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Perturbation of Ornstein-Uhlenbeck Semigroups

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Dedicated to Pierre Grisvard

Introduction

In this paper we consider the Ornstein-Uhlenbeck process $Z(t,x)$, solution of the following differential stochastic equation in a Hilbert space H:

$$dZ = AZdt + dW(t),\ Z(0) = x.$$

Here W is a cylindrical Wiener process on H and A is the infinitesimal generator of an exponentially stable analytic semigroup e^{tA} in H. Under this hypothesis it is well known that the process $Z(t,x)$ has a unique invariant measure μ, see e.g. [7].

Let us denote by $\mathcal{A}$ the infinitesimal generator of the transition semigroup

$$R_t\varphi(x) = \mathbb{E}[\varphi(Z(t,x))],\ t \geq 0,$$

defined in the space $L^2(H;\mu)$. $\mathcal{A}$ can be written formally as

$$\mathcal{A}\varphi = \frac{1}{2}\,\mathrm{Tr}\,[D^2\varphi(x)] + \langle Ax, D\varphi(x)\rangle.$$

In G. Da Prato and J. Zabczyk see [9], it was proved that $\mathcal{A}$ is an m-dissipative operator on $L^2(H;\mu)$. Moreover, in that paper we also

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studied perturbations of $\mathcal{A}$ of the form

$$\langle F(x), D\varphi(x)\rangle, \tag{0.1}$$

where $F : H \to H$ is a continuous and bounded mapping.

The main result of the present paper is a precise characterization, under suitable assumptions, of the domain $D(\mathcal{A})$ of $\mathcal{A}$, as a subspace of $W^{2,2}(H;\mu)$.

We notice that the operator $\mathcal{A}$ has been extensively studied using the Theory of Dirichlet forms, see Z. M. Ma and M. Röckner [17]. Using this method one can show that, in several situations, the operator $\mathcal{A}$ is variational, and consequently one can conclude that $D(\mathcal{A})$ is a subspace of $W^{1,2}(H;\mu)$. Knowing that $D(\mathcal{A}) \subset W^{2,2}(H;\mu)$, will allow us to consider perturbations of $\mathcal{A}$ more general than (0.1).

Our method is based on a generalization of the well known L. Nirenberg's proof about H^2 regularity of second order elliptic equations, see e.g. [2]. We establish a basic identity for functions belonging to $D(\mathcal{A})$, that, under suitable assumptions (see Hypotheses 1.1 and 3.1), yields a characterization of $D(\mathcal{A})$. These assumptions are in particular fulfilled when $\mathcal{A}$ is self-adjoint and when H is finite-dimensional.

We notice that, when A is self-adjoint, a characterization of $D(\mathcal{A})$ could also be obtained by using the spectral decomposition of $\mathcal{A}$ written in terms of Hermite polynomial, see [7]. Moreover, when H is finite-dimensional, our characterization coincides with that proved earlier by A. Lunardi, see [16], by a completely different method involving interpolatory arguments.

In section §1 we recall several known results, proved for instance in [7], about transition semigroups R_t, $t \geq 0$, defined in space of continuous functions.

Section §2 is devoted to the description of the transition semigroup R_t, $t \geq 0$, in $L^2(H;\mu)$. Here we recall several results proved earlier in [9] and [12], and we give some improvements that will be used later.

In §3 we present a characterization of the domain of $\mathcal{A}$. This characterization is exploited in §4 to study different perturbations of $\mathcal{A}$.

1. Notation and setting of the problem

We are given a separable Hilbert space H (norm $|\cdot|$, inner product $<\cdot\,,\,\cdot>$), and a differential stochastic equation in H

$$\begin{cases} dZ(t) = AZ(t)dt + dW(t) \\ \\ Z(0) = x \in H, \end{cases} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a linear operator and $W(t), t \geq 0$, is a cylindrical Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, see e.g. [7].

We shall assume that

HYPOTHESIS 1.1.

(i) A is the infinitesimal generator of an analytic semigroup e^{tA} in H. There exist $M \geq 1$ and $\omega > 0$ such that

$$\|e^{tA}\| \leq Me^{-\omega t},\ t \geq 0.$$

(ii) For any $t > 0$, $e^{tA} \in \mathcal{L}_2(H)$ ([1]) and, setting

$$Q_t x = \int_0^t e^{sA} e^{sA^*} x\, dt,\ x \in H, \tag{1.2}$$

we have

$$\text{Tr}\,[Q_t] < +\infty,\ \forall\, t > 0.$$

The following result is proved in [7].

PROPOSITION 1.1. *Assume that Hypothesis* 1.1 *holds.*

(i) Problem (1.1) *has a unique mild solution given by*

$$Z(t,x) = e^{tA}x + \int_0^t e^{(t-s)A} dW(s),\ x \in H,\ t \geq 0. \tag{1.3}$$

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[1] $\mathcal{L}(H)$ is the Banach algebra of all linear bounded operators on H, endowed with the sup norm $\|\cdot\|$. By $\mathcal{L}_1(H)$ (norm $\|\cdot\|_{\mathcal{L}_1(H)}$) we mean the Banach space of all trace-class operators on H, and by $\mathcal{L}_2(H)$ (norm $\|\cdot\|_{\mathcal{L}_2(H)}$) the Hilbert space of all Hilbert-Schmidt operators in H. If $T \in \mathcal{L}_1(H)$, the trace of T is denoted by Tr T.

Moreover $Z(t,x)$ is a Gaussian random variable $\mathcal{N}(e^{tA}x, Q_t)$, for all $t \ge 0$ and all $x \in H$. ([2])

(ii) There exists a unique probability measure μ on $(H, \mathcal{B}(H))$ that is invariant for the process $Z(t,x)$, that is such that

$$\int_H R_t\varphi(x)\mu(dx) = \int_H \varphi(x)\mu(dx),\ \forall\, \varphi \in C_b(H), ([3])$$

where $R_t,\ t \ge 0$ is the transition semigroup

$$R_t\varphi(x) = \int_H \varphi(y)\mathcal{N}(e^{tA}x, Q_t)(dy),\ \varphi \in C_b(H),\ t \ge 0,\ x \in H. \tag{1.4}$$

Moreover $\mu = \mathcal{N}(0, Q)$, where

$$Qx = \int_0^{+\infty} e^{tA}e^{tA^*}x dt,\ x \in H. \tag{1.5}$$

One can easily check that Q is a solution to the following *Lyapunov* equation

$$2\langle A^*x, Qx\rangle + |x|^2 = 0,\ x \in D(A^*). \tag{1.6}$$

We end this section by recalling some properties of the semigroup $R_t,\ t \ge 0$, in the space $C_b(H)$.

The following result is proved in [7].

