phi_1$ and u_α be as above. Then*

i. for $\alpha + 2 > \beta_\psi$ we have $u_\alpha \simeq \phi_1$ *on* $\frac{1}{2}D(\psi)$,

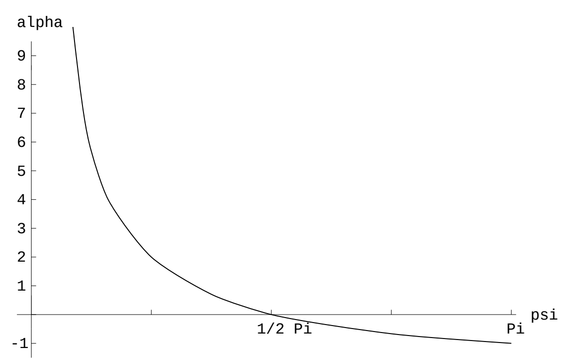
ii. for $\alpha + 2 < \beta_\psi$ we have $u_\alpha \simeq r^{\alpha+2-\beta_\psi}\,\phi_1$ *on* $\frac{1}{2}D(\psi)$,

iii. for $\alpha + 2 = \beta_\psi$ we have $u_\alpha \simeq (-\ln r)\;\phi_1$ *on* $\frac{1}{2}D(\psi)$.

Remark 1: From ii) and iii) it follows that for $\alpha + 2 \leq \beta_\psi$ one has

$$u_\alpha \not\preceq \phi_1 \text{ on } \tfrac{1}{2}D(\psi).$$

Remark 2: The relation between the opening angle $\psi \in (0,\pi)$ and the critical growth $\alpha \in (-1,\infty)$ is $\alpha = \frac{\pi}{\psi} - 2$ or reversed $\psi = \frac{\pi}{\alpha+2}$. For the semilinear problem $-\Delta u = f(u)$ with $f(0) < 0$ the angle $\frac{1}{2}\pi$ is critical for the existence of a positive solution, see [24].



Proof. In all of the parts we will use the maximum principle as follows. If $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$\begin{cases} -\Delta u \le -\Delta v & \text{in } \tfrac{1}{2}D(\psi), \\ u \le v & \text{on } \partial\big(\tfrac{1}{2}D(\psi)\big), \end{cases}$$

then

$$u \le v \qquad \text{in } \tfrac{1}{2}D(\psi).$$

Note that

$$\phi_1 \preceq u_\alpha \preceq \phi_1 \qquad \text{on } \tfrac{1}{2}\Gamma_\psi \;\left(= \partial\big(\tfrac{1}{2}D(\psi)\big) \cap D(\psi)\right).$$

We will compare u_α with several w_i. All functions w_i satisfy $w_i = 0$ on S_ψ.

i) above. We set $\beta = \min\left(\alpha + 2, 2\beta_\psi\right)$. Then it follows that $\alpha+2 \ge \beta > \beta_\psi$ and hence that $r^{\alpha+2} \le r^\beta \le r^{\beta_\psi}$ for $r \in [0,1]$. Now we compare with

$$w_1 = k\, r^{\beta_\psi} \cos \beta_\psi \varphi - r^\beta \left(\cos(\beta\varphi) - \cos\left(\beta/\beta_\psi \tfrac{1}{2}\pi\right)\right), \quad (11)$$

where we take k large enough such that $w_1 \ge 0$ on $\tfrac{1}{2}D(\psi)$. Then, using the fact that $\cos\left(\beta/\beta_\psi \tfrac{1}{2}\pi\right) < 0$, we find

$$-\Delta w_1 = -\beta^2\, r^{\beta-2} \cos\left(\beta/\beta_\psi \tfrac{1}{2}\pi\right) \succeq r^{\beta-2} \quad \text{on } \tfrac{1}{2}D(\psi)$$

and $w_1 \preceq r^{\beta_\psi} \cos \beta_\psi \varphi$ on $\tfrac{1}{2}D(\psi)$. Since

$$\begin{cases} r^\alpha \preceq -\Delta w_1 & \text{on } \tfrac{1}{2}D(\psi), \\ 0 = u_\alpha \preceq w_1 & \text{on } \partial\left(\tfrac{1}{2}D(\psi)\right), \end{cases}$$

the previous lemma and the maximum principle imply

$$u_\alpha \preceq w_1 \preceq \phi_1 \quad \text{on } \tfrac{1}{2}D(\psi).$$

i) below. We compare with

$$w_2 = r^{\beta_\psi}\left(1 - r^{\alpha+2-\beta_\psi}\right) \cos \beta_\psi \varphi. \qquad (12)$$

It follows that

$$\begin{cases} -\Delta w_2 = \left((\alpha+2)^2 - \beta_\psi^2\right) r^\alpha \cos \beta_\psi \varphi \preceq r^\alpha & \text{on } D(\psi), \\ 0 = w_2 = u_\alpha & \text{on } \partial D(\psi), \end{cases}$$

and hence by the previous lemma and the maximum principle we find

$$\phi_1 \preceq w_2 \preceq u_\alpha \quad \text{on } \tfrac{1}{2} D(\psi).$$

ii) above. Set

$$w_3 = r^{\alpha+2} \left(\cos\left((\alpha+2)\varphi\right) - \cos\left(\frac{\alpha+2}{\beta_\psi} \tfrac{1}{2}\pi \right)\right). \tag{13}$$

One finds $-\Delta w_3 = r^\alpha \ (\alpha+2)^2 \cos\left(\frac{\alpha+2}{\beta_\psi} \tfrac{1}{2}\pi\right)$ and hence

$$\begin{cases} r^\alpha \preceq -\Delta w_3 & \text{on } D(\psi), \\ u_\alpha \preceq w_3 & \text{on } \partial\left(\tfrac{1}{2} D(\psi)\right), \end{cases}$$

from which it follows that

$$u_\alpha \preceq w_3 \preceq r^{\alpha+2-\beta_\psi} \phi_1 \quad \text{on } \tfrac{1}{2} D(\psi).$$

ii) below. Taking

$$w_4 = r^{\alpha+2} \cos \beta_\psi \varphi \tag{14}$$

we obtain

$$\begin{cases} -\Delta w_4 = \left(\beta_\psi^2 - (\alpha+2)^2\right) r^\alpha \cos \beta_\psi \varphi \preceq r^\alpha & \text{on } D(\psi), \\ 0 \preceq w_4 \preceq u_\alpha & \text{on } \partial\left(\tfrac{1}{2} D(\psi)\right), \end{cases}$$

and hence

$$r^{\alpha+2-\beta_\psi} \phi_1 \preceq w_4 \preceq u_\alpha \quad \text{on } \tfrac{1}{2} D(\psi).$$

iii) For $\alpha + 2 = \beta_\psi$ we compare with

$$w_5 = r^{\beta_\psi} \left((k - \ln r) \ \cos\left(\beta_\psi \varphi\right) + \varphi \sin\left(\beta_\psi \varphi\right) - \frac{\pi}{2\beta_\psi} \right) \tag{15}$$

with k large enough such that $w_5 > 0$ on $D(\psi)$. Since $-\Delta w_5 = \beta_\psi \, r^{\beta_\psi - 2} \, \frac{\pi}{2}$ we can use this function for both sides of the estimate. From

$$\begin{cases} -\Delta w_5 \preceq r^\alpha \preceq -\Delta w_5 & \text{in } D(\psi), \\ w_5 \preceq u_\alpha \preceq w_5 & \text{on } \partial\left(\frac{1}{2}D(\psi)\right), \end{cases}$$

it follows that

$$w_5 \preceq \; u_\alpha \preceq w_5 \quad \text{in } \tfrac{1}{2}D(\psi).$$

Since

$$w_5 \preceq -r^{\beta_\psi} \; \ln r \; \cos\left(\beta_\psi \varphi\right) \preceq w_5 \quad \text{in } \tfrac{1}{2}D(\psi)$$

the previous lemma implies the last estimate of the theorem. □

COROLLARY 6. *Fix $\psi \in (0, \pi)$ and let u be a solution of*

$$\begin{cases} -\Delta u = f & \text{in } D(\psi), \\ u = 0 & \text{on } S_\psi, \\ u = g & \text{on } \Gamma_\psi, \end{cases}$$

with $0 \leq g \in C(\Gamma_\psi)$ and $0 \leq f \in C(D(\psi))$, such that for some θ and M one has

$$0 \neq f(x_1, x_2) \leq M\, |x_1|^\vartheta \quad \text{for } x \in D(\psi).$$

If $\vartheta > \dfrac{\pi}{\psi} - 2$, then

$$u \simeq \phi_1 \;\; \text{on } \tfrac{1}{2}D(\psi).$$

Remark 3: Note that if we do not assume a fixed sign for f or g we find that there is $k > 0$ such that

$$|u(x)| \leq M \;\, \phi_1(x) \;\text{ for all } x \in \tfrac{1}{2}D(\psi).$$

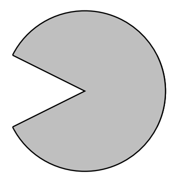
Remark 4: The conditions on f imply that $f \in L^2(D(\psi))$. For $\alpha \geq 0$ one has $f \in L^\infty(D(\psi))$. For $-1 < \alpha < 0$ one finds in fact that $f \in L^p(D(\psi))$ for all $1 \leq p < \frac{2}{-\alpha}$ and hence $f \in L^2(D(\psi))$:

$$\int_{D(\psi)} |f|^p \, dx \leq M^p \int_{x_1=0}^{1} \int_{x_2=-cx_1}^{cx_1} x_1^{p\alpha} \; dx_2 dx_1 = \frac{2cM^p}{p\alpha + 2}.$$

Hence there is a solution $u \in W_0^{1,2}(D(\psi))$. Regularity results, see [8], show that $u \in W^{2,2}(\Omega')$ for any $\Omega' \subset D(\psi)$ with $\bar{\Omega}'$ not containing the corner points. In fact, since $D(\psi)$ is convex it follows from [12] that $u \in W^{2,2}(D(\psi))$; see Theorem 3.2.1.2 of [9].

