Nonlinear Infinite Dimensional Optimal Control Problems with State Constraints and Unbounded Control Sets

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Summary. - *Using nonlinear programming theory in Banach spaces we derive a version of Pontryagin's maximum principle that includes state constraints and allows unbounded control sets. We discuss applications.*

1. Introduction

Let A be the infinitesimal generator of a strongly continuous semigroup $S(t)$ in a Banach space E. Consider the optimal control problem for the system

$$y'(t) = Ay(t) + f(t, y(t), u(t)), \quad y(0) = \zeta \qquad (1.1)$$

in a fixed or variable interval $0 \le t \le \bar{t}$, with cost functional

$$y_0(t, u) = \int_0^t f_0(\tau, y(\tau), u(\tau)) \, d\tau , \qquad (1.2)$$

with control constraint

$$u(t) \in U \quad = \quad \textbf{control set}, \qquad (1.3)$$

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and with state constraint and target conditions

$$y(t) \in M \quad = \quad \textbf{state constraint set} \quad (0 \le t \le \bar{t}),$$
$$y(\bar{t}) \in Y \quad = \quad \textbf{target set}. \tag{1.4}$$

Under suitable conditions, Pontryagin's maximum (or minimum) principle holds under the form

$$z_0 f_0(t, y(t, \bar{u}), \bar{u}(t)) + \langle \bar{z}(t), f(t, y(t, \bar{u}), \bar{u}(t)) \rangle$$
$$= \min_{v \in U} \{ z_0 f_0(t, y(t, \bar{u}, v) + \langle \bar{z}(t), f(t, y(t, \bar{u}), v) \rangle \} \tag{1.5}$$

a.e. in $[0, \bar{t}]$, where $\bar{u}(t)$ is an optimal control, $y(t, \bar{u})$ is the corresponding optimal trajectory and $\bar{z}(t) \in E^*$ is the costate. The minimum principle puts especially drastic requirements on the optimal control $\bar{u}(t)$ and the costate $\bar{z}(t)$ when the control set U is unbounded. For instance, if $f(t, y, u) = f(t, y) + u$ and $U = E$ then (1.5) guarantees that $z_0 \ne 0$ or that $\bar{z}(t) = 0$ a.e. (otherwise the minimum would be $-\infty$ in a set of positive measure, contradicting existence of optimal controls). The same contradiction with existence is obtained (whether or not $z_0 \ne 0$) if $f_0(t, y, u)$ does not contain u explicitly or, more generally, if it has less than linear growth in u. Unboundedness of the control set can be exploited in many other ways. In fact, when given a choice (such as in parametric finite dimensional problems) taking the control set U unbounded usually leads to stronger results and/or shorter and more elegant proofs; for a beautiful treatment of several classical problems of the calculus of variations in this vein see [8].

The maximum principle for general infinite dimensional optimal control problems with unbounded control sets seem to have received scant attention except in particular cases such as the linear-quadratic problem, where an enormous literature exists. For general problems, two early contributions are [6] and [7], the first admitting point targets but no state constraints, the second (for the Navier−Stokes equations modelled as abstract differential equations) not including either point targets or state constraints. However, the proof in [6] is incorrect and that in [7] pays insufficient attention to several key points. We present in this paper a general minimum principle for control problems described by abstract differential equations that

includes state constraints and point targets and allows unbounded control sets.

2. The abstract model

We consider the control system (1.1), where A generates a strongly continuous semigroup $S(t)$ in a Banach space E. The control set U is arbitrary, and the admissible control space $C_{\text{ad}}(0, T; U)$ consists of $U-$valued functions $u(\,\cdot\,)$ defined in $0 \leq t \leq T$; $C_{\text{ad}}(0, T; U)$ is equipped with the distance

$$d(u(\,\cdot\,), v(\,\cdot\,)) = |\{t \in [0, T]; u(t) \neq v(t)\}|, \qquad (2.1)$$

where $|\,\cdot\,|$ indicates exterior measure (the set between bars in (2.1) may not measurable). The function $f : [0, T] \times E \times U \to E$ has a Fréchet derivative $\partial_y f(t, y, u) \in (E, E)$ with respect to y $((E, F)$ is the space of all linear bounded operators from a Banach space E into a Banach space F equipped with the uniform operator norm) and satisfies:

(a) For every t, u and $z \in E$ fixed the functions $y \to f(t, y, u)$ and $y \to \partial_y f(t, y, u)z$ are continuous in y,

(b) For every $u(\,\cdot\,) \in C_{\text{ad}}(0, T; U)$ and y, z fixed the functions $t \to f(t, y, u(t))$ and $t \to \partial_y f(t, y, u(t))z$ are strongly measurable,

(c) For every $u(\,\cdot\,) \in C_{\text{ad}}(0, T; U)$ there exist functions $K_u(\,\cdot\,, c)$, $L_u(\,\cdot\,, c)$ (in general depending on u) such that

$$\|f(t, y, u(t))\|_E \leq K_u(t, c), \quad \|\partial_y f(t, y, u(t))\|_{(E, E)} \leq L_u(t, c)$$
$$(0 \leq t \leq T, \ \|y\|_E \leq c). \quad (2.2)$$

State and target conditions are given by (1.4), with the state constraint set M and the target set Y closed in E. The cost functional is given by (1.2) with assumptions on $f_0 : [0, T] \times E \times U \to I\!R$ that parallel those on $f(t, y, u)$; the Fréchet derivative $\partial_y f_0(t, y, u) \in E^*$ exists and

(d) For every t, u fixed the functions $y \to f_0(t, y, u)$ and $y \to \partial_y f_0(t, y, u)$ are continuous in y,

(e) For every $u(\,\cdot\,) \in C_{\mathrm{ad}}(0,T;U)$ and y fixed the functions $t \to$
$f_0(t,y,u(t))$ and $\partial_y f_0(t,y,u(t))$ are strongly measurable,

(f) For every $u(\,\cdot\,) \in C_{\mathrm{ad}}(0,T;U)$ there exist functions $K_u^0(\,\cdot\,,c)$,
$L_u^0(\,\cdot\,,c)$ (in general depending on u) such that

$$\|f_0(t,y,u(t))\|_E \le K_u^0(t,c), \quad \|\partial_y f_0(t,y,u(t))\|_{E^*} \le L_u^0(t,c)$$
$$(0 \le t \le T, \|y\|_E \le c). \quad (2.3)$$

A subspace $C_{\mathrm{bad}}(0,T;U) \subseteq C_{\mathrm{ad}}(0,T;U)$ is (f,f_0)-**bounded** if estimates (2.2), (2.3) are independent of $u(\,\cdot\,) \in C_{\mathrm{bad}}(0,T;U)$, that is, there exist functions $K(t,c), L(t,c), K^0(t,c), L^0(t,c)$ such that (2.2) and (2.3) hold for all $u(\,\cdot\,) \in C_{\mathrm{bad}}(0,T;U)$. Obviously, a (f,f_0)-bounded subspace $C_{\mathrm{bad}}(0,T;U)$, remains (f,f_0)-bounded (with different functions on the right sides of (2.2) and (2.3)) after adjoining any finite number of elements of $C_{\mathrm{ad}}(0,T;U)$.