Proposition 1.2. *Assume that Hypothesis* 1.1 *holds.*

(i) For all $t > 0$ we have $e^{tA}(H) \subset Q_t^{1/2}(H)$. Moreover the linear operator $\Gamma(t) := Q_t^{-1/2}e^{tA}$ belongs to $\mathcal{L}_2(H)$ and the following estimate holds

$$\|\Gamma(t)\| \le t^{-1/2},\ t > 0. \tag{1.7}$$

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[2] For any $z \in H$, and any positive operator $L \in \mathcal{L}_1(H)$, we denote by $\mathcal{N}(z, L)$ the Gaussian measure on $(H, \mathcal{B}(H))$, (where $\mathcal{B}(H)$ is the family of all Borel subsets of H), with mean z and covariance operator L.)

[3] $C_b(H)$ is the Banach space of all uniformly continuous and bounded mappings from H into $\mathbb{R}$, endowed with the norm $\|\varphi\|_0 = \sup_{x\in H} |\varphi(x)|$.

(ii) For all $t > 0$ and all $\varphi \in C_b(H)$, we have $R_t\varphi \in C_b^1(H)$ (4) and

$$\langle DR_t\varphi(x), h\rangle = \tag{1.8}$$
$$= \int_H \langle \Gamma(t)h, Q_t^{-1/2}y\rangle\, \varphi(e^{tA}x + y)\, \mathcal{N}(0, Q_t)(dy),\ h \in H.$$

We notice that, when A is not identically 0, the semigroup $R_t,\ t \geq 0$ is never strongly continuous, see [3]. Moreover its restriction to the "subspace of continuity":

$$\{\varphi \in C_b(H) :\ t \to R_t\varphi \text{ is continuous in } C_b(H)\},$$

is not an analytic semigroup, see [5].

Proceeding as in S. Cerrai [3], we define the infinitesimal generator $\mathcal{A}$ of $R_t,\ t \geq 0$, through its resolvent, by setting

$$R(\lambda, \mathcal{A})\varphi(x) = \int_0^{+\infty} e^{-\lambda t} R_t\varphi(x)dt,\ x \in H,\ \varphi \in C_b(H). \tag{1.9}$$

To give a description of the infinitesimal generator $\mathcal{A}$, it is convenient to introduce the space $\mathcal{E}$ of all finite linear combinations of the exponential functions $\varphi_h = e^{i\langle h,x\rangle},\ x \in H,\ h \in D(A^*)$.

2. Transition semigroup in $L^2(H;\mu)$

In this section we first recall the definition and some properties of the Sobolev spaces $W^{1,2}(H;\mu)$ and $W^{2,2}(H;\mu)$. Then we show, following [7], that the semigroup $R_t,\ t \geq 0$ can be uniquely extended as a contraction semigroup to $L^2(H;\mu)$, and we state several properties of it, needed in the sequel.

2.1 Sobolev spaces

First of all we remark that, as easily checked, the linear space $\mathcal{E}$ of exponential functions, as introduced in §1, is dense in $L^2(H;\mu)$. Moreover we denote by $\{e_k\}$ a complete orthonormal system in H of

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[4] $C_b^1(H)$ is the set of all functions in $C_b(H)$ that are uniformly continuous and bounded together with their Fréchet derivative.

eigenvectors of Q_∞, and by $\{\lambda_k\}$, the corresponding set of eigenvalues:

$$Qe_k = \lambda_k e_k, \ k \in \mathbb{N}.$$

For any $k \in \mathbb{N}$ we denote by $D_k\varphi$ the derivative of φ in the direction of e_k, and we set $x_k = \langle x, e_k\rangle, \ x \in H$.

The following lemma and proposition are well known, see e.g. [12]. However, we give a sketch of proofs for the reader's convenience.

LEMMA 2.1. *Let $\varphi, \psi \in \mathcal{E}$ and $h \in \mathbb{N}$. Then we have*

$$\int_H D_h\varphi(x)\ \psi(x)\mu(dx) + \int_H D_h\psi(x)\ \varphi(x)\mu(dx) = \\ = \frac{1}{\lambda_h} \int_H x_h\varphi(x)\ \psi(x)\mu(dx). \tag{2.1}$$

Proof. Since $\mathcal{E}$ is dense in $L^2(H;\mu)$, it is enough to prove (2.1) for

$$\varphi(x) = e^{i\langle\alpha,x\rangle}, \ \psi(x) = e^{i\langle\beta,x\rangle}, \ \alpha,\beta \in H.$$

In this case we have ([5]):

$$\int_H D_h\varphi(x)\ \psi(x)\mu(dx) + \int_H D_h\psi(x)\ \varphi(x)\mu(dx) = \\ = i(\alpha_h - \beta_h)e^{-\frac{1}{2}\langle Q(\alpha-\beta),\alpha-\beta\rangle}. \tag{2.2}$$

Moreover

$$\begin{aligned}
\int_H x_h\varphi(x)\ \psi(x)\mu(dx) &= \int_H x_h e^{i\langle\alpha-\beta,x\rangle}\mu(dx) \\
&= -i\frac{d}{d\lambda}\int_H e^{i\langle\alpha-\beta+\lambda e_h,x\rangle}\mu(dx)\Big|_{\lambda=0} \\
&= -i\frac{d}{d\lambda}\ e^{-\frac{1}{2}\langle Q(\alpha-\beta+\lambda e_h),\alpha-\beta g+\lambda e_h\rangle}\Big|_{\lambda=0} \\
&= i\,e^{-\frac{1}{2}\langle Q(\alpha-\beta),\alpha-\beta\rangle}(\alpha_h - \beta_h)\lambda_h.
\end{aligned} \tag{2.3}$$

Now the conclusion follows. □

[5]If $\nu = \mathcal{N}(0,Q)$ is a Gaussian measure on H, then the characteristic function of ν is defined as $F(h) = \int_H e^{i\langle h,x\rangle}\nu(dx)$. One can easily show that $F(h) = e^{-\frac{1}{2}\langle Qh,h\rangle}$.

From Lemma 2.1 we have

PROPOSITION 2.2. *For any $h \in \mathbb{N}$ the linear operator*

$$D_h : \mathcal{E} \subset L^2(H;\mu) \to L^2(H;\mu), \ \varphi \to D_h\varphi,$$

is closable in $L^2(H;\mu)$.

We shall still denote by D_h the closure of D_h.

Proof. Let $\{\varphi_n\}$ be a sequence in $\mathcal{E}$ and let $g \in L^2(H;\mu)$ such that

$$\varphi_n \to 0, \ D_h\varphi_n \to g, \ \text{ in } L^2(H;\mu), \text{ as } n \to \infty.$$

We have to show that $g = 0$.

