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Poisson Estimates and Maximal Regularity for Evolutionary Integral Equations in L_p-Spaces

JAN PRÜSS (*)

In Memory of Pierre Grisvard

SUMMARY. - *Consider the evolutionary integral equation*

$$u(t) + \int_0^t b(s)Au(t-s)ds = f(t), \quad t \in \mathbb{R}_+,$$

in the spaces $X = L_r(\Omega; \mathbb{R}^N)$, $1 < r < \infty$, *where* $\Omega \subset \mathbb{R}^n$ *denotes a Lipschitz-domain,* $b \in L_{1,loc}(\mathbb{R}_+)$, $f \in L_p(\mathbb{R}_+; X)$, *and* A *is such that* $(z+A)^{-1}$ *admits a kernel representation*

$$[(z+A)^{-1}g](x) = \int_\Omega \gamma_z(x,y)g(y)dy, \quad x \in \Omega,$$

with kernel $\gamma_z(x,y)$ *satisfying a Poisson estimate in a suitable sector* $z \in \Sigma_\phi$ *of the complex plane. Assuming that the equation in question is parabolic, it is shown that its fundamental solution admits also a kernel representation with a kernel subject to a Poisson estimate, and that the equation has the maximal regularity property in* $L_p(\mathbb{R}_+; L_r(\Omega; \mathbb{R}^N))$, *for* $1 < p, r < \infty$.

(*) Indirizzo dell'Autore: Fachbereich Mathematik und Informatik, Martin-Luther-Universität Halle-Wittenberg, Theodor-Lieser Str. 5 – D-60120 Halle (Germany).

1. Introduction

Let X be a Banach space, A a closed linear, but in general unbounded operator in X, $b \in L_{1,loc}(\mathbb{R}_+)$, and $f \in L_p(\mathbb{R}_+; X)$, $1 \le p \le \infty$. In this paper we consider the evolutionary integral equation

$$u(t) + \int_0^t b(s)Au(t-s)ds = f(t), \quad t \in \mathbb{R}_+. \tag{1.1}$$

Problems of this type have attracted much interest during the last decades, due to their various applications in mathematical physics like viscoelasticity, thermodynamics, or electrodynamics with memory. For a recent comprehensive presentation of the state of the art in the theory for (1.1) we refer to the author's monograph [11]. In applications, the operator A typically is a differential operator acting in spatial variables, like the Laplacian, the Stokes operator, or the elasticity operator, which implies that X is a space of vector-functions on a domain $\Omega \subset \mathbb{R}^n$ like $L_r(\Omega; \mathbb{R}^N)$, $1 \le r \le \infty$. $b(t)$ should be thought of as a kernel like $b(t) = e^{-\eta t} t^{\beta - 1}/\Gamma(\beta)$, $\eta \ge 0$, $\beta \in (0, 2)$. We are here interested in the *parabolic case*, and want to study in particular *maximal regularity* properties of (1.1). Recall the (1.1) has the maximal regularity property w.r.t. a function space $\mathcal{F}(\mathbb{R}_+; X)$ if (1.1) admits a unique solution $u \in \mathcal{F}(\mathbb{R}_+; X)$, for any given $f \in \mathcal{F}(\mathbb{R}_+; X)$.

To be able to apply Laplace transform methods to (1.1) we assume in the sequel that b is of *subexponential growth*, which means

$$\int_0^\infty |b(t)|e^{-\varepsilon t} dt < \infty \qquad \text{for each } \varepsilon > 0.$$

Recall that (1.1) is called *parabolic* if $\hat{b}(\lambda) \ne 0$ for $\mathrm{Re}\lambda > 0$, $-1/\hat{b}(\lambda) \in \rho(A)$, and there is a constant $M > 0$ such that

$$|(I + \hat{b}(\lambda)A)^{-1}| \le M \qquad \text{for } \mathrm{Re}\lambda \ge 0. \tag{1.2}$$

Here the hat indicates the Laplace transform. It has been shown in [11] that parabolicity of (1.1) is a necessary condition for (1.1) to have the maximal regularity property of type $L_p(\mathbb{R}_+; X)$, for any $p \in [1, \infty]$.

In this paper we are concerned with the converse of this statement, i.e. when does parabolicity imply maximal regularity of type $L_p(\mathbb{R}_+; X)$. At the time being there are essentially three different methods known to prove such results which we briefly describe now.

(i) Multiplier techniques
This method is based on the representation

$$\hat{u}(\lambda) = (I + \hat{b}(\lambda)A)^{-1}\hat{f}(\lambda), \qquad \text{for } \mathrm{Re}\lambda > 0, \tag{1.3}$$

which follows from (1.1) by taking Laplace transforms and using parabolicity. If X is a Hilbert space, it is well known that the Laplace transform induces an isomorphism from $L_2(\mathbb{R}_+; X)$ onto $\mathcal{H}_2(\mathbb{C}_+; X)$, the Hardy space of power 2. Hence without any further assumptions on A or on the kernel $b(t)$, (1.2) and (1.3) imply maximal regularity of $L_2(\mathbb{R}_+; X)$.

This result can be extended in two different ways. Firstly, if again X is a Hilbert space, the Mikhlin multiplier theorem is known to be valid in the vector-valued case as well. Therefore, if in addition with $M(\lambda) = (I + \hat{b}(\lambda)A)^{-1}$

$$|M(\lambda)| + |\lambda M'(\lambda)| \leq C \qquad \text{for } \mathrm{Re}\lambda \geq 0. \tag{1.4}$$

then (1.1) has maximal regularity of type $L_p(\mathbb{R}_+; X)$, $1 < p < \infty$. Since

$$M'(\lambda) = -\hat{b}'(\lambda)A(I + \hat{b}(\lambda)A)^{-2},$$

(1.4) follows from parabolicity of (1.1), provided the kernel $b(t)$ satisfies an additional mild regularity assumption, namely if b is *1-regular*. Recall that b is *k-regular* ($k \in \mathbb{N}$) if

$$|\lambda^j \hat{b}^{(j)}(\lambda)| \leq c|\hat{b}(\lambda)| \qquad \text{for } \mathrm{Re}\lambda > 0, 1 \leq j \leq k. \tag{1.5}$$

For a discussion of k-regularity and for the results stated above we refer to [11]. On the other hand, it has been shown by Pisier that the Mikhlin multiplier theorem holds for the spaces $L_p(\mathbb{R}; X)$ *only* if X is a Hilbert space.

The second extension is based on the fact, that a Mikhlin multiplier theorem remains valid in the vector-valued case for the Besov-spaces $\overset{\circ}{B}{}^s_{p,q}(\mathbb{R}_+; X)$, $p, q \in [1, \infty]$, where X is an *arbitrary* Banach space; see Amann [1] or Weis [16]. The kernel $b(t)$ then should be 2-regular. However, results for these spaces have been obtained before in [11] by the following method.

(ii) Singular convolutions
This method relies on the variation of parameters formula

$$u(t) = f(t) + \int_0^t \dot{S}(t-s)f(s)ds, \quad t > 0 \tag{1.6}$$

for the solution of (1.1). Here $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ denotes the *resolvent family* of (1.1) (or *fundamental solution*) defined by

$$S(t) + A\int_0^t b(t-\tau)S(\tau)d\tau = I, \quad t > 0. \tag{1.7}$$

$S(t)$ is known to exist in case (1.1) is parabolic, and if b is 2-regular we have in addition the estimates

$$|S(t)| + |t\dot{S}(t)| \leq M, \quad t > 0, \tag{1.8}$$

and

$$|t^2\dot{S}(t) - s^2\dot{S}(s)| \leq M|t-s|[1 + \log\frac{t}{t-s}], \quad t > s \geq 0. \tag{1.9}$$

Observe that (1.6) contains a singular convolution since $\dot{S}(t)$ will not be integrable at 0, unless A is a bounded operator which is the trivial case. However, based on estimates (1.8) and (1.9), it has been proved in [11] that (1.1) has the maximal regularity property for the Besov spaces $\overset{\circ}{B}{}^s_{p,q}(\mathbb{R}_+;X)$, where $p,q \in [1,\infty]$, $s \notin \mathbb{N}_0$, and X is an arbitrary Banach space.

(iii) Operator sums
Fix any function space $\mathcal{F}(\mathbb{R}_+;X)$ and define operators in this space by means of

$$(\mathcal{A}u)(t) = Au(t), \quad t \geq 0, \tag{1.10}$$

with domain $D(\mathcal{A}) = \{u \in Y : \mathcal{A}u \in Y\}$, and via Laplace transforms

$$(\mathcal{B}u)\widehat{\ }(\lambda) = \frac{1}{\hat{b}(\lambda)}\hat{u}(\lambda), \quad \mathrm{Re}\lambda > 0, \tag{1.11}$$

with $D(\mathcal{B})$ appropriate. Then (1.1) can be rewritten as

$$\mathcal{A}u + \mathcal{B}u = \mathcal{B}f, \tag{1.12}$$

hence the solution u is formally given by

$$u = \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1} f, \tag{1.13}$$

since $\mathcal{A}$ and $\mathcal{B}$ commute. Thus, if it can be shown that $\mathcal{A} + \mathcal{B}$ with domain $D(\mathcal{A} + \mathcal{B}) = D(\mathcal{A}) \cap D(\mathcal{B})$ is invertible, in particular closed, then (1.1) has the maximal regularity property of type $\mathcal{F}(\mathbb{R}_+; X)$.

This approach was used for the first time in the fundamental paper of Da Prato and Grisvard [4] for evolution equations (i.e. $b(t) \equiv 1$) more than 20 years ago. For evolutionary integral equations it has been used by Clement and Da Prato [3]; see also Lunardi [10] and Pugliese [13]. Without going into details let us mention that for 1-regular kernels b by means of the Da Prato-Grisvard theorem one can show that the spaces $L_p(\mathbb{R}_+; D_A(\alpha, q))$ are spaces with the maximal regularity property for (1.1), where $p \in [1, \infty)$, $q \in [1, \infty]$, $\alpha \notin \mathbb{N}_0$. Here $D_A(\alpha, q)$ denote the real interpolation spaces between X and $X_A = (X, |\cdot|_A)$, $|x|_A = |x| + |A_x|$.

The Da Prato-Grisvard theorem was lateron improved by Dore and Venni [6] and by Prüss and Sohr [12]. Imposing a condition on the Banach space X, namely ζ-convexity or equivalently the UMD-property, and restricting the class of operators A under consideration to $A + \omega_A \in BIP(X)$ these results imply that (1.1) has the maximal regularity property for the spaces $L_p(\mathbb{R}_+; X)$, $1 < p < \infty$, provided b is assumed to be 1-regular. This result is worked out in §8 of [11], and it generalizes to the vector-valued fractional Sobolev spaces $\overset{\circ}{H}{}^s_p(\mathbb{R}_+; X)$, $s \geq 0$, $1 < p < \infty$. □

In this paper we want to introduce another method for proving maximal regularity of (1.1) in spaces $L_p(\mathbb{R}_+; L_r(\Omega; \mathbb{R}^N))$, $1 < p, r < \infty$, when A is an elliptic differential operator such that its resolvent satisfies *Poisson estimates*, i.e. $(z + A)^{-1}$ is represented as

$$((z + A)^{-1} g)(x) = \int_\Omega \gamma_z(x, y) g(y) dy, \quad x \in \Omega, \tag{1.14}$$

and

$$|\gamma_z(x,y)| \leq C|z|^{\frac{n}{m}-1} p(|x-y||z|^{\frac{1}{m}}), \quad x,y \in \Omega, \tag{1.15}$$

where $z \in \mathbb{C}$ is restricted to a suitable sector. The function $p : (0,\infty) \to (0,\infty)$ is assumed to be continuous, nonincreasing, and such that

$$\int_0^\infty p(r) r^n \frac{dr}{r} < \infty, \tag{1.16}$$

or instead of (1.16) satisfies the stronger condition

$$p(r)(r^{n+\delta} + r^{n-\delta}) \leq M < \infty, \quad r > 0, \tag{1.17}$$

for some $\delta > 0$. Here $\Omega \subset \mathbb{R}^n$, and m refers to the order of the elliptic differential operator under consideration; see Section 2 for more details. We show that, in case (1.1) is parabolic, the fundamental solution $S(t)$ of (1.1) admits a kernel representation, too, and the kernel satisfies certain Poisson estimates as well. This is well known in the case of analytic semigroups but seems to be new for evolutionary integral equations of the form (1.1).