The (f,f_0)-**completion** $[\![C_{\mathrm{bad}}(0,T;U)]\!]$ of a bounded subspace $C_{\mathrm{bad}}(0,T;U) \subseteq C_{\mathrm{ad}}(0,T;U)$ is obtained by adjoining to $C_{\mathrm{bad}}(0,T;U)$ all other controls $u(\,\cdot\,) \in C_{\mathrm{ad}}(0,T;U)$ that satisfy (2.2) and (2.3). Obviously, the completion of a (f,f_0)-bounded subspace will depend not only on f, f_0 but on the particular functions K, L, K^0, L^0 on the right hand side of (2.2) and (2.3), so that a more precise notation for $[\![C_{\mathrm{bad}}(0,T;U)]\!]$ would be

$$[\![C_{\mathrm{bad}}(0,T;U)]\!]_{(f,f_0,K,L,K^0,L^0)}.$$

We will gloss over this subtlety.

A sequence $u_n(\,\cdot\,) \subseteq C_{\mathrm{ad}}(0,T;U)$ is **stationary** if there exists a null set e such that, for every $t \in [0,T]\backslash e$ there exists n (depending on t) such that $u_n(t) = u_{n+1}(t) = u_{n+2}(t) = \ldots$. A subspace $C_{\mathrm{sad}}(0,T;U) \subseteq C_{\mathrm{ad}}(0,T;U)$ is **saturated** if the pointwise limit of every stationary sequence in $C_{\mathrm{sad}}(0,T;U)$ belongs to $C_{\mathrm{sad}}(0,T;U)$. Saturation implies completeness of $C_{\mathrm{sad}}(0,T;U)$ with respect to the metric (2.1).

Let $u(\,\cdot\,) \in C_{\mathrm{ad}}(0,T;U), \mathbf{v}(\,\cdot\,) = (v_1(\,\cdot\,),\ldots,v_m(\,\cdot\,))$ be an arbitrary collection of m elements of $C_{\mathrm{ad}}(0,T;U)$, $\mathbf{e} = (e_1,\ldots,e_m)$ a collection of m pairwise disjoint sets in $[0,T]$. The **patch perturbation** $u_{\mathbf{e},\mathbf{v}}(\,\cdot\,)$ of $u(\,\cdot\,)$ corresponding to $\mathbf{e}, \mathbf{v}$ is $u_{\mathbf{e},\mathbf{v}}(t) = v_j(t)$ ($t \in e_j, j = 1,\ldots,m), u_{\mathbf{e},\mathbf{v}}(t) = u(t)$ elsewhere. A subspace $C_{\mathrm{cad}}(0,T;U)$

$\subseteq C_{\mathrm{ad}}(0, T; U)$ is **patch complete** if every patch perturbation of a control $u(\,\cdot\,)$ with $v_j(\,\cdot\,) \in C_{\mathrm{cad}}(0, T; U)$ belongs to $C_{\mathrm{cad}}(0, T; U)$. If $C_{\mathrm{bad}}(0, T; U)$ is a (f, f_0)−bounded subspace of $C_{\mathrm{ad}}(0, T; U)$, then every patch perturbation $u_{\mathbf{e},\mathbf{v}}(\,\cdot\,)$ of a control $u(\,\cdot\,) \in C_{\mathrm{bad}}(0, T; U)$ with $v_j(\,\cdot\,) \in C_{\mathrm{bad}}(0, T; U)$ belongs to the completion $[\![C_{\mathrm{bad}}(0, T; U)]\!]$.

We add the following three assumptions on the control set and the control space:

(g) There exists a sequence $U_1 \subseteq U_2 \subseteq \ldots \subseteq U$ of subsets of U such that

$$U = \bigcup_{m=1}^{\infty} U_m \qquad\qquad (2.4)$$

and such that, for each m, the subspace $C_{\mathrm{ad}}(0, T; U_m)$ consisting of all $u(\,\cdot\,) \in C_{\mathrm{ad}}(0, T; U)$ with $u(t) \in U_m$ is (f, f_0)−bounded and satisfies

(h) $C_{\mathrm{ad}}(0, T; U_m)$ is saturated.

(k) $C_{\mathrm{ad}}(0, T; U_m)$ is patch complete

Note that we do *not* assume that the entire admissible control space $C_{\mathrm{ad}}(0, T; U)$ is saturated; the reason why is illustrated in an example in §4.

Our approach to the minimum principle goes as follows. Let $\bar{u}(\,\cdot\,)$ be an optimal control (with respect to the entire admissible control space $C_{\mathrm{ad}}(0, T; U)$). Then $\bar{u}(\,\cdot\,)$ is optimal in any subspace of $C_{\mathrm{ad}}(0, T; U)$, in particular, in $[\![\{\bar{u}(\,\cdot\,)\} \cup C_{\mathrm{ad}}(0, T; U_m)]\!]$ for each m. Since this space is (f, f_0)−bounded, estimates (2.2) and (2.3) hold for an arbitrary element $u(\,\cdot\,) \in [\![\{\bar{u}(\,\cdot\,)\} \cup C_{\mathrm{ad}}(0, T; U_m)]\!]$ with functions $K(\,\cdot\,, c)$, $L(\,\cdot\,, c)$, $K^0(\,\cdot\,, c)$, $L^0(\,\cdot\,, c)$ independent of the control $u(\,\cdot\,)$. Under these conditions, Pontryagin's minimum principle including state constraints has been shown to hold in [5]. In its integral form, it reads

$$z_{0m} \int_0^{\bar{t}} \{f_0(s, y(s, \bar{u}), v(s)) - f_0(s, y(s, \bar{u}), \bar{u}(s))\} ds$$

$$+ \int_0^{\bar{t}} \langle \bar{z}_m(s), f(s, y(s, \bar{u}), v(s)) - f(s, y(s, \bar{u}), \bar{u}(s)) \rangle ds \geq 0 \quad (2.5)$$

and holds in a set $e_m \subseteq [0, T]$ of full measure for arbitrary $v(\,\cdot\,) \in [\![\{\bar{u}(\,\cdot\,)\} \cup C_{\mathrm{ad}}(0, T; U_m)]\!]$, where the costate $\bar{z}_m(s)$ solves the adjoint

variational equation

$$dz̄_m(s) = -\{A^* + \partial_y f(s, y(s, \bar{u}), \bar{u}(s))^*\}\bar{z}_m(s)ds$$
$$-z_{0m}\partial_y f_0(s, y(s, \bar{u}), \bar{u}(s))ds - \mu_m(ds), \qquad \bar{z}_m(\bar{t}) = z_m. \quad (2.6)$$

