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Imaginary Powers and Interpolation Spaces of Unbounded Operators

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SUMMARY. - *We show that there exists an operator A of type $(0,2)$ with domain $D(A)$ on the Hilbert space L^2, such that A^{is} is bounded for each $s \in \mathbb{R}$ and that for every $0 < \theta < 1$, we have $(D(A), L^2)_\theta \neq (D(A^*), L^2)_\theta$.*

1. Introduction

Let H be a Hilbert space and let A, B be two closed operators on H with domain $D(A)$ and $D(B)$, positive, resolvent commuting and of respective types (w_A, M_A) and (w_B, M_B) such that $w_A + w_B < \pi$. It is shown by G. Da Prato and P. Grisvard [D-G] that in this context, the sum $A + B$ is closable. However, J.B. Baillon and Ph. Clément [B-C] have constructed such a pair (A, B), such that the sum $A + B$ is not closed.

G. Da Prato and P. Grisvard in 1975 (see [D-G], Th. 3.14) were interested in finding a condition to ensure that the sum $A + B$ is closed. They proved that if the interpolation spaces $(D(A), H)_\theta$ and $(D(A^*), H)_\theta$ coincide for some $0 < \theta < 1$ (where A^* is the adjoint of A), then the sum $A + B$ is closed.

Later, in [D-V] Remark 2.11, G. Dore and A. Venni found another condition to ensure the closedness of $A + B$, namely that the

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imaginary powers of A, A^{is} are bounded on H for all $s \in \mathbb{R}$ and $s \to A^{is}$ is a strongly continuous group on $\mathbb{R}$. It was shown later in [V] that the second condition boundedness is in fact a consequence of the first one strong continuity.

G. Dore and A. Venni extended this result in [D-V], Th. 2.1, to the case where X is a UMD space, (see [Bo] and [Bu] for a definition), under the stronger condition that the imaginary powers of B are also bounded and there exist positive numbers θ_A , θ_B and C so that $\|A^{is}\| \leq Ce^{\theta_A|s|}$, $\|B^{is}\| \leq Ce^{\theta_B|s|}$ for every $s \in \mathbb{R}$ and $\theta_A + \theta_B < \pi$.

This last result was particularly interesting because it solved positively in this context the question of the L_p-regularity of the X-valued Cauchy problem

$$\begin{cases} u'(t) + Au(t) = f(t) \\ u(0) = 0, \end{cases}$$

for $1 < p < \infty$ and $f \in L_p(X)$, (which is still open in its full generality):

If X is a UMD space, A is a positive operator on X such that $\|A^{is}\| \leq Ce^{w_A|s|}$ for every $s \in \mathbb{R}$ with $w_A < \pi/2$, then there exists a constant $C' > 0$ such that:

$$\forall f \in L_p([0,T];X) =: L_p(X)\,, \|Au\|_{L_p(X)} \leq C'\|f\|_{L_p(X)}.$$

Indeed, denote by $\frac{d}{dt}$ the closed operator on $L_p(X)$ with domain

$$W_0^{1,p}(X) = \{u \in L_p(X) : u' \in L_p(X), u(0) = 0\}$$

such that $\frac{d}{dt}(u) = u'$.

$\frac{d}{dt}$ is a positive operator, for every $s \in \mathbb{R}$, $(\frac{d}{dt})^{is}$ is bounded and there exists a constant $M > 0$, such that $\|(\frac{d}{dt})^{is}\| \leq M(1+s^2)e^{\pi|s|/2}$ for every $s \in \mathbb{R}$.

If $D(A)$ is the domain of A, let us also still denote by A, the operator on $L_p(X)$ with domain $L_p(D(A)) := L_p([0,T];D(A))$ defined by $A(u)(t) = A(u(t))$. It is easy to see that A is a positive operator on $L_p(X)$ which verifies again the condition $\|A^{is}\| \leq Ce^{w_A|s|}$ for every $s \in \mathbb{R}$ with the same $w_A < \pi/2$

To prove the L_p-regularity of the X-valued Cauchy problem, it was known before that it is sufficient to prove that the sum of the two operators $\frac{d}{dt}$ and A is closed on $L_p(X)$ (see [D-G]). This is clearly a consequence of G.Dore and A.Venni's result: indeed, since A acts on X and $\frac{d}{dt}$ acts on the variable $t \in [0,T]$, these two operators are automatically resolvent commuting . Thus $\frac{d}{dt}$ and A verify the G.Dore and A.Venni's condition, $\frac{d}{dt}+A$ is closed and so the X-valued Cauchy problem is L^p-regular.

