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Generation of Strongly Continuous Semigroups by Elliptic Operators with Unbounded Coefficients in $L^p(\mathbb{R}^n)$

Alessandra Lunardi and Vincenzo Vespri (*)

1. Introduction

This paper deals with generation of contraction semigroups by elliptic operators in divergence form in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. The main novelty with respect to the previous literature is that the coefficients of the first order derivatives are allowed to be unbounded, with (not more than) linear growth at ∞. Precisely, we consider a differential operator $\mathcal{A}$ in $\mathbb{R}^n$ of the type

$$\begin{aligned}(\mathcal{A}u)(x) \;=\; & \sum_{i,j=1}^{n} D_i(q_{ij}(x)D_j u(x)) + \sum_{i=1}^{n} D_i(a_i(x)u(x)) + \\ & + \sum_{i=1}^{n} b_i(x)D_i u(x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{1.1}$$

The coefficients q_{ij} are assumed throughout to be measurable and bounded in $\mathbb{R}^n$, and to satisfy the ellipticity condition

$$\sum_{i,j=1}^{n} q_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2, \quad \forall x, \ \xi \in \mathbb{R}^n, \tag{1.2}$$

(*) Indirizzi degli Autori: A. Lunardi: Dipartimento di Matematica, Università di Parma, Via D'Azeglio 85/A, 43100 Parma (Italy).
e-mail lunardi@prmat.math.unipr.it
V. Vespri: Dipartimento di Matematica Pura e Applicata, Università dell'Aquila, Via Vetoio, 67010 Coppito, L'Aquila (Italy). *e-mail* vespri@ing.univaq.it

with $\nu > 0$. The coefficients a_i and b_i are Lipschitz continuous, possibly unbounded, in $\mathbb{R}^n$.

We show that the realization of $\mathcal{A}$ in $L^p(\mathbb{R}^n)$ generates a strongly continuous contraction semigroup, which is not analytic in general but it enjoys further smoothing properties, which will be the object of a subsequent paper.

In the case of bounded a_i, b_i, uniformly continuous and bounded q_{ij}, and $p \geq 2$, generation of analytic semigroups was proved by Cannarsa, Vespri [4]. The same papers deal also with unbounded coefficients, but their operator $\widetilde{\mathcal{A}}$ is of the type $\widetilde{\mathcal{A}}u = \mathcal{A}u + vu$, where the potential v is unbounded in such a way that it balances in a certain sense the unboundedness of a_i and b_i. In this context, see also Aronson, Besala [1, 2].

In the case $v = 0$, analytic semigroups are generated by certain operators where a_i, b_i grow superlinearly (see Davies [5]) or where a_i, b_i grow linearly but $L^p(\mathbb{R}^n)$ is replaced by a suitably weighted L^p space (see [10] for the Ornstein-Uhlenbeck operator).

The lack of continuity of q_{ij} gives additional technical difficulties, even in the definition of the realization A_p of $\mathcal{A}$ in $L^p(\mathbb{R}^n)$. To define such a realization we introduce the bilinear form associated to $\mathcal{A}$,

$$\begin{aligned} a(u,\varphi) \;=\; & -\int_{\mathbb{R}^n} \sum_{i,j=1}^n q_{ij}(x) D_j u(x) D_i \varphi(x)\,dx \\ & -\int_{\mathbb{R}^n} \sum_{i=1}^n a_i(x) u(x) D_i \varphi(x)\,dx \\ & +\int_{\mathbb{R}^n} \sum_{i=1}^n b_i(x) D_i u(x) \varphi(x)\,dx, \end{aligned} \tag{1.3}$$

for every u, φ such that the above integrals make sense.

If the coefficients q_{ij} are uniformly continuous, the definition of $D(A_p)$ is the standard one: we set

$$\begin{aligned} D(A_p) \;=\; & \Big\{ u \in W^{1,p}_{loc}(\mathbb{R}^n) : \exists C > 0 \text{ such that} \\ & |a(u,\varphi)| \leq C \|\varphi\|_{L^{p'}} \ \forall \varphi \in W^{1,p'}_0(\mathbb{R}^n) \Big\}, \end{aligned} \tag{1.4}$$

where p' is the conjugate exponent of p and $W^{1,p'}_0(\mathbb{R}^n)$ is the subspace of $W^{1,p'}(\mathbb{R}^n)$ consisting of the functions with compact support.

If the coefficients q_{ij} are not continuous, the definition of $D(A_p)$ is more complicated. If $p > 2$ we have to replace the condition $u \in W^{1,p}_{loc}(\mathbb{R}^n)$ by $u \in W^{1,2}_{loc}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. If $p < 2$ we must add further conditions in order to prove that the resolvent set of A_p is not empty, precisely to get uniqueness of the solution of $\lambda u - A_p u = f$ for every $f \in L^p(\mathbb{R}^n)$ and λ sufficiently large. See Sections 3, 4.

In any case, since $W^{1,p'}_0(\mathbb{R}^n)$ is dense in $L^{p'}(\mathbb{R}^n)$, for every $u \in D(A_p)$ the mapping $\varphi \mapsto a(u,\varphi)$ may be continuously extended to $L^{p'}(\mathbb{R}^n)$ so that there exists a unique $f \in L^p(\mathbb{R}^n)$ such that $a(u,\varphi) = \langle f,\varphi\rangle_{L^p\times L^{p'}}$. Then we set

$$A_p u = f. \tag{1.5}$$

Therefore, fixed any $\lambda \in \mathbb{R}$, $f \in L^p(\mathbb{R}^n)$, a function $u \in D(A_p)$ is a solution of the resolvent equation

$$\lambda u - A_p u = f \tag{1.6}$$

if for each $\varphi \in W^{1,p'}_0(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \left(\sum_{i,j=1}^n q_{ij} D_j u D_j \varphi + \sum_{i=1}^n a_i u D_i \varphi - \sum_{i=1}^n b_i D_i u \varphi + \lambda u \varphi \right) dx = \\ = \int_{\mathbb{R}^n} f(x)\varphi(x)dx,$$

that is, if u is a distributional solution of

$$\lambda u - \mathcal{A} u = f. \tag{1.7}$$

Similarly, fixed any $\lambda \in \mathbb{R}$, $f_i \in L^p(\mathbb{R}^n)$, $i = 0, \dots, n$, a function $u \in W^{1,p}_{loc}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ (if $p \le 2$), $u \in W^{1,2}_{loc}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ (if $p \ge 2$) is said to be a solution of

$$\lambda u - \mathcal{A} u = f_0 + \sum_{i=1}^n D_i f_i \tag{1.8}$$

if for each $\varphi \in W^{1,p'}_0(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \left(\sum_{i,j=1}^n q_{ij} D_j u D_j \varphi + \sum_{i=1}^n a_i u D_i \varphi - \sum_{i=1}^n b_i D_i u \varphi + \lambda u \varphi \right) dx = \\ = \int_{\mathbb{R}^n} \left(f\varphi - \sum_{i=1}^n f_i D_i \varphi \right) dx,$$

that is, if u is a distributional solution of (1.8).

Things are a bit different in the case $p = \infty$. We can still prove that for λ large enough (precisely, for $\lambda > \lambda_\infty = \sum_{i=1}^n \|D_i a_i\|_\infty$) and for every $f \in L^\infty(\mathbb{R}^n)$ problem (1.6) has a unique solution $u \in L^\infty(\mathbb{R}^n) \cap H^1_{loc}(\mathbb{R}^n)$, and that the estimate

$$\|u\|_\infty \le \frac{1}{\lambda - \lambda_\infty} \|f\|_\infty$$

holds. However the domain of the realization A_∞ of $\mathcal{A}$ in $L^\infty(\mathbb{R}^n)$ is not dense in general, so that we cannot conclude that A_∞ generates a strongly continuous semigroup. Even if we replace $L^\infty(\mathbb{R}^n)$ by $UCB(\mathbb{R}^n)$, the space of the uniformly continuous and bounded functions, the domain of the realization A of $\mathcal{A}$ in $UCB(\mathbb{R}^n)$ fails to be dense in general. Neverthless, under further regularity assumptions on q_{ij} we have proved in [11] that A generates (in a suitable sense) a semigroup $T(t)$ which enjoys nice smoothing properties.

2. The case p=2

The main result of this section concerns unique solvability of (1.8), with $p = 2$, for λ large enough. The generation theorem will be a byproduct of this one.