Here, $(z_{0m}, \mu_m, z_m) \in I\!\!R \times \Sigma(0, \bar{t}; E^*) \times E^*$, where $\Sigma(0, T; E^*)$ is the space of all E^*-valued countably additive measures μ of bounded variation defined in the field generated by the closed sets in $[0, T]$, the space $\Sigma(0, \bar{t}; E^*)$ endowed with the total variation norm; we have $\Sigma(0, \bar{t}; E^*) = C(0, T; E)^*$ with duality product

$$\langle f, \mu \rangle = \int_0^{\bar{t}} \langle f(s), \mu(ds) \rangle$$

(see [5]). We also have

$$z_{0m} \geq 0, \;\; \mu_m \in (\liminf_{n \to \infty} T_{\mathbf{M}(\bar{t})}(\tilde{y}_m^n(\cdot)))^-, \;\; z_m \in (\liminf_{n \to \infty} T_Y(\tilde{y}_m^n))^-,$$
$$(2.7)$$

where $\mathbf{M}(\bar{t}) \subseteq C(0, \bar{t}; E)$ is defined by

$$\mathbf{M}(\bar{t}) = \{y(\cdot) \in C(0, \bar{t}; E); \; y(t) \in M \;\; (0 \leq t \leq \bar{t})\},$$

and where $\{\tilde{y}_m^n(\cdot)\}$ (resp.$\{\tilde{y}_m^n\}$) is a sequence in $\mathbf{M}(\bar{t})$ (resp. Y) such that $\tilde{y}_m^n(\cdot) \to y(\cdot, \bar{u})$ (resp. $\tilde{y}_m^n \to y(\bar{t}, \bar{u})$); $T_X(x)$ denotes the Clarke tangent cone to X at $x \in X$. The **polar cone** $X^- \subseteq E^*$ to a set $X \subseteq E$ is the set of all $z \in E^*$ such that $\langle z, x \rangle \leq 0 \; (x \in X)$, and $\liminf_{n\to\infty} X_n$ is the set of all $x = \lim_{n\to\infty} x_n$ with $x_n \in X_n$.

The solution $\bar{z}_m(s)$ of (2.6) is understood as follows: if $S(t, s; \bar{u})$ is the solution operator of the variational equation

$$\xi'(t) = \{A + \partial_y f(t, y(t, \bar{u}), \bar{u}(t))\}\xi(t)$$

and

$$\nu_m(ds) = \delta_{\bar{t}}(ds)z + z_{0m}\partial_y f_0(s, y(s, \bar{u}), \bar{u}(s))ds + \mu_m(ds)$$

($\delta_{\bar{t}}(ds)$ the Dirac delta centered at $\bar{t}$) then

$$\bar{z}_m(s) = \int_s^{\bar{t}} S(\sigma, s; \bar{u})^* \nu_m(d\sigma), \qquad (2.8)$$

which is shorthand for: $\bar{z}_m(s)$ is the only element of E^* satisfying

$$\langle \bar{z}_m(s), y \rangle = \int_s^{\bar{t}} \langle \nu_m(d\sigma), S(\sigma, s; \bar{u})y \rangle \qquad (2.9)$$

for all $y \in E$. No assumptions on the adjoint semigroup $S(t)^*$ or the adjoint solution operator are necessary. In general, $\bar{z}_m(\cdot)$ is everywhere defined, $E-$weakly continuous and strongly measurable when E^* is separable; moreover,

$$\|\bar{z}_m(s)\|_{E^*} \leq C(|z_{0m}| + \|z_m\|_{E^*} + \|\mu_m\|_{\Sigma(0,\bar{t},E^*)}) \quad (0 \leq t \leq \bar{t}). \ (2.10)$$

where C is a bound independent of m. Throughout, the subindex m on the various entities is a reminder that we are applying the maximum principle in the space $[\![\{\bar{u}(\cdot)\} \cup C_{\mathrm{ad}}(0, T; U_m)]\!]$ rather than in the full space $C_{\mathrm{ad}}(0, \bar{t}; U)$. See [5] for proofs and further details.

Let e (resp. e_0) be the set of all Lebesgue points of all functions

$$f(\cdot, y(\cdot, \bar{u}), \bar{u}(\cdot)), \ f(\cdot, y(\cdot, \bar{u}), v) \qquad \forall v \in U_m \,,$$

(respectively

$$f_0(\cdot, y(\cdot, \bar{u}), \bar{u}(\cdot)), \ f_0(\cdot, y(\cdot, \bar{u}), v) \qquad \forall v \in U_m \ ).$$

Assuming that

    (j) $e \cap e_0$ has full measure in $[0, \bar{t}]$,

    (k) $C_{\mathrm{ad}}(0, T; U)$ contains all constant controls $u(t) \equiv v \in U$,

we may put *spike perturbations* $v(\sigma) = v_{\sigma,h}(\sigma)$ in (2.5) $(v_{\sigma,h}(\sigma) = v$ in $s - h \leq \sigma \leq s$, $v_{\sigma,h}(\sigma) = \bar{u}(\sigma)$ elsewhere) and take limits as $h \to 0$. The result is the *pointwise* minimum principle

$$
\begin{aligned}
z_{0m}\{f_0(s, y(s, \bar{u}), v) &- f_0(s, y(s, \bar{u}), \bar{u}(s))\} \\
+ \langle \bar{z}_m(s), f(s, y(s, \bar{u}), v) &- f(s, y(s, \bar{u}), \bar{u}(s))\} \rangle \geq 0 \qquad (2.11) \\
&(v \in U_m)
\end{aligned}
$$

for $t \in e \cap e_0 \cap e_m$ (that is, a.e. in $[0, \bar{t}]$). Since the U_m are an expanding sequence and every $v \in U$ belongs to some U_m, (2.11) will be verified for every $v \in U$ and m sufficiently large (depending on v.)

Thus, if we can show that $(z_{0m}, \bar{z}_m(s))$ is $I\!\!R \times E-$weakly convergent to $(z_0, \bar{z}(s)) \in I\!\!R \times E^*$, an $m-$independent version of (2.11) will be obtained. Of course, the essential difficulty is to show that $\bar{z}(s)$ is the costate corresponding to a nontrivial (that is, nonzero) multiplier $(z_0, \mu, z) \in I\!\!R \times \Sigma(0, \bar{t}; E^*) \times E^*$.