By using (2.1) with $\varphi = \varphi_n$ and with ψ being any element in $\mathcal{E}$, we have in fact

$$\int_H D_h\varphi_n(x)\ \psi(x)\mu(dx) + \int_H D_h\psi(x)\ \varphi_n(x)\mu(dx) = \\ = \frac{1}{\lambda_h} \int_H x_h\varphi_n(x)\ \psi(x)\mu(dx).$$

Letting n tend to ∞ we have by the hypothesis

$$\int_H g(x)\psi(x)\mu(dx) = 0,$$

that yields $g = 0$ due to the density of $\mathcal{E}$ and the arbitrariness of ψ. This completes the proof. □

We can now define Sobolev spaces. We denote by $W^{1,2}(H;\mu)$ the linear space of all functions $\varphi \in L^2(H;\mu)$ such that $D_k\varphi \in L^2(H;\mu)$ for all $k \in \mathbb{N}$ and

$$\int_H |D\varphi(x)|^2\mu(dx) = \sum_{k=1}^{\infty} \int_H |D_k\varphi(x)|^2\mu(dx) < +\infty.$$

$W^{1,2}(H;\mu)$, endowed with the inner product

$$\langle\varphi,\psi\rangle_1 = \int_H \varphi(x)\psi(x)\mu(dx) + \int_H \langle D\varphi(x), D\psi(x)\rangle\mu(dx),$$

is a Hilbert space. We recall that the embedding of $W^{1,2}(H;\mu)$ into $L^2(H;\mu)$ is compact, see [6], [19], [7].

In a similar way we can define the Sobolev space $W^{2,2}(H;\mu)$ consisting of all functions $\varphi \in W^{1,2}(H;\mu)$ such that $D_hD_k\varphi \in L^2(H;\mu)$ for all $h,k \in \mathbb{N}$ and $D^2\varphi(x) \in \mathcal{L}_2(H)$ for all $x \in H$.

$W^{2,2}(H;\mu)$, endowed with the inner product

$$\begin{aligned}\langle\varphi,\psi\rangle_2 &= \langle\varphi,\psi\rangle_1 + \sum_{h,k=1}^{\infty}\int_H D_hD_k\varphi(x)\ D_hD_k\psi(x)\mu(dx)\\ &= \langle\varphi,\psi\rangle_1 + \int_H \langle D^2\varphi(x), D^2\psi(x)\rangle^2_{\mathcal{L}_2(H)}\mu(dx)\end{aligned}$$

is a Hilbert space. Notice that, when H is infinite-dimensional, the embedding of $W^{2,2}(H;\mu)$ into $W^{1,2}(H;\mu)$ is not compact, see [9].

Now from Lemma 2.1 and Proposition 2.2 the following integration by parts formula follows, see [12].

Proposition 2.3. *Let $\psi_1,\psi_2 \in W^{1,2}(H,\mu)$ and $\alpha \in H$. Then we have*

$$\begin{aligned}\int_H \langle D\psi_1(x), Q\alpha\rangle\ \psi_2(x)\mu(dx) + \int_H \langle D\psi_2(x), Q\alpha\rangle\psi_1(x)\mu(dx) =&\\ = \int_H \psi_1(x)\psi_2(x)\langle\alpha,x\rangle\mu(dx).&\qquad (2.4)\end{aligned}$$

We finish this subsection by proving some useful properties of the spaces $W^{1,2}(H,\mu)$ and $W^{2,2}(H,\mu)$.

Proposition 2.4. ([12]) *Let $\zeta \in W^{1,2}(H,\mu)$ and $\alpha \in H$. Then the function*

$$x \to \langle x,\alpha\rangle\zeta(x),$$

belongs to $L^2(H,\mu)$ and the following inequality holds.

$$\begin{aligned}\int_H |\langle\alpha,x\rangle|^2\zeta^2(x)\mu(dx) &\le 2|Q^{1/2}\alpha|^2\int_H \zeta^2(x)\mu(dx) +\\ &\quad + 16|Q\alpha|^2\int_H |D\zeta(x)|^2\mu(dx). \qquad (2.5)\end{aligned}$$

Proof. It is enough to prove (2.5) when $\zeta \in \mathcal{E}$. We apply the integration by parts formula (2.4) with

$$\psi_1(x) = \langle \alpha, x \rangle,\ \psi_2(x) = \zeta^2(x).$$

Since

$$D\psi_1(x) = \alpha,\quad D\psi_2(x) = 2\zeta(x)D\zeta(x),\ x \in H,$$

we obtain, using Hölder's inequality

$$\begin{aligned}
&\int_H |\langle \alpha, x\rangle|^2 \zeta^2(x)\mu(dx) = \\
&\quad = \int_H \langle Q\alpha, \alpha\rangle \zeta^2(x)\mu(dx) + 2\int_H \langle \alpha, x\rangle\langle D\zeta(x), Q\alpha\rangle\ \zeta(x)\mu(dx) \\
&\quad \le |Q^{1/2}\alpha|^2 \|\zeta\|^2_{L^2(\mu,H)} + \\
&\qquad + 2\left[\int_H |\langle \alpha, x\rangle|^2\zeta^2(x)\mu(dx)\right]^{1/2}\ \left[\int_H |\langle Q\alpha, D\zeta(x)\rangle|^2\mu(dx)\right]^{1/2} \\
&\quad \le |Q^{1/2}\alpha|^2 \|\zeta\|^2_{L^2(\mu,H)} + \\
&\qquad + \frac{1}{2}\int_H |\langle \alpha, x\rangle|^2\zeta^2(x)\mu(dx) + 8\int_H |\langle Q\alpha, D\zeta(x)\rangle|^2\mu(dx),
\end{aligned}$$

that yields (2.5). □

By Proposition 2.4 it follows the result

COROLLARY 2.5. *Let $\zeta \in W^{1,2}(H,\mu)$. Then the function*

$$H \to \mathbb{R},\ x \to |x|\zeta(x),$$

belongs to $L^2(H,\mu)$ and the following estimate holds

$$\begin{aligned}
\int_H |x|^2\zeta^2(x)\mu(dx) \quad &\le \quad 2\ \mathrm{Tr}\ Q\ \int_H \zeta^2(x)\mu(dx) + \\
&\quad + 16\ \mathrm{Tr}\ [Q^2]\ \int_H |D\zeta(x)|^2\mu(dx). \quad (2.6)
\end{aligned}$$

Proof. Let $k \in \mathbb{N}$; setting in (2.5) $\alpha = e_k$, we find

$$\int_H x_k^2\ \zeta^2(x)\mu(dx) \le 2\lambda_k \int_H \zeta^2(x)\mu(dx) + 16\lambda_k^2 \int_H |D\zeta(x)|^2\mu(dx).$$

Summing up on k, the inequality (2.6) follows. □

We now consider functions ζ in $W^{2,2}(H,\mu)$.