THEOREM 2.1. *Set*

$$\lambda_2 = \frac{1}{2} \sum_{i=1}^n \|D_i(b_i - a_i)\|_{L^\infty}. \tag{2.1}$$

then for every $\lambda > \lambda_2$ and for every $f_0, \dots, f_n \in L^2(\mathbb{R}^n)$, problem (1.8) has a unique solution $u \in H^1(\mathbb{R}^n)$. There is $C(\lambda) > 0$, independent of f_i, $i = 0, \dots, n$, such that

$$\|u\|_{H^1} \le C(\lambda) \sum_{i=0}^n \|f_i\|_{L^2}. \tag{2.2}$$

Proof. We approximate the coefficients a_i and b_i by bounded ones. For $m \in \mathbb{N}$ we define

$$a_i^{(m)}(x) = \begin{cases} a_i(x) & \text{if } |x| \le m, \\ a_i(mx/|x|) & \text{otherwise}, \end{cases}$$
$$b_i^{(m)}(x) = \begin{cases} b_i(x) & \text{if } |x| \le m, \\ b_i(mx/|x|) & \text{otherwise}. \end{cases} \tag{2.3}$$

Note that the Lipschitz seminorms of $a_i^{(m)}$, $b_i^{(m)}$ are less or equal to the ones of a_i, b_i, respectively. Consider the operators $\mathcal{A}_m$ defined as the operator $\mathcal{A}$, with a_i replaced by $a_i^{(m)}$ and b_i replaced by $b_i^{(m)}$. For every $\lambda > \lambda_2$ and $f_0, \ldots, f_n \in L^2(\mathbb{R}^n)$, the equation

$$\lambda u_m - \mathcal{A}_m u_m = f_0 + \sum_{i=1}^n D_i f_i$$

has a unique solution $u_m \in H^1(\mathbb{R}^n)$ thanks to the Lax-Milgram theorem. Indeed, the bilinear form a_m, defined as a with a_i replaced by $a_i^{(m)}$ and b_i replaced by $b_i^{(m)}$, is obviously continuous in $H^1(\mathbb{R}^n)$ and it is coercive, as it is easy to check. Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^n} \Bigg(\sum_{i,j=1}^n q_{ij}(x) D_i u_m(x) D_j u_m(x) - \\ &- \sum_{i=1}^n (b_i^{(m)}(x) - a_i^{(m)}(x)) u_m(x) D_i u_m(x) + \lambda u_m^2(x) \Bigg) dx = \\ &\qquad = \int_{\mathbb{R}^n} \Bigg(f_0(x) u_m(x) - \sum_{i=1}^n f_i(x) D_i u_m(x) \Bigg) dx. \end{aligned} \tag{2.4}$$

Thanks to the ellipticity condition (1.2) we get

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^n q_{ij}(x) D_i u_m(x) D_j u_m(x) dx \ge \nu \|D u_m\|_{L^2}^2.$$

Moreover,

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\sum_{i=1}^{n}(b_i^{(m)}(x)-a_i^{(m)}(x))u_m(x)D_iu_m(x)dx\right| &= \\ &= \left|\frac{1}{2}\int_{\mathbb{R}^n}\sum_{i=1}^{n}(b_i^{(m)}(x)-a_i^{(m)}(x))D_i(u_m^2)(x)dx\right| \\ &= \left|\frac{1}{2}\int_{\mathbb{R}^n}u_m^2(x)\sum_{i=1}^{n}D_i(b_i^{(m)}-a_i^{(m)})(x)dx\right| \\ &\leq \frac{1}{2}\sum_{i=1}^{n}\|D_i(b_i^{(m)}-a_i^{(m)})\|_{L^\infty}\|u_m\|_{L^2}^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\nu\|Du_m\|_{L^2}^2+\left(\lambda-\frac{1}{2}\sum_{i=1}^{n}\|D_i(b_i-a_i)\|_{L^\infty}\right)\|u_m\|_{L^2}^2 \leq \\ \leq \|u_m\|_{L^2}\|f_0\|_{L^2}+\sum_{i=1}^{n}\|D_iu_m\|_{L^2}\|f_i\|_{L^2} \qquad (2.5)\end{aligned}$$

so that

$$\begin{aligned}\nu\|Du_m\|_{L^2}^2+(\lambda-\lambda_2)\|u_m\|_{L^2}^2 \leq \varepsilon\|u_m\|_{L^2}^2+\frac{1}{4\varepsilon}\|f_0\|_{L^2}^2+ \\ +\frac{\nu}{2}\|Du_m\|_{L^2}^2+\frac{1}{2\nu}\sum_{i=1}^{n}\|f_i\|_{L^2}^2,\quad \forall\varepsilon>0.\end{aligned}$$

Taking ε such that $\lambda-\lambda_2-\varepsilon>0$ we get

$$\frac{\nu}{2}\|Du_m\|_{L^2}^2+(\lambda-\lambda_2-\varepsilon)\|u_m\|_{L^2}^2 \leq \frac{1}{4\varepsilon}\|f_0\|_{L^2}^2+\frac{1}{2\nu}\sum_{i=1}^{n}\|f_i\|_{L^2}^2. \quad (2.6)$$

In particular, the functions u_m are equibounded in $H^1(\mathbb{R}^n)$. Hence there exists a subsequence $u_{m_k}^{(1)}$ converging weakly in $H^1(B(0,1))$. From this subsequence it is possible to extract another one $u_{m_k}^{(2)}$ converging weakly in $H^1(B(0,2))$. Iterating this procedure and defining $v_s=u_{m_s}^{(s)}$, the subsequence v_s converges weakly to a function u in

$H^1(K)$, for every compact set $K \subset \mathbb{R}^n$. It follows easily that u is a solution of (1.8) and satisfies (2.6), so that it satisfies (2.2).

It remains to prove uniqueness of the solution of (1.8). Let $z \in H^1(\mathbb{R}^n)$ be such that $\lambda z - \mathcal{A}z = 0$.

For every $k \geq 1$ let θ_k be a smooth cutoff function such that

$$\begin{cases} \theta_k(x) = 1 \ \text{ if } |x| \leq k, \ \ \theta_k(x) = 0 \ \text{ if } |x| \geq 2k, \ \ 0 \leq \theta_k(x) \leq 1, \\ \\ \|D_i\theta_k(x)\|_{L^\infty} \leq c/k \ \ \forall x \in \mathbb{R}^n, \ \ i = 1, \ldots, n, \end{cases} \tag{2.7}$$

where c is a constant independent on k. It is easy to check that $\theta_k z$ satisfies

$$\lambda\theta_k z - \mathcal{A}_m(\theta_k z) = -\sum_{i=1}^{n} D_i(q_{ij} z D_j\theta_k) + \sum_{i=1}^{n}(b_i^{(m)} - a_i^{(m)}) z D_i\theta_k,$$

provided m is large enough ($m > 2k$), so that $b_i^{(m)} = b_i$, $a_i^{(m)} = a_i$ on the support of θ_k. Estimate (2.6) gives then

$$\|\theta_k z\|_{H^1} \leq C(\lambda)\left(\sum_{i=1}^{n}\|q_{ij} z D_j\theta_k\|_{L^2} + \sum_{i=1}^{n}\|(b_i - a_i) z D_i\theta_k\|_{L^2}\right). \tag{2.8}$$

Let $\widetilde{B}_k$ be the complement of $B(0,k)$ in $\mathbb{R}^n$. Then for every i, $j = 1, \ldots, n$

$$\|q_{ij} z D_j\theta_k\|_{L^2(\mathbb{R}^n)} \leq \frac{c}{k}\|q_{ij}\|_{L^\infty}\|z\|_{L^2(\widetilde{B}_k)},$$

and for every $i = 1, \ldots, n$

$$\|(b_i - a_i) z D_i\theta_k\|_{L^2(\mathbb{R}^n)} \leq \frac{c}{k}\|b_i - a_i\|_{L^\infty(B(0,2k))}\|z\|_{L^2(\widetilde{B}_k)}.$$

Since a_i and b_i have at most linear growth there exists c_1 such that

$$\frac{c}{k}\sum_{i=1}^{n}\|a_i + b_i\|_{L^\infty(B(0,2k))} \leq c_1, \quad \forall k \in \mathbb{N}.$$

Therefore from (2.8) we get

$$\|\theta_k z\|_{H^1} \leq C(\lambda)\left[\frac{c}{k}\sum_{i,j=1}^{n}\|q_{ij}\|_{L^\infty} + c_1\right]\|z\|_{L^2(\widetilde{B}_k)}.$$

The right hand side goes to 0 when $k \to \infty$. Therefore, $z \equiv 0$. □

REMARK 2.2. The above proof shows in fact uniqueness of the solution in $H^1_{loc}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Define $D(A_2)$ as in the case of smooth coefficients, that is

$$\begin{cases} D(A_2) = \{u \in H^1(\mathbb{R}^n) : \exists C > 0 \text{ such that} \\ \qquad\qquad |a(u,\varphi)| \le C\|\varphi\|_{L^2}\ \forall \varphi \in H^1_0(\mathbb{R}^n)\}, \\ A_2 u = f, \end{cases}$$

where f is the unique element of $L^2(\mathbb{R}^n)$ such that $a(u,\varphi) = \langle f, \varphi\rangle$ for every $\varphi \in H^1_0(\mathbb{R}^n)$.