3.   Nonvanishing of multipliers

Let $u(\,\cdot\,) \in C_{\mathrm{ad}}(0, \bar{t}, U)$ be a control such that $y(t, u)$ exists in $0 \leq t \leq \bar{t}$. Given a probability vector $\mathbf{p} = (p_1, \ldots, p_k)$ and $\mathbf{v}(\,\cdot\,) = (v_1(\,\cdot\,), \ldots, v_k(\,\cdot\,))$ $(v_j(\,\cdot\,) \in C_{\mathrm{ad}}(0, \bar{t}; U))$ we denote by $\xi(t, u, \mathbf{v}, \mathbf{p})$ the solution $\xi(t)$ of the inhomogeneous variational equation

$$
\begin{aligned}
\xi'(t) &= \{A + \partial_y f(t, y(t, u), u(t))\} \xi(t) \\
&\quad + \sum_{j=1}^{k} p_j \{f(t, y(t, u), v_j(t)) - f(t, y(t, u), u(t))\}, \quad (3.1)
\end{aligned}
$$

$$
\xi(0) = 0.
$$

Given $\tilde{t} \in [0, \bar{t}]$ we denote by $\Xi(\bar{t}, u)(\tilde{t}) \subseteq C(0, \bar{t}; E) \times E$ the set of all pairs $(\xi(\cdot, u, \mathbf{v}, \mathbf{p}), \xi(\tilde{t}, u, \mathbf{v}, \mathbf{p}))$ for all possible $\mathbf{p}$ and $\mathbf{v}(\,\cdot\,)$; clearly, $\Xi(\bar{t}, u)(\tilde{t})$ is convex. If each of the components $v_j(\,\cdot\,)$ of $\mathbf{v}(\,\cdot\,)$ belong to $[\![\{\bar{u}(\,\cdot\,)\} \cup C_{\mathrm{ad}}(0, \bar{t}; U_m)]\!]$ ($\{U_m\}$ the sequence in (i), Section 2), the corresponding set is denoted by $\Xi_m(\bar{t}, u)(\tilde{t})$. Using these sets, we can give a condition that guarantees that, for each m, the multiplier $(z_{0m}, \mu_m, z_m) \in I\!\!R \times \Sigma(0, \bar{t}; E^*) \times E^*$ in the adjoint variational equation (2.6) is nontrivial, i.e.

$$
(z_{0m}, \mu_m, z_m) \neq 0 \quad (m = 1, 2, \ldots). \tag{3.2}
$$

Call a sequence $\{Q_n\}$ of subsets of a Banach space **precompact** if every sequence $\{q_n\}$, $q_n \in Q_n$ has a convergent subsequence. A constant sequence $Q, Q, \ldots$ is precompact if and only if Q is a precompact subset.

LEMMA 3.1.   *Assume that for each m there exists $\rho_m > 0$ and a precompact sequence $\{\mathbf{Q}_m^n\}$, $\mathbf{Q}_m^n \subseteq C(0, \bar{t}; E) \times E$ such that*

$$
\bigcap_{n=n_0}^{\infty} \left\{ \Xi_m(\bar{t}, \tilde{u}_m^n)(\tilde{t}) - (T_{\mathbf{M}(\bar{t})}(\tilde{y}_m^n(\,\cdot\,)) \times T_Y(\tilde{y}_m^n)) \cap \mathbf{B}(0, \rho_m) + \mathbf{Q}_m^n \right\}
$$

$$
\tag{3.3}
$$

contains an interior point in the space $C(0, \bar{t}; E) \times E$ for n_0 large enough, where the sequences

$$\{\tilde{u}_m^n(\,\cdot\,)\} \subseteq [\![\{\bar{u}(\,\cdot\,)\} \cup C_{\mathrm{ad}}(0, T; U_m)]\!]$$

and

$$\{(\tilde{y}_m^n(\,\cdot\,), \tilde{y}_m^n)\} \subseteq \mathbf{M}(\bar{t}) \times Y$$

are such that $\tilde{u}_m^n(\,\cdot\,) \to \bar{u}(\,\cdot\,)$, $(\tilde{y}_m^n(\,\cdot\,), \tilde{y}_m^n) \to (y(\cdot, \bar{u}), y(\bar{t}, \bar{u}))$, and $\mathbf{B}(0, \rho_m)$ is the ball of center 0 and radius ρ_m in $C(0, \bar{t}; E) \times E$. Then (3.2) holds.

For the time optimal problem, we replace $\Xi_m(\bar{t}, \tilde{u}^n)(\bar{t})$ by

$$\Xi_m(\bar{t}, \tilde{u}_m^n)(t_n),$$

where $\{t_n\}$ is an arbitrary sequence in $[0, t_n)$ with $t_n \to \bar{t}$.

For a proof of Lemma 3.1 see [3, Lemma 2.5]. Note that Lemma 3.1 is an independent result for each m; it requires that (3.3) contain a ball $\mathbf{B}((y_m(\,\cdot\,), y_m), \epsilon_m)$ for each m, but no interaction between the balls is postulated.

Let F be a Banach space. A closed set $X \subseteq F$ is $T-$**full at** $\bar{x} \in X$ if, for every sequence $\{x^n\} \subseteq X$ such that $x^n \to \bar{x}$ there exists $\rho > 0$ and a precompact sequence $\{Q_n\}, Q_n \subseteq F$ such that

$$\bigcap_{n=n_0}^{\infty} \{T_X(x^n) \cap B(0, \rho) + Q_n\} \qquad (3.4)$$

contains an interior point for n_0 large enough. Note that when the sequence $\{Q_n\}$ is bounded (in particular, when $Q_n = Q$) we may dispense with $B(0, \rho)$ in the intersection (3.4); however, unbounded precompact sequences exist (example: $E, \{0\}, E, \{0\}, \ldots$). The set X is **strongly** $T-$**full at** $\bar{x}$ if (3.4) contains an interior point for $Q_n = \{0\}$ (here $B(0, \rho)$ can be dropped). Finally, a set X is $T-$**full** if it is $T-$full at each $\bar{x} \in X$, with a corresponding definition for strongly $T-$full. Examples of strongly $T-$full sets are closed convex sets with nonempty interior. Fullness (obviously not strong fullness) survives after cutting with finitely many smooth hypersurfaces:

LEMMA 3.2. *Let $X_0 \subseteq F$ be convex and closed with nonempty interior, and let $\varphi_1, \ldots, \varphi_n$ be continuously Fréchet differentiable functionals in F,*

$$X = X_0 \cap \{x; \varphi_j(x) = c_j; \; j = 1, \ldots, n\}. \qquad (3.5)$$

Then, if $\bar{x} \in X$ and the Fréchet derivatives $\partial \varphi_1(\bar{x}), \ldots, \partial \varphi_n(\bar{x})$ are linearly independent, X is $T-$full at $\bar{x}$.

For proofs and additional details see [4], [5]. It is an immediate consequence of the definition that the set (3.3) will contain an interior point for n_0 large enough if $\mathbf{M}(\bar{t}) \times Y$ is $T-$full in $C(0, \bar{t}; E) \times E$, which is equivalent to $T-$fullness of $\mathbf{M}(\bar{t})$ and Y in their respective spaces; no attention is paid to $\Xi_m(\bar{t}, \tilde{u}^n)(\bar{t})$ here. On the other hand, the following result does not scorn help from $\Xi_m(\bar{t}, \tilde{u}^n)(\bar{t})$, if only in the second coordinate of $C(0, \bar{t}; E) \times E$.