At this point we want to draw the readers attention to an important difference between the case of evolution equations and the case of general kernels. If $b(t) \equiv 1$ then the kernel bound for the fundamental solution $S(t) = e^{-At}$ is a bounded function for fixed $t > 0$, while for general $b(t)$ this is not the case. It can be shown that for $\Omega = \mathbb{R}^n$, $n > 1$, $N = 1$, $A = -\Delta$, $b(t) = t^{\beta-1}/\Gamma(\beta)$, where $\beta \in (0,2)$, the kernel representing $S(t)$ is bounded if and only if $\beta = 1$.

Section 3 contains the main result of this paper, namely the maximal regularity of (1.1) in $L_p(\mathbb{R}_+; L_r(\Omega; \mathbb{R}^N))$. For the case of evolution equations, where $b(t) \equiv 1$, this was proved recently in Hieber and Prüss [9]. The proof of the main result is carried out in Sections 3 and 4. It is similar in spirit to that of Hieber and Prüss, which in turn was inspired by the recent paper of Duong and Robinson [7]. But it is different in a number of details, due to the fact that the kernel bound $q(r)$ for the fundamental solution $S(t)$ is not a bounded function. The appendix, Section 5, contains some estimates for the iterates of γ_z as well as the sup-inf inequality which are needed in the proof of the main result.

2. Poisson Estimates

Let $X = L_r(\Omega; \mathbb{R}^N)$, $1 \le r \le \infty$, where $\Omega \subset \mathbb{R}^n$ is an open domain. Let A be a closed linear operator in X which is *sectorial* in the sense that $\rho(A) \supset (-\infty, 0)$ and there is a constant $M > 0$ such that

$$|(t+A)^{-1}| \le \frac{M}{t} \qquad \text{for all } t > 0. \tag{2.1}$$

Then the resolvent set $\rho(-A)$ contains a sector of the form $\Sigma_\phi = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \phi\}$. The *spectral angle* ϕ_A of A is defined by

$$\phi_A := \inf\{\phi \in [0,\pi) : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{z \in \Sigma_{\pi-\phi}} |z(z+A)^{-1}| < \infty\}.$$

Then $\sigma(A) \subset \overline{\Sigma}_{\phi_A}$ and we let

$$M_{\pi-\phi} := \sup\{|z(z+A)^{-1}| : z \in \Sigma_{\pi-\phi}\}.$$

By X_A we denote the domain $D(A)$ of A equipped with the graph norm $|\cdot|_A$ of A; note that X_A is also a Banach space, by closedness of A. Concerning the kernel b we assume that b is ϕ_b-*sectorial* in the sense that

$$\hat{b}(\lambda) \neq 0 \qquad \text{for all } \mathrm{Re}\lambda > 0, \ \text{ and}$$
$$\phi_b := \sup\{|\arg \hat{b}(\lambda)| : \ \mathrm{Re}\lambda > 0\} < \infty \tag{2.2}$$

Then equation (1.1) is *parabolic* provided $\phi_A + \phi_b < \pi$, and the Laplace transform $H(\lambda)$ of the fundamental solution of (1.1) is given by

$$H(\lambda) = \frac{1}{\lambda}(1 + \hat{b}(\lambda)A)^{-1}, \quad \mathrm{Re}\lambda > 0. \tag{2.3}$$

The main hypothesis concerning A is contained in the following definition.

DEFINITION 1. *Let A be a sectorial operator in X. A is said to belong to the* **Poisson class** $\mathcal{P}(X)$ *if the resolvent $(z+A)^{-1}$ of A admits a kernel representation*

$$[(z+A)^{-1}f](x) = \int_\Omega \gamma_z(x,y)f(y)dy, \quad x \in \Omega, \tag{2.4}$$

for $z \in \Sigma_{\pi-\phi}$*, and the measurable kernel* γ_z *satisfies the* **Poisson estimate**

$$|\gamma_z(x,y)| \leq C|z|^{\frac{n}{m}-1}p(|x-y||z|^{1/m}), \quad x,y \in \Omega,\ z \in \Sigma_{\pi-\phi}, \quad (2.5)$$

where $p : (0,\infty) \to (0,\infty)$ *is continuous nonincreasing and such that*

$$\int_0^\infty p(r)r^{n-1}dr < \infty.$$

The **Poisson angle** ϕ_A^P *of* A *is defined as the infimum of all* $\phi > 0$ *such that (2.4) and (2.5) are valid.*

EXAMPLE 1. As a typical example for an operator A belonging to the class $\mathcal{P}(X)$ consider $\Omega = \mathbb{R}^n$ and A the L_r-realization of a system of differential operator of order m with constant coefficients, i.e.

$$(Au)(x) = (\mathbb{A}(D)u)(x) = \Sigma_{|\alpha|=m}a_\alpha D^\alpha u(x), \quad x \in \mathbb{R}^n, \qquad (2.6)$$

where $a_\alpha \in \mathbb{C}^{N\times N}$. If $\mathbb{A}$ is *elliptic* in the sense that $\sigma(\mathbb{A}(i\xi)) \subset \overline{\Sigma}_{\phi_A}\setminus\{0\}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, then $A \in \mathcal{P}(X)$ with

$$p(r) = \int_0^\infty e^{-\kappa r(1+s)}\frac{s^{n-2}ds}{(1+s)^{m-1}}, \quad r > 0, \qquad (2.7)$$

where $\kappa > 0$ is a constant, and the Poisson angle ϕ_A^P equals ϕ_A. □

This result extends to many other (systems of) differential operators with nonconstant coefficients on domains $\Omega \neq \mathbb{R}^n$, for example to boundary value problems of the Agmon-Douglis-Nirenberg type on sufficiently smooth domains, with sufficiently smooth coefficients. There is a large literature on Poisson estimates, however, here we refer only to the recent monographs of Davies [5] and Robinson[14], to the recent paper of Arendt and ter Elst [2], and the references given there. See also Hieber and Prüss [9] for further examples and discussions.

The purpose of this section is to obtain Poisson estimates for the fundamental solution $S(t)$ of (1.1) in case A belongs to the class $\mathcal{P}(X)$. The result reads as follows.

THEOREM 1. *Let $X = L_r(\Omega;\mathbb{R}^N)$, $1 \le r < \infty$, suppose $A \in \mathcal{P}(X)$, and let $b \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth, ϕ_b-sectorial and 1-regular. Assume $\phi_A^P + \phi_b < \pi$, in particular (1.1) is parabolic, and suppose*

$$c|\lambda|^{-\beta} \le |\widehat{b}(\lambda)| \le c^{-1}|\lambda|^{-\beta}, \quad \mathrm{Re}\lambda > 0, \tag{2.8}$$

for some constants $c > 0$ and $\beta \in (0,2)$.

Then (1.1) has a fundamental solution $\{S(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ which admits the kernel representation

$$(S(t)f)(t) = \int_\Omega \sigma_t(x,y)f(y)dy, \quad x \in \Omega,\ t > 0, \tag{2.9}$$

for each $f \in X$, where σ_t is measurable and subject to the Poisson estimate

$$|\sigma_t(x,y)| \le t^{-\beta n/m}q(|x-y|t^{-\beta/m}), \quad x,y \in \Omega, x \ne y,\ t > 0, \tag{2.10}$$

with a continuous noincreasing function $q : (0,\infty) \to (0,\infty)$ satisfying

$$\int_0^\infty q(r)r^{n-1}dr < \infty. \tag{2.11}$$

In particular, the estimate

$$|S(t)|_{\mathcal{B}(X)} \le \int_0^\infty q(r)r^{n-1}dr < \infty$$

shows that $\{S(t)\}_{t>0}$ is uniformly bounded in $\mathcal{B}(X)$.

Proof. Since $\phi_A^P \ge \phi_A$ and b is 1-regular, Theorem 3.1 of Prüss [11] shows that the fundamental solution $S(t)$ of (1.1) exists, is strongly continuous on $\mathbb{R}_+$, and locally Hölder-continuous of any order $\alpha \in [0,1)$ on the open halfline $(0,\infty)$. It is given by the representation formula

$$S(t) = \frac{-1}{2\pi i t}\int_\Gamma H'(\lambda)e^{\lambda t}d\lambda, \quad t > 0,$$

where Γ denotes any contour $\gamma + is$, $s \in \mathbb{R}$, with $\gamma > 0$. Since for $\mathrm{Re}\lambda > 0$ we have

$$\begin{aligned} -H'(\lambda) &= \lambda^{-2}(I + \widehat{b}(\lambda)A)^{-1} + \lambda^{-1}\widehat{b}'(\lambda)A(I + \widehat{b}(\lambda)A)^{-2} \\ &= \lambda^{-2}(I + \widehat{b}(\lambda)A)^{-1}[1 + \frac{\lambda\widehat{b}'(\lambda)}{\widehat{b}(\lambda)}(1 - (I + \widehat{b}(\lambda)A)^{-1})], \end{aligned}$$

we obtain the following representation for the kernel $\sigma_t(x,y)$ of $S(t)$.

$$\sigma_t(x,y) = \frac{-1}{2\pi i t}\int_\Gamma \Big[\gamma_{1/\widehat{b}(\lambda)}(x,y)\Big(1+\frac{\lambda\widehat{b}'(\lambda)}{\widehat{b}(\lambda)}\Big) - \\ - \gamma^1_{1/\widehat{b}(\lambda)}(x,y)\frac{\lambda\widehat{b}'(\lambda)}{\widehat{b}^2(\lambda)}\Big]e^{\lambda t}\frac{d\lambda}{\lambda^2\widehat{b}(\lambda)},$$

where $\gamma^1_z(x,y) = \int_\Omega \gamma_z(x,\xi)\gamma_z(\xi,y)d\xi$, denotes the first iterate of γ_z. By 1-regularity of the kernel $b(t)$, this representation yields the estimate

$$|\sigma_t(x,y)| \le \frac{C}{t}\int_\Gamma |e^{\lambda t}|[|\gamma_{1/\widehat{b}(\lambda)}(x,y)| + |\widehat{b}(\lambda)^{-1}\gamma^1_{1/\widehat{b}(\lambda)}(x,y)|]\frac{|d\lambda|}{|\lambda^2\widehat{b}(\lambda)|}.$$