THEOREM 2.3. *The operator A_2 defined above generates a strongly continuous contraction semigroup in $L^2(\mathbb{R}^n)$. Specifically, $\rho(A_2) \supset \{\lambda \in \mathbb{R} : \lambda > \lambda_2\}$, λ_2 being defined by (2.1), and*

$$\|R(\lambda, A_2)f\|_{L^2(\mathbb{R}^n)} \le \frac{1}{\lambda - \lambda_2}\,\|f\|_{L^2(\mathbb{R}^n)}, \quad \lambda > \lambda_2, \tag{2.9}$$

$$\|DR(\lambda, A_2)f\|_{L^2(\mathbb{R}^n)} \le \frac{1}{\nu^{1/2}(\lambda - \lambda_2)^{\frac{1}{2}}}\,\|f\|_{L^2(\mathbb{R}^n)}, \quad \lambda > \lambda_2. \tag{2.10}$$

Proof. $D(A_2)$ is dense in L^2, since it contains $C_0^\infty(\mathbb{R}^n)$. Taking $\lambda > \lambda_2$, $f_0 = f \in L^2(\mathbb{R}^n)$, $f_i = 0$, $i = 1, \dots, n$, Theorem 2.1 implies that the resolvent equation

$$\lambda u - A_2 u = f$$

has a unique solution $u \in D(A_2)$. To prove estimates (2.9) and (2.10) let us revisit the proof of Theorem 2.1. For every $m \in \mathbb{N}$ we get from (2.5)

$$\nu\|Du_m\|^2_{L^2} + (\lambda - \lambda_2)\|u_m\|^2_{L^2} \le \|u_m\|_{L^2}\|f\|_{L^2},$$

so that

$$(\lambda - \lambda_2)\|u_m\|_{L^2} \le \|f\|_{L^2}, \quad (\nu(\lambda - \lambda_2))^{\frac{1}{2}}\|Du_m\|_{L^2} \le \|f\|_{L^2},$$

which implies (2.9) and (2.10). By the Hille-Yosida Theorem, A_2 generates a strongly continuous semigroup. □

COROLLARY 2.4. *$H^1(\mathbb{R}^n)$ belongs to the class $J_{1/2}$ between $L^2(\mathbb{R}^n)$ and $D(A_2)$. Specifically,*

$$\|Du\|_{L^2} \le \frac{2}{\nu^{1/2}} \|u\|_{L^2}^{1/2} \|(A_2 - \lambda_2 I)u\|_{L^2}^{1/2}, \quad \forall u \in D(A_2). \tag{2.11}$$

Proof. Fix $u \in D(A_2)$. By estimate (2.10) for every $\lambda > \lambda_2$ we have

$$\begin{aligned} \|Du\|_{L^2} &\le \frac{1}{\nu^{1/2}(\lambda-\lambda_2)^{1/2}} \|\lambda u - A_2 u\| \\ &\le \frac{(\lambda-\lambda_2)^{1/2}}{\nu^{1/2}} \|u\|_{L^2} + \frac{1}{\nu^{1/2}(\lambda-\lambda_2)^{1/2}} \|\lambda_0 u - A_2 u\|_{L^2}. \end{aligned}$$

If $\lambda_2 u - A_2 u = 0$, then $\|Du\|_{L^2} \le \nu^{-1/2}(\lambda-\lambda_2)^{1/2}\|u\|_{L^2}$ for every $\lambda > \lambda_2$, so that $u = 0$ and (2.11) holds. If $Au - \lambda_2 u \neq 0$, then $u \neq 0$. Take $\lambda > \lambda_2$ such that $(\lambda-\lambda_2)^{1/2} = \|Au - \lambda_2 u\|_{L^2}^{1/2}/\|u\|_{L^2}^{1/2}$. Then $\|Du\|_{L^2} \le 2\nu^{-1/2}\|u\|_{L^2}^{1/2}\|(A-\lambda_2 I)u\|_{L^2}^{1/2}$ and (2.11) is proved. □

REMARK 2.5. Similar results hold if the bilinear form a is replaced by

$$\begin{aligned} \widetilde{a}(u,\varphi) = a(u,\varphi) + \sum_{i=1}^n \int_{\mathbb{R}^n} \Big(&- \widetilde{a}_i(x)u(x)D_i\varphi(x) + \\ &+ \widetilde{b}_i D_i u(x)\varphi(x) + a_0(x)u(x)\Big) dx, \end{aligned}$$

provided the coefficients a_0, $\widetilde{a}_i$, $\widetilde{b}_j$, $i = 1, \dots, n$, belong to $L^\infty(\mathbb{R}^n)$. It is not hard to check that in this case the constant λ_2 has to be replaced by

$$\widetilde{\lambda}_2 = \lambda_2 + \|a_0^+\|_{L^\infty} + \frac{1}{2\varepsilon} \sum_{i=1}^n \|\widetilde{a}_i - \widetilde{b}_i\|_\infty,$$

where ε is any positive number such that

$$\varepsilon \max\{\|\widetilde{a}_i - \widetilde{b}_i\|_\infty : \ i = 1, \dots, n\} \le \frac{\nu}{2},$$

and $a_0^+(x) = \max\{a_0(x), 0\}$. Indeed, estimating $\|u_m\|_{H^1}$ as in the proof of Theorem 2.1 we get the additional term

$$\int_{\mathbb{R}^n} \sum_{i=1}^n (\widetilde{a}_i(x) - \widetilde{b}_i(x)) u_m(x) D_i u_m(x) dx - \int_{\mathbb{R}^n} a_0(x) u_m^2(x) dx.$$

The modulus of the first integral is less or equal to

$$\sum_{i=1}^n \|\widetilde{a}_i - \widetilde{b}_i\|_{L^\infty} \left(\frac{\varepsilon}{2} \|D_i u_m\|_{L^2}^2 + \frac{1}{2\varepsilon} \|u_m\|_{L^2}^2 \right)$$

for every $\varepsilon > 0$. The second integral is greater or equal to

$$-\int_{\mathbb{R}^n} a_0^+(x) u_m^2(x) dx \geq -\|a_0^+\|_{L^\infty} \|u_m\|_{L^2}^2,$$

and the statement follows.

The result of Theorem 2.3 may be extended to the case of suitably weighted L^2 spaces. Precisely, let $\psi \geq 0$ be a smooth function such that

$$\sum_{i=1}^n \left| \frac{D_i \psi(x)}{\psi(x)} \right| + \sum_{i,j=1}^n \left| \frac{D_{ij} \psi(x)}{\psi(x)} \right| \leq C(1 + |x|)^{-1}, \quad x \in \mathbb{R}^n. \tag{2.12}$$

We say that a function f belongs to $L^2_\psi(\mathbb{R}^n)$, ($H^1_\psi(\mathbb{R}^n)$, respectively) if $\|\psi\|_{L^2_\psi(\mathbb{R}^n)} = \|\psi f\|_{L^2(\mathbb{R}^n)}$ (respectively, $\|\psi\|_{H^1_\psi(\mathbb{R}^n)} = \|\psi f\|_{H^1(\mathbb{R}^n)}$) is finite.

The natural domain of the realization $A_{2,\psi}$ of $\mathcal{A}$ in $L^2_\psi(\mathbb{R}^n)$ is

$$\begin{aligned} D(A_{2,\psi}) \quad = \quad & \{u \in H^1_{loc}(\mathbb{R}^n) : \exists C > 0 \text{ such that} \\ & |a(u, \varphi)| \leq C \|\varphi\|_{L^2_\psi} \ \forall \varphi \in H^1_0(\mathbb{R}^n)\}. \end{aligned}$$

Proposition 2.6. *The operator $A_{2,\psi}$ generates a contraction semigroup in $L^2_\psi(\mathbb{R}^n)$. Moreover $D(A_{2,\psi}) \subset H^1_\psi(\mathbb{R}^n)$ and there is $C > 0$ such that for λ sufficiently large, say $\lambda > \lambda_\psi$,*

$$\|DR(\lambda, A)f\|_{L^2_\psi} \leq \frac{C}{\lambda^{1/2}} \|f\|_{L^2_\psi}, \quad \forall f \in L^2_\psi(\mathbb{R}^n).$$

Proof. If $f \in L^2_\psi(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$, the equation

$$\lambda u - A_{2,\psi} u = f \tag{2.13}$$

is equivalent (through the changement of unknown $v = \psi u$) to

$$\lambda v - B_2 v = \psi f, \tag{2.14}$$

where B_2 is the realization in $L^2(\mathbb{R}^n)$ of the operator associated to the bilinear form

$$b(v,\varphi) = a(v,\varphi) + \int_{\mathbb{R}^n} \sum_{i,j=1}^n q_{ij} \left(\frac{D_i\psi}{\psi} D_j v - \frac{D_i\psi}{\psi}\frac{D_j\psi}{\psi} v \right) \varphi\, dx$$

$$- \int_{\mathbb{R}^n} \sum_{i,j=1}^n q_{ij} \frac{D_i\psi}{\psi} v D_i\varphi\, dx + \int_{\mathbb{R}^n} \sum_{i=1}^n (b_i + a_i) \frac{D_i\psi}{\psi} v\varphi\, dx.$$

Since the coefficients b_i have at most linear growth and ψ satisfies (2.12), the form b satisfies the assumptions of Remark 2.5. Therefore, the operator B_2 generates a contraction semigroup in $L^2(\mathbb{R}^n)$, and consequently the operator $A_{2,\psi}$ generates a contraction semigroup in $L^2_\psi(\mathbb{R}^n)$.