LEMMA 3.3. *Assume $\mathbf{M}(\bar{t})$ is $T-$full in $C(0, \bar{t}; E)$ and that for each m there exists $\rho_m > 0$ and a precompact sequence $\{Q_m^n\}$, $Q_m^n \subseteq E$ such that*

$$\bigcap_{n=n_0}^{\infty} \left\{ \Pi(\Xi_m(\bar{t}, \tilde{u}_m^n)(\bar{t})) - T_Y(\tilde{y}_m^n) \cap B(0, \rho_m) + Q_m^n \right\} \qquad (3.6)$$

contains an interior point for n_0 large enough, where

$$\{\tilde{u}_m^n\} \subseteq [\{\bar{u}(\,\cdot\,)\} \cup C_{\mathrm{ad}}(0, T; U_m)] \qquad and \qquad \{\tilde{y}_m^n\} \subseteq Y$$

are such that $\tilde{u}_m^n(\,\cdot\,) \to \bar{u}(\,\cdot\,)$ and $\tilde{y}_m^n \to y(\bar{t}, \bar{u})$ and Π is the projection of $C(0, \bar{t}; E) \times E$ into E. Then there exists $\rho_m > 0$ and a precompact sequence $\{\mathbf{Q}_m^n\}$, $\mathbf{Q}_m^n \subseteq C(0, \bar{t}; E) \times E$ such that (3.3) has an interior point in $C(0, \bar{t}; E) \times E$ for n_0 large enough.

For the proof, note that (3.3) will be satisfied (with a different ρ_m) if there exist $\rho_m > 0$, an integer n_0, and precompact sequences $\{Q_{m,c}^n\}$ in $C(0, \bar{t}; E)$ and $\{Q_m^n\}$ in E such the set $\mathbf{\Delta}_n \subseteq C(0, \bar{t}; E) \times E$ of all points with coordinates

$$\bar{t}^{-1} \xi(\cdot, \tilde{u}_m^n, \mathbf{p}, \mathbf{v}) - T_{\mathbf{M}(\bar{t})}(\tilde{y}_m^n(\,\cdot\,)) \cap B_c(0, \rho_m) + Q_{m,c}^n \qquad (3.7)$$

$$\bar{t}^{-1} \xi(\bar{t}, \tilde{u}_m^n, \mathbf{p}, \mathbf{v}) - T_Y(\tilde{y}_m^n) \cap B(0, \rho_m) + Q_m^n \qquad (3.8)$$

contain a common polyball

$$B_c(y_m(\cdot), \epsilon_m) \times B(y_m, \epsilon_m) \quad \text{in} \quad C(0, \bar{t}; E) \times E$$

for $n \geq n_0$. Here (for each m, n) $\mathbf{p}$ ranges over all probability vectors, $\mathbf{v}$ over all vectors of elements of $[\{\bar{u}(\,\cdot\,)\} \cup C_{\mathrm{ad}}(0, T; U_m)]$ and $\mathbf{p}$ and $\mathbf{v}$ are the same in (3.7) and (3.8).

Under the present assumptions, formula (3.1) determines that

$$\|\xi(t, \tilde{u}_m^n, \mathbf{p}, \mathbf{v})\| \leq C_m \quad (0 \leq t \leq \bar{t}) \tag{3.9}$$

for all $\mathbf{p}$, $\mathbf{v}$. The fact that $\mathbf{M}(\bar{t})$ is T-full implies the existence of $\sigma_m > 0$ and of a precompact sequence $\{\tilde{Q}_{m,c}^n\}$ in $C(0, \bar{t}; E)$ such that the sets

$$T_{\mathbf{M}(\bar{t})}(\tilde{y}_m^n(\,\cdot\,)) \cap B(0, \sigma_m) + \tilde{Q}_{m,c}^n$$

contain a common ball $B_c(y_m(\,\cdot\,), \epsilon_m)$ for $n \geq n_0$ large enough. Replacing $\tilde{Q}_{m,c}^n$ by $\tilde{Q}_{m,c}^n - y_m(\,\cdot\,)$ we shift the ball to $B_c(0, \epsilon_m)$. Accordingly,

$$\bigcap_{n=n_0}^{\infty} \left\{ T_{\mathbf{M}(\bar{t})}(\tilde{y}_m^n(\,\cdot\,)) \cap B_c(0, r_m \sigma_m) + r_m(\tilde{Q}_{m,c}^n - y_m(\,\cdot\,)) \right\}$$

contains the ball $B_c(0, r_m \epsilon_m) \subseteq C(0, \bar{t}; E)$ for $n \geq n_0$ large enough (recall that $T_{\mathbf{M}(\bar{t})}(\tilde{y}_m^n(\,\cdot\,))$ is a cone). Taking r_m so huge that $r_m \epsilon_m > 2 C_m \bar{t}^{-1}$ (C the constant in (3.9)) and defining

$$Q_{m,c}^n = r_m(\tilde{Q}_{m,c}^n - y_m(\,\cdot\,))$$

we deduce that the sets (3.7) contain $B_c(0, C_m)$ for $n \geq n_0$ *for any choice of* $\mathbf{p}$, $\mathbf{v}$, thus we are free to choose $\mathbf{p}$, $\mathbf{v}$ at will in (3.8); all we have to do is to check that the sets (3.8) contain a common ball in E for n large enough. This is insured by the assumptions of Lemma 3.1, since we may always suppose that $r_m \sigma_m > \rho_m$.

See the comments after Lemma 3.1 for the time optimal problem. The proofs of all the results above are based on the following result ([2], [3]), where $\{\Delta_n\}$ is a sequence of subsets of a Banach space F.

LEMMA 3.4. *Let $\{z_n\}$ be a sequence in F^* such that*

$$0 < c \leq \|z_n\|_{F^*} \leq C, \qquad \langle z_n, y \rangle \geq -\epsilon_n \quad (y \in \Delta_n), \qquad (3.10)$$

with $\epsilon_n \to 0$. Assume there exists a precompact sequence $\{Q_n\}$ such that

$$\Delta = \bigcap_{n=n_0}^{\infty} \{\overline{\mathrm{conv}}(\Delta_n) + Q_n\} \qquad (3.11)$$

contains an interior point ($\overline{\mathrm{conv}} = $ closed convex hull). Then every $E-$weakly convergent subsequence of $\{z_n\}$ has a nonzero limit.

For the proof, see [2, Lemma 2.5].