Proposition 2.6. *Let* $\zeta \in W^{2,2}(H,\mu)$ *and* $\alpha \in H$. *Then the function* $x \to |\langle x,\alpha\rangle|^2\zeta(x)$ *belongs to* $L^2(H;\mu)$ *and*

$$\begin{aligned}\int_H |\langle x,\alpha\rangle|^4\zeta^2(x)\mu(dx) \ \leq\ & 4\,(|Q^{1/2}\alpha|^4 + 8|\alpha|^2|Q\alpha|^2)\int_H \zeta^2(x)\mu(dx)\\ &+ 96\,|Q\alpha|^2|Q^{1/2}\alpha|^2\int_H |D\zeta(x)|^2\mu(dx)\\ &+ 512\,|Q\alpha|^4\int_H \|D^2\zeta(x)\|^2_{\mathcal{L}_2(H)}\mu(dx)\end{aligned} \tag{2.7}$$

Proof. Setting $\eta(x) = \langle x,\alpha\rangle\zeta(x)$, we have by Proposition 2.4 that $\eta \in L^2(H;\mu)$ and

$$\begin{aligned}\int_H \eta^2(x)\mu(dx) \ \leq\ & 2|Q^{1/2}\alpha|^2\int_H \zeta^2(x)\mu(dx)\\ &+ 16|Q\alpha|^2\int_H |D\zeta(x)|^2\mu(dx).\end{aligned} \tag{2.8}$$

Moreover, for any $i \in \mathbb{N}$, we have

$$D_i\eta(x) = \alpha_i\zeta(x) + \langle x,\alpha\rangle D_i\zeta(x).$$

Thus, by Proposition 2.4, $D_i\eta \in L^2(H;\mu)$ and

$$\begin{aligned}\int_H |D_i\eta(x)|^2\mu(dx) \ \leq\ & 2|\alpha_i|^2\int_H \zeta^2(x)\mu(dx) +\\ &+ 2\int_H |\langle x,\alpha\rangle|^2|D_i\zeta(x)|^2\mu(dx)\\ \leq\ & 2|\alpha_i|^2\int_H \zeta^2(x)\mu(dx) +\\ &+ 4|Q^{1/2}\alpha|^2\int_H |D_i\zeta(x)|^2\mu(dx) +\\ &+ 32|Q\alpha|^2\int_H |DD_i\zeta(x)|^2\mu(dx).\end{aligned}$$

Summing up on i we have

$$\begin{aligned}\int_H |D\eta(x)|^2\mu(dx) \quad &\leq \quad 2|\alpha|^2\int_H \zeta^2(x)\mu(dx) + \\ &+ 4|Q^{1/2}\alpha|^2\int_H |D\zeta(x)|^2\mu(dx) \\ &+ 32|Q\alpha|^2\int_H \|D^2\zeta(x)\|^2_{\mathcal{L}_2(H)}\mu(dx). \quad (2.9)\end{aligned}$$

This shows that $\eta \in W^{1,2}(H;\mu)$. Now, applying once again Proposition 2.4, we have that $g = \langle x,\alpha\rangle\eta \in L^2(H;\mu)$ and

$$\begin{aligned}\int_H |\langle x,\alpha\rangle|^4\zeta^2(x)\mu(dx) \quad &\leq \quad 2|Q^{1/2}\alpha|^2\int_H \eta^2(x)\mu(dx) + \\ &+ 16|Q\alpha|^2\int_H |D\eta(x)|^2\mu(dx). \quad (2.10)\end{aligned}$$

By substituting (2.8) and (2.9) into (2.10) we obtain the conclusion (2.7). □

In a similar way we prove the following result.

Proposition 2.7. *Let $\zeta \in W^{2,2}(H,\mu)$. Then the function $x \to (1+|x|^2)\zeta(x)$ belongs to $L^2(H;\mu)$ and*

$$\begin{aligned}&\int_H (1+|x|^2)^2\zeta^2(x)\mu(dx) \leq \\ &\quad [32 \text{ Tr } Q^2 + (1+2 \text{ Tr } Q)^2]\int_H \zeta^2(x)\mu(dx) + \\ &\quad + 48 \text{ Tr } [Q^2](1+2 \text{ Tr } Q)\int_H |D\zeta(x)|^2\mu(dx) + \\ &\quad + 512 \,(\text{Tr } [Q^2])^2\int_H \|D^2\zeta(x)\|^2_{\mathcal{L}_2(H)}\mu(dx). \qquad (2.11)\end{aligned}$$

Proof. Setting $\rho(x) = \sqrt{1+|x|^2}\ \zeta(x)$, we have by (2.6) that $\rho \in L^2(H;\mu)$ and

$$\begin{aligned}&\int_H \rho^2(x)\mu(dx) = \int_H \zeta^2(x)\mu(dx) + \int_H |x|^2\zeta^2(x)\mu(dx) \leq \\ &\leq (1+2 \text{ Tr } Q)\int_H \zeta^2(x)\mu(dx) + 16 \text{ Tr } [Q^2]\int_H |D\zeta(x)|^2\mu(dx). \\ &\hfill (2.12)\end{aligned}$$

For any $i \in \mathbb{N}$ we have

$$D_i\rho(x) = x_i(1+|x|^2)^{-1/2}\zeta(x) + (1+|x|^2)^{1/2}D_i\zeta(x),$$

so that

$$\begin{aligned}\int_H |D_i\rho(x)|^2\mu(dx) \quad &\leq \quad 2\int_H \frac{x_i^2}{1+|x|^2}\zeta^2(x)\mu(dx)\\ &\quad + 2\int_H |D_i\zeta(x)|^2\mu(dx)\\ &\quad + 2\int_H |x|^2|D_i\zeta(x)|^2\mu(dx).\end{aligned}$$

Consequently, by (2.6) it follows that $D_i\rho \in L^2(H;\mu)$ and

$$\begin{aligned}\int_H |D_i\rho(x)|^2\mu(dx) \quad &\leq \quad 2\int_H \frac{x_i^2}{1+|x|^2}\zeta^2(x)\mu(dx)\\ &\quad + 2\int_H |D_i\zeta(x)|^2\mu(dx)\\ &\quad + 4 \text{ Tr } Q\int_H |D_i\zeta(x)|^2\mu(dx)\\ &\quad + 32 \text{ Tr } [Q^2]\int_H |DD_i\zeta(x)|^2\mu(dx).\end{aligned}$$

Summing up on i we obtain

$$\begin{aligned}\int_H |D\rho(x)|^2\mu(dx) &\leq 2\int_H \zeta^2(x)\mu(dx) +\\ &+ (2+4 \text{ Tr } Q)\int_H |D\zeta(x)|^2\mu(dx)\\ &+ 32 \text{ Tr } Q^2\int_H \|D^2\zeta(x)\|^2_{\mathcal{L}_2(H)}\mu(dx),\end{aligned} \tag{2.13}$$

that yields $\rho \in W^{1,2}(H;\mu)$. Finally by (2.6) it follows

$$\begin{aligned}\int_H (1+|x|^2)^2\zeta^2(x)\mu(dx) \quad &\leq \quad \int_H \rho^2(x)\mu(dx) + \int_H |x|^2\rho^2(x)\mu(dx)\\ &\leq \quad (1+2 \text{ Tr } Q)\int_H \rho^2(x)\mu(dx) +\\ &\quad + 16 \text{ Tr } [Q^2]\int_H |D\rho(x)|^2\mu(dx).\end{aligned} \tag{2.14}$$

By substituting (2.12) and (2.13) into (2.14) we complete the proof.
□

2.2 Transition semigroup

The following result was proved in [7], see also [8]. We give however a sketch of the proof for the reader's convenience.