4.3 Cones that are concave near 0

In this section we recall some results for domains with an entrant corner. We will use $D(\psi)$ that have a corner at $(0,0)$ with angle $\psi \in (\pi, 2\pi)$.



$D(\psi)$ for $\psi \in (\pi, 2\pi)$.

From results of Grisvard [11], see also Lemma 4.4.3.1 and Theorem 4.4.3.7 in [9], one has for $f \in L^p(D(\psi))$, with $p \geq 2$ and $\frac{2\psi}{q\pi} \notin \mathbb{N}$, $\frac{1}{p}+\frac{1}{q}=1$, that there exists a solution $u \in W_0^{1,2}(D(\psi))$ and it satisfies

$$u = \bar{u}_f + \sum_{m=1}^{n_{p,\psi}} c_{m,f}\ r^{m\beta_\psi}\ \chi_m(\varphi)$$

where

$$\begin{aligned} &\bar{u}_f \in W^{2,p}(\Omega), \\ &c_{m,f} \in \mathbb{R}, \\ &\chi_m(\varphi) = \sin\left(m\beta_\psi\left(\varphi - \tfrac{1}{2}\psi\right)\right), \\ &n_{p,\psi} = \left[\frac{2\psi}{q\pi}\right], \text{ the entier of } \frac{2\psi}{q\pi}. \end{aligned}$$

Taking $p = 2+\varepsilon$ in the above, with $0 < \varepsilon < \dfrac{2\pi-\psi}{\psi-\pi}$, one finds

$$u = \bar{u}_f + c_{1,f}\ r^{\beta_\psi}\ \cos\left(\beta_\psi\varphi\right)$$

with $\bar{u}_f \in W^{2,2+\varepsilon}(D(\psi))$. Since $W^{2,2+\varepsilon}(D(\psi)) \subset C^1\left(\overline{D(\psi)}\right)$ holds in a two dimensional domain with Lipschitz boundary we find that $|\bar{u}_f(x)| \preceq d(x,\partial\Omega) \preceq r^{\beta_\psi} \cos(\beta_\psi \varphi)$ on $D(\psi)$. Hence $|u(x)| \preceq r^{\beta_\psi} \cos(\beta_\psi \varphi)$ on $D(\psi)$. Note that the solution of (8) is given by $v = r^{\beta_\psi} \cos(\beta_\psi \varphi)$. Next we will show that the solutions of (8), (9) and (10) with $\alpha > -1$, have the same behavior near 0.

LEMMA 7. *Let v and ϕ_1 be the solutions of respectively (8) and (9) on $D(\psi)$ with concave corner. Then*

$$v \simeq \phi_1 \quad \text{on } \tfrac{1}{2}D(\psi).$$

Proof. See Lemma 4. □

COROLLARY 8. *Let $D(\psi)$ and ϕ_1 be as above. Let $f \in L^p(D(\psi)) \cap C(D(\psi))$ with $p > 2$ such that $0 \neq f \geq 0$, and let $0 \leq g \in C(\Gamma_\psi)$. Then the solution u of*

$$\begin{cases} -\Delta u = f & \text{in } D(\psi), \\ u = 0 & \text{on } S_\psi, \\ u = g & \text{on } \Gamma_\psi, \end{cases}$$

satisfies

$$u \simeq \phi_1 \text{ on } \tfrac{1}{2}D(\psi). \tag{16}$$

Remark 5: Note that $f(r) = r^\alpha$, with $-1 < \alpha < 0$, is in $L^p(D(\psi))$ with $p \in \left(2, -2\,\alpha^{-1}\right)$. If u_α is the solution of (10) then u_α satisfies $\phi_1 \preceq u_\alpha \preceq \phi_1$ on $\frac{1}{2}D(\psi)$.

Remark 6: Again, if we skip the sign condition for f and g, we find that for some $k > 0$

$$|u(x)| \leq k\ \phi_1(x) \text{ for all } x \in \tfrac{1}{2}D(\psi).$$

Proof. By the results of Grisvard we find

$$u \preceq r^{\beta_\psi} \cos(\beta_\psi \varphi) \qquad \text{on } D(\psi).$$

By the explicit formula for ϕ_1 we obtain

$$r^{\beta_\psi} \cos\left(\beta_\psi \varphi\right) \preceq \phi_1 \quad \text{on } \tfrac{1}{2}D\left(\psi\right),$$

which shows the estimate from above. The estimate from below follows by

$$\begin{cases} -\Delta v = 0 \preceq -\Delta u & \text{in } \tfrac{1}{2}D\left(\psi\right), \\ v \preceq u & \text{on } \partial\left(\tfrac{1}{2}D\left(\psi\right)\right), \end{cases}$$

the maximum principle and the previous lemma. □

5. The basic result in general domains with corners

The domains that we will consider satisfy the following assumptions.

CONDITION 9. *The domain Ω is an open bounded subset of $\mathbb{R}^2$ such that:*

i. Ω is the inside of a (closed) Jordan curve Γ, say

$$\Gamma = \left\{\gamma\left(e^{i\varphi}\right); \varphi \in [0, 2\pi]\right\};$$

ii. γ is Dini smooth except in finitely many points $\left\{e^{it_j}\right\}_{j=1}^{k+m}$;

iii. at every point $y^{(j)} = \gamma\left(e^{it_j}\right)$ the boundary $\partial\Omega$ has a Dini smooth corner;

iv. the corresponding angles,

$$\psi_j = \lim_{\varepsilon \downarrow 0} \angle\left(\gamma'(e^{i(t_j-\varepsilon)}), \gamma'(e^{i(t_j+\varepsilon)})\right)$$

which are measured from inside, lie in $(0, 2\pi)$.

We will assume that $\psi_j \in (0, \pi)$ for $1 \le j \le k$, and $\psi_j \in (\pi, 2\pi)$ for $k+1 \le j \le k+m$.

Remark 1: Γ is called a Jordan curve if $\Gamma = \gamma(\partial \boldsymbol{B}_1(0))$ with $\gamma : \partial \boldsymbol{B}_1(0) \to \Gamma$ continuous and one-one.

A function is called Dini smooth if the derivative exists and is Dini continuous. A (part of the) boundary is called Dini smooth if there exists a Dini smooth parameterization γ with $\gamma' \neq 0$.

The curve Γ has a Dini smooth corner at $\gamma\left(e^{it_j}\right)$ if

$$\Gamma_{j,\varepsilon}^{-} = \left\{\gamma\left(e^{it}\right); t \in (t_j - \varepsilon, t_j]\right\} \ , \ \Gamma_{i,\varepsilon}^{+} = \left\{\gamma\left(e^{it}\right); t \in [t_j, t_j + \varepsilon)\right\}$$

are Dini smooth arcs for some small $\varepsilon > 0$.

Remark 2: The condition implies that Ω is simply connected. This is not necessary. Most results in this paper have an obvious extension to bounded domains which boundary consists of finitely many non intersecting Jordan curves, all of which satisfy the items ii), iii) and vi) in Condition 9.

THEOREM 10. *Suppose Ω satisfies Condition 9 and that $f \in L^p(\Omega) \cap C(\Omega)$ with $p > 2$ and $0 \neq f \geq 0$. Let $u \in W_0^{1,2}(\Omega) \cap W_{loc}^{2,p}(\Omega)$ be the solution of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$u \succeq \phi_1 \quad \text{on } \Omega. \tag{17}$$

If moreover, f is such that for every $i \in \{1, \ldots, k\}$ there exist $\vartheta_i > \dfrac{\pi}{\psi_i} - 2$ and $\varepsilon_i, M_i > 0$ with

$$f(x) \leq M_i \left|x - y^{(i)}\right|^{\vartheta_i} \quad \text{on } B_{\varepsilon_i}\left(y^{(i)}\right) \cap \Omega, \tag{18}$$

then

$$u \simeq \phi_1 \quad \text{on } \Omega. \tag{19}$$

Proof. On $\Omega \backslash \bigcup_{i=1}^{k+m} B_\delta\left(y^{(i)}\right)$ the estimates in (17) and (19) follow by the strong maximum principle. Hence it remains to show (17) and (19) on $\Omega \cap \bigcup_{i=1}^{k+m} B_\delta\left(y^{(i)}\right)$. By Corollary A.5 there is an appropriate holomorphic mapping $\boldsymbol{h}_i$ from $\boldsymbol{\Omega}$ to $\mathbb{C}$, such that $\left|\boldsymbol{h}'\right|$ is bounded away from 0 and ∞, and for some $c > 0$

$$c^{-1} D(\psi_i) \subset h\left(B_\varepsilon\left(y^{(i)}\right) \cap \Omega\right) \subset c\, D(\psi_i).$$

By Corollaries 6 and 8 we find that the solution u_h of

$$\begin{cases} -\Delta u_h(x) = f\left(h^{inv}(x)\right) & \text{for } x \in h(\Omega), \\ u_h = 0 & \text{on } \partial h(\Omega), \end{cases}$$

satisfies, respectively without and with condition (18)

$$\left[\begin{array}{l} u_h \succeq \phi_{1,c^{-1}D(\psi_i)} \\ u_h \simeq \phi_{1,c^{-1}D(\psi_i)} \end{array}\right. \quad \text{on } \tfrac{1}{2}c^{-1}D(\psi_i). \tag{20}$$

Since (20) also holds when u_h is replaced by $\phi_{1,h(\Omega)}$, we find respectively that

$$\left[\begin{array}{l} u_h \succeq \phi_{1,h(\Omega)} \\ u_h \simeq \phi_{1,h(\Omega)} \end{array}\right. \quad \text{on } \tfrac{1}{2}c^{-1}D(\psi_i).$$

Lemma A.1 shows for some $\varepsilon' > 0$ that

$$\left[\begin{array}{l} u \succeq \phi_{1,\Omega} \\ u \simeq \phi_{1,\Omega} \end{array}\right. \quad \text{on } B_{\varepsilon'}\left(y^{(i)}\right) \cap \Omega.$$

□

6. A semilinear equation

Consider the equation

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{21}$$

with $p \in (-1, 0)$.