A natural question was then to compare G. Da Prato and P. Grisvard's condition and that given by G. Dore and A. Venni, to ensure that $A + B$ is closed in a Hilbert space.

A. Yagi [Y] proved the following result:

THEOREM (YAGI). *If* $(D(A), H)_\theta = (D(A^*), H)_\theta$ *for some* $0 < \theta < 1$ *then* A^{is} *is bounded for every* $s \in \mathbb{R}$.

Thus G. Dore and A. Venni's result is stronger than G. Da Prato and P. Grisvard's one.

In this paper, we exhibit an example which shows that the converse of Yagi's theorem is false. Then, in the case of Hilbert spaces, G. Dore and A. Venni's result is strictly stronger than that given by G. Da Prato and P. Grisvard.

2. The Example

Let us first recall some notions. Let $(X, \|.\|)$ be a complex Banach space, and let $A : D(A) \subset X \to X$ be a closed and densely defined operator with domain $D(A)$, we denote the resolvent set of A by $\rho(A)$ and its spectrum by $\sigma(A)$.

The operator A is called positive (see [D-V] [T]) if

(i) $(-\infty, 0] \subset \rho(A)$,

(ii) there exists $M \geq 1$ such that

$$\|(t+A)^{-1}\| \leq \frac{M}{1+t} \text{ , for every } t \geq 0.$$

For $\theta \in [0,\pi)$, we define the sector Σ_θ by

$$\Sigma_\theta = \{z \in \mathbb{C} \setminus \{0\} : \ |arg(z)| \leq \theta\}.$$

The positive operator A is said to be of type (ω, M) (see [Ta]), if there exists $0 \leq \omega < \pi$ and $M \geq 1$ such that

(i) $\sigma(A) \subset \Sigma_\omega$;

(ii) for every $\theta \in (\omega, \pi]$, there exists $M(\theta) \geq 1$ with $M(0) = M$, such that:

$$\forall z \in \mathbb{C} \ , \ |arg(z)| \geq \theta \Rightarrow \|(A-z)^{-1}\| \leq \frac{M(\theta)}{1+|z|}.$$

It is known that every positive operator is of type (ω, M) for some $0 \leq \omega < \pi$ and $M \geq 1$ [K].

If A is a positive operator and if there exists $0 \leq \omega < \pi$ and $C \geq 1$ so that A^{is} is bounded and $\|A^{is}\| \leq Ce^{\omega|s|}$ for every $s \in \mathbb{R}$, then A is of type (ω, M) for some $M \geq 1$ and for the same ω [D-V].

Two positive operators A and B are called resolvent commuting if

$$\forall \lambda \in \rho(A), \forall \mu \in \rho(B) \ , \ (\lambda - A)^{-1}(\mu - B)^{-1} = (\mu - B)^{-1}(\lambda - A)^{-1}.$$

If A is a positive operator, then the complex power of A denoted by A^z with $|z| < 1$ is defined as the closure of the operator I^z, where

$$I^z(x) = \frac{sin\pi z}{\pi}\{z^{-1}x - (I+A)^{-1}A^{-1}(x) + \int_0^1 t^{z+1}(A+t)^{-1}A^{-1}(x)dt$$

$$+ \int_1^{+\infty} t^{z-1}(t+A)^{-1}A(x)dt\} \ \text{ for } z \neq 0, \ x \in D(A) \cap R(A)$$

$$I^0(x) = x, \ x \in X,$$

where $R(A) = \{Ax : x \in D(A)\}$. For more informations about the complex powers of unbounded operators, we refer to [T] and [D-V].