Lemma 3.4 will also serve to justify the taking of limits in (2.5) and (2.11). To this end, we begin by observing that (2.5) can be written in the form

$$z_{0m}\xi_0(\bar{t}, \bar{u}, \mathbf{p}, \mathbf{v}) + \langle \mu_m, \xi(\cdot, \bar{u}, \mathbf{p}, \mathbf{v}) \rangle_c + \langle z_m, \xi(\bar{t}, \bar{u}, \mathbf{p}, \mathbf{v}) \rangle \geq 0, \quad (3.12)$$

for arbitrary $\mathbf{p}$, $\mathbf{v}$, where

$$v_j(\,\cdot\,) \in [\![\{\bar{u}(\,\cdot\,)\} \cup C_{\mathrm{ad}}(0, \bar{t}; U_{m_1})]\!]$$

and $\xi(t, \bar{u}, \mathbf{p}, \mathbf{v})$ is given by (3.1),

$$\xi_0(t, \bar{u}, \mathbf{p}, \mathbf{v}) = \int_0^t \langle \partial_y f_0(\tau, y(\tau, \bar{u}), \bar{u}(\tau)), \xi(\tau, \bar{u}, \mathbf{p}, \mathbf{v}) \rangle d\tau$$

$$+ \sum_{j=1}^m p_j \int_0^t \{f_0(\tau, y(\tau, \bar{u}), v_j(\tau)) - f_0(\tau, y(\tau, \bar{u}), \bar{u}(\tau))\} d\tau, \quad (3.13)$$

and $\langle \,\cdot\,, \cdot\, \rangle$ (resp. $\langle \,\cdot\,, \cdot\, \rangle_c$) indicates the duality of E and E^* (resp. the duality of $C(0, \bar{t}; E)$ and $\Sigma(0, \bar{t}, E^*)$). For a proof of the equivalence of both inequalities see [5] where the reverse route is followed; (3.12) is obtained first as a consequence of nonlinear programming theory and then transformed into (2.5).

Assuming that (3.3) holds each multiplier (z_{0m}, μ_m, z_m) is non-zero, thus we may complement (3.12) with the condition

$$z_{0m}^2 + \|\mu_m\|_{\Sigma(0, \bar{t}, E^*)}^2 + \|z_m\|_{E^*}^2 = 1 \quad (m = 1, 2, \ldots). \qquad (3.14)$$

Selecting a subsequence, we may assume that (z_{0m}, μ_m, z_m) is $(I\!\!R \times C(0, \bar{t}; E) \times E)$—weakly convergent to $(z_0, \mu, z) \in I\!\!R \times \Sigma(0, \bar{t}; E^*) \times E^*$, and the objective is to show that

$$(z_0, \mu, z) \neq 0. \qquad (3.15)$$

To this end, we combine (3.12) with (2.7), obtaining

$$z_{0m} \xi_0(\bar{t}, \bar{u}, \mathbf{p}, \mathbf{v}) + \langle \mu_m, \xi(\cdot, \bar{u}, \mathbf{p}, \mathbf{v}) - y_m(\cdot) \rangle_c$$
$$+ \langle z_m, \xi(\bar{t}, \bar{u}, \mathbf{p}, \mathbf{v}) - y_m \rangle \geq 0, \qquad (3.16)$$

for $(y_m(\cdot), y_m) \in \Delta_m = \mathcal{M}_m \times \mathcal{Y}_m$ where

$$\mathcal{M}_m = \liminf_{n \to \infty} T_{\mathbf{M}(\bar{t})}(\tilde{y}_m^n(\cdot)) \subseteq C(0, \bar{t}, E),$$
$$\mathcal{Y}_m = \liminf_{n \to \infty} T_Y(\tilde{y}_m^n)) \subseteq E. \qquad (3.17)$$

If $z_0 = \lim_{m \to \infty} z_{0m} \neq 0$, there is nothing to prove. On the other hand, if $z_{0m} \to 0$, (3.16) and uniform boundedness of $\xi_0(\bar{t}, \bar{u}, \mathbf{p}, \mathbf{v})$ (recall that $v_j(\cdot,) \in [\![\{\bar{u}(\cdot)\} \cup C_{\mathrm{ad}}(0, \bar{t}; U_m)]\!]$ imply that

$$\langle \mu_m, \xi(\cdot, \bar{u}, \mathbf{p}, \mathbf{v}) - y_m(\cdot) \rangle_c + \langle z_m, \xi(\bar{t}, \bar{u}, \mathbf{p}, \mathbf{v}) - y_m \rangle \geq -\epsilon_m \to 0,$$
$$(3.18)$$

for arbitrary $\mathbf{p}$, $\mathbf{v}$, $(y_m(\cdot), y_m) \in \Delta_m$. Accordingly, the result below follows from Lemma 3.4.

LEMMA 3.5. *Assume that there exists $\rho > 0$ and a precompact sequence $\{\boldsymbol{Q}_m\}$, $\boldsymbol{Q}_m \subseteq C(0, \bar{t}; E) \times E$ such that*

$$\bigcap_{m=m_0}^{\infty} \{ \Xi_m(\bar{t}, \bar{u})(\bar{t}) - (\mathcal{M}_m \times \mathcal{Y}_m) + \boldsymbol{Q}_m \} \qquad (3.19)$$

contains an interior point for m_0 large enough. Then (3.12) holds.

   Our objective below is to give conditions on the sets $\Xi_m(\bar{t}; u)(\bar{t})$, $\mathbf{M}(\bar{t})$ and Y that guarantee the assumptions of Lemma 3.1 and Lemma 3.5 at the same time. If for each m the intersection (3.3) contains an interior point for n_0 large enough we have (3.2), and we can count on (3.14) and (3.16); then we use Lemma 3.5 to show that $(z_0, \mu, z) \neq 0$, which is the final aim of this paper.

Consider an arbitrary Banach space F, a closed subset $X \subseteq F$, a point $\bar{x} \in X$ and a double sequence $\{x_m^n; m, n = 1, 2, \ldots\} \subseteq X$ such that

$$\lim_{n \to \infty} x_m^n = \bar{x} \quad (m = 1, 2, \ldots)$$

LEMMA 3.6. *Let X be strongly $T-$full at $\bar{x}$, and let*

$$\mathcal{X}_m = \liminf_{n \to \infty} T_X(x_m^n).$$

Then

$$\bigcap_{m=m_0}^{\infty} \mathcal{X}_m \qquad (3.20)$$

contains an interior point for m_0 large enough.

Proof. Obviously, for each m, the definition of $\mathcal{X}_m$ is oblivious to deletion of finitely many terms of $\{x_m^n; n = 1, 2, \ldots\}$ thus we can assume that $\|x_m^n - \bar{x}\| \leq 1/m$ $(n = 1, 2, \ldots)$. Arrange the sequence $\{x_m^n\}$ in an infinite matrix

$$
\begin{array}{cccc}
x_1^1 & x_1^2 & x_1^3 & \ldots \\
x_2^1 & x_2^2 & x_2^3 & \ldots \\
x_3^1 & x_3^2 & x_3^3 & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{array}
$$

and take the Cantor zig-zag sequence

$$x_1^1, \ x_1^2, \ x_2^1, \ x_3^1, \ x_2^2, \ x_1^3, \ \ldots$$

It is plain that this sequence converges to $\bar{x}$, thus by definition of strong $T-$fullness there exists m_0 such that

$$\bigcap_{m+n \geq m_0} T_X(x_m^n) \qquad (3.21)$$

contains a ball $B(x, \epsilon)$ for m_0 large enough. This means: for each m, given $w \in B(x, \epsilon)$ there exists $w_m^n \in T_X(x_m^n)$ such that $w = w_m^n$ $(n \geq \max(0, m_0 - m))$. This is much stronger than the claim of Lemma 3.6. $\qquad \square$

In the following result (Lemma 3.7) we have a double sequence $\{\tilde{y}_m^n; m, n = 1, 2, \ldots\} \subseteq Y$ such that $\lim_{n \to \infty} \tilde{y}_m^n = \bar{y}$ for all m and $\mathcal{M}_m, \mathcal{Y}_m$ are defined by (3.17).