THEOREM 11. *Let Ω be a bounded domain in $\mathbb{R}^2$ that satisfies Condition 9. Let ψ_0 be the angle of the smallest corner of $\partial\Omega$. If $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, with $q > 2$, is a solution of (21), then $p > -\frac{2\psi_0}{q\pi}$.*

Proof. We find $-\Delta u \in L^q(\Omega)$ and hence, if ψ denotes the angle of the boundary at 0, we find that there is $c > 0$ such that near 0 we have

$$c_1 \; |x|^{\frac{\pi}{\psi}-1} d(x) \le u(x) \le c_2 \; |x|^{\frac{\pi}{\psi}-1} d(x).$$

Then $u^p \in L^q(\Omega \cap B_\varepsilon(0))$ for small ε, if and only if $pq\frac{\pi}{\psi} + 1 > -1$.

□

7. A generalized maximum principle

The classical maximum principle for the Dirichlet problem not only holds with a positive potential but also with a potential V with $V > -\lambda_1$, where λ_1 is the first eigenvalue.

- ***A version of the classical maximum principle.*** *Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^2 boundary. Suppose that $V \in L^\infty(\bar{\Omega})$ with $V > -\lambda_1$, and that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies*

$$\begin{cases} -\Delta u + Vu \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}$$

then $u \geq 0$.

If Ω is a $C^{0,1}$-domain one is able, by using the Hardy inequality

$$\int_\Omega \left(\frac{|u(x)|}{d(x)}\right)^2 dx \leq c_H \int_\Omega |\nabla u(x)|^2 \, dx \text{ for all } u \in W_0^{1,2}(\Omega), \tag{23}$$

(see [15] or [7]) to generalize the result above to potentials V that are unbounded near the boundary $\partial\Omega$. Although the result seems to be standard we have not been able to locate a reference[2].

THEOREM 12. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a $C^{0,1}$-boundary. Let c_H denote the best constant in (23). Suppose $V \in C(\Omega)$ is such that $-c_H^{-1} < -c \leq V\, d(\cdot)^2$. If $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies (22) and $u \in W_0^{1,2}(\Omega)$ then $u \geq 0$.*

Proof. Set $V_\pm = \frac{1}{2}(|V| \pm V)$ and $u_\pm = \frac{1}{2}(|u| \pm u)$. Set

$$\Omega^* = \{x \in \Omega; u(x) < 0\}$$

and suppose that Ω^* is nonempty. For $x \in \Omega^*$ we have $-\Delta u - V_- u \geq -V_+ u \geq -V_+ u_+ = 0$ which implies for all $\phi \in C_0^\infty(\Omega^*)$ with $\phi \geq 0$ that

$$\int_\Omega (\nabla u \cdot \nabla \phi - V_- u\, \phi) \, dx \geq 0.$$

[2] Added in proof: [30]

Since $u_- \in W_0^{1,2}(\Omega)$ we find with (23) that

$$0 \le \int_\Omega (\nabla u \cdot \nabla u_- - V_- \, u \, u_-) \, dx \le \left(-1 + \frac{c}{c_H}\right) \int_\Omega |\nabla u_-|^2 \, dx \le 0.$$

Hence $u_- = 0$. □

THEOREM 13. *Let Ω be a bounded domain in $\mathbb{R}^2$ that satisfies Condition 9. Suppose that $V \in C(\Omega)$ is such that there are $\varepsilon, \delta, K > 0$ for which the following holds.*

$$\begin{aligned} &i) \quad -\delta \le d(x)^2 \, V(x) \quad \text{for } x \in \Omega, \\ &ii) \quad -K \le d(x)^{2-\varepsilon} \, V(x) \quad \text{for } x \in \Omega. \end{aligned}$$

If $\delta < c_H$ then a function $u \in C(\bar{\Omega}) \cap C^2(\Omega) \cap W_0^{1,2}(\Omega)$ that satisfies (22) is positive and moreover either $u \equiv 0$ or there is $c > 0$ with

$$u(x) \ge c \, \phi_1(x) \;\; \text{for } x \in \Omega.$$

where ϕ_1 is the first eigenfunction of $-\Delta$.

For smooth domains and V bounded, optimal results in comparing u and ϕ_1 are found in [27].

Proof. By Theorem 12 it follows that for δ small the function u satisfies $u \ge 0$. Then we have $-\Delta u + V_+ u = V_- u \ge 0$ in Ω. By the standard strong maximum principle one finds $u \equiv 0$ or $u > 0$ for every domain Ω' with $\overline{\Omega'} \subset \Omega$ and hence in Ω. It remains to show the boundary behavior. With similar arguments as we used in the proof of Theorem 10, which are the results stated in the appendix, it is sufficient to show the boundary behavior in a neighborhood of 0 for $\Omega = D(\psi)$ with the appropriate ψ. First we consider the case where $\psi = \pi$. We may assume that $\varepsilon \le 1$ and we fix $\gamma = \frac{1}{2}\varepsilon$.

i) $\psi = \pi$. We use the function

$$w_6(x_1, x_2) = x_1 \left(1 + x_1^\gamma - 2x_1\right) - 2x_2^2. \tag{24}$$

Then one has on the set where $w_6 > 0$ holds that

$$-\Delta w_6 + V_+ \, w_6 \le -\Delta w_6 + K x_1^{\varepsilon-2} \, w_6 \le$$

$$\leq x_1^{\gamma-1}\left(-\gamma(\gamma+1)+8x_1^{1-\gamma}+2Kx_1^{\varepsilon-\gamma}\right)$$

which is negative for $x_1 \in (0,\rho)$ with

$$\rho := \min\left(\frac{1}{2}, \sqrt[\gamma]{\frac{\gamma(\gamma+1)}{2K+8}}\right).$$

Set $S = \left\{x \in D(\pi)\,; x_2^2 < x_1 < \rho\right\}$. Then $w_6 \leq 0$ on the part of ∂S where $x_2^2 = x_1$. Fix δ in $\left(\sqrt{\rho}, \sqrt{1-\rho^2}\right)$. Since $u > 0$ in $D(\pi)$ the number

$$\tau = \max\left\{\frac{w_6(\rho,x_2)}{u(\rho,x_2)}; |x_2| \leq \delta\right\} \tag{25}$$

is well defined and positive. Now we are able to compare u and $\tau w_6 - t$ on S.

For large $t > 0$ we have $u > \tau w_6 - t$ in $\bar{S}$ and for every $t > 0$ we find $u > \tau w_6 - t$ on ∂S. So either $u - \tau w_6 \geq 0$ in S or there is a smallest $t^* > 0$ such that for some $x^* \in S$

$$\begin{aligned} u &\geq \tau w_6 - t^* \quad \text{in } S, \\ u(x^*) &= \tau w_6(x^*) - t^*. \end{aligned}$$

Suppose that the second possibility holds. Since $u > 0$ on S and hence $\tau w_6(x^*) - t = u(x^*) > 0$ there exists S^*, with $\overline{S^*} \subset S$ and ∂S^* smooth, such that $\tau w_6 - t \geq 0$ on S^*. Then we find on S, and hence on S^* that

$$(-\Delta + V_+)\left(u - (\tau w_6 - t^*)\right) \geq$$

$$\geq V_- u - \tau(-\Delta + V_+) w_6 + V_+ t^* \geq 0.$$

It follows by the strong maximum principle that $u > \tau w_6 - t^*$ on S^* and hence a contradiction. That is

$$u - \tau w_6 \geq 0 \text{ in } S.$$

By another application of the sweeping principle of McNabb ([16]), now a shift of $u - (\tau w_6 - t)$ of at most $\delta - \sqrt{\rho}$ in the x_2-direction and repeating the argument above, we find that $u(x_1+s,x_2) \geq \tau w_6(x_1,x_2)$ in S when $|s| \leq \delta - \sqrt{\rho}$. Hence we have

$$u(x_1,x_2) \geq \tau w_6(x_1,0) \text{ for } 0 < x_1 < \rho,\ |x_2| < \delta - \sqrt{\rho}.$$

The estimate in the theorem follows since $w_6(x_1,0) \geq c\,x_1$ for some $c>0$ and all $x_1 \in \left(0,\frac{1}{2}\right)$

ii) $\psi \neq \pi$. With similar arguments as before we use

$$w_7(r,\varphi) = r^{\beta_\psi}\left(1+r^\gamma-2r\right)\cos\left(\beta_\psi\varphi\right). \tag{26}$$