PROPOSITION 2.8. *(i) Assume that Hypothesis* 1.1 *holds. Then, for any* $t > 0$*, the operator* R_t*, defined by* (1.4)*, has a unique extension to a linear bounded operator in* $L^2(H;\mu)$*, that we still denote by* R_t*. Moreover* R_t*,* $t \geq 0$ *is a strongly continuous semigroup of contractions in* $L^2(H;\mu)$*, and*

$$R_t\varphi(x) = \int_H \varphi(e^{tA}x + y)\mathcal{N}(0, Q_t)(dy),$$
$$t \geq 0,\ x \in H,\ \varphi \in L^2(H;\mu). \tag{2.15}$$

(ii) $\mathcal{E} \subset D(\mathcal{A})$ *and*

$$\mathcal{A}(e^{i\langle h,\cdot\rangle})(x) = \left(\langle A^*h, x\rangle - \frac{1}{2}|h|^2\right) e^{i\langle h,x\rangle},\ x \in H. \tag{2.16}$$

Moreover, $\mathcal{E}$ *is a core for the infinitesimal generator* $\mathcal{A}$ *of* R_t*,* $t \geq 0$*.*

(iii) For all $t > 0$ *and all* $\varphi \in L^2(H;\mu)$*, one has* $R_t\varphi \in W^{1,2}(H;\mu)$ *and*

$$\langle DR_t\varphi(x), h\rangle = \int_H \langle \Gamma(t)h, Q_t^{-1/2}y\rangle\ \varphi(e^{tA}x + y)\ \mathcal{N}(0, Q_t)(dy). \tag{2.17}$$

Consequently, R_t *is compact on* $L^2(H;\mu)$ *for all* $t > 0$*.*

Proof. Let $\varphi \in C_b(H)$, then by (1.4) and Hölder's estimate, we have

$$|R_t\varphi(x)|^2 \leq \int_H \varphi^2(e^{tA}x + y)\mathcal{N}(0, Q_t)(dy) = R_t(\varphi^2)(x).$$

Using the invariance of μ, it follows that

$$\int_H |R_t\varphi(x)|^2\mu(dx) \le \int_H |\varphi(x)|^2\mu(dx),$$

that proves (i).

(ii) Notice first that, in view of (2.15), for all $h \in H$ we have

$$R_t e^{i\langle h,\cdot\rangle}(x) = e^{i\langle e^{tA^*}h,x\rangle - \frac{1}{2}\langle Q_y h,h\rangle}.$$

Thus, for any $t > 0$, R_t maps $\mathcal{E}$ into itself. Since clearly $\mathcal{E} \subset D(\mathcal{A})$, we have that $\mathcal{E}$ is a core for $\mathcal{A}$, see [11, Theorem 1.9].

Let us prove (iii). Let $\varphi \in C_b(H)$ and $h \in H$. By (1.8) and the Hölder inequality we have

$$\begin{aligned}
&|\langle DR_t\varphi(x),h\rangle|^2 \le \\
&\le \int_H |\langle \Gamma(t)h, Q_t^{-1/2}y\rangle^2 \int_H |\varphi(e^{tA}x+y)|^2 \mathcal{N}(0,Q_t)(dy) \\
&= |\Gamma(t)h|^2 R_t(\varphi^2)(x).
\end{aligned}$$

Integrating on x and using the invariance of μ, we find

$$\int_H |\langle DR_t\varphi(x),h\rangle|^2\mu(dx) \le |\Gamma(t)h|^2 \int_H |\varphi(x)|^2\mu(dx).$$

Setting $h = e_k$, $k \in \mathbb{N}$, summing up on k, and recalling that by Proposition 1.2-(i), $\Gamma(t) \in \mathcal{L}_2(H)$, we obtain

$$\int_H |DR_t\varphi(x)|^2\mu(dx) \le \text{ Tr }[\Gamma(t)\Gamma^*(t)] \int_H |\varphi(x)|^2\mu(dx).$$

The conclusion follows from the density of $C_b(H)$ in $L^2(H;\mu)$. □

The following propositions were proved in [12], see also [1]. Before stating it we need some preliminary results.

LEMMA 2.9. *For any $\varphi,\psi \in \mathcal{E}$ the following identity holds.*

$$\int_H (\mathcal{A}\varphi)(x)\psi(x)\mu(dx) = \int_H \langle QD\psi(x), A^*D\varphi(x)\rangle\mu(dx). \qquad (2.18)$$

Proof. It is enough to prove (2.18) for

$$\varphi(x) = e^{i\langle x,\alpha\rangle},\ \psi(x) = e^{i\langle x,\beta\rangle},\ x \in H,\ \alpha,\beta \in D(A^*).$$

In this case we have, by a simple computation,

$$\int_H (\mathcal{A}\varphi)(x)\psi(x)\mu(dx) = \\ = -\left(\langle A^*\alpha, Q(\alpha-\beta)\rangle + \frac{1}{2}|\alpha|^2\right) e^{-\frac{1}{2}\langle Q(\alpha-\beta),\alpha-\beta\rangle}, \tag{2.19}$$

and

$$\int_H \langle QD\psi(x), A^*D\varphi(x)\rangle\mu(dx) = \langle A^*\alpha, Q\beta\rangle e^{-\frac{1}{2}\langle Q(\alpha-\beta),\alpha-\beta\rangle}. \tag{2.20}$$

Taking into account (2.19) and (2.20), we see that equality (2.18) is equivalent to

$$2\langle A^*\alpha, Q\alpha\rangle + |\alpha|^2 = 0,$$

that coincides with Lyapunov equation (1.6). □

The lemma yields now the result

PROPOSITION 2.10. *For any $\varphi \in D(\mathcal{A})$ and any $\psi \in W^{1,2}(H;\mu)$ the following identity holds.*

$$\int_H (\mathcal{A}\varphi)(x)\psi(x)\mu(dx) = \int_H \langle QD\psi(x), A^*D\varphi(x)\rangle\mu(dx). \tag{2.21}$$

Finally, taking $\phi = \psi$, and using again the Lyapunov equation we have

PROPOSITION 2.11. *Assume that Hypothesis* 1.1 *holds. Then for any $\varphi \in D(\mathcal{A})$ one has $\varphi \in W^{1,2}(H,\mu)$ and the following identity holds.*

$$\int_H (\mathcal{A}\varphi)(x)\varphi(x)\mu(dx) = -\frac{1}{2}\int_H |D\varphi(x)|^2\mu(dx). \tag{2.22}$$

The following corollary is an immediate consequence of Proposition 2.11.