The interpolation spaces that we are using here is the complex interpolation which can be found in [B-L].

We shall need the following known result:

PROPOSITION. *Let (E, F) and (E', F) be two interpolation couples and suppose that for some $0 < \theta < 1$, we have $(E, F)_\theta = (E', F)_\theta$, then for all $\theta \leq \alpha < 1$, $(E, F)_\alpha = (E', F)_\alpha$.*

We shall also use the following result due to A. Yagi:

THEOREM 1. *Let H be a Hilbert space, $A : D(A) \subset H \to H$ be a positive operator and let $0 < r < 1$, then the following two statements are equivalent:*

1. $(D(A), H)_\theta = (D(A^*), H)_\theta$ for every $r < \theta < 1$.
2. $D(A^\theta) = D(A^{*\theta})$ for every $0 < \theta < 1 - r$.

Let us now state that the converse of Yagi's theorem, stated in the introduction, is false:

THEOREM 2. *There exists a positive operator $A : D(A) \subset L^2 \to L^2$ of type $(0, 2)$, such that for every $s \in \mathbb{R}$, the operator A^{is} is bounded, and for every $0 < \theta < 1$, $(D(A), L^2)_\theta \neq (D(A^*), L^2)_\theta$.*

Proof of Theorem 2. Let γ be the normalized Lebesgue measure on $[0, 2\pi]$ and let ϕ be the function on $[0, 2\pi]$ defined by

$$\phi(t) = \begin{cases} 1 & \text{if } t \in [0, \pi] \\ 2 & \text{if } t \in (\pi, 2\pi]. \end{cases}$$

Define for each $n \in \mathbb{Z}$ and $t \in [0, 2\pi]$,

$$f_n(t) = \phi(t)e^{int} \text{ and } f_n^*(t) = \phi(t)^{-1}e^{int},$$

It is easy to see that $(f_n)_{n\in\mathbb{Z}}$ and $(f_n^*)_{n\in\mathbb{Z}}$ are unconditional basis of $L^2([0, 2\pi]; d\gamma)$, $(f_n^*)_{n\in\mathbb{Z}}$ being the dual basis of $(f_n)_{n\in\mathbb{Z}}$ and

$$(\sum_{n\in\mathbb{Z}} |c_n|^2)^{1/2} \leq \| \sum_{n\in\mathbb{Z}} c_n f_n \|_2 \leq 2(\sum_{n\in\mathbb{Z}} |c_n|^2)^{1/2} \tag{$*$}$$

$$\frac{1}{2}(\sum_{n\in\mathbb{Z}} |c_n|^2)^{1/2} \leq \| \sum_{n\in\mathbb{Z}} c_n f_n^* \|_2 \leq (\sum_{n\in\mathbb{Z}} |c_n|^2)^{1/2}. \tag{$**$}$$

The above inequalities imply in particular that

$$L^2([0, 2\pi]; d\gamma) = \{\sum_{n\in\mathbb{Z}} c_n f_n : \sum_{n\in\mathbb{Z}} |c_n|^2 < \infty\}$$

$$= \{\sum_{n\in\mathbb{Z}} c_n f_n^* : \sum_{n\in\mathbb{Z}} |c_n|^2 < \infty\}.$$

Let A be the unbounded operator on $L^2([0,2\pi]; d\gamma)$ defined by

$$D(A) = \Big\{ \sum_{n\in\mathbb{Z}} c_n f_n : \sum_{n\in\mathbb{Z}} e^{2|n|}|c_n|^2 < \infty \Big\} \quad \text{and}$$
$$A\Big(\sum_{n\in\mathbb{Z}} c_n f_n\Big) = \sum_{n\in\mathbb{Z}} e^{|n|} c_n f_n.$$

The adjoint operator A^* of A is given by

$$D(A^*) = \Big\{ \sum_{n\in\mathbb{Z}} d_n f_n^* : \sum_{n\in\mathbb{Z}} e^{2|n|}|d_n|^2 < \infty \Big\} \quad \text{and}$$
$$A^*\Big(\sum_{n\in\mathbb{Z}} d_n f_n^*\Big) = \sum_{n\in\mathbb{Z}} e^{|n|} d_n f_n^*.$$

Indeed, $D(A^*)$ is the set of all $g = \sum_{n\in\mathbb{Z}} d_n f_n^* \in L^2$ such that the linear functional defined on L^2 by $f \to < Af, g >$ is continuous.