Then

$$\begin{aligned}-\Delta w_7 + V_+ w_7 &\leq \\ &\leq r^{\beta_\psi-2+\gamma}\Big(-\left(\left(\beta_\psi+\gamma\right)^2-\beta_\psi^2\right)+ \\ &\qquad +2\left(\left(\beta_\psi+1\right)^2-\beta_\psi^2\right)r^{1-\gamma}\Big)\cos\left(\beta_\psi\varphi\right)+ \\ &\qquad\qquad +K\,r^{\beta_\psi-2+\varepsilon}\left(1+r^\gamma-2r\right)\cos\left(\beta_\psi\varphi\right)\leq\end{aligned}$$

$$\leq r^{\beta_\psi-2+\gamma}\left(-\left(2\gamma\beta_\psi+\gamma^2\right)+2\left(2\beta_\psi^2+\beta_\psi\right)r^{1-\gamma}+2Kr^\gamma\right)\cos\left(\beta_\psi\varphi\right)$$

which is negative on $D(\psi)$ for $0<r<r_0 := \sqrt[\gamma]{\frac{2\gamma\beta_\psi+\gamma^2}{4\beta_\psi^2+2\beta_\psi+2K}}$. We replace S by $\tilde{S} = r_0 D(\psi)$ and τ with

$$\tilde{\tau} = \max\left\{\frac{w_7(r_0,\varphi)}{u(r_0,\varphi)};\, |\varphi| \leq \tfrac{1}{2}\psi\right\}.$$

The result of part i) shows that the quotient remains bounded when $\varphi \to \pm\frac{1}{2}\psi$ from inside. Finishing the argument as before we find that $u \geq \tilde{\tau} w_7$, implying the estimate of the theorem.

8. The anti-maximum principle

Clément and Peletier showed in [6] a result that reads for the Laplacian with zero Dirichlet boundary conditions as follows.

- *Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^2 boundary. Suppose $f \in L^p(\Omega)$, $p>n$, such that $0 \neq f \geq 0$, and suppose u_λ satisfies the equation*

$$\begin{cases} -\Delta u - \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{27}$$

Then there exists $\delta>0$, depending on f, such that if $\lambda_1<\lambda<\lambda_1+\delta$, *i.* $u_\lambda(x) < 0$ *for all* $x \in \Omega$,

ii. $\dfrac{\partial u_\lambda}{\partial n}(x) > 0$ *for all* $x \in \partial\Omega$,

where n *is the outward normal.*

Birindelli recently ([4]) extended the anti-maximum principle to general domains but only for right hand sides f which have its support outside of the non smooth boundary. We allow less general domains and more general f. Our estimate will be optimal.

THEOREM 14. *Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *that satisfies Condition 9. Suppose* $f \in L^p(\Omega) \cap C(\Omega)$, *with* $p > 2$ *and such that* $0 \neq f \geq 0$.
We assume that for all $i \in \{1, \dots, k\}$ *there exists* $\vartheta_i > \dfrac{\pi}{\psi_i} - 2$ *and* $\epsilon > 0$ *with*

$$|f(x)| \leq M \left| x - y^{(i)} \right|^{\vartheta_i} \quad \textit{for all } x \in B_\epsilon\left(y^{(i)}\right) \cap \Omega, \tag{28}$$

Suppose u_λ *satisfies the equation in (27). Then there exists* $\delta > 0$, *depending on* f, *such that for* $\lambda_1 < \lambda < \lambda_1 + \delta$, *there exists* $c_1, c_2 > 0$ *with*

$$-c_1 \phi_1(x) \geq u_\lambda(x) \geq -c_2 \phi_1(x) \quad \textit{for all } x \in \Omega.$$

Remark 1: Without loss of generality we may suppose that

$$p < 2 + \frac{2\pi - \psi_i}{\psi_i - \pi} \quad \text{for all } i \in \{k+1, \dots, k+m\}. \tag{29}$$

The result is also optimal in the following sense.

PROPOSITION 15. *Let* $\Omega = D(\psi)$ *for some* $\psi \in (0, \pi)$ *and take* $\vartheta \in \left(-1, \dfrac{\pi}{\psi} - 2\right)$ *with* $\vartheta \leq 0$. *For* $f = r^\vartheta$ *we find that for all* $\lambda \in (\lambda_1, \lambda_2)$ *the solution* u_λ *of (27) changes sign.*

Remark 2: Note that $f = r^\vartheta \in L^p(\Omega)$ for some $p > 2$. For $\psi < \frac{1}{2}\pi$ one may take $f = 1$.

The proof basically follows [6]. First we suppose that Ω not only satisfies Condition 9 but even that $\partial\Omega$ is $C^{2,\alpha}$ smooth except for $C^{2,\alpha}$ smooth corners. It means that the boundary consists of finitely many curves $\Gamma_i = \gamma_i([0,1])$ with $\gamma_i \in C^{2,\alpha}[0,1]$ and $\gamma_i' \neq 0$. For such a domain one can solve (1) for $f \in L^p(\Omega)$, with $2 < p \leq 2+\varepsilon$ and ε small, in the following way.

LEMMA 16. *Let Ω be as above. Then there is $\varepsilon > 0$ and there exist $\{\zeta_i\}_{i=k+1}^{k+m}$ such that*

i. $\Delta\zeta_i \in C^2\left(\bar{\Omega}\right)$

ii. $\operatorname{support}(\zeta_i) \subset B_\varepsilon\left(y^{(i)}\right) \cap \bar{\Omega}$;

iii. $-\phi_1(x) \leq \zeta_i(x) \leq \phi_1(x)$ *for* $x \in B_{\frac{1}{2}\varepsilon}\left(y^{(i)}\right) \cap \bar{\Omega}$;

iv. for all $f \in L^{2+\varepsilon}$ *the solution* $u \in W_0^{1,2}(\Omega)$ *of (1) satisfies*

$$u = \bar{u} + \sum_{i=k+1}^{k+m} c_i\ \zeta_i,$$

with $\bar{u} \in W^{2,2+\varepsilon}(\Omega)$ *and* $c_i \in \mathbb{R}$.

Proof. By the remark following Corollary A.5 one finds holomorphic mappings $\boldsymbol{h}_i$ that maps a neighborhood of $y^{(i)}$ in Ω onto $cD(\psi_i)$ and moreover $h_i \in C^{2,\alpha}\left(\bar{\Omega}\right)$. This implies that $x \mapsto u(x) \in W^{2,p}(\Omega)$ is equivalent with $x \mapsto u\left(h_i^{inv}(x)\right) \in W^{2,p}(h_i(\Omega))$ and even

$$\|u\|_{W^{2,p}(\Omega)} \simeq \left\|u \circ h_i^{inv}\right\|_{W^{2,p}(h(\Omega))} \quad \text{for } u \in W^{2,p}(\Omega).$$

Since we assume (29) we find by [9] for all $i \in \{k+1, \dots, k+m\}$ a neighborhood of 0 $(= h_i\left(y^{(i)}\right))$ with

$$u\left(h_i^{inv}(x)\right) = \tilde{u}_i(x) + c_i\ \xi_i(x)$$

where $\tilde{u}_i \in W^{2,p}(cD(\psi_i))$ and $\xi_i(x) = r_i^{\frac{\pi}{\psi_i}} \cos\left(\frac{\pi}{\psi_i}\varphi_i\right)\eta_i(x)$. The function η_i is chosen such that it localizes ξ_i, that is $\eta_i \in C^\infty\left(\mathbb{R}^2\right)^+$

and for some $0 < \delta_1 < \delta_2 < c$ one has $\eta_i \equiv 1$ on $B_{\delta_1}(0)$ and $\eta_i \equiv 0$ on $\mathbb{R}^2 \backslash B_{\delta_2}(0)$. For u we find that

$$u = \bar{u} + \sum_{i=k+1}^{k+m} c_i \; \xi_i \circ h_i \quad \text{on } \Omega,$$

with $\bar{u} \in W^{2,p}(\Omega)$. Direct calculus shows $\Delta \xi_i \in C^\infty(h(\Omega))$. The estimate in 3) follows from Corollary 8. □

We will replace the function e that is used in [6] by

$$e(x) = \phi_1(x) + d(x).$$

Remember that one has

$$\phi_1(x) \preceq d(x) \not\succeq \phi_1(x) \quad \text{for } x \text{ near a 'convex' corner,}$$
$$\phi_1(x) \succeq d(x) \not\preceq \phi_1(x) \quad \text{for } x \text{ near a 'concave' corner.}$$

The following Banach space (even a Banach lattice) will be used:

$$C_e = \left\{ u \in C_0\left(\bar{\Omega}\right); |u| \preceq e \right\}$$

with norm

$$\|u\|_e = \sup \left\{ \left| \frac{u(x)}{e(x)} \right| ; x \in \Omega \right\}.$$

Since $e(\cdot) \succeq d(\cdot)$ we find that $C^1\left(\bar{\Omega}\right) \cap C_0\left(\bar{\Omega}\right)$ is continuously imbedded in C_e. Since $e(\cdot) \succeq \phi_1(\cdot)$ we find $\zeta_i \in C_e$. Denote $Y = L^p(\Omega)$ (p as above) with its standard norm and

$$X = \left\{ \bar{u} \in W^{2,p}(\Omega); \bar{u} = 0 \text{ on } \partial\Omega \right\} \oplus [\![\zeta_i]\!]_{i=k+1}^{k+m}$$

with norm

$$\|u\|_X = \|\bar{u}\|_{W^{2,p}(\Omega)} + \sum_{i=k+1}^{k+m} |c_i|,$$

where $u = \bar{u} + \sum_{i=k+1}^{k+m} c_i \zeta_i$. Since the set $\{\zeta_i\}$ is independent and $\zeta_i \notin W^{2,p}(\Omega)$ this norm is well defined. Theorem 4.3.2.4 of [9] shows that

$$\|\bar{u}\|_{W^{2,p}(\Omega)} \leq c \left(\|\Delta \bar{u}\|_{L^p(\Omega)} + \|\bar{u}\|_{L^p(\Omega)} \right). \tag{30}$$