Corollary 2.12. *Assume that Hypothesis* 1.1 *holds. Then for any* $\varepsilon > 0$ *one has*

$$\int_H |D\varphi(x)|^2 \mu(dx) \le \varepsilon \int_H |\mathcal{A}\varphi)(x)|^2 \mu(dx) + \frac{4}{\varepsilon} \int_H |\varphi(x)|^2 \mu(dx). \tag{2.23}$$

Remark 2.13. If $M = 1$ (6), one can prove that the semigroup R_t $t \ge 0$ is analytic in $L^2(H;\mu)$, see [12], [9].

3. Characterization of $D(\mathcal{A})$

In this section we want to characterize the domain of $\mathcal{A}$. From now on we shall assume that

$$\{e_k\} \subset D(A). \tag{3.1}$$

Then we set

$$A_{h,k} = \langle Ae_k, e_h \rangle,\ h,k \in \mathbb{N},$$

and we write $\mathcal{A}$ on $\mathcal{E}$ as

$$(\mathcal{A}\varphi)(x) = \frac{1}{2}\sum_{h=1}^{\infty} D_h^2\varphi(x) + \sum_{h,k=1}^{\infty} A_{h,k}x_k D_h\varphi(x),\ \varphi \in \mathcal{E}. \tag{3.2}$$

We start with a basic identity.

Proposition 3.1. *Assume that Hypotheses* 1.1 *and* (3.1) *hold. Let* $\varphi \in \mathcal{E}$ *and let* $f = \mathcal{A}\varphi$. *Then the following identity holds:*

$$\begin{aligned}\frac{1}{2}\int_H &\|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)}\ \mu(dx) - \int_H \langle D\varphi(x), A^* D\varphi(x)\rangle \mu(dx)\\ &= 2\int_H |f(x)|^2 \mu(dx) - 2\int_H f(x)\langle Ax + \frac{1}{2}Q^{-1}x, D\varphi(x)\rangle \mu(dx).\end{aligned} \tag{3.3}$$

---

[6] M is the constant in Hypothesis 1.1

Proof. By differentiating (3.2) with respect to x_j, we obtain

$$\mathcal{A}(D_j\varphi)(x) + \sum_{h=1}^{\infty} A_{h,j} D_h\varphi(x) = D_j f(x).$$

Multiplying both sides by $D_j\varphi(x)$ and integrating with respect to μ it follows

$$\int_H \mathcal{A}(D_j\varphi)\ D_j\varphi\ \mu(dx) + \sum_{h=1}^{\infty} \int_H A_{h,j} D_h\varphi D_j\varphi\ \mu(dx) = $$

$$= \int_H D_j\varphi D_j f\ \mu(dx).$$

Recalling (2.22) we see that the above equality is equivalent to

$$\frac{1}{2}\int_H |DD_j\varphi(x)|^2\mu(dx) \quad - \quad \sum_{h=1}^{\infty}\int_H A_{h,j} D_h\varphi(x) D_j\varphi(x)\ \mu(dx)$$

$$= \quad -\int_H D_j\varphi(x) D_j f(x)\mu(dx).$$

By (2.1) we get

$$\frac{1}{2}\ \int_H |DD_j\varphi(x)|^2\mu(dx) - \sum_{h=1}^{\infty}\int_H A_{h,j} D_h\varphi(x) D_j\varphi(x)\ \mu(dx)$$

$$= \int_H f(x) D_j^2\varphi(x)\ \mu(dx) - \int_H \frac{x_j}{\lambda_j} f(x) D_j\varphi(x)\mu(dx).$$

Summing up on j we find

$$\frac{1}{2}\ \int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)}\ \mu(dx) - \int_H \langle D\varphi(x), A^* D\varphi(x)\rangle\mu(dx)$$

$$= \int_H f(x) \left\{\mathrm{Tr}\,[D^2\varphi(x)] - \langle Q^{-1}x, D\varphi(x)\rangle\right\} \mu(dx),$$

and the conclusion follows. □

In order to characterize $D(\mathcal{A})$ we need some further assumptions.

HYPOTHESIS 3.1. *(i) $D(A)\cap Q(H)$ is dense in H and the linear operator*

$$\begin{cases} D(K) = D(A)\cap Q(H), \\ \\ Kx := Ax + \dfrac{1}{2}\, Q^{-1}x,\ x\in D(K), \end{cases} \tag{3.4}$$

is bounded in H.

(ii) There exists $\eta>0$ such that

$$\langle Ax,x\rangle \le -\eta |x|^2,\ x\in D(A). \tag{3.5}$$

If Hypothesis 3.1 holds, we shall denote by K the unique extension of the operator K to H. Notice that if Hypothesis 1.1 holds with $M=1$, then (3.5) holds with $\eta=\omega$.

In the following we denote by H_A the Banach space obtained by taking the completion of $D(A)$ with respect to the norm

$$|x|^2_{H_A} = -\langle Ax,x\rangle,\ \ x\in D(A).$$

THEOREM 3.2. *Assume that Hypotheses* 1.1, 3.1 *and* (3.1) *hold. Let $\mathcal{A}$ be the infinitesimal generator of the semigroup R_t, $t\ge 0$, defined by* (2.15). *Then we have*

$$\begin{aligned} D(\mathcal{A}) \ &= \ \Big\{\varphi\in W^{2,2}(H;\mu): \\ &\qquad |D\varphi(x)|\in H_A,\ \mu \text{ a.e.},\ |D\varphi(\cdot)|_{H_A}\in L^2(H;\mu)\Big\} \end{aligned} \tag{3.6}$$

Proof. We first prove that

$$\begin{aligned} D(\mathcal{A}) \ &\subset \ \Big\{\varphi\in W^{2,2}(H;\mu): \\ &\qquad D\varphi(x)\in H_A,\ \mu \text{ a.e. } |D\varphi(\cdot)|_{H_A}\in L^2(H;\mu)\Big\}. \end{aligned} \tag{3.7}$$

For this, recalling that $D(\mathcal{A}) \subset W^{1,2}(H;\mu)$ by Proposition 2.11, it suffices to prove that for any $\varphi \in D(\mathcal{A})$ the following estimate holds

$$\begin{aligned} &\frac{1}{4} \int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)}\mu(dx) + \int_H |D\varphi(x)|^2_{H_A}\mu(dx) \\ &\quad \le 2(1 + 128\|K\|^2 \text{ Tr } [Q^2]) \int_H |f(x)|^2\mu(dx) + \\ &\quad\quad + \frac{\text{Tr } Q}{32 \text{ Tr } [Q^2]} \int_H |D\varphi(x)|^2\mu(dx). \end{aligned} \tag{3.8}$$