By Definition 1 we have

$$|\gamma_z(x,y)| \le C|z|^{n/m-1}p(|x-y||z|^{1/m}), \quad z\in\Sigma_{\pi-\phi},\ x,y\in\Omega,$$

hence from the definition of γ^1_z we obtain

$$\begin{aligned}
|z\gamma^1_z(x,y)| &\le |z|\int_\Omega |\gamma_z(x,\xi)||\gamma_z(\xi,y)|d\xi\\
&\le C|z|^{2n/m-1}\int_{\mathbb{R}^n} p(|x-\xi||z|^{1/m})p(|\xi-y||z|^{1/m})d\xi\\
&= C|z|^{n/m-1}\int_{\mathbb{R}^n} p(|x|z|^{1/m}-\xi|)p(|\xi-y|z|^{1/m}|)d\xi\\
&= C|z|^{n/m-1}p_1(|x-y||z|^{1/m})\\
&\le C|z|^{n/m-1}p(c|x-y||z|^{1/m}),
\end{aligned}$$

with some constants $C,c>0$, by Proposition 1 of the Appendix, Section 5. Therefore we can deduce with $z=1/\widehat{b}(\lambda)$ and $\phi^P_A+\phi_b<\pi$

$$|\sigma_t(x,y)| \le \frac{C}{t}\int_\Gamma |e^{\lambda t}||\widehat{b}(\lambda)|^{1-n/m}p(c|x-y||\widehat{b}(\lambda)|^{-1/m})\frac{|d\lambda|}{|\lambda^2\widehat{b}(\lambda)|}.$$

Choosing $\gamma = 1/t$ and using estimate (2.8) as well as monotonicity of p we get

$$|\sigma_t(x,y)| \le \frac{C}{t}\int_\Gamma |e^{\lambda t}||\lambda|^{\beta n/m}p(c|x-y||\lambda|^{\beta/m})\frac{|d\lambda|}{|\lambda^2|}$$

$$\begin{aligned}
&= \frac{C}{t}\int_{-\infty}^{\infty} |e^{(1/t+is)t}||1/t+is|^{\beta n/m} \\
&\qquad\qquad p(c|x-y||1/t+is|^{\beta/m})\frac{ds}{|1/t+is|^2} \\
&= Ct^{-\beta n/m}\int_0^{\infty} p(c|x-y|t^{-\beta/m}[1+s^2]^{\beta/2m}) \\
&\qquad [1+s^2]^{\beta n/2m-1}ds \\
&= Ct^{-\beta n/m}q(|x-y|t^{-\beta/m}),
\end{aligned}$$

where

$$q(r)=\int_0^{\infty} p(cr[1+s^2]^{\beta/2m})[1+s^2]^{\beta n/2m-1}ds, \quad r>0.$$

Obviously, $q:(0,\infty)\to(0,\infty)$ is nonincreasing since p has this property, and

$$\begin{aligned}
\int_0^{\infty} q(r)r^{n-1}dr &= \int_0^{\infty}\int_0^{\infty} p(cr[1+s^2]^{\beta/2m})r^{n-1} \\
&\qquad\qquad [1+s^2]^{\beta n/2m-1}ds\,dr \\
&= \int_0^{\infty}\Big[\int_0^{\infty} p(c\rho)\rho^{n-1}d\rho\Big]\frac{ds}{1+s^2} \\
&\le \int_0^{\infty}\Big[\int_0^{\infty} p(c\rho)\rho^{n-1}d\rho\Big]\frac{ds}{1+s^2}<\infty.
\end{aligned}$$

This proves Theorem 1. □

In a quite similar way we can also treat the *integral resolvent family* (or *fundamental solution of second kind*) $R(t)$ which is defined as the solution of the convolution equation

$$R(t)+A\int_0^t b(\tau)R(t-\tau)d\tau=b(t), \quad t>0. \tag{2.12}$$

It can been shown as in the author's monograph [11] for $S(t)$ that $\{R(t)\}_{t>0}\subset\mathcal{B}(X)$ exists and is continuous in $\mathcal{B}(X)$, provided A is ϕ_A-sectorial, b is 1-regular and ϕ_b-sectorial, and $\phi_A+\phi_b<\pi$, which implies that (1.1) is parabolic. The Laplace transform of R is given by

$$\widehat{R}(\lambda)=\widehat{b}(\lambda)(I+\widehat{b}(\lambda)A)^{-1}, \quad \mathrm{Re}\lambda>0,$$

hence $R(t)$ is represented by the complex integral

$$R(t) = \frac{1}{2\pi i t}\int_\Gamma \widehat{b}'(\lambda)(I+\widehat{b}(\lambda)A)^{-2}e^{\lambda t}d\lambda, \quad t>0,$$

where as before Γ denotes any contour $\lambda = \gamma + is$, $s \in \mathbb{R}$, with $\gamma > 0$. If we let again denote by γ_z^1 the first iterate of γ_z, i.e. $\gamma_z^1(x,y) = \int_\Omega \gamma_z(x,\xi)\gamma_z(\xi,y)d\xi$, then the kernel $\rho_t(x,y)$ of $R(t)$ is given by

$$\rho_t(x,y) = \frac{1}{2\pi i t}\int_\Gamma \frac{\widehat{b}'(\lambda)}{\widehat{b}^2(\lambda)}\gamma^1_{1/\widehat{b}(\lambda)}(x,y)e^{\lambda t}d\lambda, \quad t>0,\ x,y\in\Omega.$$

We can then obtain a Poisson estimate for $\rho_t(x,y)$ in a similar way as in the proof of Theorem 1.

$$\begin{aligned}
|\rho_t(x,y)| &\le \frac{C}{t}\int_\Gamma |e^{\lambda t}||\widehat{b}(\lambda)|^{1-n/m}p(c|x-y||\widehat{b}(\lambda)|^{-1/m})\frac{|d\lambda|}{|\lambda|} \\
&\le \frac{C}{t}\int_\Gamma |e^{\lambda t}||\lambda|^{\beta(n/m-1)}p(c|x-y||\lambda|^{\beta/m})\frac{|d\lambda|}{|\lambda|} \\
&= \frac{C}{t}\int_{-\infty}^\infty |e^{(1/t+is)t}||1/t+is|^{\beta(n/m-1)} \\
&\qquad\qquad p(c|x-y||1/t+is|^{\beta/m})\frac{ds}{|1/t+is|} \\
&= Ct^{-\beta(n/m-1)-1}\int_0^\infty p(c|x-y|t^{-\beta/m}[1+s^2]^{\beta/2m}) \\
&\qquad\qquad [1+s^2]^{\beta n/2m-(1+\beta)/2}ds \\
&= Ct^{-\beta(n/m-1)-1}q(|x-y|t^{-\beta/m}),
\end{aligned}$$

where this time

$$q(r) = \int_0^\infty p(cr[1+s^2]^{\beta/2m})[1+s^2]^{\beta n/2m-(1+\beta)/2}ds, \quad r>0.$$

Note that also in this case $q : (0,\infty) \to (0,\infty)$ is nonincreasing and satisfies

$$\int_0^\infty q(r)r^{n-1}dr \le (\int_0^\infty p(\rho)\rho^{n-1}d\rho)(\int_0^\infty \frac{ds}{[1+s^2]^{(1+\beta)/2}} < \infty.$$

As a result we obtain

THEOREM 2. *Let $X = L_r(\Omega;\mathbb{R}^N)$, $1 \le r < \infty$, suppose $A \in \mathcal{P}(X)$, and let $b \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth, ϕ_b-sectorial and 1-regular. Assume $\phi_A^P + \phi_b < \pi$, in particular (1.1) is parabolic, and suppose (2.8) is valid. Then (1.1) admits the fundamental solution of second kind $\{R(t)\}_{t>0} \subset \mathcal{B}(X)$ defined by (2.12) which has the kernel representation*

$$(R(t)f)(t) = \int_\Omega \rho_t(x,y)f(y)dy, \quad x \in \Omega,\ t>0, \tag{2.13}$$

for each $f \in X$, where ρ_t is measurable and subject to the Poisson estimate

$$|\rho_t(x,y)| \le t^{\beta(1-n/m)-1} q(|x-y|t^{-\beta/m}), \quad x,y \in \Omega, x \ne y,\ t>0, \tag{2.14}$$

with a continuous noincreasing function $q : (0,\infty) \to (0,\infty)$ satisfying (2.11). In particular, the estimate

$$|R(t)|_{\mathcal{B}(X)} \le t^{\beta-1} \int_0^\infty q(r) r^{n-1} dr < \infty$$

shows that $\{R(t)\}_{t>0}$ is locally integrable in $\mathcal{B}(X)$ on $\mathbb{R}_+$.

In quite analoguous manner one can derive kernel representations also for the derivative of $S(t)$ as well as for $AS(t)$, provided $b(t)$ is 2-regular. Observe the relation $AR(t) = -\dot{S}(t)$.

COROLLARY 1. *Let $X = L_r(\Omega;\mathbb{R}^N)$, $1 \le r \le \infty$, suppose $A \in \mathcal{P}(X)$, and let $b \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth, ϕ_b-sectorial and 2-regular. Assume $\phi_A^P + \phi_b < \pi$, in particular (1.1) is parabolic, and suppose (2.8) is valid.*

Then the fundamental solutions $\{S(t)\}_{t>0} \subset \mathcal{B}(X)$ is of class C^1 on $(0,\infty)$, and $\dot{S}(t) = -AR(t)$ admits a kernel representation with kernel $\dot{\sigma}_t(x,y)$. We have

$$|\dot{\sigma}_t(x,y)| \le t^{-\beta n/m-1} q(|x-y|t^{-\beta/m}), \quad x,y \in \Omega, x \ne y,\ t>0, \tag{2.15}$$

with a continuous noincreasing function $q : (0,\infty) \to (0,\infty)$ satisfying (2.11). In particular, $|\dot{S}(t)|_{\mathcal{B}(X)} \le C/t$ for all $t>0$.

Corollary 2. *Let $X = L_r(\Omega;\mathbb{R}^N)$, $1 \le r \le \infty$, suppose $A \in \mathcal{P}(X)$, and let $b \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth, ϕ_b-sectorial and 2-regular. Assume $\phi_A^P + \phi_b < \pi$, in particular (1.1) is parabolic, and suppose (2.8) is valid. Then the fundamental solutions $\{S(t)\}_{t>0} \subset \mathcal{B}(X, X_A)$ is continuous on $(0,\infty)$, and $AS(t)$ admits kernel representation with kernel $\sigma_t^A(x,y)$. We have*

$$|\sigma_t^A(x,y)| \le t^{-\beta n/m-\beta} q(|x-y|t^{-\beta/m}), \quad x,y \in \Omega, x \neq y,\ t > 0, \tag{2.16}$$

with a continuous noincreasing function $q : (0,\infty) \to (0,\infty)$ satisfying (2.11). In particular, $|AS(t)|_{\mathcal{B}(X)} \le Ct^{-\beta}$ for all $t > 0$.

Some further remarks are in order.