Since $W^{2,p}(\Omega)$ is compactly imbedded in $L^p(\Omega)$ we find from Theorem 6.2 of [23] that the operator

$$\Delta : \left\{\bar{u} \in W^{2,p}(\Omega); \bar{u} = 0 \text{ on } \partial\Omega\right\} \to L^p(\Omega)$$

is a semi-Fredholm operator; that is, it has a closed range and a finite dimensional null space. Since $\Delta\zeta_i \in L^p(\Omega)$ we also find that

$$A = \Delta : X \to Y$$

is a semi-Fredholm operator. Theorem 4.4.3.7 of [9] implies that $A \in \mathcal{L}(X;Y)$ has an inverse (and hence A is a Fredholm operator of index 0). We will denote this inverse by T. Being the inverse of a bounded linear operator on Banach spaces (see Theorem 4.1 of [23]) it is bounded. Hence we find

$$\|Tf\|_X \simeq \|f\|_Y \quad \text{for } f \in L^p(\Omega).$$

By Theorem 7.26 of [8] it follows that the imbedding $W^{2,p}(\Omega) \to C^1\left(\bar{\Omega}\right)$ is compact. Hence the imbedding $X \to C^1\left(\bar{\Omega}\right) \oplus [\![\zeta_i]\!]_{i=k+1}^{k+m}$ is compact. Since $\zeta_i \in C_e$ and since $C^1\left(\bar{\Omega}\right) \cap C_0\left(\bar{\Omega}\right)$ is continuously imbedded in C_e we find that the operator $T_e := T : C_e \to C_e$ is well defined and compact. We summarize.

LEMMA 17. *The following imbedding results hold.*

i. $X \to \left(C^1\left(\bar{\Omega}\right) \cap C_0\left(\bar{\Omega}\right)\right) \oplus [\![\zeta_i]\!]_{i=k+1}^{k+m}$ *is compact.*

ii. $\left(C^1\left(\bar{\Omega}\right) \cap C_0\left(\bar{\Omega}\right)\right) \oplus [\![\zeta_i]\!]_{i=k+1}^{k+m} \to C_e$ *is continuous.*

iii. $C_e \to Y$ *is continuous.*

Since $T \in \mathcal{L}(Y;X)$ *the operator* $T_e \in \mathcal{L}(C_e;C_e)$ *is compact.*

Up to now we used that the boundary of Ω consists of piecewise $C^{2,\alpha}$-curves. We may replace $C^{2,\alpha}$ by Dini smoothness using a transformation h as in Corollary A.6. The function h is a diffeomorphism from Ω to a piecewise smooth domain $h(\Omega)$ which has the same corners as Ω and with $0 < c \leq |\nabla h| \leq c^{-1}$ for some $c > 0$. Instead of (27) one considers

$$\left\{\begin{array}{ll} -\Delta u_h - \lambda |\nabla h|^{-2} u_h = |\nabla h|^{-2} f_h & \text{in } h(\Omega), \\ u_h = 0 & \text{on } \partial h(\Omega). \end{array}\right. \tag{31}$$

By the strong maximum principle one finds that T_e is positive and irreducible. Since the operator T_e is compact, positive and irreducible we may use the Krein-Rutman Theorem with the De Pagter Theorem ([20], see also [25]) and find an analogy of Lemma 2 in [6]. The last statement of the next lemma follows from Theorem 10. Indeed, $\mu u - T_e u = g$ is solved by

$$u = \mu^{-1} \left(I - \mu^{-1} T_e\right)^{-1} g = \mu^{-1} \sum_{k=0}^{\infty} \left(\mu^{-1} T_e\right)^k g,$$

and $u \geq \mu^{-2} T_e g \succeq \phi_1$.

LEMMA 18. *We have:*

i. *The spectral radius $r(T_e)$ is positive (and $r(T_e) = \lambda_1^{-1}$).*

ii. *$r(T_e)$ is a simple eigenvalue of T_e with eigenvector ϕ_1, and ϕ_1 is the only eigenvector with fixed sign.*

iii. *$r(T_e)$ is a simple eigenvalue of T_e^* with eigenvector ϕ_1^*, defined by*

$$\phi_1^*(u) = \int_\Omega \phi_1(x)\, u(x)\, dx.$$

iv. *For every $g \in C_e$, with $0 \leq g \neq 0$ and $\mu > r(T_e)$, there exists exactly one solution u of $\mu u - T_e u = g$ and it satisfies $u \succeq \phi_1$.*

Remark 3: Note that in contrary to Lemma 2 of [6] we do not find $T_e f \succeq e$ for $f \in C_e$ with $0 \leq f \neq 0$. This implies that we do not obtain strong positivity of T_e in the sense of [1]. The operator T_e would be strongly positive if $T_e(P_e \backslash \{0\}) \subset \overset{\circ}{P_e}$ where P_e is the positive cone in C_e.

Similarly as in [6] one has the decomposition as in their Lemmata 2 and 3.

LEMMA 19. *The space Y satisfies $Y = [\![\phi_1]\!] \oplus R(A - \lambda_1 I)$. For every $f \in Y$ there exists $f_1 \in R(A - \lambda_1 I)$ such that*

$$f = \alpha\phi_1 + f_1 \tag{32}$$

with $\alpha = \lambda_1^{-1}\phi_1^*(Tf)/\phi_1^*(\phi_1)$.

Let $u \in X$. Since $X \subset Y$ there is a unique decomposition

$$u = \beta\phi_1 + u_1 \tag{33}$$

with $u_1 \in X \cap R(A - \lambda_1 I)$. If u is a solution of (27) with f as in (32) we find

$$\beta = \frac{-\alpha}{\lambda - \lambda_1} \tag{34}$$

$$Au_1 - \lambda u_1 = f_1. \tag{35}$$

Note that $A - \lambda I : X \cap R(A - \lambda_1 I) \subset X \to R(A - \lambda_1 I) \subset Y$ is an isomorphism for $|\lambda - \lambda_1|$ small. Hence there are constants δ, M_{f_1}, not depending on λ, such that

$$\|u_1\|_X < M_{f_1} \quad \text{for all } \lambda \in [\lambda_1 - \delta, \lambda_1 + \delta]. \tag{36}$$

We shall need an additional result for the inverse of this restriction of $A - \lambda I$.

LEMMA 20. *Suppose that $f_1 \in R(A - \lambda_1 I)$ satisfies (28). Then there is M'_{f_1} such that for $|\lambda - \lambda_1| < \delta$ the solution u_1 of (35) satisfies*

$$|u_1(x)| \le M'_{f_1}\ \phi_1(x) \quad \text{for } x \in \Omega. \tag{37}$$

Proof. First note that for all $f \in Y$ which satisfy the conditions of Theorem 14, there exists M_ν such that for all $\lambda \in [0, \lambda_1 - \nu]$, with $\nu > 0$, the solution u of (27) satisfies

$$|u(x)| \le M_\nu\ \phi_1(x). \tag{38}$$

Indeed, denoting by u^λ the solution for λ, we find

$$(-\Delta - \lambda)\left(u^\lambda - u^0\right) = \lambda u^0 \le (\lambda_1 - \nu)\, u^0 \preceq \phi_1$$

and hence by the maximum principle we have, uniformly for $\lambda \in [0, \lambda_1 - \nu]$, that there are c, M_ν such that

$$u^\lambda \leq u^0 + c\ \phi_1 \leq M_\nu\ \phi_1 \quad \text{in } \Omega. \tag{39}$$

From (36) it follows that

$$|u_1(x)| \leq c\ M_1\ e(x) \quad \text{in } \Omega. \tag{40}$$

Since the first eigenvalue λ_{1,Ω^*} on $\Omega^* = B_\varepsilon\left(y^{(i)}\right) \cap \Omega$ can be chosen large for ε small, we may solve

$$\begin{cases} (-\Delta + \lambda)\, u_1 = f_1 & \text{in } \Omega^*, \\ u_1 = u_1 & \text{on } \partial\Omega^*, \end{cases}$$

for $\lambda < \lambda_1 + \delta$ and use (38) and the bound (28) for f_1 near $y^{(i)}$ on Ω^* to find

$$|u_1| \leq M^*_\nu\ \phi_{1,\Omega^*} \leq c\ \phi_1 \quad \text{in } B_{\frac{1}{2}\varepsilon}\left(y^{(i)}\right) \cap \Omega. \tag{41}$$

Together (40) and (41) show the estimate. □

Proof of Theorem 14. Let $u \in X$ be decomposed as in (33) with β and u_1 as in (34-35). If f satisfies (28) then f_1 satisfies (28). For $\lambda \in (\lambda_1, \lambda_1 + \delta)$ we find by Lemma 20 that

$$u \leq \left(\frac{-\alpha}{\lambda - \lambda_1} + M'_{f_1}\right)\ \phi_1$$

The result follows for $0 < \lambda - \lambda_1$ small. □

Proof of Proposition 15. Fix $\lambda \in (\lambda_1, \lambda_2)$. The function u_λ solves $-\Delta u_\lambda = \lambda u_\lambda + r^\vartheta$. First we show that there is $c > 0$ and $r_0 > 0$ such that

$$u_\lambda \geq -c\phi_1 \text{ for } r < r_0. \tag{42}$$

Since $u_\lambda \in C_0\left(\overline{D(\psi)}\right)$ there is $r_0 > 0$ such that $\lambda u_\lambda + r^\vartheta > \frac{1}{2}r^\vartheta$ for $r < r_0$. By the standard Hopf's boundary point Lemma there is $c > 0$ such that $u_\lambda + c\phi_1 > 0$ on $D(\psi) \cap \{r = r_0\}$. Hence we find

$$\begin{cases} -\Delta(u_\lambda + c\phi_1) = \lambda u_\lambda + r^\vartheta + c\lambda_1\phi_1 \geq 0 & \text{in } r_0 D(\psi), \\ u_\lambda + c\phi_1 \geq 0 & \text{on } \partial(r_0 D(\psi)), \end{cases}$$

which implies that $u_\lambda + c\phi_1 > 0$ on $r_0 D(\psi)$.