Since $\mathcal{E}$ is a core for $\mathcal{A}$, it is enough to prove (3.8) for all $\varphi \in \mathcal{E}$. Let $a > 0$ be a positive number to be fixed later. By (3.3) it follows

$$\begin{aligned} &\frac{1}{2} \int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)}\ \mu(dx) + \int_H |D\varphi(x)|^2_{H_A}\mu(dx) \le \\ &\quad \le (2+4a) \int_H |f(x)|^2\mu(dx) + \frac{\|K\|^2}{a} \int_H |x|^2|D\varphi(x)|^2\mu(dx). \end{aligned}$$

Taking into account (2.6) we find

$$\begin{aligned} &\frac{1}{2} \int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)}\mu(dx) + \int_H |D\varphi(x)|^2_{H_A}\mu(dx) \le \\ &\quad \le (2+4a) \int_H |f(x)|^2\mu(dx) + \\ &\quad\quad + 2\,\frac{\|K\|^2\text{Tr } Q}{a} \int_H |D\varphi(x)|^2\mu(dx) + \\ &\quad\quad + 16\,\frac{\|K\|^2 \text{ Tr } [Q^2]}{a} \int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)}\ \mu(dx). \end{aligned}$$

Choosing finally a such that

$$a = 64\|K\|^2 \text{ Tr } Q^2$$

(3.8) and consequently (3.7) follows.

We now prove that

$$\begin{aligned} D(\mathcal{A}) \supset \Big\{&\varphi \in W^{2,2}(H;\mu) : \\ &D\varphi(x) \in H_A,\ \mu \text{ a.e.},\ |D\varphi(\cdot)|_{H_A} \in L^2(H;\mu)\Big\}. \end{aligned} \tag{3.9}$$

Let $\varphi \in \mathcal{E}$ and set

$$L = \frac{1}{2} \int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)} \, \mu(dx) + \int_H |D\varphi(x)|^2_{H_A} \mu(dx),$$

then from (3.3) we have

$$2\int_H |\mathcal{A}\varphi(x)|^2 \mu(dx) \le L + 2\|K\| \int_H |\mathcal{A}\varphi(x)| \, |x| \, |D\varphi(x)| \mu(dx)$$

$$\le L + \int_H |\mathcal{A}\varphi(x)|^2 \mu(dx) + 4\|K\|^2 \int_H |x|^2 \, |D\varphi(x)|^2 \mu(dx),$$

and so

$$\int_H |\mathcal{A}\varphi(x)|^2 \mu(dx) \le L + 4\|K\|^2 \int_H |x|^2 \, |D\varphi(x)|^2 \mu(dx).$$

By (2.6) it follows

$$\begin{aligned}\int_H |\mathcal{A}\varphi(x)|^2 \mu(dx) \quad &\le \quad L + 8 \text{ Tr } Q \, \|K\|^2 \int_H |D\varphi(x)|^2 \mu(dx) \\ &+ \quad 64 \text{ Tr } [Q^2] \int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)} \, \mu(dx),\end{aligned}$$

Taking into account (2.23), for any $\varepsilon > 0$ we have

$$\int_H |\mathcal{A}\varphi(x)|^2 \mu(dx) \le L + 8\varepsilon \|K\|^2 \text{ Tr } Q \int_H |\mathcal{A}\varphi(x)|^2 \mu(dx)$$

$$+ \frac{32\|K\|^2 \text{ Tr } Q}{\varepsilon} \int_H |\varphi(x)|^2 \mu(dx).$$

Now choosing

$$\varepsilon = \frac{1}{16 \text{ Tr } Q \, \|K\|^2},$$

we have

$$\begin{aligned}\frac{1}{2} \int_H |\mathcal{A}\varphi(x)|^2 \mu(dx) \le L \quad &+ \quad 512 \, (\text{ Tr } Q)^2 \, \|K\|^4 \int_H |\varphi(x)|^2 \mu(dx) \\ &+ \quad 64 \text{ Tr } [Q^2] \int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)} \, \mu(dx).\end{aligned} \tag{3.10}$$

From (3.10) and the density of $\mathcal{E}$ it follows that if φ is such that L is bounded, then $\varphi \in D(\mathcal{A})$. This proves the inclusion (3.9).

The proof is complete. $\square$

REMARK 3.3. It is well known that when A is a variational operator and $D(A) = D(A^*)$, then H_A coincides with $D_A\left(\frac{1}{2}, 2\right)$, the real interpolation space consisting of all $x \in H$ such that

$$|x|^2_{D_A\left(\frac{1}{2},2\right)} := \int_0^\infty |Ae^{tA}x|^2 dt < +\infty,$$

see [13]. Thus in this case, if Hypotheses 1.1, 3.1 and (3.1) hold, then the domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \Big\{\varphi \in W^{2,2}(H;\mu) \quad : \quad D\varphi(x) \in D_A\left(\frac{1}{2}, 2\right),\ \mu \text{ a.e.},$$
$$|D\varphi(\cdot)|_{D_A\left(\frac{1}{2},2\right)} \in L^2(H;\mu)\Big\}. \quad (3.11)$$

REMARK 3.4. Assume that Hypotheses 1.1, and (3.1) hold and that A is self-adjoint. In this case from (1.5) we have

$$Qx = \int_0^{+\infty} e^{2tA}xdt = -\frac{1}{2}\, A^{-1}x,\ x \in H,$$

that obviously implies $K = 0$. Consequently Hypotheses 3.1 holds and, from Theorem 3.2 it follows that

$$D(\mathcal{A}) = \Big\{\varphi \in W^{2,2}(H;\mu) \quad : \quad D\varphi(x) \in D((-A)^{1/2}),\ \mu \text{ a.e.},$$
$$(-A)^{1/2}D\varphi \in L^2(H;\mu)\Big\}. \quad (3.12)$$

REMARK 3.5. Assume that H is finite-dimensional and that A is of negative type. Then Hypotheses 1.1, 3.1 and (3.1) obviously hold. Then from Theorem 3.2 it follows that

$$D(\mathcal{A}) = W^{2,2}(H;\mu). \quad (3.13)$$

This result was earlier proved by a different method based on interpolation, by A. Lunardi, see [16].