If $f = \sum_{n\in\mathbb{Z}} c_n f_n$, we have

$$< Af, g >= \sum_{n\in\mathbb{Z}} e^{|n|} c_n \overline{d_n},$$

so $g = \sum_{n\in\mathbb{Z}} d_n f_n^* \in D(A^*)$ if and only if $\sum_{n\in\mathbb{Z}} e^{2|n|}|d_n|^2 < \infty$.

Moreover, by $< Af, g >=< f, A^*g >$, we have

$$A^*(\sum_{n\in\mathbb{Z}} d_n f_n^*) = \sum_{n\in\mathbb{Z}} e^{|n|} d_n f_n^*,$$

for $\sum_{n\in\mathbb{Z}} d_n f_n^* \in D(A^*)$.

It is easy to verify that A is closed and densely defined. Let us first show that A is of type $(0,2)$. For every $t \geq 0$,

$$(t + A)^{-1}(\sum_{n\in\mathbb{Z}} c_n f_n) = \sum_{n\in\mathbb{Z}} \frac{c_n}{e^{|n|} + t} f_n,$$

so $(-\infty, 0] \subset \rho(A)$ and $\|(t+A)^{-1}\| \leq \dfrac{2}{1+t}$ by the inequality (*). A is therefore a positive operator.

For every $\theta \in (0,\pi]$ and $z \in \mathbb{C}, |arg(z)| \geq \theta$, we have

$$(A-z)^{-1}(\sum_{n\in\mathbb{Z}} c_n f_n) = \sum_{n\in\mathbb{Z}} \frac{c_n}{e^{|n|}-z} f_n,$$

and so

$$\|(A-z)^{-1}\| \leq 2/\inf_{n\in\mathbb{Z}} |e^{|n|}-z|$$

by the inequality (*).

To show that A is of type $(0,2)$, it will suffice to show that for every $\theta \in (0,\pi]$ there exists $M(\theta) > 0$, such that for all $z \neq 0, |arg(z)| \geq \theta$, we have

$$\inf_{n\in\mathbb{Z}} |e^{|n|}-z| \geq M(\theta)(1+|z|)$$

This inequality is trivially verified with the constant $M(\theta) = 1/3$ when $|z| < 1/2$.

Suppose that $|z| \geq 1/2$ and $|arg(z)| \geq \pi/2$, we have $Re(z) \leq 0$, hence

$$\inf_{n\in\mathbb{Z}} |e^{|n|}-z| \geq 1+|z|.$$

If $|z| \geq 1/2$ and $\theta \leq |arg(z)| < \pi/2, z = a+ib$, then $|a| \leq |tg(\theta)|^{-1}|b|$, and so

$$|z| \leq (1+|tg(\theta)|^{-1})|b| \leq (1+|tg(\theta)|^{-1}) \inf_{n\in\mathbb{Z}} |e^{|n|}-z|,$$

but for $|z| \geq 1/2$, one has $1+|z| \leq 3|z|$, hence

$$3(1+|tg(\theta)|^{-1}) \inf_{n\in\mathbb{Z}} |e^{|n|}-z| \geq 1+|z|.$$

A is therefore a positive operator of type $(0,2)$.