LEMMA 3.7. *Assume that* $\mathbf{M}(\bar{t})$ *is strongly* $T-$*full in* $C(0, \bar{t}; E)$ *and that for each m and every sequence*

$$\{\tilde{u}_m^n\} \subseteq [\![\{\bar{u}(\,\cdot\,)\} \cup C_{\mathrm{ad}}(0, T; U_m)]\!]$$

with $\tilde{u}_m^n(\,\cdot\,) \to \bar{u}(\,\cdot\,)$ *the set*

$$\bigcap_{n=n_0}^{\infty} \Pi(\Xi_m(\bar{t}, \tilde{u}_m^n)(\bar{t})) \tag{3.22}$$

contains an interior point for n_0 large enough. Then: (a) the assumptions of Lemma 3.1 hold; (b) for every m_1 the set

$$\bigcap_{m=m_0}^{\infty} \{ \Xi_{m_1}(\bar{t}, \bar{u})(\bar{t}) - (\mathcal{M}_m \times \{0\}) \} \tag{3.23}$$

contains an interior point for m_0 large enough, so that the assumptions of Lemma 3.5 hold as well.

The requirements on $\Xi_m(\bar{t}, \tilde{u}_m^n)(\bar{t})$ are stronger than those in Lemma 3.3, thus (a) follows. To show (b) it is enough to prove that the set $\boldsymbol{\Delta} \subseteq C(0, \bar{t}; E) \times E$ of points with coordinates

$$\xi(\,\cdot\,, \bar{u}, \mathbf{p}, \mathbf{v}) - \mathcal{M}_m, \qquad \xi(\bar{t}, \bar{u}, \mathbf{p}, \mathbf{v}) \tag{3.24}$$

contains a polyball $B_c(0, \epsilon) \times B(0, \epsilon)$ for $m \geq m_0$ large enough, where $\mathbf{v}(\,\cdot\,)$ is such that $v_j(\,\cdot\,) \in [\![\{\bar{u}(\,\cdot\,)\} \cup C_{\mathrm{ad}}(0, \bar{t}; U_{m_1})]\!]$. To service the first coordinate we use Lemma 3.6 and then an argument similar to that used after (3.8) to show that $\{\xi(\,\cdot\,, \bar{u}, \mathbf{p}, \mathbf{v}) - \mathcal{M}_m\}$ contains a fixed ball irrespective of the choice of $\mathbf{p}, \mathbf{v}(\,\cdot\,)$. For the second coordinate, we just note that the assumptions on $\Pi(\Xi_m(\bar{t}, \tilde{u}_m^n)(\bar{t})$ particularized to $\tilde{u}_m^n = \bar{u}$ imply that $\{\xi(\bar{t}, \bar{u}, \mathbf{p}, \mathbf{v})\}$ contains an interior point. The fact that our conclusions imply those of Lemma 3.5 (that is, that the intersection (3.19) contains an interior point (3.22) does) follows from the fact that $\{\Xi_m(\bar{u}, \bar{t})(\bar{t})\}$ is an increasing sequence of sets.

COROLLARY 3.8. *Let* $\mathbf{M}(\bar{t})$ *be strongly* $T-full$ *and assume that, either* (a) *The target set* Y *is strongly* $T-full$, *or* (b) (3.22) *contains an interior point for each* m *and* n_0 *large enough for sequences* $\{\tilde{u}_m^n(\,\cdot\,)\}$ *satisfying the assumptions in Lemma 3.7. Then* (3.15) *holds.*

Proof. In case (a) the set $X = \mathbf{M}(\bar{t}) \times Y$ is strongly $T-full$ in $F = C(0, \bar{t}; E) \times E$, thus Lemma 3.6 applies. We have

$$T_{(\mathbf{M}(t) \times Y)}(\tilde{y}_m^n(\,\cdot\,), y_m^n) = T_{\mathbf{M}(t)}(\tilde{y}_m^n(\,\cdot\,)) \times T_Y(y_m^n),$$

so the intersection (3.21) is

$$\bigcap_{m+n \geq m_0} (T_{\mathbf{M}(t)}(\tilde{y}_m^n(\,\cdot\,)) \times T_Y(y_m^n))$$

and contains an interior point for m_0 large enough; this obviously implies the assumptions of Lemma 3.1. On the other hand, (3.20) is

$$\bigcap_{m=m_0}^{\infty} (\mathcal{M}_m \times \mathcal{Y}_m)$$

and contains as well an interior point for m_0 large enough, so that the requirements of Lemma 3.5 also hold. In case (b) the assumptions of Lemma 3.5 are satisfied. $\square$

4. The minimum principle

We assume that hypotheses (a) to (k) in §2 hold, and denote by $\bar{u}(\,\cdot\,)$ an optimal control.

THEOREM 4.1. *Let the assumptions of Corollary 3.8 be satisfied. Then the minimum principle*

$$z_0 f_0(s, y(s, \bar{u}), \bar{u}(s)) + \langle \bar{z}(s), f(s, y(s, \bar{u}), \bar{u}(s)) \rangle$$
$$= \min_{v \in U} \{z_0 f_0(s, y(s, \bar{u}), v) + \langle \bar{z}(s), f(s, y(s, \bar{u}), v) \rangle\} \qquad (4.1)$$

holds in a set $e \subseteq [0, T]$ *of full measure; the costate* $\bar{z}(s)$ *solves the adjoint variational equation*

$$d\bar{z}(s) = -\{A^* + \partial_y f(s, y(s, \bar{u}), \bar{u}(s))^*\}\bar{z}(s)ds$$
$$-z_0 \partial_y f_0(s, y(s, \bar{u}), \bar{u}(s))ds - \mu(ds), \qquad \bar{z}(\bar{t}) = z \qquad (4.2)$$

with $(z_0, \mu, z) \in I\!R \times \Sigma(0, \bar{t}; E^*) \times E^*$,

$$(z_0, \mu, z) \neq 0, \qquad (4.3)$$

$$z_0 \geq 0, \quad \mu \in (\liminf_{m \to \infty} \liminf_{n \to \infty} T_{\mathbf{M}(\bar{t})}(\tilde{y}_m^n(\cdot)))^-,$$
$$z \in (\liminf_{m \to \infty} \liminf_{n \to \infty} T_Y(\tilde{y}_m^n))^-, \qquad (4.4)$$

where $\{\tilde{y}_m^n(\cdot)\}$ (resp. $\{\tilde{y}_m^m\}$) is a double sequence in $C(0, \bar{t}; E)$ (resp. E) such that $\tilde{y}_m^n(\cdot) \to y(\cdot, \bar{u})$ (resp. $\tilde{y}_m^n \to y(\bar{t}, \bar{u})$) for all $m = 1, 2, \ldots$