Note that by Remark 2.5 the solution of (2.14) is unique in $L^2(\mathbb{R}^n) \cap H^1_{loc}(\mathbb{R}^n)$. Therefore the solution of (2.13) is unique in $L^2_\psi(\mathbb{R}^n) \cap H^1_{loc}(\mathbb{R}^n)$. □

The result of proposition 2.6, apart from its intrinsic interest, will be used later to study the case $p \neq 2$.

3. The case $p > 2$

For $p > 2$ we set

$$\begin{cases} D(A_p) = \{u \in H^1_{loc}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) : \exists C > 0 \text{ such that} \\ \qquad\qquad |a(u,\varphi)| \le C\|\varphi\|_{L^{p'}}\ \forall \varphi \in W^{1,p'}_0(\mathbb{R}^n)\}, \\ A_p u = f, \end{cases} \tag{3.1}$$

where f is the unique element of $L^p(\mathbb{R}^n)$ such that

$$a(u,\varphi) = \langle f, \varphi \rangle_{L^p \times L^{p'}} \text{ for every } \varphi \in W^{1,p'}_0(\mathbb{R}^n).$$

In the definition of $D(A_p)$ we cannot replace $u \in H^1_{loc}(\mathbb{R}^n)$ by $u \in W^{1,p}_{loc}(\mathbb{R}^n)$. Indeed, due to well-known counterexamples with bounded and measurable coefficients (see [6]), for $p > 2$ the estimate $|a(u,\varphi)| \leq C\|\varphi\|_{L^{p'}}$ for all $\varphi \in W_0^{1,p'}(\mathbb{R}^n)$ is not enough to guarantee that $u \in W^{1,p}_{loc}(\mathbb{R}^n)$.

The main result of this section is similar to Theorem 2.3.

THEOREM 3.1. *Let $2 < p < \infty$ and set*

$$\lambda_p = \frac{1}{p}\sum_{i=1}^{n} \|D_i(b_i - (p-1)a_i)\|_{L^\infty}. \tag{3.2}$$

Then every $\lambda > \lambda_p$ belongs to $\rho(A_p)$, and for every $f \in L^p(\mathbb{R}^n)$ we have

$$\|R(\lambda, A_p)f\|_{L^p} \leq \frac{1}{\lambda - \lambda_p}\,\|f\|_{L^p}. \tag{3.3}$$

In particular, A_p generates a strongly continuous contraction semigroup. Moreover for every $f \in L^p(\mathbb{R}^n)$ $|R(\lambda, A_p)f|^{p/2} \in H^1_{loc}(\mathbb{R}^n)$ and there is $C(\lambda) > 0$, independent of f, such that

$$\|D|R(\lambda, A_p)f|^{p/2})\|_{L^2} \leq C(\lambda)\|f\|^{p/2}_{L^p}. \tag{3.4}$$

Proof. Let ψ be a fixed weight function satisfying (2.12) and such that $L^p(\mathbb{R}^n) \subset L^2_\psi(\mathbb{R}^n)$ for every $p > 2$; for instance we may take $\psi(x) = (1 + \sum_{i=1}^n x_i^2)^{-n}$. We consider first the case where $\lambda > \max\{\lambda_p,\ \lambda_\psi\}$.

We approximate again the coefficients a_i and b_i by the bounded coefficients $a_i^{(m)}$, $b_i^{(m)}$ given by (2.3), and we approximate the coefficients q_{ij} by smooth ones, defined by

$$q_{ij}^{(m)}(x) = \int_{\mathbb{R}^n} q_{ij}(y-x)\eta_m(y)dy, \quad x \in \mathbb{R}^n, \tag{3.5}$$

where η_1 is a smooth function with support contained in $B(0,1)$ and with integral 1, and $\eta_m(x) = m^n\eta_1(mx)$. Then by (1.2)

$$\sum_{i,j=1}^{n} q_{ij}^{(m)}(x)\xi_i\xi_j \geq \nu|\xi|^2, \quad x,\ \xi \in \mathbb{R}^n. \tag{3.6}$$

Consider again the operators $\mathcal{A}_m$ defined as the operator $\mathcal{A}$, with coefficients replaced by $q_{ij}^{(m)}$, $a_i^{(m)}$, $b_i^{(m)}$ respectively.

For every $f \in L^p(\mathbb{R}^n)$, the equation

$$\lambda u_m - \mathcal{A}_m u_m = f$$

has a unique solution $u_m \in H^1_\psi(\mathbb{R}^n)$ due to Proposition 2.6. It belongs to $W^{1,p}(\mathbb{R}^n)$ thanks to classical regularity results (see e.g. [8]). Since $u_m|u_m|^{p-2} \in W^{1,p'}(\mathbb{R}^n)$ we may take it as a test function, getting

$$\begin{aligned}&\int_{\mathbb{R}^n}(p-1)\sum_{i,j=1}^n q_{ij}^{(m)}|u_m|^{p-2}D_iu_mD_ju_m dx\\ &-\int_{\mathbb{R}^n}\sum_{i=1}^n(b_i^{(m)}-a_i^{(m)})u_m|u_m|^{p-2}D_iu_m dx\\ &\qquad+\int_{\mathbb{R}^n}\lambda u_m^2(x)dx=\int_{\mathbb{R}^n}f(x)u_m(x)dx.\end{aligned}$$

Thanks to the ellipticity condition (3.6) we get

$$\int_{\mathbb{R}^n}(p-1)\sum_{i,j=1}^n q_{ij}^{(m)}|u_m|^{p-2}D_iu_mD_ju_m \geq \frac{4\nu(p-1)}{p^2}\|D(|u_m|^{p/2})\|^2_{L^2}.$$

Moreover,

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\sum_{i=1}^n(b_i^{(m)}-(p-1)a_i^{(m)})u_m|u_m|^{p-2}D_iu_m dx\right| =\\ &\qquad= \left|\frac{1}{p}\int_{\mathbb{R}^n}\sum_{i=1}^n(b_i^{(m)}-(p-1)a_i^{(m)})D_i(|u_m|^p)dx\right|\\ &\qquad= \left|\frac{1}{p}\int_{\mathbb{R}^n}\sum_{i=1}^n D_i(b_i^{(m)}-(p-1)a_i^{(m)})|u_m|^p dx\right|\\ &\qquad\leq \frac{1}{p}\sum_{i=1}^n\|D_i(b_i^{(m)}-(p-1)a_i^{(m)})\|_{L^\infty}\|u_m\|^p_{L^p}.\end{aligned}$$

Therefore,

$$\frac{4\nu(p-1)}{p^2}\|D|u_m|^{p/2}\|^2_{L^2} +$$

$$\left(\lambda - \frac{1}{p}\sum_{i=1}^{n} \|D_i(b_i - (p-1)a_i)\|_{L^\infty}\right)\|u_m\|_{L^p} \leq$$
$$\leq \|u_m\|_{L^p}^{p-1}\|f\|_{L^p}, \tag{3.7}$$

so that

$$(\lambda - \lambda_p)\|u_m\|_{L^p} \leq \|f\|_{L^p} \tag{3.8}$$

and

$$\left(\frac{4\nu(p-1)}{p^2}\right)^{1/p}(\lambda - \lambda_p)^{(p-1)/p}\|D|u_m|^{p/2}\|_{L^2}^{2/p} \leq \|f\|_{L^p}. \tag{3.9}$$

We shall show that a subsequence of u_m converges weakly to a function u in $L^p(\mathbb{R}^n)$ and in $H^1_\psi(\mathbb{R}^n)$, where ψ is any weight function satisfying (2.12) and such that $L^p(\mathbb{R}^n) \subset L^2_\psi(\mathbb{R}^n)$.

By (3.9) the sequence $\{u_m : m \in \mathbb{N}\}$ is bounded in $L^p(\mathbb{R}^n)$, so that a subsequence u_{m_k} converges weakly to a function $u \in L^p(\mathbb{R}^n)$, which satisfies

$$(\lambda - \lambda_p)\|u\|_{L^p} \leq \|f\|_{L^p}.$$

Moreover, since $f \in L^2_\psi(\mathbb{R}^n)$, by proposition 2.6 the sequence u_{m_k} is bounded in $H^1_\psi(\mathbb{R}^n)$, so that a subsequence u_{m_h} converges weakly to a function $v \in H^1_\psi(\mathbb{R}^n)$; obviously we have $v = u$.

Let us prove that u is a distributional solution of (1.7).