Remark 1. Assumption (2.8) can be weakend considerably in case one is only interested in finite intervals $J = [0,T]$ instead of the halfline $\mathbb{R}_+$. In fact, the proof of Theorem 1 shows that (2.8) is then only needed on the halfspace $\mathrm{Re}\lambda > 1/T > 0$, and since $\widehat{b}(\lambda) \neq 0$ on $\mathbb{C}_+$, (2.8) then reduces to

$$0 < c \le \liminf_{|\lambda|\to\infty} |\lambda^\beta \widehat{b}(\lambda)| \le \limsup_{|\lambda|\to\infty} |\lambda^\beta \widehat{b}(\lambda)| \le c^{-1}, \tag{2.17}$$

for some positive constant c. Moreover, it has been shown in Prüss [11] that 1-regular kernels b satisfy

$$c|\widehat{b}(|\lambda|)| \le |\widehat{b}(\lambda)| \le c^{-1}|\widehat{b}(|\lambda|)|, \quad \mathrm{Re}\lambda > 0.$$

Therfore, (2.8) is equivalent to

$$0 < c \le \lambda^\beta |\widehat{b}(\lambda)| \le c^{-1}, \quad \lambda > 0, \tag{2.18}$$

and its local version (2.17) is equivalent to

$$0 < c \le \liminf_{0<\lambda\to\infty} \lambda^\beta |\widehat{b}(\lambda)| \le \limsup_{0<\lambda\to\infty} \lambda^\beta |\widehat{b}(\lambda)| \le c^{-1}. \tag{2.19}$$

By a wellknown Abelian theorem the latter is implied by

$$0 < \liminf_{t\to 0+} t^{-\beta} |\int_0^t b(s)ds| \le \limsup_{t\to 0+} t^{-\beta} |\int_0^t b(s)ds| < \infty,$$

and this condition is even equivalent to (2.18) if $b(t)$ is real and nonnegative, by Karamata's theorem; see e.g. Widder [17].

3. Maximal regularity

Let $S(t)$ denote the fundamental solution of (1.1). Then for "nice" forcing functions $f(t)$ the solution $u(t)$ of (1.1) is given by the variation of parameters formula

$$\begin{aligned} u(t) &= \frac{d}{dt}\int_0^t S(t-s)f(s)ds \\ &= S(t)f(0) + \int_0^t S(t-s)\dot{f}(s)ds \\ &= f(t) + \int_0^t \dot{S}(t-s)f(s)ds, \end{aligned} \tag{3.1}$$

where the singular convolution $\dot{S} * f$ should be read as

$$\int_0^t \dot{S}(t-s)f(s)ds = \int_0^t \dot{S}(t-s)(f(s)-f(t))ds + (S(t)-I)f(t),\ t \geq 0.$$

If we are in the situation described in Section 2, say if A is sectorial with spectral angle ϕ_A, b is ϕ_b-sectorial and 2-regular, $\phi_A + \phi_b < \pi$, then the solution operator is well-defined, say for $f \in C_0^\infty(\mathbb{R}_+; X)$. To obtain maximal regularity in $L_p(\mathbb{R}_+; X)$ we need a stronger version of Poisson bounds.

DEFINITION 2. *Let A be a sectorial operator in X. A is said to belong to the* **strong Poisson class** $\mathcal{P}_s(X)$ *if the resolvent* $(z+A)^{-1}$ *of A admits a kernel representation*

$$[(z+A)^{-1}f](x) = \int_\Omega \gamma_z(x,y)f(y)dy, \quad x \in \Omega, \tag{3.2}$$

for $z \in \Sigma_{\pi-\phi}$, and the measurable kernel γ_z satisfies a **Poisson estimate**

$$|\gamma_z(x,y)| \leq C|z|^{\frac{n}{m}-1}p(|x-y||z|^{1/m}), \quad x, y \in \Omega,\ z \in \Sigma_{\pi-\phi}, \tag{3.3}$$

where $p : (0,\infty) \to (0,\infty)$ is continuous nonincreasing and such that

$$p(r)(r^{n-\delta} + r^{n+\delta}) \leq M < \infty,$$

for some constants $C, M, \delta > 0$. The **Poisson angle** ψ_A^P, *of A is defined as the infimum of all $\psi > 0$ such that (3.2) and (3.3) are valid.*

The main result of this section is the following result.

Theorem 3. *Let $X = L_r(\Omega; \mathbb{R}^N)$, $1 < r < \infty$, suppose $A \in \mathcal{P}_s(X)$, and let $b \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth, 2-regular, and ϕ_b-sectorial. Assume $\psi_A^P + \phi_b < \pi$, in particular (1.1) is parabolic, and suppose*

$$c|\lambda|^{-\beta} \leq |\widehat{b}(\lambda)| \leq c^{-1}|\lambda|^{-\beta}, \quad \mathrm{Re}\lambda > 0, \tag{3.4}$$

for some constants $c > 0$ and $\beta \in (0, 2)$.

Then (1.1) has the maximal regularity property with respect to the spaces $L_p(\mathbb{R}_+; X)$ for $1 < p < \infty$, i.e. for each $f \in L_p(\mathbb{R}_+; X)$ there is a unique solution $u = Gf \in L_p(\mathbb{R}_+; X)$,in the sense that $b * u \in L_{p,loc}(\mathbb{R}_+; X_A)$ and $u + Ab * u = f$ a.e. on $\mathbb{R}_+$. The solution operator $G : L_p(\mathbb{R}_+; X) \to L_p(\mathbb{R}_+; X)$ is bounded.

A number of additional remarks are in order.

Remark 2. *(i)* Although the kernel b does not belong to $L_1(\mathbb{R}_+)$, the convolution $b * u$ is well-defined since we are working on the halfline $\mathbb{R}_+$. Under the assumptions of Theorem 3 we do not obtain $b * u \in L_p(\mathbb{R}_+; X)$ but only locally. However, $Ab * u \in L_p(\mathbb{R}_+; X)$ by (1.1), hence if in addition A is invertible, then $b * u \in L_p(\mathbb{R}_+; X)$ as well.

(ii) We shall prove at the same time that the solution operator G is also bounded in $L_p(\mathbb{R}; X)$. Observe that in this case the convolution $b * u$ on $\mathbb{R}$ is not defined pointwise. It is here more convenient to invert the convolution and define an operator $\mathcal{B}$ on $L_p(\mathbb{R}; X)$ as in (1.11) and to consider (1.12) instead of (1.1); see Prüss [11], Section 8 for details.

(iii) Concerning maximal regularity on finite intervals $J = [0, T]$ one can relax (2.8) as explained in Remark 1.

Proof of Theorem 3. The aim is to prove that the solution map for (1.1)

$$u(t) = (Gf)(t) = \frac{d}{dt}\int_0^t S(t-s)f(s)ds, \quad t \geq 0, \tag{3.5}$$

which is well-defined say for $f \in C_0^\infty(\mathbb{R}_+;X)$, $X = L_r(\Omega;\mathbb{R}^N)$, is bounded in $L_p(\mathbb{R}_+;X)$, for all $1 < p, r < \infty$. This will be achieved in four steps.

Step 1 Taking Laplace transforms in (1.1) and in (3.5) we obtain the representation

$$(Gf)\widehat{\ }(\lambda) = (I + \widehat{b}(\lambda)A)^{-1}\widehat{f}(\lambda), \quad \mathrm{Re}\lambda > 0. \tag{3.6}$$

Extending f and u by 0 to all of $\mathbb{R}$ this implies in terms of Fourier transforms (indicated by a tilde)

$$(Gf)\widetilde{\ }(\rho) = (I + \widehat{b}(i\rho)A)^{-1}\widetilde{f}(\rho), \quad \rho \in \mathbb{R},\ \rho \neq 0. \tag{3.7}$$

In fact, since b is 2-regular by assumption, its Laplace transform $\widehat{b}(\lambda)$ admits boundary values $\widehat{b}(i\cdot) \in W^2_{\infty,loc}(\mathbb{R}\setminus\{0\})$, and by parabolicity $H(\lambda) = (I+\widehat{b}(\lambda)A)^{-1}$ extends continuously to $\overline{\mathbb{C}}_+\setminus\{0\}$ in $\mathcal{B}(X, X_A)$, and satisfies $|H(\lambda)|_{\mathcal{B}(X)} \leq M < \infty$ there; see [11] for these properties of b and H. By means of the vector-valued Parseval theorem we obtain therefore $G \in L_2(\mathbb{R}; L_2(\Omega;\mathbb{R}^N))$, i.e. the claim of Theorem 3 holds for $p = r = 2$.

Step 2 In this step we prove that G is bounded from

$$L_1(\mathbb{R}; L_1(\Omega;\mathbb{R}^N)) \cong L_1(\mathbb{R}\times\Omega;\mathbb{R}^N)$$

to $L_{1,weak}(\mathbb{R}\times\Omega;\mathbb{R}^N)$. This is the most difficult part of the proof which will be carried out in Section 4. It is again divided into three parts, and it is there where the Poisson estimates for the kernel representing $(z+A)^{-1}$ are used.

Step 3 By means of the Marcienkiewicz interpolation theorem, cf. e.g. [8], Steps 1 and 2 yield boundedness of G in $L_p(\mathbb{R}; L_p(\Omega;\mathbb{R}^N)) \cong L_p(\mathbb{R}\times\Omega;\mathbb{R}^N)$ for $1 < p \leq 2$. Since the dual A^* of A is again sectorial, invertible with the same constants and belongs to $\mathcal{P}(X)$ as well, with the same function $p(r)$, the representation

$$\begin{aligned}(G^*f)(t) &= -\frac{d}{dt}\int_0^\infty S^*(\tau)f(t+\tau)d\tau \qquad (3.8)\\ &= R\frac{d}{dt}\int_0^\infty S^*(\tau)(Rf)(t-\tau)d\tau, \quad t \in \mathbb{R},\end{aligned}$$

with $(Rf)(t) = f(-t)$ shows that G^* is also bounded in the space $L_p(\mathbb{R}; L_p(\Omega; \mathbb{R}^N))$ for $1 < p \le 2$. Therefore by duality G is bounded in $L_p(\mathbb{R}; L_p(\Omega; \mathbb{R}^N))$ also for all $p \in [2, \infty)$. This proves the assertion of Theorem 3 for all $p = r \in (1, \infty)$.

Step 4 By means of the theorem of Benedek, Calderon, and Panzone, cf. e.g. [8], we extend the result for $p = r \in (1, \infty)$ to the general case. In this step the Poisson estimates for the kernel of $(z + A)^{-1}$ are not used, it is enough to employ estimates (1.8) and (1.9) taken from [11], Section 3. In fact, these estimates imply the Hörmander condition

$$\int_{|t|>2|s|} |K(t-s) - K(t)|_{\mathcal{B}(X)} dt \le M < \infty, \quad t \in \mathbb{R},$$

where $K(t) = \dot{S}(t)$ for $t > 0$, $K(t) = 0$ for $t \le 0$, and so Theorem V.3.4 of Garcia-Cuerva and Rubio de Francia [8] yields the assertion of Theorem 3 for arbitrary $p, r \in (1, \infty)$. □

4. Proof of the Theorem 3: main part

We turn now to the proof of the claim in **Step 2**. On the set $\mathbb{R} \times \Omega$ we introduce the quasi-distance

$$d((t,x),(s,y)) = (|t-s|^\beta + |x-y|^m)^{1/m}, \quad t, s \in \mathbb{R}, ; x, y \in \Omega.$$

Observe that only in case $m \ge 1$ and $\beta \ge 1$ d is a metric. Let $f \in L_1(\mathbb{R}\times\Omega; \mathbb{R}^N) \cap L_\infty(\mathbb{R}\times\Omega; \mathbb{R}^N)$ be given and choose a Calderon-Zygmund decomposition for $|f|$ of level $\alpha > 0$ w.r.t. the quasi-distance d; cf. Stein [15], Theorem I.4.2. This means that we obtain balls $B_i := \{(s, y) \in \mathbb{R} \times \Omega : d((s, y), (s_i, y_i)) < \rho_i\}$ and functions g, h_i such that

1. $f = g + h$, where $h = \sum_i h_i$,
2. $|g|_\infty := \operatorname{esssup}\{|g(t,x)| : \ t \in \mathbb{R}, \ x \in \Omega\} \leq \alpha c$;
3. $\operatorname{supp} h_i \subset B_i$, and $\int_{\mathbb{R}} \int_\Omega h_i(t,x)dxdt = 0$;
4. $|h_i|_1 := \int_{\mathbb{R}} \int_\Omega |h_i(t,x)|dxdt \leq \alpha c \cdot \operatorname{mes} B_i$;
5. $\sum_i \operatorname{mes} B_i \leq c|f|_1/\alpha$.