4. Perturbation results

We assume here that A is self-adjoint and fulfills Hypotheses 1.1 and 3.1. We will be concerned with some perturbations of the operator $\mathcal{A}$, the infinitesimal generator of the semigroup $R_t,\ t \geq 0$, in $L^2(H;\mu)$, defined in §2. We recall that $\mathcal{A}$ is m-dissipative and that the domain of $\mathcal{A}$ is defined by (3.12). Then the graph norm of $\mathcal{A}$ can be defined as

$$\|\varphi\|^2_{D(\mathcal{A})} = \|\varphi\|^2_{W^{2,2}(H;\mu)} + \|(-A)^{-1/2}D\varphi\|^2_{L^2(H;\mu)},\ \varphi \in D(\mathcal{A}). \quad (4.1)$$

4.1 Relatively bounded perturbations

Let $F : H \to H$, be a Borel mapping such that

HYPOTHESIS 4.1. *$(-A)^{-1/2}F$ is bounded.*

We set

$$a = \text{sup ess}\ \{|(-A)^{-1/2}F(x)| :\ x \in H\}.$$

Now we define a mapping $\mathcal{F}$ in $L^2(H;\mu)$ by setting

$$\begin{cases} D(\mathcal{F}) = \left\{\varphi \in W^{1,2}(H;\mu) :\ (-A)^{1/2}D\varphi \in L^2(H;\mu)\right\} \\ \mathcal{F}\varphi(x) = \langle F(x), D\varphi(x)\rangle = -\langle(-A)^{-1/2}F(x), (-A)^{1/2}D\varphi(x)\rangle, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\ \varphi \in D(\mathcal{F}). \end{cases} \quad (4.2)$$

The following proposition concerns the operator $\mathcal{A} + \mathcal{F}$, defined in $D(\mathcal{A})$.

PROPOSITION 4.1. *Assume that Hypotheses* 1.1, 3.1, *and* 4.1 *hold, and let $\mathcal{F}$ be defined by* (4.2).

(i) If $a < 1$ then $\mathcal{A} + \mathcal{F}$ is m-dissipative in $L^2(H;\mu)$.

(ii) If $a = 1$ then $\mathcal{A} + \mathcal{F}$ is closable and its closure is m-dissipative in $L^2(H;\mu)$.

Proof. We first note that by (3.13) we have $D(\mathcal{F}) \subset D(\mathcal{A})$. Moreover for any $\varphi \in D(\mathcal{A})$ we have

$$\begin{aligned}
\|\mathcal{F}\varphi\|^2_{L^2(H;\mu)} &= \int_H |\langle F(x), D\varphi(x)\rangle|^2 \mu(dx) \\
&= \int_H |\langle (-A)^{-1/2}F(x), (-A)^{1/2}D\varphi(x)\rangle|^2 \mu(dx) \\
&\leq a^2 \int_H |(-A)^{1/2}D\varphi(x)|^2 \mu(dx) \leq a^2 \|\mathcal{A}\varphi\|^2_{L^2(H;\mu)}.
\end{aligned}$$

This implies that $\mathcal{F}$ is relatively bounded with respect to $\mathcal{A}$. By a well-known perturbation result, see e. g. [18], the conclusion follows. □

EXAMPLE 4.2. We take $H = L^2([0, 2\pi])$ and define a linear operator A on H by setting

$$\begin{cases} D(A) = \{x \in H^2(0, 2\pi) : \ x(0) = x(2\pi), \ D_\xi x(0) = D_\xi x(2\pi)\}, \\ Ax(\xi) = D^2_\xi x(\xi) - x(\xi), \ \xi \in [0, 2\pi], \ x \in D(A). \end{cases} \tag{4.3}$$

A is clearly self-adjoint and fulfills Hypothesis 1.1 with $M = 1$ and $\omega = 1$, and Hypothesis 3.1, since the eigenvectors of A are given by

$$e_k(\xi) = \frac{1}{2\pi}\, e^{ik\xi}, \ \xi \in [0, 2\pi], \ k \in \mathbb{Z}.$$

Let L be a positive number, and set

$$F(x)(\xi) = L\frac{d}{d\xi}\sin x(\xi), \ \xi \in [0, 2\pi]. \tag{4.4}$$

Then

$$(-A)^{1/2}F(x)(\xi) = L \sin x(\xi), \ \xi \in [0, 2\pi].$$

so that Hypothesis 4.1 holds. Thus by Proposition 4.1 it follows that if $L < 1$, then the operator $\mathcal{B}$:

$$\mathcal{B}\varphi(x)(\xi) := \mathcal{A}\varphi(x) + k\left\langle \frac{d}{d\xi}\sin x(\xi), D\varphi(x)\right\rangle, \ \varphi \in D(\mathcal{A})$$

is m-dissipative in $L^2(H;\mu)$, whereas if $L = 1$ then $\mathcal{B}$ is closable and its closure is m-dissipative in $L^2(H;\mu)$.

4.2 Perturbation by a potential

We are given a nonnegative Borel function $V : H \to \mathbb{R}$, and we define a mapping $\mathcal{V}$ in $L^2(H;\mu)$ by setting

$$\begin{cases} D(\mathcal{V}) = \{\varphi \in L^2(H;\mu) : \ V\varphi \in L^2(H;\mu)\} \\ \\ \mathcal{V}\varphi(x) = -V(x)\varphi(x), \ \forall\, \varphi \in D(\mathcal{V}). \end{cases} \tag{4.5}$$

Next proposition concerns the operator $\mathcal{A} + \mathcal{V}$ with domain $D(\mathcal{A})$.

PROPOSITION 4.3. *Let $\mathcal{V}$ be defined by* (4.5), *and assume that there are numbers $a > 0$ and $\beta \in [0,1[$ such that*

$$V(x) \le a|x|^{1+\beta}, \ x \in H. \tag{4.6}$$

Then $\mathcal{A} + \mathcal{V}$, is self-adjoint in $L^2(H;\mu)$.

Proof. Let $\varepsilon > 0$ to be chosen later, and let $C(\varepsilon,\beta) > 0$ such that

$$a^2|x|^{2+2\beta} \le \varepsilon|x|^4 + C(\varepsilon,\beta), \ x \in H.$$

Let $\varphi \in D(\mathcal{A})$, then we have

$$\int_H V^2(x)\varphi^2(x)\mu(dx) \le \varepsilon \int_H |x|^4\varphi^2(x)\mu(dx) + C(\varepsilon,\beta)\int_H \varphi^2(x)\mu(dx).$$

Consequently, in view of Proposition 2.7, we have $\varphi \in D(\mathcal{V})$ and

$$\begin{aligned} &\int_H V^2(x)\varphi^2(x)\mu(dx) \le \\ &\quad [32\varepsilon \ \mathrm{Tr}\ Q^2 + \varepsilon(1 + 2\ \mathrm{Tr}\ Q)^2 + C(\varepsilon,\beta)] \int_H \varphi^2(x)\mu(dx) + \\ &\quad + \varepsilon(48\ \mathrm{Tr}\ [Q^2](1 + 2\ \mathrm{Tr}\ Q) + 512\ (\mathrm{Tr}\ [Q^2])^2)\|\mathcal{A}\varphi\|^2_{L^2(\mu;H)}. \end{aligned}$$

So, by choosing ε sufficiently small, we see that $\mathcal{V}$ is relatively bounded with respect to $\mathcal{A}$, and the conclusion follows by the quoted result in [18]. □

REMARK 4.4. If (4.6) is fulfilled with $\beta = 1$, then the argument above works with a sufficiently small.

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