Let $\varphi \in C_0^\infty(R^n)$ and let m_0 be so large that the ball $B(0, m_0)$ contains the support of φ. Hence for each $m \geq m_0$, u_m satisfies

$$\int_{R^n}\left(\sum_{i,j=1}^{n} q_{ij}D_j u_m D_j\varphi - \sum_{i=1}^{n} b_i D_i u_m \varphi + \sum_{i=1}^{n} a_i u_m D_i\varphi + \lambda u_m \varphi\right)dx$$

$$= \int_{R^n}\left(f\varphi + \sum_{i,j=1}^{n}(q_{ij} - q_{ij}^{(m)})D_j u_m D_j\varphi\right)dx.$$

Note that

$$\left|\int_{B(0,m_0)}\sum_{i,j=1}^{n}(q_{ij}(x) - q_{ij}^{(m)})D_j u_m(x) D_j\varphi(x)dx\right| \leq$$

$$\leq \left(\int_{R^n} (\sum_{i,j=1}^n |q_{ij}(x) - q_{ij}^{(m)}|^4 dx\right)^{1/4} \cdot$$
$$\cdot \left(\int_{B(0,m_0)} \varphi^4(x)dx\right)^{1/4} \left(\int_{B(0,m_0)} u_m^2(x)dx\right)^{1/2}$$

which goes to 0 when $m \to +\infty$. Hence for each $\varphi \in C_0^\infty(R^n)$ the function v satisfies

$$\int_{R^n} \left(\sum_{i,j=1}^n q_{ij}(x)D_jv(x)D_j\varphi(x) - \right.$$
$$\left. - \sum_{i=1}^n b_i(x)D_iv(x)\varphi(x) + \lambda v(x)\varphi(x)\right) dx = \int_{R^n} f(x)\varphi(x)dx.$$

By the density of $C_0^\infty(R^n)$ such equality holds for each $\varphi \in H_0^1(\mathbb{R}^n)$. Therefore, u is a distributional solution of (1.8).

Uniqueness of the solution in L^p follows from uniqueness in L^2_ψ.

Let us consider now the case where $\lambda_\psi > \lambda_p$ and $\lambda \in (\lambda_p, \lambda_\psi]$. Fixed any μ such that $\lambda + \mu > \lambda_\psi$, the resolvent equation

$$\lambda u - A_p u = f \tag{3.10}$$

is equivalent to

$$(\lambda + \mu)u - A_p u = f + \mu u,$$

that is

$$u = R(\lambda + \mu, A_p)(f + \mu u).$$

The operator $u \mapsto \Gamma u = R(\lambda + \mu, A_p)(f + \mu u)$ is a contraction in $L^p(\mathbb{R}^n)$ since, by estimate (3.3),

$$\|R(\lambda + \mu, A_p)\mu u\|_{L^p} \leq \frac{\mu}{\lambda + \mu - \lambda_p} \|u\|_{L^p},$$

and $\mu/(\lambda + \mu - \lambda_p) < 1$. Therefore Γ has a unique fixed point in $L^p(\mathbb{R}^n)$, which is the unique solution of (3.10), and

$$\|u\|_{L^p} \leq \left(1 - \frac{\mu}{\lambda + \mu - \lambda_p}\right)^{-1} \frac{1}{\lambda + \mu - \lambda_p} \|f\|_{L^p} = \frac{1}{\lambda - \lambda_p} \|f\|_{L^p},$$

so that u satisfies (3.3). Moreover by (3.4)

$$\begin{aligned}\|D(|u|^{p/2})\|_{L^2} &\le C(\lambda+\mu)\|f+\mu u\|_{L^p}^{p/2}\\ &\le C(\lambda+\mu)(1+\mu/(\lambda-\lambda_p))^{p/2}\|f\|_{L^p}^{p/2}\\ &= C_1(\lambda)\|f\|_{L^p}^{p/2},\end{aligned}$$

so that u satisfies also (3.4). □

As a corollary of Theorem 3.1 we get a similar result for $p=\infty$.

COROLLARY 3.2. *Set*

$$\lambda_\infty = \sum_{i=1}^n \|D_i a_i\|_{L^\infty}. \tag{3.11}$$

Then for every $\lambda > \lambda_\infty$ *and for every* $f \in L^\infty(\mathbb{R}^n)$ *the equation*

$$\lambda u - \mathcal{A}u = f$$

has a unique solution $u \in H^1_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *and*

$$\|u\|_{L^\infty} \le \frac{1}{\lambda-\lambda_\infty}\|f\|_{L^\infty}. \tag{3.12}$$

Proof. Set again $\psi(x) = (1+\sum_{i=1}^n x_i^2)^{-n}$, and fix $\lambda > \max\{\lambda_\infty,\ \lambda_\psi\}$. Since $L^\infty(\mathbb{R}^n) \subset L^2_\psi(\mathbb{R}^n)$, the equation $\lambda u - \mathcal{A}u = f$ has a solution $u \in H^1_\psi(\mathbb{R}^n)$, and the solution is unique in $L^2_\psi(\mathbb{R}^n) \cap H^1_{loc}(\mathbb{R}^n)$, by Proposition 2.6.

For every $k \in \mathbb{N}$ set $f_k = f\chi_{B(0,k)}$. Then $f_k \in L^p(\mathbb{R}^n)$ for every p. Taking p large enough so that $\lambda > \lambda_p$ and setting $u_k = R(\lambda, A_p)f_k$, by Theorem 3.1 we have

$$\|u_k\|_{L^p(\mathbb{R}^n)} \le \frac{1}{\lambda-\lambda_p}\|f_k\|_{L^p(\mathbb{R}^n)} \le \frac{1}{\lambda-\lambda_p}\|f\|_{L^\infty(\mathbb{R}^n)}.$$

Therefore $u_k \in L^\infty(\mathbb{R}^n)$ and letting $p \to \infty$ we get

$$\|u_k\|_{L^\infty(K)} \le \frac{1}{\lambda-\lambda_\infty}\|f\|_{L^\infty(\mathbb{R}^n)},$$

and since K is arbitrary,

$$\|u_k\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{\lambda - \lambda_\infty}\|f\|_{L^\infty(\mathbb{R}^n)}.$$

Since $f_k \to f$ in $L^2_\psi(\mathbb{R}^n)$, then $u_k \to u$ in $L^2_\psi(\mathbb{R}^n)$, and a subsequence converges to u almost everywhere. It follows that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{\lambda - \lambda_\infty}\|f\|_{L^\infty(\mathbb{R}^n)}.$$

The case where $\lambda_\psi > \lambda_\infty$, $\lambda \in (\lambda_\infty, \lambda_\psi)$ can be treated as in the proof of Theorem 3.1. □

We cannot conclude that the realization of $\mathcal{A}$ in $L^\infty(\mathbb{R}^n)$ generates a strongly continuous semigroup because its domain is not dense in general, not even in the case of constant q_{ij} and linear b_i (see e.g. [7]).

4. The case $1 < p < 2$

For $1 < p < 2$ the solution of a divergence form equation with measurable and bounded coefficients q_{ij},

$$\lambda u - \sum_{i,j=1}^{n} D_i(q_{ij} D_j u) = f,$$

and $f \in L^p(\mathbb{R}^n)$, is not unique in general.

In the case of a bounded domain Ω with Dirichlet boundary condition Meyers [12] proved the existence of $\varepsilon > 0$ such that for $2 - \varepsilon < p < 2$ there is a unique solution in $W^{1,p}(\Omega)$. Serrin [15] (see also Prignet [14]) proved non uniqueness in the case $n > 2$ and $1 \le p \le n/(n-1)$ (see also the contribution of Boccardo et al. [3]). For the general case $n/(n-1) < p < 2-\varepsilon$ uniqueness is still an open question.

In our case ($\Omega = \mathbb{R}^n$, unbounded coefficients) it is possible to prove uniqueness of the solution of (1.6) in $W^{1,p}(\mathbb{R}^n)$ for λ large provided the coefficients q_{ij} are uniformly continuous.

Proposition 4.1. *Let q_{ij} be uniformly continuous and bounded, let a_i, b_i be Lipschitz continuous. Let $1 < p < 2$ and let $\lambda > \lambda_{p'}$. Then problem (1.6) has at most one solution in $W^{1,p}(\mathbb{R}^n)$.*

Proof. Assume that $z \in W^{1,p}(\mathbb{R}^n)$ is a solution of (1.6) with $f = 0$. Then for each $\varphi \in W_0^{1,p'}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \left(\lambda z\varphi + \sum_{i,j=1}^{n} q_{ij} D_j z D_j \varphi - \sum_{i=1}^{n} b_i D_i z \varphi + \sum_{i=1}^{n} a_i z D_i \varphi + \right) dx = 0. \tag{4.1}$$

We recall that

$$\|z\|_{L^p(\mathbb{R}^n)} = sup_{k\in\mathbb{N},\, g\in L^{p'}(\mathbb{R}^n):\, \|g\|_{L^{p'}}=1} \int_{\mathbb{R}^n} \theta_k g z dx.$$

For each $g \in L^{p'}(\mathbb{R}^n)$ such that $\|g\|_{L^{p'}} = 1$ let w be the unique solution of $\lambda w - A'_{p'} w = g$, where $A'_{p'}$ is the realization of the formal adjoint $\mathcal{A}'$ of $\mathcal{A}$ in $L^{p'}(\mathbb{R}^n)$,

$$\mathcal{A}'\varphi = \sum_{i,j=1}^{n} D_i(q_{ij} D_j \varphi) - \sum_{i=1}^{n} (D_i(b_i \varphi) - a_i D_i \varphi)). \tag{4.2}$$

Then for each $\varphi \in W_0^{1,p}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \left(\lambda w\varphi + \sum_{i,j=1}^{n} q_{ij} D_j w D_j \varphi - \right.$$
$$\left. - \sum_{i=1}^{n} b_i w(x) D_i \varphi + \sum_{i=1}^{n} a_i D_i w \varphi \right) dx \;=\; \int_{\mathbb{R}^n} f\varphi dx. \tag{4.3}$$

Since the coefficients q_{ij} are uniformly continuous and bounded, $w \in W_{loc}^{1,p'}(\mathbb{R}^n)$. Let θ_k be the cutoff functions defined by (2.7). Then $\theta_k z \in W_0^{1,p}(\mathbb{R}^n)$ may be taken as a test function in (4.3), and $\theta_k w \in W_0^{1,p'}(\mathbb{R}^n)$ may be taken as a test function in (4.1). Comparing we get

$$\begin{aligned} \int_{\mathbb{R}^n} g\theta_k z\, dx \;&=\; \int_{\mathbb{R}^n} \left(\sum_{i,j=1}^{n} q_{ij} z D_i w D_j \theta_k + \sum_{i,j=1}^{n} q_{ij} w D_i z D_j \theta_k \right) dx \\ &+ \int_{\mathbb{R}^n} \sum_{i=1}^{n} (a_i - b_i) w z D_i \theta_k\, dx. \end{aligned} \tag{4.4}$$

It is easy to see that all the addenda in the right hand side of (4.4) go to 0 as k goes to ∞, except perhaps

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^{n} q_{ij} z D_i w D_j \theta_k \, dx.$$

The difficulty is due to the fact that w does not necessarily belong to $W^{1,p'}(\mathbb{R}^n)$ but only to $W^{1,p'}_{loc}(\mathbb{R}^n)$.