Here c denotes a fixed constant depending only on $\Omega \subset \mathbb{R}^n$, and "mes" indicates Lebesgue's measure on $\mathbb{R} \times \Omega$. Observe that at this point we need a Lipschitz boundary for the domain Ω since the result applied here uses the so-called "doubling property" for the quasi-distance d. Observe the inequality $|g|_1 + |h|_1 \leq (1 + 2c^2)|f|_1$, in particular g belongs also to L_1 and therefore also to L_2.

To show boundedness of G from $L_1(\mathbb{R} \times \Omega; \mathbb{R}^N)$ to $L_{1,weak}(\mathbb{R} \times \Omega; \mathbb{R}^N)$, we have to prove that there is a constant $C > 0$, independent of f, such that

$$\operatorname{mes}\{(t,x) \in \mathbb{R} \times \Omega : \ |(Gf)(t,x)| > \alpha\} \leq C\frac{|f|_1}{\alpha}, \qquad \text{for each } \alpha > 0.$$

For this purpose we decompose the bad functions h_i further into

$$h_i = k_i * T_i h_i + (h_i - k_i * T_i h_i),$$

where $T_i = (1 + r_i A)^{-k}$, $k_i(t) = \exp(-|t|/t_i)/2t_i$ for $t \in \mathbb{R}$, where r_i and t_i will be chosen later, and the $*$ here indicates the convolution over $\mathbb{R}$. Accordingly, we decompose $u = Gf$ as

$$u = u_0 + \sum_i u_i = u_0 + v + w,$$

where $u_0 = Gg$, $u_i = Gh_i$, $v = \sum_i v_i$, $v_i = Gk_i * T_i h_i$, $w = \sum_i w_i$, and $w_i = u_i - v_i$. Then the estimate

$$\begin{aligned} \operatorname{mes}\{|u(t,x)| > 3\alpha\} \leq \operatorname{mes}\{|u_0(t,x)| > \alpha\} + \\ + \operatorname{mes}\{|v(t,x)| > \alpha\} + \operatorname{mes}\{|w(t,x)| > \alpha\} \end{aligned}$$

shows that it is enough to estimate u_0, v, and w separately.

(i) This is fairly simple for u_0. In fact, by Tschebyscheff's inequality and by Step 1

$$\begin{aligned}
\mathrm{mes}\{(t,x):\ |(Gg)(t,x)|>\alpha\} &\le \frac{1}{\alpha^2}|Gg|_2^2\\
&\le \frac{|G|_{L_2}^2}{\alpha^2}|g|_2^2 \le \frac{C}{\alpha^2}|g|_1|g|_\infty\\
&\le \frac{C}{\alpha^2}|f|_1\alpha = C\frac{|f|_1}{\alpha}.
\end{aligned}$$

(ii) Next we estimate the function $w=\sum_i w_i$. We have the representation

$$\begin{aligned}
w_i(t) &= \int_{-\infty}^t \dot S(t-s)\Big[h_i(s)-\\
&\quad -\int_{-\infty}^\infty (2t_i)^{-1}e^{-|s-r|/t_i}(I+r_iA)^{-k}h_i(r)dr\Big]ds\\
&= \int_{-\infty}^\infty K_i(t-s)h_i(s)ds,
\end{aligned}$$

where the operator-valued kernel $K_i(t)$ is given by

$$\begin{aligned}
K_i(t) &= \dot S(t)\chi_0(t)-\int_0^\infty \dot S(s)(2t_i)^{-1}e^{-|t-s|/t_i}(I+r_iA)^{-k}ds\\
&= (2\pi i)^{-1}\int_{\Gamma_\mu} k_\mu(t)(\mu-A)^{-1}d\mu,
\end{aligned}$$

a Dunford integral along a contour $\Gamma_\mu=(\infty,0]e^{-i\phi}\cup[0,\infty)e^{i\phi}$ with $\phi>\phi_A^P$. Here the function $k_\mu(t)$ is given by

$$k_\mu(t)=\dot s_\mu(t)\chi_0(t)-\int_0^\infty \dot s_\mu(r)(2t_i)^{-1}e^{-|t-r|/t_i}(1+r_i\mu)^{-k}dr.$$

By $s_\mu(t)$ we denote the solution of the scalar Volterra equation depending on the parameter $\mu\in\mathbb{C}$

$$s(t)+\mu\int_0^t b(t-r)s(r)dr=1,\quad t>0.$$

We proceed now as follows. Let

$$B_i^2=\{(t,x)\in\mathbb{R}\times\Omega: d((s,y),(s_i,y_i))<2\kappa\rho_i\},$$

where κ denotes the quasi-distance constant, i.e.

$$d((t,x),(s,y)) \le \kappa(d((t,x),(r,z)) + d((r,z),(s,y));$$

then with the characteristic function χ_i of the ball B_i^2 we have $w_i = w_i\chi_i + w_i(1-\chi_i)$. With the doubling property of d, 3., and 4. we get

$$\begin{aligned} \mathrm{mes}\{|w(t,x)| > \alpha\} &= \mathrm{mes}\{|\sum_i [w_i\chi_i + w_i(1-\chi_i)| > \alpha\} \\ &\le \mathrm{mes}\{|\sum_i w_i\chi_i| > \alpha/2\} + \\ &\qquad + \sum_i 2|w_i(1-\chi_i)|_1/\alpha \\ &\le \sum_i \mathrm{mes}(B_i^2) + \sum_i 2|w_i(1-\chi_i)|_1/\alpha \\ &\le C|f|_1/\alpha + \sum_i C|h_i|_1/\alpha \\ &\le C|f|_1/\alpha, \end{aligned}$$

provided we can show $|w_i(1-\chi_i)|_1 \le C|h_i|_1$, for some constant $C > 0$, independent of i and f.

For these terms we begin the estimation in the following way, employing the above representation of the kernel $K(t)$ as well as the Poisson estimate for the kernel γ_z of $(z+A)^{-1}$. Recall supp $h_i \subset B_i$.

$$\begin{aligned} |w_i(1-\chi_i)|_1 &\le C|h_i|_1 \sup_{(s,y)\in B_i} \iint_{(B_i^2)^c} |K_i(t-s)(x,y)|dxdt \\ &\le C|h_i|_1 \sup_{(s,y)\in B_i} \iint_{(B_i^2)^c} \int_{\Gamma_\mu} |k_\mu(t-s)| \cdot \\ &\qquad \cdot|\gamma_{-\mu}(x,y)||d\mu|dxdt \\ &\le C|h_i|_1 \sup_{(s,y)\in B_i} \iint_{(B_i^2)^c} \int_{\Gamma_\mu} |k_\mu(t-s)| \cdot \\ &\qquad \cdot|\mu|^{n/m-1} p(|x-y||\mu|^{1/m})|d\mu|dxdt \\ &\le C|h_i|_1 \iint_{(B_i^0)^c} \int_{\Gamma_\mu} |k_\mu(t)||\mu|^{n/m-1} \cdot \end{aligned}$$

$$\cdot p(|x||\mu|^{1/m})|d\mu|dxdt,$$

since $d((t,x),(s_i,y_i)) \geq 2\kappa\rho_i$ and $d((s,y),(s_i,y_i)) \leq \rho_i$ imply

$$d((t,x),(s,y)) \geq \rho_i\,;$$

by B_i^0 we denote the balls $d((t,x),(0,0)) = (|t|^\beta + |x|^m)^{1/m} \leq \rho_i$ in $\mathbb{R} \times \mathbb{R}^n$. By means of radial symmetry we then obtain with $r = |x|$ and the scaling $r|\mu|^{1/m} \to r$

$$\begin{aligned}
&|w_i(1-\chi_i)|_1 \ \leq \\
&\leq \ C|h_i|_1 \int_{\Gamma_\mu} \iint_{|t|^\beta + r^m \geq \rho_i^m} |k_\mu(t)||\mu|^{n/m-1} p(r|\mu|^{1/m}) dxdt|d\mu| \\
&= \ C|h_i|_1 \int_{\Gamma_\mu} \iint_{|t|^\beta + r^m \geq \rho_i^m} |k_\mu(t)||\mu|^{n/m-1} p(r|\mu|^{1/m}) r^{n-1} drdt|d\mu| \\
&= \ C|h_i|_1 \int_{\Gamma_\mu} \iint_{|t|^\beta|\mu| + r^m \geq |\mu|\rho_i^m} |k_\mu(t)||\mu|^{-1} p(r) r^{n-1} drdt|d\mu| \\
&\leq \ C|h_i|_1 \int_{\Gamma_\mu} [\int_{|t|\geq t_i/2} |k_\mu(t)|dt \int_0^\infty p(r) r^{n-1} dr \\
&\qquad + \int_{|t|<t_i/2}^\infty |k_\mu(t)|dt \int_{r^m \geq (\rho_i^m - |t_i/2|^\beta)|\mu|} p(r) r^{n-1} drdt] \frac{|d\mu|}{|\mu|} \\
&= \ C|h_i|_1 [I_1 + I_2],
\end{aligned}$$

where

$$I_1 = \int_{\Gamma_\mu} [\int_{|t|\geq t_i/2} |k_\mu(t)|dt] \frac{|d\mu|}{|\mu|},$$

and

$$I_2 = \int_{\Gamma_\mu} [\int_{|t|\leq t_i/2} |k_\mu(t)|dt \int_{r^m \geq |\mu r_i|} p(r) r^{n-1} dr] \frac{|d\mu|}{|\mu|},$$

with the choice $r_i = (t_i/2)^\beta = \rho_i^m/2$. This choice for r_i and t_i will be fixed from now on. It therefore remains to show that both integrals I_1 and I_2 are finite, with a bound independent of i. For this purpose we use a representation formula for the function $k_\mu(t)$ which is the content of the following

Proposition 1. *Let $b \in L_{1,loc}$ be of subexponential growth, 2-regular, ϕ_b-sectorial, and let $\mu \in \mathbb{C}\setminus\{0\}$ be such that $|\arg(\mu)| < \phi_0 < \pi - \phi_b$.*

Then there is a uniformly bounded holomorhic function $\varphi : \mathbb{C}_+ \times \Sigma_{\phi_0} \to \mathbb{C}$ *such that*

$$k_\mu(t) = \int_{\Gamma_\sigma} \varphi(\lambda,\mu) \frac{\mu\widehat{b}(\lambda)}{(1+\mu\widehat{b}(\lambda))^2} g_{r_i\mu}(t_i\lambda, t/t_i) \frac{d\lambda}{t^2\lambda^2}, \tag{4.1}$$

for any contour Γ_σ *of the form* $\sigma + is$, $s \in \mathbb{R}$, *with* $0 < \sigma < 1/t_i$. *The function* $g_\mu(\lambda, t)$ *is given by*

$$\begin{aligned} g_\mu(\lambda,t) \quad = \quad & \chi_0(t)e^{\lambda t}[1-(1+\mu)^{-k}(1-\lambda^2)^{-1}] + \\ & + e^{-|t|}[(1-\chi_0(t))(1-\lambda)^{-1} - \\ & - \chi_0(t)(1+\lambda)^{-1}](1+\mu)^{-k}/2. \end{aligned}$$

$g_\mu(\lambda, t)$ *satisfies the estimate*

$$|g_\mu(\lambda,t)| \leq C\{e^{\mathrm{Re}\lambda|t|}[\frac{|\mu|}{1+|\mu} + \frac{|\lambda|^2}{(1+|\lambda|)^2}] + e^{-|t|}\frac{1}{|1-\lambda|}\},$$

for all $t \in \mathbb{R}$, $\mu \in \Sigma_{\phi_0}$, $\lambda \in \mathbb{C}_+$, $0 \leq \mathrm{Re}\lambda \leq 1/2$.