Now let v solve

$$\begin{cases} -\Delta v = \lambda u_\lambda & \text{in } D(\psi), \\ v = 0 & \text{on } \partial D(\psi). \end{cases}$$

Due to (42) it follows that $-\Delta v \geq -c'\phi_1$ in $D(\psi)$. By the maximum principle and Theorem 5 we find that $v \geq -c^*\phi_1$ and that there is $c_1 > 0$ such that $u_\lambda - v \geq c_1 r^{\vartheta+2-\frac{\pi}{\psi}}\phi_1$. Hence, since $\vartheta + 2 - \frac{\pi}{\psi} < 0$ and

$$u_\lambda \geq \left(c_1 r^{\vartheta+2-\frac{\pi}{\psi}} - c^*\right)\phi_1$$

we find that u_λ is positive near 0. □

9. Green function estimate and 3G-Theorem

Zhao in [29] obtained a two sided estimate for the Green function for $-\Delta$ on a 2-dimensional domain. His result is the following.

- *There exist $C > 0$ such that for all $x, y \in \Omega$*

$$C^{-1}\ G(x,y) \leq \ln\left(1 + \frac{d(x)\, d(y)}{|x-y|^2}\right) \leq C\ G(x,y). \tag{43}$$

See also [26]. This result is not true for Lipschitz domains. Zhao's proof needs Dini smooth boundary ([5]).

THEOREM 21. *Let Ω be a bounded domain in $\mathbb{R}^2$ that satisfies Condition 9. Let $\varepsilon > 0$ be such that $\min\left|y^{(i)} - y^{(j)}\right| = 2\varepsilon$ and let w be defined by*

$$w(x,y) = \begin{cases} \min\left(\left(\frac{d(x)}{\varphi_1(x)}\right)^2, \left(\frac{d(y)}{\varphi_1(y)}\right)^2\right) \\ \qquad \text{for } (x,y) \in \left(B_\varepsilon\left(y^{(i)}\right) \cap \Omega\right)^2 \text{ with } 1 \leq i \leq k, \\ \max\left(\left(\frac{d(x)}{\varphi_1(x)}\right)^2, \left(\frac{d(y)}{\varphi_1(y)}\right)^2\right) \\ \qquad \text{for } (x,y) \in \left(B_\varepsilon\left(y^{(i)}\right) \cap \Omega\right)^2 \text{ with } k+1 \leq i \leq k+m, \\ 1 \qquad \text{elsewhere.} \end{cases}$$

Then there exist $C > 0$ such that for all $x, y \in \Omega$

$$C^{-1}\ G(x,y) \leq \ln\left(1 + w(x,y)\,\frac{\phi_1(x)\,\phi_1(y)}{|x-y|^2}\right) \leq C\ G(x,y)\,. \quad (44)$$

Remark 1: The properties of w are such that one can define a function $\tilde{w} \in C^\infty(\Omega^2)$ with

$$w \preceq \tilde{w} \preceq w \quad \text{on } \Omega^2.$$

Remark 2: Theorem 7.4 in [18] gives an asymptotic expansion of the Green function near a conical boundary point.

Proof. By Koebe's distortion Theorem, see Corollary 1.4 of [21], one finds the following. Let $\boldsymbol{h}$ map $\boldsymbol{\Omega}$ conformally to $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$. Then one has

$$\frac{1}{4}\left(1 - |h(x)|^2\right) \leq d(x)\left|\boldsymbol{h}'(\boldsymbol{x})\right| \leq \left(1 - |h(x)|^2\right)$$

Since

$$G_\Omega(x,y) = G_{B_1(0)}(h(x), h(y))$$

it follows that

$$\frac{1}{C} G_\Omega(x,y) \leq \ln\left(1 + \frac{d(x)\,d(y)\left|\boldsymbol{h}'(\boldsymbol{x})\right|\left|\boldsymbol{h}'(\boldsymbol{y})\right|}{|h(x)-h(y)|^2}\right) \leq C\ G_\Omega(x,y)$$

Stretching a corner, say in 0, with angle $\psi \in (0, 2\pi)$ one uses $h : \Omega \to \mathbb{C}$ defined by $\boldsymbol{h}(z) = z^\alpha$ with $\alpha = \dfrac{\pi}{\psi}$. One finds for some $c_i > 0$ that

$$c_2^{-1}\frac{\phi_1(x)}{d(x)} \leq c_1^{-1}\,|x|^{\alpha-1} \leq \left|\boldsymbol{h}'(\boldsymbol{x})\right| \leq c_1\ \ |x|^{\alpha-1} \leq c_2\frac{\phi_1(x)}{d(x)}.$$

To find estimates for $|h(x) - h(y)|$ we distinguish three cases.

i. Both x and y are near a convex corner $y^{(j)}$ $(1 \leq j \leq k)$. That is $\alpha = \dfrac{\pi}{\psi} > 1$. Then by Lemma A.3

$$c_3^{-1}\,|x-y| \leq \left(\frac{\varphi_1(x)}{d(x)} + \frac{\varphi_1(y)}{d(y)}\right)^{-1} |h(x)-h(y)| \leq c_3\,|x-y|\,. \quad (45)$$

and

$$\frac{d(x)}{\varphi_1(x)} \to \infty \quad \text{when } x \underset{\text{nontangentially}}{\longrightarrow} y^{(j)} \text{ (corner).}$$

Hence one finds

$$c_6^{-1} \frac{\phi_1(x)\,\phi_1(y)}{|x-y|^2} \min\left(\left(\tfrac{d(x)}{\varphi_1(x)}\right)^2, \left(\tfrac{d(y)}{\varphi_1(y)}\right)^2\right) \le$$

$$\le \frac{d(x)\,d(y)\,|\boldsymbol{h}'(\boldsymbol{x})|\,|\boldsymbol{h}'(\boldsymbol{y})|}{|h(x)-h(y)|^2} \le$$

$$\le c_6 \frac{\phi_1(x)\,\phi_1(y)}{|x-y|^2} \min\left(\left(\tfrac{d(x)}{\varphi_1(x)}\right)^2, \left(\tfrac{d(y)}{\varphi_1(y)}\right)^2\right),$$

ii. Both x and y are near a concave corner $y^{(i)}$ $(k+1 \le j \le k+m)$. That is $\alpha = \dfrac{\pi}{\psi} < 1$. Then by Lemma A.3

$$c_3^{-1}|x-y| \le \left(\frac{d(x)}{\varphi_1(x)} + \frac{d(y)}{\varphi_1(y)}\right)|h(x)-h(y)| \le c_3|x-y|. \tag{46}$$

and

$$\frac{d(x)}{\varphi_1(x)} \to 0 \quad \text{when } x \underset{\text{nontangentially}}{\longrightarrow} y^{(i)} \text{ (corner).}$$

Hence

$$c_4^{-1} \frac{\phi_1(x)\,\phi_1(y)}{|x-y|^2} \max\left(\left(\tfrac{d(x)}{\varphi_1(x)}\right)^2, \left(\tfrac{d(y)}{\varphi_1(y)}\right)^2\right) \le$$

$$\le c_3^{-1} \frac{\phi_1(x)\,\phi_1(y)}{|x-y|^2} \left(\tfrac{d(x)}{\varphi_1(x)} + \tfrac{d(y)}{\varphi_1(y)}\right)^2 \le$$

$$\le \frac{d(x)\,d(y)\,|\boldsymbol{h}'(\boldsymbol{x})|\,|\boldsymbol{h}'(\boldsymbol{y})|}{|h(x)-h(y)|^2} \le$$

$$\le c_4 \frac{\phi_1(x)\,\phi_1(y)}{|x-y|^2} \max\left(\left(\tfrac{d(x)}{\varphi_1(x)}\right)^2, \left(\tfrac{d(y)}{\varphi_1(y)}\right)^2\right).$$

iii. x is near $y^{(i)}$ and y is near $y^{(j)} \ne y^{(i)}$. Then

$$c_7^{-1} \frac{\phi_1(x)\,\phi_1(y)}{|x-y|^2} \le \frac{d(x)\,d(y)\,|\boldsymbol{h}'(\boldsymbol{x})|\,|\boldsymbol{h}'(\boldsymbol{y})|}{|h(x)-h(y)|^2} \le c_7 \frac{\phi_1(x)\,\phi_1(y)}{|x-y|^2}.$$

□

A. Auxiliary results related with the Riemann Mapping

Remember that one may use a holomorphic mapping to transform one domain to another domain with possibly little change in the estimates we are interested in. We will start by stating such result. Next we will recall the relation between smoothness of the boundary and smoothness of related conformal mappings. A excellent reference for results in the last direction is the book ([21]) by Ch. Pommerenke.