To prove that also the above integral goes to 0 as $k \to \infty$ it is sufficient to show that for every $i = 1, \dots, n$, $x \mapsto (1+|x|^2)^{-1/2} D_i w(x) \in L^{p'}(\mathbb{R}^n)$. Indeed, setting

$$M_j = \sup_{x \in \mathbb{R}^n,\, k \in \mathbb{N}} |D_j \theta_k(x)|(1+|x|^2)^{1/2},$$

we have in that case

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} q_{ij} z D_i w D_j \theta_k \, dx \right| \le \\
&\qquad \le \sum_{i,j=1}^{n} \|q_{ij}\|_{L^\infty} M_j \left(\int_{\mathbb{R}^n} \frac{|D_i w(x)|^{p'}}{(1+|x|^2)^{p'/2}} dx \right)^{1/p'} \cdot \\
&\qquad\qquad \cdot \left(\int_{k \le |x| \le 2k} |z(x)|^p dx \right)^{1/p'}.
\end{aligned}$$

which goes to 0 as $k \to \infty$. Hence $\|z\|_{L^p(\mathbb{R}^n)} = 0$ and $u = v$.

The proof of the fact that $x \mapsto (1+|x|^2)^{-1/2} D_i w(x) \in L^{p'}(\mathbb{R}^n)$ for every $i = 1, \dots, n$ is rather lenghty.

Let θ_k, $k \in \mathbb{N}$, be the cutoff function considered in (2.7), and set $\chi_k = \theta_{2^k} - \theta_{2^{k-2}}$ for $k \ge 2$, $\chi_1 = \theta_1$. It is easy to check that the function $\chi_k w$ satisfies

$$\begin{aligned}
\lambda \chi_k w - \mathcal{A}'(\chi_k w) &= -\sum_{i,j=1}^{n} D_i(q_{ij} w D_j \chi_k)) - \sum_{i,j=1}^{n} q_{ij} D_i w D_j \chi_k + \\
&\quad + \sum_{i=1}^{n} (a_i + b_i) w D_i \chi_k + g \chi_k
\end{aligned}$$

so that the function v defined by

$$v(x) = \chi_k(2^{-k}x) w(2^{-k}x)$$

satisfies

$$\begin{aligned} 4^{-k}\lambda v - \sum_{i,j=1}^{n} D_i(\widetilde{q}_{ij}D_j v) + 2^{-k}\sum_{i=1}^{n}\widetilde{a}_i D_i v + \\ + 2^{-k}\sum_{i=1}^{n} D_i(\widetilde{b}_i v) \quad = \quad \phi_0 + \sum_{i=1}^{n} D_i\phi_i, \end{aligned}$$

where

$$\begin{aligned} \phi_0 \quad &= \quad -2^{-k}\sum_{i,j=1}^{n}\widetilde{q}_{ij}D_i\widetilde{w}D_j\chi_k(2^{-k}\cdot) \\ &\quad + 2^{-k}\sum_{i=1}^{n}(\widetilde{a}_i + \widetilde{b}_i)\widetilde{w}D_i\chi_k(2^{-k}\cdot) + 4^{-k}\widetilde{g}\chi_k(2^{-k}\cdot), \end{aligned}$$

$$\phi_i \quad = \quad -2^{-k}\sum_{j=1}^{n}\widetilde{q}_{ij}\widetilde{w}D_j\chi_k(2^{-k}\cdot), \qquad i = 1,\ldots,n,$$

and $\widetilde{q}_{ij}(y) = q_{ij}(2^{-k}y)$, $\widetilde{a}_i(y) = 2^{-k}a_i(2^{-k}y)$, $\widetilde{b}_i(y) = 2^{-k}b_i(2^{-k}y)$, $\widetilde{g}(y) = g(2^{-k}y)$ $\widetilde{w}(y) = w(2^{-k}y)$. The coefficients $\widetilde{a}_i$ and $\widetilde{a}_i$ are bounded by a constant independent of k; the coefficients $\widetilde{q}_{ij}$ are uniformly continuous with modulus of continuity bounded by a modulus of continuity independent of k. Therefore we may apply the classical regularity results (see e.g. [13, Thm. 7.4.1(iii) p. 297]), which give

$$\|v\|_{W^{1,p'}(\mathbb{R}^n)} \leq C\left(\sum_{i=0}^{n}\|\phi_i\|_{L^{p'}(\mathbb{R}^n)} + \|v\|_{L^{p'}(\mathbb{R}^n)}\right)$$

with constant C independent of k. It is not hard to see that there exists $C_1 > 0$ such that

$$\begin{aligned} \sum_{i=0}^{n}\|\phi_i\|_{L^{p'}(\mathbb{R}^n)} \quad \leq \quad C_1\Big(2^{-k}\|\widetilde{w}\|_{L^{p'}(B(0,2^{2k+1})\setminus B(0,2^{2k-2})} + \\ + 4^{-k}\|\widetilde{g}\|_{L^{p'}(B(0,2^{2k+1})\setminus B(0,2^{2k-2})}\Big). \end{aligned}$$

Recalling that $v = \widetilde{w}$ on $B(0,2^{2k}) \setminus B(0,2^{2k-1})$ and coming back to w we get

$$\sum_{i=1}^{n}\int_{B(0,2^k)\setminus B(0,2^{k-1})} |(1+|x|^2)^{-1/2}D_i w(x)|^{p'}dx \ \leq$$

$$\leq \; C_2 2^{-kp'} \int_{B(0,2^{k+1})\setminus B(0,2^{k-2})} (|(1+|x|^2)^{-1/2} D_i w(x)|^{p'} + \\ + |w(x)|^{p'} + |g(x)|^{p'})dx.$$

Summing up for $k \geq k_0$ we get

$$\sum_{i=1}^{n} \int_{\mathbb{R}^n \setminus B(0,2^{k_0-1})} |(1+|x|^2)^{-1/2} D_i w(x)|^{p'} dx \; \leq \\ \leq \; C_3 2^{-kp'} \int_{\mathbb{R}^n \setminus B(0,2^{k_0-2})} (|(1+|x|^2)^{-1/2} D_i w(x)|^{p'} + \\ + |w(x)|^{p'} + |g(x)|^{p'})dx.$$

Taking k_0 large enough we get

$$\sum_{i=1}^{n} \int_{\mathbb{R}^n \setminus B(0,2^{k_0-1})} |(1+|x|^2)^{-1/2} D_i w(x)|^{p'} dx \leq \\ \leq C_4 \int_{\mathbb{R}^n \setminus B(0,2^{k_0-2})} (|w(x)|^{p'} + |g(x)|^{p'})dx. \tag{4.5}$$

From the general regularity theory of elliptic differential equations (see e.g. [13, Thm. 7.4.1(iii)]) we get also

$$\sum_{i=1}^{n} \|D_i w(x)\|_{L^{p'}(B(0,2^{k_0-1}))} dx \leq \\ \leq C_5 \Big(\|w\|_{H^1(B(0,2^{k_0}))} + \|g(x)\|_{L^{p'}(B(0,2^{k_0}))} \Big). \tag{4.6}$$

Note that $\|w\|_{H^1(B(0,2^{k_0}))}$ is finite thanks to Proposition 2.6: it is sufficient to consider a weight ψ satisfying (2.12) and such that $L^p(\mathbb{R}^n) \subset L^2_\psi(\mathbb{R}^n)$. (4.5) and (4.6) imply now that $x \mapsto (1+|x|^2)^{-1/2} D_i w(x) \in L^{p'}(\mathbb{R}^n)$ for every $i = 1, \ldots, n$. □

In the case of measurable and bounded q_{ij} we have to look for a solution to (1.6) satisfying additional conditions if we want the solution to be unique.

In the paper [16], Stampacchia introduced a restricted class of solutions of Dirichlet problems in bounded domains in which he was able to prove uniqueness. We consider now a similar class when $\Omega = \mathbb{R}^n$.