Proof of Proposition 1. Let $\mu \in \Sigma_{\phi_0}$ be fixed. Then the Laplace transform $\theta(\lambda)$ of $\dot{s}_\mu(t)$ is given by

$$\theta(\lambda) = \mu\widehat{b}(\lambda)(1+\mu\widehat{b}(\lambda)^{-1}.$$

A simple calculation yields

$$\theta''(\lambda) = 2\pi i\mu\widehat{b}(\lambda)(1+\mu\widehat{b}(\lambda))^{-2}\lambda^{-2}\varphi(\lambda,\mu),$$

where

$$\begin{aligned} \varphi(\lambda,\mu) \quad = \quad & (2\pi i)^{-1}\{2(\lambda\widehat{b}'(\lambda)/\widehat{b}(\lambda))^2(1+\mu\widehat{b}(\lambda))^{-1} + \\ & + \lambda^2\widehat{b}''(\lambda)/\widehat{b}(\lambda) - 2(\lambda\widehat{b}'(\lambda)/\widehat{b}(\lambda))^2\}. \end{aligned}$$

Note that in view of the assumptions of Proposition 1 the function $\varphi(\lambda, \mu)$ is holomorphic on $\mathbb{C}_+ \times \Sigma_{\phi_0}$ and uniformly bounded. With this representation of $\theta(\lambda)$ we obtain

$$\dot{s}_\mu(t) = \int_{\Gamma_\sigma} \varphi(\lambda,\mu)\frac{\mu\widehat{b}(\lambda)}{(1+\mu\widehat{b}(\lambda))^2} e^{\lambda t}\frac{d\lambda}{t^2\lambda^2}, \quad t>0,$$

where Γ_σ as in the statement of Proposition 1. Inserting this representation into the definition of $k_\mu(t)$ and applying Fubini's theorem, we obtain

$$\begin{aligned} k_\mu(t) &= \dot{s}_\mu(t)\chi_0(t) - \int_0^\infty \dot{s}_\mu(r)(2t_i)^{-1}e^{-|t-r|/t_i}(1+r_i\mu)^{-k}ds \\ &= \int_{\Gamma_\sigma} \varphi(\lambda,\mu)\frac{\mu\widehat{b}(\lambda)}{(1+\mu\widehat{b}(\lambda))^2}\Big\{\chi_0(t)e^{\lambda t} - \\ &\quad - \int_0^\infty [e^{\lambda s}e^{-|t-s|/t_i}/2t_i]ds(1+r_i\mu)^{-k}\Big\}\frac{d\lambda}{t^2\lambda^2} \\ &= \int_{\Gamma_\sigma} \varphi(\lambda,\mu)\frac{\mu\widehat{b}(\lambda)}{(1+\mu\widehat{b}(\lambda))^2} g_{r_i\mu}(t_i\lambda, t/t_i)\frac{d\lambda}{t^2\lambda^2}, \end{aligned}$$

with $g_\mu(\lambda,t)$ as in Proposition 1. The estimate is straightforward. □

We proceed now with the estimates for I_1. By means of Proposition 1, I_1 now takes the following form.

$$I_1 = \int_{\Gamma_\mu}\int_{|t|\geq t_i/2}\left|\int_{\Gamma_{\sigma t_i}}\frac{\mu\widehat{b}(\lambda)}{(1+\mu\widehat{b}(\lambda))^2}g_{r_i\mu}(\lambda t_i, t/t_i)\varphi(\lambda,\mu)\frac{d\lambda}{\lambda^2t^2}\right|dt\frac{|d\mu|}{|\mu|},$$

which by means of (2.8), parabolicity and boundedness of φ can be estimated by

$$I_1 \leq C\int_{\Gamma_\mu}\int_{|t|\geq t_i/2}\int_{\Gamma_{\sigma t_i}}\frac{|\lambda|^{-\beta}}{(1+|\mu||\lambda|^{-\beta})^2}|g_{r_i\mu}(\lambda t_i, t/t_i)|\frac{|d\lambda|}{|\lambda|^2t^2}dt|d\mu|.$$

Next we introduce the scaling $\mu r_i \to \mu$, $\lambda t_i \to \lambda$, $t/t_i \to t$, $\sigma t_i \to \sigma$, which by the relation $r_i = (t_i/2)^\beta$ leads to

$$I_1 \leq C\int_{\Gamma_\mu}\int_{|t|\geq 1/2}\int_{\Gamma_\sigma}\frac{|\lambda|^{\beta}}{(|\lambda|^\beta+|\mu|)^2}|g_\mu(\lambda,t)|\frac{|d\lambda|}{|\lambda|^2t^2}dt|d\mu|.$$

Thus we have a bound which is uniform in i, once the right hand side of the last inequality is finite. Next we parametrize the curves Γ_μ by $\mu = t^{-\beta}\rho e^{\pm i\phi}$ and Γ_σ by $\lambda = (1/4+is)/|t|$, in particular

$\mathrm{Re}\lambda = \sigma = 1/4|t| \leq 1/2$, use symmetry and the estimate for $g_\mu(\lambda, t)$ proved in Proposition 1.

$$\begin{aligned} I_1 \quad &\leq \quad C \int_{1/2}^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{(1+s)^\beta}{((1+s)^\beta + \rho)^2} \Big\{ \frac{\rho}{t^\beta + \rho} + \\ &\qquad + \frac{(1+s)^2}{(1+s+t)^2} + e^{-t} \Big\} [ds/(1+s)^2] d\rho dt/t. \end{aligned}$$

Write the right hand side of the last inequality as $J_1 + J_2 + J_3$. Then integrating first w.r.t. ρ we obtain

$$J_2 = \int_{1/2}^{\infty} \Big[\int_1^{\infty} \frac{ds}{(s+t)^2} \Big] dt/t = \int_{1/2}^{\infty} dt/t(1+t) < \infty,$$

as well as

$$J_3 = \Big[\int_{1/2}^{\infty} e^{-t} dt/t \Big] \Big[\int_1^{\infty} ds/s^2 \Big] < \infty.$$

On the other hand, with $(s^\beta + \rho)^{-2} \leq s^{-\beta/2}(1+\rho)^{-3/2}$ for $s \geq 1$ we get

$$\begin{aligned} J_1 \quad &\leq \quad \int_0^{\infty} \int_1^{\infty} \Big[\int_{1/2}^{\infty} \frac{\rho}{t^\beta + \rho} dt/t \Big] s^{\beta/2-2} ds \Big] \frac{d\rho}{(1+\rho)^{3/2}} \\ &= \quad \Big[\int_1^{\infty} s^{\beta/2-2} ds \Big] \int_0^{\infty} \log(1 + 2^\beta \rho) \frac{d\rho}{(1+\rho)^{3/2}} \\ &= \quad (1-\beta/2)^{-1} \int_0^{\infty} \log(1+2^\beta \rho) \frac{d\rho}{(1+\rho)^{3/2}} < \infty \end{aligned}$$

The second term I_2 must be treated somewhat differently. Here we first decompose $k_\mu(t)$ into three parts, namely $k_\mu(t) = k_\mu^1(t) + k_\mu^2(t) + k_\mu^3(t)$, where

$$k_\mu^1(t) = \chi_0(t) \Big[\dot{s}_\mu(t) - \int_0^t \dot{s}_\mu(r)(2t_i)^{-1} e^{-(t-r)/t_i} (1 + r_i \mu)^{-k} dr \Big],$$

$$k_\mu^2(t) = -\chi_0(t) \int_t^{\infty} \dot{s}_\mu(r)(2t_i)^{-1} e^{(t-r)/t_i} (1 + r_i \mu)^{-k} dr,$$

and

$$k_\mu^3(t) = -(1 - \chi_0(t)) e^{-|t|/t_i} \int_0^{\infty} \dot{s}_\mu(r)(2t_i)^{-1} e^{-r/t_i} (1 + r_i \mu)^{-k} dr.$$

Then $I_2 \le I_{21} + I_{22} + I_{23}$ where I_{2j} is defined as I_2, with k_μ replaced by k_μ^j. We estimate these terms separately.

The integral in the definition of k_μ^3 can be evaluated explicitly, namely

$$k_\mu^3(t) = (1 - \chi_0(t))[e^{-|t|/t_i}/2t_i]\frac{\mu\widehat{b}(1/t_i)}{1 + \mu\widehat{b}(1/t_i)}[1 + r_i\mu]^{-k},$$

hence by (2.8) we have with $r_i = (t_i/2)^\beta$

$$\begin{aligned} I_{23} &\le C\int_{\Gamma_\mu}\int_{|t|\le t_i/2} |k_\mu^3(t)|dt\frac{|d\mu|}{|\mu|} \\ &\le C\Big[\int_0^{t_i/2} e^{-t/t_i}dt/t_i\Big]\Big[\int_0^\infty (1 + r_i\rho)^{-k-1}r_i d\rho\Big] \\ &= C\Big[\int_0^{1/2} e^{-s}ds\Big]\Big[\int_0^\infty (1+\rho)^{-k-1}d\rho\Big] < \infty. \end{aligned}$$

To deal with k_μ^2 we employ Hardy's inequality for the function $\varphi(t) = \chi_0(t)\dot{s}_\mu(t)e^{-t/t_i}$, which reads

$$\int_0^\infty |\varphi(t)|dt \le \frac{1}{2}\int_{\Gamma_\sigma} |\widehat{\dot{s}}_\mu(\lambda)||d\lambda|,$$

where $\sigma = 1/t_i$. By means of (2.8) we then obtain with $\rho = |\mu|$

$$\begin{aligned} I_{22} &\le C\int_{\Gamma_\mu}\Big[\int_0^{t_i/2} |k_\mu^2(t)|dt\Big]\frac{|d\mu|}{|\mu|} \\ &\le C\int_0^\infty \Big[\int_0^\infty |\dot{s}_\mu(r)e^{-r/t_i}|dr\Big] \\ &\qquad \Big[\int_0^{t_i/2} e^{t/t_i}dt/t_i\Big](1 + r_i\rho)^{-k}d\rho/\rho \\ &\le C\int_0^\infty \Big[\int_{\Gamma_\sigma} |\widehat{\dot{s}}_\mu(\lambda)||d\lambda|\Big](1 + r_i\rho)^{-k}d\rho \\ &\le C\int_0^\infty \Big[\int_1^\infty \frac{\rho t_i^\beta}{(s^\beta + t_i^\beta\rho)^2}s^{\beta-1}ds\Big](1 + r_i\rho)^{-k}d\rho/\rho \\ &= C\int_0^\infty \Big[\int_1^\infty \frac{s^{\beta-1}}{(s^\beta + r)^2}ds\Big](1 + r)^{-k}dr \\ &\le C\Big[\int_1^\infty s^{-\beta/2-1}ds\Big]\Big[\int_0^\infty (1 + r)^{-k-1/2}dr\Big] < \infty, \end{aligned}$$

where we again used the relation $r_i = (t_i/2)^\beta$ and proper scaling.