A.1 Conformal transformation with bounded derivative

A mapping $z \longmapsto \boldsymbol{h}(z)$ that is conformal on $\boldsymbol{\Omega}$ and continuous on $\overline{\boldsymbol{\Omega}}$, changes problem (1) in

$$\begin{cases} -\Delta(u(h^{inv}(x))) = \left|\boldsymbol{h}'\left(\boldsymbol{h}^{inv}(\boldsymbol{x})\right)\right|^{-2} f\left(h^{inv}(x)\right) & \text{for } x \in h(\Omega), \\ u\left(h^{inv}(x)\right) = 0 & \text{on } \partial h(\Omega). \end{cases} \tag{47}$$

If $\left|\boldsymbol{h}'\right|$ is bounded away from 0 and ∞ we can compare solutions of (1) and

$$\begin{cases} -\Delta u_h(x) = f\left(h^{inv}(x)\right) & \text{for } x \in h(\Omega), \\ u_h = 0 & \text{on } \partial h(\Omega). \end{cases} \tag{48}$$

in a uniform way. Similarly we may compare the eigenvalue problems. Let us denote by $\phi_{1,A}$, $\lambda_{1,A}$ the first eigenfunction respectively eigenvalue on A. Since

$$-\Delta\left(\phi_{1,h(\Omega)}(h(x))\right) = \lambda_{1,h(\Omega)} \left|\boldsymbol{h}'(\boldsymbol{x})\right|^2 \phi_{1,h(\Omega)}(h(x)) \quad \text{for } x \in \Omega$$

we have:

LEMMA A.1. *Let $\boldsymbol{h} : \boldsymbol{\Omega} \to \mathbb{C}$ be conformal and satisfying*

$$0 < c_1 \leq \inf_{x\in\Omega} \left|\boldsymbol{h}'(\boldsymbol{x})\right| \leq \sup_{x\in\Omega} \left|\boldsymbol{h}'(\boldsymbol{x})\right| \leq c_2 < \infty. \tag{49}$$

Let u respectively u_h be the solutions of (1) and (48) for some $f \in L^p(\Omega)$ with $p > 2$ and $f \geq 0$. Then

$$c_1^2 u_h(x) \leq u\left(h^{inv}(x)\right) \leq c_2^2 u_h(x) \quad \textit{for } x \in h(\Omega) \tag{50}$$

and

$$\phi_{1,h(\Omega)}(x) \simeq \phi_{1,\Omega}\left(h^{inv}(x)\right) \quad \textit{for } x \in h(\Omega), \tag{51}$$

$$c_2^{-2}\ \lambda_{1,\Omega} \le \lambda_{1,h(\Omega)} \le c_1^{-2}\ \lambda_{1,\Omega}$$

A.2 Holomorpic mappings on domains with corners

We will use conformal mappings $\boldsymbol{h}$ from $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ onto $\boldsymbol{\Omega}$. The smoothness of such a conformal mapping $\boldsymbol{h}$ is directly related with the smoothness of $\partial\Omega$. We refer to a theorem of Kellogg-Warschawski, Theorem 3.6 on page 49 of [21]. Moreover, if $\partial\Omega$ is Dini-smooth, then $\boldsymbol{h}'$ has a continuous extension to $\overline{\mathbb{D}}$ which is nowhere equal to 0 on $\overline{\mathbb{D}}$. See Theorem 3.5 in [21]. In that case the inverse of $\boldsymbol{h}$ has the same regularity as $\boldsymbol{h}$. Remember that Hölder continuity implies Dini continuity.

For the domains that we are interested in we use a Theorem by Lindelöf and an extension by Warschawski. Both results are also found in [21], see Theorem 3.9. The domain Ω satisfies the assumptions of Condition 9.

For $\boldsymbol{h} : \boldsymbol{\Omega} \to \mathbb{C}$ without uniformly bounded derivative the following consequence of Koebe's distortion Theorem holds.

LEMMA A.2. *Suppose Ω satisfies Condition 9. Let $\boldsymbol{h} : \boldsymbol{\Omega} \to \mathbb{C}$ be conformal with $h(\partial\Omega) = \partial h(\Omega)$. Then it follows that*

$$\tfrac{1}{4} d(x, \partial\Omega) \left|\boldsymbol{h}'(\boldsymbol{x})\right| \le d(h(x), \partial h(\Omega)) \le 4 d(x, \partial\Omega) \left|\boldsymbol{h}'(\boldsymbol{x})\right|. \tag{52}$$

Proof. Corollary 1.4 of [21] states that for conformal $\boldsymbol{f} : \mathbb{D} \to \mathbb{C}$

$$\tfrac{1}{4}\left(1 - |z|^2\right) \left|\boldsymbol{f}'(z)\right| \le d(\boldsymbol{f}(z), \partial\boldsymbol{f}(\mathbb{D})) \le \left(1 - |z|^2\right) \left|\boldsymbol{f}'(z)\right|. \tag{53}$$

By Condition 9.1 and the Riemann Mapping Theorem there exists a conformal mapping $\boldsymbol{f} : \mathbb{D} \to \mathbb{C}$ with $f(\mathbb{D}) = \boldsymbol{\Omega}$ and $\boldsymbol{f}(\partial\mathbb{D}) = \partial\boldsymbol{\Omega}$. The claim follows by using (53) for $\boldsymbol{f}$ and $\boldsymbol{h} \circ \boldsymbol{f}$. □

In order to handle the cones we need an estimate for $\boldsymbol{h} : \mathbb{D} \to \mathbb{C}$ defined by $\boldsymbol{h}(\boldsymbol{z}) = \boldsymbol{z}^\alpha$ with $\alpha \in (0, 1)$.

Lemma A.3. *Fix $\delta > 0$. For $\alpha \in (0,1)$ there exist $c > 0$ such that for all $x, y \in D\left(2\left(\pi - \delta\right)\right)$:*

$$c\ \left|x - y\right| \leq \left(\left|x\right|^{1-\alpha} + \left|y\right|^{1-\alpha}\right)\left|\boldsymbol{x}^{\alpha} - \boldsymbol{y}^{\alpha}\right| \leq c^{-1}\ \left|x - y\right|.$$

For $\alpha \in (1,\infty)$ there exist $c > 0$ such that for all $x, y \in D\left(\frac{2(\pi-\delta)}{\alpha}\right)$:

$$c\ \left|x - y\right| \leq \left(\left|x\right|^{\alpha-1} + \left|y\right|^{\alpha-1}\right)^{-1}\left|\boldsymbol{x}^{\alpha} - \boldsymbol{y}^{\alpha}\right| \leq c^{-1}\ \left|x - y\right|.$$

Proof. The result for $\alpha > 1$ is a direct consequence of the result for $\alpha \in (0,1)$. Hence we assume $\alpha \in (0,1)$. For $z \in D\left(2\pi\right)$ the function $\boldsymbol{z} \mapsto \boldsymbol{z}^{\alpha}$ is well defined by $\left(re^{i\varphi}\right)^{\alpha} = r^{\alpha}e^{i\alpha\varphi}$ with $\left|\varphi\right| < \pi$. Assume without loss of generality that $\left|x\right| \leq \left|y\right|$. We set $\boldsymbol{w} = \boldsymbol{x}\ \boldsymbol{y}^{-1}$ and $\boldsymbol{w}_{\alpha} = \boldsymbol{x}^{\alpha}\left(\boldsymbol{y}^{\alpha}\right)^{-1}$. Notice that $\boldsymbol{w}^{\alpha}$ is not well defined in general and if well defined it not necessarily equals $\boldsymbol{w}_{\alpha}$. However, in a small neighborhood of 1 it behaves properly. Indeed, for $\boldsymbol{w} \in K$ with $K = \left\{\boldsymbol{w} \in \mathbb{C}; \left|\arg \boldsymbol{w}\right| \leq \delta, \frac{1}{2} \leq \left|\boldsymbol{w}\right| \leq \frac{3}{2}\right\}$ we find $\left|\arg\left(\boldsymbol{x}\ \boldsymbol{y}^{-1}\right)\right| < 2\delta$ and hence $\boldsymbol{w}^{\alpha} = \boldsymbol{x}^{\alpha}\left(\boldsymbol{y}^{\alpha}\right)^{-1}$. Since $\boldsymbol{w} \mapsto \boldsymbol{w}^{\alpha}$ is conformal on a neighborhood of K there is $c_1 > 0$ such that for $\boldsymbol{w} \in K$:

$$c_1\left|\boldsymbol{w} - 1\right| \leq \left|\boldsymbol{w}_{\alpha} - 1\right| \leq c_1^{-1}\left|\boldsymbol{w} - 1\right|. \tag{54}$$

If $\boldsymbol{w} \in \overline{\mathbb{D}}\backslash K$ then both $\left|\boldsymbol{w} - 1\right|$ and $\left|\boldsymbol{w}_{\alpha} - 1\right|$ are bounded away from 0 and bounded from above. Hence (54) is satisfied for some (other) c_1. Using again $0 < \alpha < 1$ we finish by

$$\left(\left|x\right|^{1-\alpha} + \left|y\right|^{1-\alpha}\right)\left|\boldsymbol{x}^{\alpha} - \boldsymbol{y}^{\alpha}\right| \leq 2\left|y\right|\left|\boldsymbol{w}_{\alpha} - 1\right| \leq 2c_1^{-1}\left|y\right|\ \left|\boldsymbol{w} - 1\right|$$

and

$$\left(\left|x\right|^{1-\alpha} + \left|y\right|^{1-\alpha}\right)\left|\boldsymbol{x}^{\alpha} - \boldsymbol{y}^{\alpha}\right| \geq \left|y\right|\left|\boldsymbol{w}_{\alpha} - 1\right| \geq c_1\left|y\right|\left|\boldsymbol{w} - 1\right|.$$