Let $1 < p < 2$, let $\mathcal{A}'$ be the formal adjoint of $\mathcal{A}$ defined in (4.2), let $\lambda > \max\{\lambda_\psi,\ \lambda_{p'}\}$, and consider the operator

$$G : L^{p'}(\mathbb{R}^n) \mapsto L^{p'}(\mathbb{R}^n)$$

defined by $G(g) = v$, $v \in D(A_{p'})$ being the solution of

$$\lambda v - \mathcal{A}' v = g,$$

which exists and is unique by theorem 3.1. Such a theorem gives also

$$(\lambda - \lambda_{p'})\|G(g)\|_{L^{p'}} \leq \|g\|_{L^{p'}}.$$

Let now $f \in L^p(\mathbb{R}^n)$. A function $u \in L^p(\mathbb{R}^n)$ is said to be a S-solution of (1.6) if for any $g \in L^{p'}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} ugdx = \int_{\mathbb{R}^n} fG(g)dx.$$

If u is a S-solution, we say that $\mathcal{A}u = \lambda u - f$ in the *$S-$sense.* Existence and uniqueness of an S-solution is a consequence of the Riesz representation theorem.

To define the domain $D(A_p)$ we consider the set

$$\begin{aligned} D_p \quad = \quad & \{u \in W^{1,p}_{loc}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) :\ \exists C > 0 \text{ such that} \\ & |a(u,\varphi)| \leq C\|\varphi\|_{L^{p'}}\ \forall \varphi \in W^{1,p'}_0(\mathbb{R}^n)\}. \end{aligned}$$

For every $u \in D_p$ the mapping $\varphi \mapsto a(u,\varphi)$ may be continuously extended to $L^{p'}(\mathbb{R}^n)$ so that there exists a unique $f = f(u) \in L^p(\mathbb{R}^n)$ such that $a(u,\varphi) = \langle f, \varphi\rangle_{L^p \times L^{p'}}$. Then we set

$$D(A_p) = \{u \in D_p :\ \mathcal{A}u = f \text{ in the S-sense}\},\ \ A_p u = f, \qquad (4.7)$$

THEOREM 4.2. *Let $1 < p < 2$, $p' = (p-1)/p$. Then every $\lambda > \lambda_p$ belongs to $\rho(A_p)$, and for every $f \in L^p(\mathbb{R}^n)$ we have*

$$\|R(\lambda, A_p)f\|_{L^p} \leq \frac{1}{\lambda - \lambda_p}\,\|f\|_{L^p}. \qquad (4.8)$$

In particular A_p generates a strongly continuous contraction semigroup in $L^p(\mathbb{R}^n)$. Moreover $R(\lambda, A_p)f \in W^{1,p}(\mathbb{R}^n)$ and there is $c(\nu, p) > 0$, independent of f and λ, such that

$$\|DR(\lambda, A_p)f\|_{L^p} \leq \frac{c(\nu,p)}{(\lambda-\lambda_p)^{1/2}}\|f\|_{L^p}. \tag{4.9}$$

Proof. We try to follow as far as possible the procedure of Theorem 3.1. First we note that is sufficient to prove that the statement holds for $\lambda > \max\{\lambda_p,\ \lambda_{p'},\ \lambda_\psi\}$, where, as usual, $\psi(x) = (1+\sum_{i=1}^n x_i^2)^{-n}$ (the general case $\lambda > \lambda_p$ can be recovered arguing as in the final part of the proof of Theorem 3.1). So, we fix $\lambda > \max\{\lambda_p,\ \lambda_{p'},\ \lambda_\psi\}$ and we approximate the coefficients q_{ij}, a_i, b_i by q_{ij}^m, $a_i^{(m)}$, $b_i^{(m)}$ given by (3.5), (2.3) respectively. The problem (1.6) with coefficients q_{ij}^m, $a_i^{(m)}$, $b_i^{(m)}$ has a unique solution $u_m \in W^{1,p}(\mathbb{R}^n)$ for λ large enough. However, now we cannot take $|u_m|^{p-2}u_m$ as a test function to get an estimate similar to (3.9) because it does not necessarily belong to $W^{1,p'}(\mathbb{R}^n)$. Indeed, its gradient may have singularities of any order at the zeroes of u_m.

We overcome this difficulty taking as a test function

$$\varphi_{h.k} = u_m\left(u_m^2 + \frac{1}{h}\right)^{(p-2)/2}\theta_k, \quad h,\ k \in \mathbb{N},$$

where θ_k is the cut-off function defined in (2.7), and then letting h, $k \to \infty$. For every h, k we have

$$\int_{\mathbb{R}^n}\left(\lambda u_m\varphi_{h.k} + \sum_{i,j=1}^n q_{ij}^{(m)} D_j u_m D_j\varphi_{h.k} + \right.$$
$$\left. + \sum_{i=1}^n a_i^{(m)} u_m D_i\varphi_{h.k} - \sum_{i=1}^n b_i^{(m)} D_i u_m \varphi_{h.k}\right)dx = \int_{\mathbb{R}^n} f\varphi_{h.k}dx.$$

The right hand side is easily estimated, for all $h, k \in \mathbb{N}$, by

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f u_m\left(u_m^2+\frac{1}{h}\right)^{(p-2)/2}\theta_k dx\right| &\leq \int_{\mathbb{R}^n}|f|\cdot|u_m|^{p-1}dx \\ &\leq \|f\|_{L^p}\|u_m\|_{L^p}^{(p-1)/p}.\end{aligned}$$

The left hand side may be splitted into the sum $\sum_{i=1}^4 I_i$, where

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n} \lambda u_m^2 \left(u_m^2 + \frac{1}{h}\right)^{(p-2)/2} \theta_k dx, \\
I_2 &= \int_{\mathbb{R}^n} \sum_{i=1}^n ((p-1)a_i^{(m)} - b_i^{(m)}) u_m \left(u_m^2 + \frac{1}{h}\right)^{(p-2)/2} \theta_k D_i u_m dx, \\
I_3 &= \int_{\mathbb{R}^n} \sum_{i=1}^n a_i^{(m)} \left(u_m^2 \left(u_m^2 + \frac{1}{h}\right)^{(p-2)/2} D_i\theta_k - \right. \\
&\quad \left. - \frac{p-2}{h} u_m \left(u_m^2 + \frac{1}{h}\right)^{(p-4)/2} \theta_k D_i u_m \right) dx, \\
I_4 &= \int_{\mathbb{R}^n} \sum_{i,j=1}^n q_{ij}^{(m)} \left((p-1)\left(u_m^2 + \frac{1}{h}\right)^{(p-2)/2} D_i u_m D_j u_m \theta_k \right. \\
&\quad - \frac{p-2}{h} \left(u_m^2 + \frac{1}{h}\right)^{(p-4)/2} D_i u_m D_j u_m \theta_k + \\
&\quad \left. + u_m \left(u_m^2 + \frac{1}{h}\right)^{(p-2)/2} D_i u_m D_j \theta_k \right) dx.
\end{aligned}$$

Letting $h \to \infty$ and then $k \to \infty$ we get easily

$$\lim_{k\to\infty} (\lim_{h\to\infty} I_1) = \lambda \|u_m\|_{L^p}^p. \tag{4.10}$$

For every h, k we have, recalling that $u_m(u_m^2 + 1/h)^{(p-2)/2} D_i u_m = D_i(u_m^2 + 1/h)^{p/2}/p$,

$$\begin{aligned}
I_2 &= \frac{1}{p} \int_{\mathbb{R}^n} \sum_{i=1}^n ((p-1)a_i^{(m)} - b_i^{(m)}) \theta_k D_i \left(u_m^2 + \frac{1}{h}\right)^{p/2} dx \\
&= \frac{1}{p} \int_{\mathbb{R}^n} \left(u_m^2 + \frac{1}{h}\right)^{p/2} \sum_{i=1}^n \theta_k D_i ((p-1)a_i^{(m)} - b_i^{(m)}) dx \\
&+ \frac{1}{p} \int_{\mathbb{R}^n} \left(u_m^2 + \frac{1}{h}\right)^{p/2} \sum_{i=1}^n ((p-1)a_i^{(m)} - b_i^{(m)}) D_i \theta_k dx.
\end{aligned}$$

Since a_i, b_i have at most linear growth, there is $C > 0$ such that $\sum_{i=1}^n \|((p-1)a_i^m - b_i^m)D_i\theta_k\|_{L^\infty} \le C$, for every m, k, so that the second integral is estimated by $C/p\|(u_m^2 + 1/h)^{p/2}\|_{L^1(\widetilde{B}_k)}$, where $\widetilde{B}_k$ is the complement of $B(0,k)$ in $\mathbb{R}^n$. Therefore,

$$\lim_{k\to\infty}(\lim_{h\to\infty} I_2) = \frac{1}{p}\int_{\mathbb{R}^n}\sum_{i=1}^n D_i((p-1)a_i^{(m)} - b_i^{(m)})|u_m|^p dx. \tag{4.11}$$