To estimate the term I_{21} we proceed as for I_1, this time based on the representation

$$k^1_\mu(t) = \int_{\Gamma_\sigma} \varphi(\lambda,\mu) \frac{\mu\widehat{b}(\lambda)}{(1+\mu\widehat{b}(\lambda))^2} g^1_{r_i\mu}(t_i\lambda, t/t_i) \frac{d\lambda}{t^2\lambda^2}, \tag{4.2}$$

where now

$$\begin{aligned} g^1_\mu(\lambda,t) &= \chi_0(t)e^{\lambda t}[1-2^{-1}(1+\lambda)^{-1}(1+\mu)^{-k}] + \\ &\quad + e^{-t}2^{-1}(1+\lambda)^{-1}(1+\mu)^{-k} \end{aligned}$$

where $\mathrm{Re}\lambda > 0$ and $\mu \in \Sigma_\phi$. For $|t| < t_i/2$ we may estimate very roughly.

$$|g^1_\mu(\lambda,t)| \le C\chi_0(t)e^{\mathrm{Re}\lambda t}.$$

Then we obtain with $\lambda = (1+is)/t$, $\nu = r_i|\mu|$ and $\tau = t/(t_i/2)$

$$\begin{aligned} I_{21} &\le C\int_{\Gamma_\mu}\int_0^{t_i/2}\int_{\Gamma_\sigma} \left|\frac{\mu\widehat{b}(\lambda)}{(1+\mu\widehat{b}(\lambda))^2}\right| e^{\mathrm{Re}\lambda t} \\ &\qquad \Big[\int_{r^m>|\mu r_i|} p(r)r^{n-1}dr\Big]\frac{|d\lambda|}{|\lambda t|^2}dt|d\mu|/|\mu| \\ &\le C\int_{\Gamma_\mu}\int_0^{t_i/2}\int_{\Gamma_\sigma} \frac{|\lambda|^\beta}{(|\lambda|^\beta+|\mu|)^2}e^{\mathrm{Re}\lambda t} \\ &\qquad \Big[\int_{r^m>|\mu r_i|} p(r)r^{n-1}dr\Big]\frac{|d\lambda|}{|\lambda t|^2}dt|d\mu| \\ &\le \int_0^\infty\int_0^1\int_1^\infty \frac{s^\beta\tau^{-\beta}}{(s^\beta\tau^{-\beta}+\nu)^2} \\ &\qquad \int_{r^m>\nu} p(r)r^{n-1}dr(ds/s^2)(d\tau/\tau)d\nu. \end{aligned}$$

Evaluating the integral over ν first, we get

$$|I_{21}| \le C\int_0^\infty p(r)r^{n-1}[\int_1^\infty\int_0^1 \frac{r^m\tau^\beta}{s^\beta+r^m\tau^\beta}(d\tau/\tau)(ds/s^2)]dr.$$

Since $s \ge 1$ we may estimate and evaluate the integrals over s and τ to the result

$$|I_{21}| \le C\int_0^\infty p(r)r^{n-1}\beta^{-1}\log(1+r^m)dr < \infty,$$

by the assumptions on $p(r)$.

(iii) Finally, we estimate the functions $v_i = Gk_i * T_i h_i$. By means of the Tschebyscheff inequality and boundedness of G in L_2 we get

$$\begin{aligned} \mathrm{mes}\{|\sum_i v_i| > \alpha\} &\le |\sum_i v_i|_2^2/\alpha^2 \\ &\le C|\sum_i k_i * T_i h_i|_2^2/\alpha^2. \end{aligned}$$

We still have the number $k \in \mathbb{N}$ for choice. Fixing at this point $k \ge n/\delta$, we know from Proposition A2 that the kernel bound $p := p_k$ for $T_i = (1 + r_i A)^{-k}$ is bounded. Then from Proposition A3 and 3. we obtain

$$\begin{aligned} |(k_i * T_i h_i)(t,x)| &\le |h_i|_1 \sup_{(s,y)\in B_i} |(k_i T_i)((t,x),(s,y))| \\ &\le |h_i|_1 \sup_{(s,y)\in B_i} \{(2t_i)^{-1} e^{-|t-s|/t_i} r_i^{-n/m} p(|x-y| r_i^{-1/m})\} \\ &\le C\alpha \,\mathrm{mes}\,(B_i) \inf_{(s,y)\in B_i} \{(2t_i)^{-1} e^{-|t-s|/t_i} r_i^{-n/m} p(|x-y| r_i^{-1/m}/2)\} \\ &\le C\alpha \int_{\mathbb{R}} \int_{\mathbb{R}^n} G_i(|t-s|,|x-y|)\chi_i(s,y)\,dyds \\ &= C\alpha (G_i \chi_i)(t,x), \end{aligned}$$

where χ_i denotes the characteristic function of the ball B_i, and G_i the convolution over $\mathbb{R} \times \mathbb{R}^n$ with kernel

$$G_i(t,x) = [e^{-|t|/t_i}/2t_i][r_i^{-n/m} p(|x| r^{-1/m}/2).$$

Let $M_2 h$ denote the maximal function of $h \in L_2$. Then the maximal inequality $|(G_i h)(t,x)| \le (M_2|h|)(t,x)$ implies for $h \in L_2(\mathbb{R} \times \mathbb{R}^n$

$$\begin{aligned} |(\sum_i k_i * T_i h_i, h)| &\le C\alpha(\sum_i G_i \chi_i, |h|) = C\alpha \sum_i (\chi_i, G_i|h|) \\ &\le C\alpha \sum_i (\chi_i, M_2|h|) \\ &\le C\alpha |\sum_i \chi_i|_2 \cdot |M_2|h||_2 \\ &\le C\alpha |M_2|_{\mathcal{B}(L_2(\mathbb{R}\times\mathbb{R}^n))} |\sum_i \chi_i|_2 |h|_2, \end{aligned}$$

by boundedness of the maximal operator in $L_2(\mathbb{R}\times\mathbb{R}^n)$. Hence 4. implies

$$\sum_i k_i * T_i h_i|_2 \le C\alpha|\sum_i \chi_i|_2 \le C\alpha|(\sum_i \text{mes } B_i)^{1/2} \le C\alpha(|f|_1/\alpha)^{1/2},$$

and combining these estimates we arrive at

$$\text{mes}\{|\sum_i v_i| > \alpha\} \le C|f|_1/\alpha,$$

what was to be shown. □

5. Appendix: Poisson kernels

Here we collect some properties of Poisson kernels which have been used in the proofs of our main results.

PROPOSITION A1. *Suppose $p : (0,\infty) \to (0,\infty)$ is continuous, nonincreasing and such that*

$$\int_0^\infty p(r)r^{n-1}dr < \infty.$$

Then the iterated kernels $p_k(r)$ defined inductively by means of

$$p_{k+1}(|x|) = \int_{\mathbb{R}^n} p_k(|x-y|)p(|y|)dy, \quad x \in \mathbb{R}^n,$$

satisfy $p_k(r) \le C_k p(\varepsilon_k r)$, $r > 0$, where $C_k > 0$ and $\varepsilon_k > 0$ denote constants only depending on $k \in \mathbb{N}$.

Proof. By induction it is evidently enough to prove the assertion for the case $k = 1$. Let $x \in \mathbb{R}^n$ be given and set $\rho = |x|$. W.l.og. we assume $n \ge 2$. Choose a rotation Q which rotates the vector x into $(\rho, 0\ldots, 0) = \rho e_1$. Then

$$\begin{aligned} p_2(\rho) &= \int_{\mathbb{R}^n} p(|\rho e_1 - y|)p(|y|)dy \\ &= \int_0^\infty \int_{|\zeta|=1} p(|\rho e_1 - r\zeta|)p(r)r^{n-1}d\zeta dr \end{aligned}$$

$$= \int_0^\infty \int_{-1}^1 \int_{|\xi|^2=1-\eta^2} p([\rho^2+r^2-2r\rho\eta]^{1/2})p(r)r^{n-1}d\xi d\eta dr$$
$$= \omega_{n-2}\int_0^\infty \int_{-1}^1 p([\rho^2+r^2-2r\rho\eta]^{1/2})[1-\eta^2]^{n/2-1}p(r)r^{n-1}d\eta dr,$$

where ω_{n-2} denotes the surface area of the $n-2$-dimensional unit sphere. By means of the variable transformation $\eta = \sin t$ we obtain

$$p_2(\rho) = \int_0^\infty \int_{-\pi/2}^{\pi/2} p(\sqrt{\rho^2+r^2-2r\rho\sin t})\cos^{n-1}(t)dtp(r)r^{n-1}dr.$$

To estimate $p_2(\rho)$ we decompose the region of integration into three parts and name the corresponding terms $p_{2j}(\rho)$.

(i) The first region restricts t to the range $-\pi/2 \le t \le t_0 < \pi/2$. For such t

$$\rho^2+r^2-2r\rho\sin t \ge \rho^2+r^2-2r\rho\sin t_0 \ge \rho^2(1-\sin t_0),$$

hence we obtain

$$p_{21}(\rho) \le p(\varepsilon\rho)[\int_0^\infty p(r)r^{n-1}dr][\int_{-\pi/2}^{\pi/2}\cos^{n-1}(t)dt] = Cp(\varepsilon\rho), \qquad \rho \in (0,\infty),$$

where $\varepsilon = \sqrt{1-\sin t_0}$.