□

Lemma A.4. *Let $\boldsymbol{f}$ map $\mathbb{D}$ conformally onto $\boldsymbol{\Omega}$ and assume that Ω satisfies Condition 9. Set $\min_{i\neq j}\left|y^{(i)} - y^{(j)}\right| = 2\varepsilon$. Define $\alpha_i = \dfrac{\pi}{\psi_i}$.*

Then

$$\left|f^{inv}(x)-f^{inv}(y)\right| \simeq \begin{cases} \left(\left|x-y^{(i)}\right|^{1-\alpha_i}+\left|y-y^{(i)}\right|^{1-\alpha_i}\right)^{-1}|x-y| \\ \qquad \text{for } (x,y)\in\left(B_\varepsilon\left(y^{(i)}\right)\cap\Omega\right)^2 \\ \qquad \text{with } 1\le i\le k, \\ \left(\left|x-y^{(i)}\right|^{\alpha_i-1}+\left|y-y^{(i)}\right|^{\alpha_i-1}\right)|x-y| \\ \qquad \text{for } (x,y)\in\left(B_\varepsilon\left(y^{(i)}\right)\cap\Omega\right)^2 \\ \qquad \text{with } k+1\le i\le k+m, \\ |x-y| \quad \text{elsewhere,} \end{cases}$$

and

$$\left|\left(\boldsymbol{f}^{inv}\right)'(\boldsymbol{x})\right| \simeq \begin{cases} \left|x-y^{(i)}\right|^{\alpha_i-1} \\ \qquad \text{for } x\in B_\varepsilon\left(y^{(i)}\right)\cap\Omega \text{ with } 1\le i\le k+m, \\ 1 \quad \text{elsewhere.} \end{cases}$$

Remark 1: Power series type expansions at a corner are established by Wigley in [28].

Proof. We start with the first estimate. For

$$(x,y)\notin \bigcup_{i=1}^{k+m}\left(\ B_\varepsilon\left(y^{(i)}\right)^2\right)\cap\Omega^2$$

one finds either $x\in B_{\frac12\varepsilon}\left(y^{(i)}\right)$ and $y\notin B_\varepsilon\left(y^{(i)}\right)$ for some i (or vice versa), or $x,y\in\Omega\backslash\left(\bigcup_{i=1}^{k+m}B_{\frac12\varepsilon}\left(y^{(i)}\right)\right)$. In the first case the estimates hold since $\left|f^{inv}(x)-f^{inv}(y)\right|$ and $|x-y|$ are bounded away from 0. In the second case the estimates follow since $u\mapsto\left|\boldsymbol{f}'(\boldsymbol{u})\right|$ is uniformly bounded away from 0 and ∞ on

$$f^{inv}\left(\overline{\Omega\backslash\left(\bigcup_{i=1}^{k+m}B_{\frac12\varepsilon}\left(y^{(i)}\right)\right)}\right).$$

The last result follows from an adaptation of Theorem 3.5 in [21].

It remains to show the estimate when both x and y belong to $B_\varepsilon\left(y^{(i)}\right)\cap\Omega$. Suppose that $B_\varepsilon\left(y^{(i)}\right)\cap\Omega\subset y^{(i)}+D(\psi)$ after a possible rotation. Then we may define $g : \bar{\Omega} \to \mathbb{R}^2$, with $\boldsymbol{g}$ holomorphic on $\boldsymbol{\Omega}$ by $\boldsymbol{g}(\boldsymbol{x}) = \left(\boldsymbol{x} - \boldsymbol{y}^{(i)}\right)^{\alpha_i}$. By a theorem of Warschawski (Theorem 3.9 in [21]) one finds that

$$|g \circ f(u) - g \circ f(v)| \simeq |u - v| \qquad \text{for } u, v \in f^{inv}\left(B_\varepsilon\left(y^{(i)}\right) \cap \Omega\right).$$

Hence and by using Lemma A.3 we find if $\alpha_i > 1$ that for $x, y \in B_\varepsilon\left(y^{(i)}\right) \cap \Omega$

$$\begin{aligned} \left|f^{inv}(x) - f^{inv}(y)\right| &\simeq |g(x) - g(y)| \simeq \\ &\simeq \left|\left(\boldsymbol{x} - \boldsymbol{y}^{(i)}\right)^{\alpha_i} - \left(\boldsymbol{y} - \boldsymbol{y}^{(i)}\right)^{\alpha_i}\right| \simeq \\ &\simeq \left(\left|x - y^{(i)}\right|^{\alpha_i - 1} + \left|y - y^{(i)}\right|^{\alpha_i - 1}\right)|x - y|. \end{aligned}$$

Similarly the result for $\alpha_i \in (0,1)$ can be shown.

The second statement of Theorem 3.9 in [21] shows that

$$\left|(\boldsymbol{g} \circ \boldsymbol{f})'(\boldsymbol{u})\right| \simeq 1 \qquad \text{for } u \in h^{inv}\left(B_\varepsilon\left(y^{(i)}\right) \cap \Omega\right)$$

and since $\alpha_i > 0$ hence

$$\left|x - y^{(i)}\right|^{\alpha_i - 1} = \frac{1}{\alpha_i}\left|\boldsymbol{g}'(\boldsymbol{x})\right| \simeq \left|\boldsymbol{f}'\left(\boldsymbol{f}^{inv}(\boldsymbol{x})\right)\right|^{-1} = \left|\left(\boldsymbol{f}^{inv}\right)'(\boldsymbol{x})\right|.$$

□

Corollary A.5. *Let Ω satisfy Condition 9. For every $i \in \{1, \ldots, k+m\}$ there is a continuous mapping $h_i : \bar{\Omega} \to \mathbb{R}^2$ such that*

i. $\boldsymbol{h}_i : \boldsymbol{\Omega} \to \mathbb{C}$ *is conformal;*

ii. $\boldsymbol{h}_i(\partial\boldsymbol{\Omega}) = \partial\boldsymbol{h}_i(\boldsymbol{\Omega})$;

iii. $h_i\left(y^{(i)}\right) = 0$;

iv. $h_i(\Omega) \cap B_1(0) = D(\psi_i)$;

v. $\boldsymbol{h}_i'$ *can be extended continuously to* $\partial\boldsymbol{\Omega}$;

vi. $0 < \left|\boldsymbol{h}_i'\right| < \infty$ *on* $\overline{\boldsymbol{\Omega}}$.

Remark 2: Assuming that $\Gamma \subset \partial\Omega$ is $C^{n,\alpha}$ with $0 < \alpha < 1$ implies that $\boldsymbol{h}_i \in C^{n,\alpha}$ on $\boldsymbol{\Gamma}$. See Theorem 3.6 of [21].

Proof. Since $\Gamma^+_{i,\varepsilon}$ and $\Gamma^-_{i,\varepsilon}$ are Dini smooth arcs γ' is uniformly bounded near the corner. Hence there exists a domain Ω^* that satisfies $\partial\Omega^* \supset \Gamma^+_{i,\varepsilon/2} \cup \Gamma^-_{i,\varepsilon/2}$, $\Omega \subset \Omega^* \neq \Omega$ and $\partial\Omega^* \setminus \left\{y^{(i)}\right\}$ is Dini smooth. By the Riemann Mapping Theorem there exist a holomorphic mapping $\boldsymbol{f}$ from $\mathbb{D}$ onto $\boldsymbol{\Omega}^*$ with $f((-1,0)) = y^{(i)}$ and $f((1,0)) \in \partial\Omega^* \setminus \partial\Omega$. Set

$$\begin{aligned} \boldsymbol{g}_1(\boldsymbol{z}) &= \frac{\boldsymbol{z}-1}{\boldsymbol{z}+1} \quad \text{for } \boldsymbol{z} \text{ with } |\arg \boldsymbol{z}| \leq \tfrac{1}{2}\pi , \\ \boldsymbol{g}_2(\boldsymbol{z}) &= \boldsymbol{z}^{\frac{\pi}{\psi_i}} \quad \text{for } \boldsymbol{z} \text{ with } |\arg \boldsymbol{z}| \leq \psi_i . \end{aligned}$$

For sufficiently large constant $c > 0$ the function h_i defined by

$$\boldsymbol{h}_i = c\, \boldsymbol{g}_2^{inv} \circ \boldsymbol{g}_1^{inv} \circ \boldsymbol{f}^{inv}$$

satisfies the assumptions above. □

In a similar way one shows:

COROLLARY A.6. *Let Ω satisfy Condition 9. Then there is a continuous mapping $h : \bar{\Omega} \to \mathbb{R}^2$ such that*

i. $\boldsymbol{h} : \boldsymbol{\Omega} \to \mathbb{C}$ *is conformal;*

ii. $h(\partial\Omega) = \partial h(\Omega)$*;*

iii. $\boldsymbol{h}'$ *can be extended continuously to* $\partial\boldsymbol{\Omega}$;

iv. $0 < |\boldsymbol{h}'| < \infty$ *on* $\overline{\boldsymbol{\Omega}}$;

v. $\partial h(\Omega) \setminus \left\{h\left(y^{(i)}\right); 1 \leq i \leq k+m\right\} \in C^\infty$*;*

vi. $\partial h(\Omega)$ *at* $h\left(y^{(i)}\right)$*, with* $1 \leq i \leq k+m$*, has a corner with the same angle as* $\partial\Omega$ *at* $y^{(i)}$*.*

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