Concerning I_3 we have

$$\begin{aligned} I_3 \quad \le \quad & \int_{\mathbb{R}^n}\left(u_m^2 + \frac{1}{h}\right)^{p/2}\sum_{i=1}^n |a_i^m D_i\theta_k|dx + \\ & + \left|\frac{1}{h}\int_{\mathbb{R}^n}\left(u_m^2 + \frac{1}{h}\right)^{(p-2)/2}\sum_{i=1}^n D_i(a_i^m\theta_k)dx\right|. \end{aligned}$$

Arguing as in the estimate for I_2 we see that the first integral goes to 0 as $h \to \infty$ and then $k \to \infty$. The second addendum is less or equal to

$$\frac{1}{h^{p/2}}\int_{k\le|x|\le 2k} D_i(a_i^m\theta_k)dx$$

which goes to 0 as $h \to \infty$ for every $k \in \mathbb{N}$. Therefore,

$$\lim_{k\to\infty}(\lim_{h\to\infty} I_3) = 0. \tag{4.12}$$

Finally, lets us consider I_4. Letting $h \to \infty$ and then $k \to \infty$ we see easily that the first addendum goes to

$$\begin{aligned} (p-1)\int_{\mathbb{R}^n}|u_m|^{p-2}\sum_{i,j=1}^n q_{ij}^m D_i u_m D_j u_m dx \ge \\ \ge (p-1)\nu\int_{\mathbb{R}^n}|u_m|^{p-2}|Du_m|^2 dx. \end{aligned}$$

The second addendum is nonnegative for every h,k. As $h \to \infty$ the third one goes to

$$\int_{\mathbb{R}^n} u_m|u_m|^{p-2}\sum_{i,j=1}^n q_{ij}^{(m)} D_i u_m D_j\theta_k dx,$$

whose modulus is less or equal to

$$\frac{C}{k}\sum_{i,j=1}^{n}\|q_{ij}\|_{L^\infty}\left(\int_{\mathbb{R}^n}|D_iu_m|^pdx\right)^{1/p}\left(\int_{\mathbb{R}^n}|u_m|^pdx\right)^{(p-1)/p}$$

which goes to 0 as $k\to\infty$. Therefore,

$$\liminf_{h,\,k\to\infty} I_4 \geq (p-1)\nu\int_{\mathbb{R}^n}|u_m|^{p-2}|Du_m|^2dx. \tag{4.13}$$

Taking into account (4.10), (4.11), (4.12), (4.13) we get

$$\begin{aligned}(p-1)\nu\int_{\mathbb{R}^n}|u_m|^{p-2}|Du_m|^2dx+\lambda\|u_m\|^p_{L^p}-\\ -\frac{1}{p}\sum_{i=1}^{n}\|D_i(b_i-(p-1)a_i)\|_{L^\infty}\|u_m\|^p_{L^p} \quad\leq\quad \|f\|_{L^p}\|u_m\|^{p-1}_{L^p}\end{aligned}$$

which coincides with (3.7), so that (3.8) and (3.9) hold. Therefore, the sequences u_m, $D|u_m|^{p/2}$, are bounded in $L^p(\mathbb{R}^n)$, so that $|u_m|^{p/2}$ is bounded in $L^2(\mathbb{R}^n)$. We prove now that u_m is bounded in $W^{1,p}(\mathbb{R}^n)$. For every m we have

$$\begin{aligned}\int_{\mathbb{R}^n}|Du_m|^pdx &= \int_{\mathbb{R}^n}|Du_m|^p|u_m|^{-(2-p)p/2}|u_m|^{(2-p)p/2}dx\\ &\leq \left(\int_{\mathbb{R}^n}|u_m|^pdx\right)^{\frac{2-p}{2}}\left(\int_{\mathbb{R}^n}|Du_m|^2u_m^{p-2}dx\right)^{\frac{p}{2}}\\ &\leq \frac{1}{(\nu(p-1))^{p(p-1)/2}}\,\frac{1}{(\lambda-\lambda_p)^{1/2}}\,\|f\|^p_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Let $f\in L^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. From Theorem 2.1 we know that problem (1.6) has a unique solution u in $D(A_2)$, obtained as the weak limit of a subsequence u_{m_k} of u_m. Since u_{m_k} is bounded in $W^{1,p}(\mathbb{R}^n)$, then $u\in W^{1,p}(\mathbb{R}^n)$.

Let us prove that u is a S-solution of (1.6). For every $g\in L^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ let $w=G(g)\in D(A_2')\cap D(A_{p'}')$ be the solution of $\lambda w-\mathcal{A}w=g$. Then

$$\begin{aligned}\int_{\mathbb{R}^n}vgdx &= \int_{\mathbb{R}^n}v(\lambda w-A_2w)dx=\\ &= \int_{\mathbb{R}^n}(\lambda v-A_2v)G(g)dx = \int_{\mathbb{R}^n}fG(g)dx.\end{aligned}$$

Since $L^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $L^{p'}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ is dense in $L^{p'}(\mathbb{R}^n)$, and the mappings $L^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\mapsto L^p(\mathbb{R}^n)$, $f\mapsto u$, and $L^{p'}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\mapsto L^{p'}(\mathbb{R}^n)$, $g\mapsto G(g)$, are continuous, then the above equality holds for every $f\in L^p(\mathbb{R}^n)$, $g\in L^{p'}(\mathbb{R}^n)$. In other words, the function u constructed by our procedure is a S-solution of (1.6). This ends the proof. □

The following corollary may be proved as Corollary 2.4.

COROLLARY 4.3. *$W^{1,p}(\mathbb{R}^n)$ belongs to the class $J_{1/2}$ between $L^p(\mathbb{R}^n)$ and $D(A_p)$.*

The semigroup generated by A_p is not in general analytic, as the following counterexample shows.

EXAMPLE 4.4. *Let $n=1$ and set $\mathcal{A}u(x)=u''(x)+xu'(x)$. Then the semigroup $T(t)$ generated by the realization of $\mathcal{A}$ in $L^p(\mathbb{R})$ is not differentiable, and consequently it is not analytic.*

Proof. We shall show that for every $t>0$, $T(t)$ does not map continuously $L^p(\mathbb{R})$ to $D(A_p)$. There is a simple representation formula for $T(t)$: indeed, for $t>0$ we have

$$(T(t)u)(x)=\frac{1}{\sqrt{2\pi(e^{2t}-1)}}\int_R e^{-\frac{y^2}{2(e^{2t}-1)}}u(e^tx-y)dy.$$

Let $u_n=\chi_{[n,n+1]}$. Then

$$T(t)u_n(x)=\frac{1}{\sqrt{2\pi(e^{2t}-1)}}\int_{e^tx-n-1}^{e^tx-n}e^{-\frac{y^2}{2(e^{2t}-1)}}dy,$$

so that

$$\begin{aligned}\frac{d}{dx}T(t)u_n(x) &= \frac{e^t}{\sqrt{2\pi(e^{2t}-1)}}\left(e^{-\frac{(e^tx-n)^2}{2(e^{2t}-1)}}-e^{-\frac{(e^tx-n-1)^2}{2(e^{2t}-1)}}\right),\\ \frac{d^2}{dx^2}T(t)u_n(x) &= \frac{e^{2t}}{\sqrt{2\pi(e^{2t}-1)^3}}\left(-(e^tx-n)e^{-\frac{(e^tx-n)^2}{2(e^{2t}-1)}}+\right.\\ &\qquad \left.+(e^tx-n-1)e^{-\frac{(e^tx-n-1)^2}{2(e^{2t}-1)}}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\frac{d^2}{dx^2}T(t)u_n\right\|_{L^p} &= \frac{e^{2t}}{\sqrt{2\pi(e^{2t}-1)^3}}\left(\int_{\mathbb{R}}\left|(e^t x-n)e^{-\frac{(e^t x-n)^2}{2(e^{2t}-1)}} -\right.\right.\\ &\quad \left.\left. -\,(e^t x-n-1)e^{-\frac{(e^t x-n-1)^2}{2(e^{2t}-1)}}\right|dx\right)^{1/p}\\ &\leq \frac{2^{p-1}e^{2t}}{\sqrt{2\pi(e^{2t}-1)^3}}\left(\int_{\mathbb{R}} e^{-t}|y|^p e^{-\frac{py^2}{2(e^{2t}-1)}}dy\right)^{1/p}\\ &= c(p)\frac{e^{t(2-1/p)}}{e^{2t}-1},\end{aligned}$$

which is bounded independently on n, and

$$\begin{aligned}\left\|x\,\frac{d}{dx}T(t)u_n\right\|^p_{L^p(R)} &= \frac{e^{pt}}{(2\pi(e^{2t}-1))^{\frac{p}{2}}}\,\cdot\\ &\quad \cdot\int_R (z+n)^p\left(e^{-\frac{z^2}{2(e^{2t}-1)}} - e^{-\frac{(z-1)^2}{2(e^{2t}-1)}}\right)^p dz\;,\end{aligned}$$

which goes to ∞ as n goes to ∞. Therefore for every $t > 0$ we have $\lim_{n\to\infty}\|A_pT(t)u_n\|_{L^p} = +\infty$, whereas $\|u_n\|_{L^p} = 1$ for every n. □

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