Proof. The multiplier (z_0, μ, z) is constructed, the same as in §3, as the $(I\!R \times C(0, \bar{t}; E) \times E)$−weak limit of (a subsequence of) the multipliers (z_{0m}, μ_m, z_m); thus (4.3) is insured by Corollary 3.8. All we have to do is to take limits in the pointwise maximum principle (2.11) and in the adjoint variational equation (2.6) for $\bar{z}_m(s)$; this means we must show that

$$\bar{z}_m(s) \to \bar{z}(s) \quad \text{weakly in } E^* \quad (0 \leq s \leq \bar{t}).$$

To show this we write the equation (2.9) for $\bar{z}_m(s)$ and keep in mind the facts that $z_{0m} \to z_0$, $z_m \to z$ E−weakly in E^* and $\mu_m(ds) \to \mu(ds)$ $C(0, \bar{t}; E)$−weakly in $\Sigma(0, \bar{t}; E^*)$, so that $\nu_m(ds) = \delta_{\bar{t}}(ds) z_m + z_{0m} \partial_y f_0(s, y(s, \bar{u}), \bar{u}(s)) ds + \mu_m(ds) \to \nu(ds) = \delta_{\bar{t}}(ds) z + z_0 \partial_y f_0(s, y(s, \bar{u}), \bar{u}(s)) ds + \mu(ds)$ $C(0, \bar{t}; E)$−weakly in $\Sigma(0, \bar{t}; E^*)$; taking limits in (4.6), the equation

$$\langle \bar{z}(s), y \rangle = \int_s^{\bar{t}} \langle \nu(d\sigma), S(\sigma, s; \bar{u}) y \rangle$$

for $\bar{z}(s)$ results.

It only remains to show the three relations (4.4). The first is obvious since $z_{m0} \to z_0$. To show the third, note that we have

$$\langle z_m, y_m \rangle \leq 0 \quad (y_m \in \liminf_{n \to \infty} T_Y(\tilde{y}_m^n))$$

hence, taking limits,

$$\langle z, y \rangle \leq 0 \quad (z \in \liminf_{m \to \infty} \liminf_{n \to \infty} T_Y(\tilde{y}_m^n)).$$

The proof of the corresponding relation for μ is the same. $\square$

REMARK 4.2 The double $\liminf$ in conditions (4.4) can be used essentially as the simple $\liminf$ in conditions (2.7). For instance, we have

THEOREM 4.3. Let $e_0 \in [0, \bar{t}]$ be the set where the state constraints are not saturated (that is, where $y(t, \bar{u}) \in \text{Int}(M)$). Then $\mu = 0$ in e_0.

Proof. We use Lemma 2.3 in [4] for the control space $C_{\text{ad}}(0, \bar{t}; U_m)$ and obtain $\mu_m = 0$ in e_0. Then we take limits.   □

All results above apply to the system

$$y'(t) = Ay(t) + f(t, y(t)) + u(t), \quad y(0) = \zeta \qquad (4.5)$$

in an arbitrary Banach space E, with A the infinitesimal generator of a strongly continuous semigroup. We use as control set an arbitrary closed subset $U = E$. The admissible control space $C_{\text{ad}}(0, T; U)$ is the subset of $L^p(0, T; E)$ $(1 \le p \le \infty)$ defined by $u(t) \in U$ a.e. ($p = 1$ is bad for existence purposes). We assume that the function $f(t, y)$ satisfies (a), (b) and (c) in §2; that these conditions are valid for $f(t, y, u(t)) = f(t, y) + u(t)$ follows from the definition of $C_{\text{ad}}(0, T; U)$. The cost functional (1.2) is required to satisfy (d), (e), (f). Conditions (g), (h), (i), (j), (k) are obvious, with $U_m = B(0, m)$ in (g). If the state constraint set M and the target set Y satisfy the assumptions in Corollary 3.8, nontriviality of the multiplier is guaranteed. Incidentally, this system illustrates why we don't ask the whole admissible control space $C_{\text{ad}}(0, \bar{t}; U)$ to be saturated; if $U = E$, $u \in E$, $u \ne 0$ the sequence $u^n(t) = 0$ $(0 \le t \le 1/n)$, $u^n(t) = u/t$ $(1/n < t \le \bar{t})$ is stationary in $C_{\text{ad}}(0, \bar{t}; U)$ but its limit is not a member of $C_{\text{ad}}(0, \bar{t}; U)$. On the other hand it makes no difference to ask patch completeness of $C_{\text{ad}}(0, \bar{t}; U_m)$ (condition (k)) or of the whole space.

Assume an optimal control exists in the case $U = E$. Then the minimum principle (4.1) holds; in particular the minimum must be a.e. finite. A number of conclusions are immediately obvious. On the one hand, $z_0 > 0$ (otherwise the minimum would be $-\infty$ or the costate trivial); this means the so-called "abnormal" case is ruled out.

On the other hand, and for the same reason, the cost functional must contain u explicitly and its growth at infinity must be at least linear. This is of course related with convexity of $f_0(t, y, u)$ with respect to u, a condition that has to do with weak lower semicontinuity of the cost functional and is needed for existence theorems.

In case A generates a strongly continuous group, it is known (see [4] and references thereof) that the assumptions of Lemma 3.7 hold, thus nothing is needed of the target set and the point target case is tractable. However, lack of compactness of $S(\,\cdot\,)$ compromises existence theorems for optimal controls.

A more interesting application is to semilinear wave equations

$$
\frac{\partial^2}{\partial t^2} y(t, x) = \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{\partial}{\partial x_j} \Big(a_{jk}(x) \frac{\partial}{\partial x_k} y(t, x) \Big)
$$

$$
+ \sum_{j=1}^{m} b_j(x) \frac{\partial}{\partial x_j} y(t, x) + c(x) y(t, x) - \phi(y(t, x)) + u(t, x) \quad (4.6)
$$

in a bounded domain $\Omega \subseteq I\!\!R^m$; the control set U is an closed subset of $L^2(\Omega)$ and $C_{\mathrm{ad}}(0, \bar{t}; U)$ is the subset of $L^2(0, \bar{t}; L^2(\Omega))$ defined by $u(t) \in U$ a.e. The point target case is tractable (see [4] for details). If (4.6) is reduced to a first order equation in the energy space

$$
\mathbf{E} = H^1(\Omega) \times L^2(\Omega)
$$

in the customary way, the corresponding semigroup $\mathbf{S}(\,\cdot\,)$ is not compact (in fact, it is a group) but existence theorems can be obtained on the basis of the compactness of the imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, guaranteed by the Rellich-Kondrachev theorem.

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