(ii) The second region restricts r to the range $|r-\rho| \ge \eta\rho > 0$, where $\eta \in (0,1)$ is fixed. For such r we have

$$\rho^2+r^2-2r\rho\sin t \ge \rho^2+r^2-2r\rho = |r-\rho|^2 \ge \eta^2\rho^2,$$

hence we obtain with positivity and monotonicity of p

$$p_{22}(\rho) \le p(\eta\rho)[\int_0^\infty p(r)r^{n-1}dr][\int_{-\pi/2}^{\pi/2}\cos^{n-1} t dt = Cp(\eta\rho).$$

(iii) The third region is $(t,r) \in [t_0,\pi/2]\times[\rho(1-\eta),\rho(1+\eta)]$. Here we estimate as follows.

$$p_{23}(\rho) = \int_{t_0}^{\pi/2}\int_{\rho(1-\eta)}^{\rho(1+\eta)} p(\sqrt{\rho^2+r^2-2r\rho\sin t})(r\cos t)^{n-1}p(r)drdt$$

$$\begin{aligned} &\leq p((1-\eta)\rho) \\ &\qquad \int_{t_0}^{\pi/2}\int_{\rho(1-\eta)}^{\rho(1+\eta)} p(\sqrt{\rho^2+r^2-2r\rho\sin t})(r\cos t)^{n-1}drdt \\ &= p((1-\eta)\rho) \\ &\qquad \int_{\rho(1-\eta)}^{\rho(1+\eta)}\int_0^{\delta} p(\sqrt{\rho^2+r^2-2r\rho\cos s})(r\sin s)^{n-1}dsdr, \end{aligned}$$

where $\delta = \pi/2 - t_0$. Since $\sin s \leq s$ and $\cos s \leq 1 - s^2/4$ for $\delta > 0$ small enough, this yields

$$\begin{aligned} p_{23}(\rho) &\leq cp(\rho(1-\eta))\int_{\rho(1-\eta)}^{\rho(1+\eta)}\int_0^{\delta} p(\sqrt{(\rho-r)^2+\rho r s^2/2})(rs)^{n-1}dsdr \\ &\leq cp(\rho(1-\eta))\int_{1-\eta}^{1+\eta}\int_0^{\infty} p(\rho s\sqrt{(1-\eta)/2})\rho^n s^{n-1}dsd\tau \\ &\leq cp(\rho(1-\eta)(\int_0^{\infty} p(t)t^{n-1}dt) \end{aligned}$$

where we used the change of variable $r = \rho\tau$ and $t = s\rho\sqrt{(1-\eta)/2}$. This completes the proof. □

In general one cannot expect that the iterated Poisson kernels become bounded eventually, which is needed for the sup-inf inequality proved in Proposition A3 below. For this we have to assume more.

PROPOSITION A2. *Suppose $p : (0,\infty) \to (0,\infty)$ is continuous nonincreasing and such that*

$$p(r)[r^{n+\delta} + r^{n-\delta}] \leq M, \quad r > 0,$$

for some constants $M > 0$ and $\delta \in (0,1)$.

Then for $k \geq n/\delta$, the iterated Poisson kernels $p_k(r)$ defined in Proposition 1 are bounded and satisfy $p_k(r) \leq M_k r^{-(n+\delta)}$ on $(0,\infty)$.

Proof. Since p subject to the assumptions of Proposition 2 satisfies the assumptions of Proposition 1 we obtain

$$p_k(r) \leq C_k p(\varepsilon_k r) \leq M_k r^{-(n+\delta)}, \quad r > 0.$$

Therefore it is enough to show that p_k is bounded near zero, for $k \geq n/\delta$.

We prove by induction

$$p_k(r) \leq M_k r^{k\delta - n}, \quad r > 0,$$

for all $k < n/\delta$. So assume that this is true for k. Then we write as in the proof of Proposition 1

$$\begin{aligned}
p_{k+1}(\rho) = & \int_0^\infty \int_{-\pi/2}^{\pi/2} p_k(\sqrt{\rho^2 + r^2 - 2r\rho \sin t}) \cos^{n-1} t dt p(r) r^{n-1} dr \\
\leq & \; M_k \int_0^\infty \int_{-\pi/2}^{\pi/2} [\sqrt{\rho^2 + r^2 - 2r\rho \sin t}]^{-n+k\delta} \cos^{n-1} t dt r^{\delta-1} dr \\
= & \; M_k \rho^{(k+1)\delta - n} \int_0^\infty \int_{-\pi/2}^{\pi/2} [\sqrt{1 + s^2 - 2s \sin t}]^{-n+k\delta} \\
& \qquad \cos^{n-1} t dt s^{\delta-1} ds,
\end{aligned}$$

where we used the scaling $r = \rho s$. The integral is treated in a way which is analoguous to the splitting in the proof of Proposition 1.

$$\begin{aligned}
\int_0^\infty \int_{-\pi/2}^{t_0} \overline{\sqrt{1 + s^2 - 2s \sin t}}^{-n+k\delta} \cos^{n-1} t dt s^{\delta-1} ds \leq \\
\leq c_1 \int_0^\infty \overline{\sqrt{1 + s^2}}^{k\delta - n} s^{\delta-1} ds,
\end{aligned}$$

which is finite provided $n > (k+1)\delta$.

$$\begin{aligned}
\int_{|s-1| \geq \eta} \int_{-\pi/2}^{\pi/2} \overline{\sqrt{1 + s^2 - 2s \sin t}}^{-n+k\delta} \cos^{n-1} t dt s^{\delta-1} ds \leq \\
\leq c_2 \int_{|s-1| \geq \eta} |s-1|^{k\delta - n} s^{\delta-1} ds,
\end{aligned}$$

which is finite iff $n > (k+1)\delta$.

$$\begin{aligned}
& \int_{1-\eta}^{1+\eta} \int_{t_0}^{\pi/2} [\sqrt{1 + s^2 - 2s \sin t}]^{-n+k\delta} \cos^{n-1} t dt s^{\delta-1} ds = \\
= & \int_{1-\eta}^{1+\eta} \int_0^{\varepsilon} [\sqrt{1 + s^2 - 2s \cos \tau}]^{k\delta - n} \sin^{n-1} \tau d\tau s^{\delta-1} ds \leq
\end{aligned}$$

$$\begin{aligned}
&\leq \quad c_3 \int_{1-\eta}^{1+\eta} \int_0^{\varepsilon} \sqrt{(1-s)^2 + s\tau^2/2}^{k\delta-n} \tau^{n-1} d\tau s^{\delta-1} ds \\
&\leq \quad c_4 \int_0^1 \tau^{k\delta-1} d\tau,
\end{aligned}$$

which is finite. This proves the desired estimate for $k < n/\delta$.

Now consider the step $k\delta \leq n < (k+1)\delta$. Here we have to estimate differently. Since

$$\rho^2 + r^2 - 2r\rho\sin t = (\rho - r)^2 \sin t + (\rho^2 + r^2)(1 - \sin t) \geq r^2(1 - \sin t),$$

we obtain

$$\begin{aligned}
&\int_0^1 \int_{-\pi/2}^{\pi/2} \sqrt{\rho^2 + r^2 - 2r\rho\sin t}^{k\delta-n} \cos^{n-1} t dt r^{\delta-1} dr \\
\leq \quad & \left(\int_0^1 r^{(k+1)\delta-n-1} dr\right) \cdot \left(\int_{-\pi/2}^{\pi/2} \sqrt{1-\sin t}^{k\delta-n} \cos^{n-1} t dt\right) < \infty,
\end{aligned}$$

for $(k+1)\delta - n > 0$. Similarly,

$$\begin{aligned}
&\int_1^{\infty} \int_{-\pi/2}^{\pi/2} \sqrt{\rho^2 + r^2 - 2r\rho\sin t}^{k\delta-n} \cos^{n-1} t dt r^{-\delta-1} dr \\
\leq \quad & \left(\int_1^{\infty} r^{k\delta-n-1-\delta} dr\right) \cdot \left(\int_{-\pi/2}^{\pi/2} \sqrt{1-\sin t}^{k\delta-n} \cos^{n-1} t dt\right) < \infty,
\end{aligned}$$

because of $k\delta \leq n$. Therefore p_{k+1} is bounded, and then we obtain by induction boundedness of p_l for all $l \geq k+1$. □

PROPOSITION A3. *Let $p \in C(\mathbb{R}_+)$ be positive and nonincreasing, let $m > 0$, $\beta \in (0,2)$, and define a quasidistance d on $\mathbb{R} \times \mathbb{R}^n$ by*

$$d((t,x),(s,y)) = [|t-s|^{\beta} + |x-y|^m]^{1/m}.$$

Let $B_0 := \{(s,y) \in \mathbb{R} \times \mathbb{R}^n : d((s,y),(s_0,y_0)) \leq \rho_0$ and set $t_0 = \rho_0^{m/\beta}$ and $r_0 = \rho_0$.

Then there is a universal constant $C > 0$ such that for all $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $\rho_0 > 0$, $(s_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$ the inequality

$$\begin{aligned}
&\sup_{(s,y)\in B_0} [(2t_0)^{-1} e^{-|t-s|/t_0} r_0^{-n/m} p(|x-y| r_0^{-m})] \leq \\
&\qquad C \inf_{(s,y)\in B_0} [(2t_0)^{-1} e^{-|t-s|/t_0} r_0^{-n/m} p(|x-y| r_0^{-m}/2)] \qquad (5.1)
\end{aligned}$$

is valid.

Proof. We may assume w.o.l.g. $t = x = 0$. Then the assertion is equivalent to

$$e^{-|s|/t_0} p(|y| {r_0}^{-1/m}) \leq C e^{-|\bar{s}|/t_0} p(|\bar{y}| {r_0}^{-1/m}/2),$$

for all $(s,y), (\bar{s},\bar{y}) \in B_0$. But the latter implies $d((s,y),(\bar{s},\bar{y})) \leq 2\kappa\rho_0$, i.e. $|s-\bar{s}|^\beta + |y-\bar{y}|^m \leq (2\kappa\rho_0)^m$, or equivalently $|s/t_0 - \bar{s}/t_0|^\beta + |y/r_0 - \bar{y}/r_0|^m \leq (2\kappa)^m$, where κ denotes the quasidistance constant. Therefore the assertion will follow if we can prove that there is a constant $C > 0$ such that

$$e^{-|s|} p(|y|) \leq C e^{-|\bar{s}|} p(|\bar{y}|/2),$$

for all $|s-\bar{s}| \leq (2\kappa)^{m/\beta}$ and $|y-\bar{y}| \leq 2\kappa$. But this in turn means

$$e^{|\bar{s}|-|s|} \leq C p(|\bar{y}|/2)/p(|y|),$$

for all such $s, \bar{s}$ and $y, \bar{y}$. Because of $||s|-|\bar{s}|| \leq |s-\bar{s}| \leq (2\kappa)^{m/\beta}$ this condition is equivalent to

$$0 < c \leq p(|\bar{y}|/2)/p(|y|),$$

for some constant $c > 0$, and with $||y|-|\bar{y}|| \leq |y-\bar{y}| \leq 2\kappa$ we have only to check the condition

$$0 < c \leq p(\bar{r}/2)/p(r), \qquad \text{for all } r, \bar{r} \geq 0,\ |r-\bar{r}| \leq 2\kappa. \tag{5.2}$$

To prove this, we consider three cases. If $\bar{r} \leq r$ then by monotonicity of p we have

$$p(r) \leq p(\bar{r}) \leq p(\bar{r}/2),$$

i.e. (5.2) holds wih $c = 1$. Assume next $2\kappa \leq r \leq \bar{r} \leq r + 2\kappa$; then $\bar{r}/2 \leq \kappa + r/2 \leq r$, hence $p(r) \leq p(\bar{r}/2)$, by monotonicity of p, i.e. (5.2) holds wih $c = 1$. Finally, if $2r \leq \kappa$, $r \leq \tilde{r} \leq r + 2\kappa$, then $\bar{r} \leq 4\kappa$, hence

$$p(r) \leq p(0) = p(2\kappa)^{-1} p(2\kappa) \leq p(2\kappa)^{-1} p(\bar{r}/2),$$

which implies that (5.2) holds with $c = \min\{1, p(2\kappa)\}$. □

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