Let us now compute A^{is}. For every $s \in \mathbb{R}$, we have

$$A^{is}(\sum_{n\in\mathbb{Z}} c_n f_n) = \sum_{n\in\mathbb{Z}} e^{i|n|s} c_n f_n \qquad (***)$$

Indeed, it is easy to verify by the definition that for every $n \in \mathbb{Z}, s \in \mathbb{R}$,

$$A^{is}(cf_n) = e^{|n|i} c f_n, \qquad c \in \mathbb{C}.$$

If we denote by L_0^2 the linear subspace of L^2 spanned by $(f_n)_{n\in\mathbb{Z}}$, for each $f = \sum_{n\in\mathbb{Z}} c_n f_n$ in L_0^2,

$$A^{is}(\sum_{n\in\mathbb{Z}} c_n f_n) = \sum_{n\in\mathbb{Z}} e^{|n|i} c_n f_n.$$

On the other hand, the linear operator on L^2 defined by

$$\sum_{n\in\mathbb{Z}} c_n f_n \to \sum_{n\in\mathbb{Z}} e^{|n|i} c_n f_n$$

is clearly closed, so to show that (***) is true, it will suffice to show that for each $f = \sum_{n\in\mathbb{Z}} c_n f_n \in L^2$, there exists a sequence $(g_n)_{n\in\mathbb{Z}}$ in L_0^2, such that $lim_{n\to\infty} g_n = f$ and $lim_{n\to\infty} A^{is}(g_n) = \sum_{n\in\mathbb{Z}} e^{|n|i} c_n f_n$. Obviously, we can take $g_n = \sum_{|k|\le n} c_k f_k$.

Thus, it is clear that $\|A^{is}\| \le 2$ by the inequality (*).

We shall show that for all $0 < \theta < 1$,

$$(D(A), L^2([0,2\pi]; d\gamma))_\theta \neq (D(A^*), L^2([0,2\pi]; d\gamma))_\theta.$$

By proposition and theorem 1, it suffices to show $D(A^\theta) \neq D(A^{*\theta})$ for all $0 < \theta < 1$.

Now let us fix $0 < \theta < 1$, then in a similar way as in (***), we have

$$D(A^\theta) = \Big\{ \sum_{n\in\mathbb{Z}} c_n f_n : \sum_{n\in\mathbb{Z}} e^{2|n|\theta} |c_n|^2 < \infty \Big\} \quad \text{and}$$

$$A^\theta\Big(\sum_{n\in\mathbb{Z}} c_n f_n\Big) = \sum_{n\in\mathbb{Z}} e^{|n|\theta} c_n f_n,$$

$$D(A^{*\theta}) = \Big\{ \sum_{n\in\mathbb{Z}} c_n f_n^* : \sum_{n\in\mathbb{Z}} e^{2|n|\theta} |c_n|^2 < \infty \Big\} \quad \text{and}$$

$$A^{*\theta}\Big(\sum_{n\in\mathbb{Z}} c_n f_n^*\Big) = \sum_{n\in\mathbb{Z}} e^{|n|\theta} c_n f_n^*.$$

By (*), we have

$$D(A^\theta) = \phi\{\sum_{n\in\mathbb{Z}} c_n e^{int} : \sum_{n\in\mathbb{Z}} e^{2|n|\theta}|c_n|^2 < \infty\}$$

and by (**), we have

$$D(A^{*\theta}) = \phi^{-1}\{\sum_{n\in\mathbb{Z}} c_n e^{int} : \sum_{n\in\mathbb{Z}} e^{2|n|\theta}|c_n|^2 < \infty\}.$$

Let $E_\theta = \{\sum_{n\in\mathbb{Z}} c_n e^{int} : \sum_{n\in\mathbb{Z}} e^{2|n|\theta}|c_n|^2 < \infty\} \subset L^2([0,2\pi];d\gamma)$, then

$$D(A^\theta) = \phi E_\theta \text{ and } D(A^{*\theta}) = \phi^{-1}E_\theta.$$

Assume that $D(A^\theta) = D(A^{*\theta})$, since the constant function 1 belongs to the subspace E_θ of $L^2([0,2\pi];d\gamma)$, the function ϕ^2 belongs to E_θ, but it is easy to compute that:

$$\phi(t)^2 = \frac{5}{2} + \sum_{n\in\mathbb{Z}} \frac{3i}{(2n+1)\pi} e^{i(2n+1)t}$$

which is not an element of E_θ.

This implies that for each $0 < \theta < 1$,

$$(D(A), L^2([0,2\pi];d\gamma))_\theta \neq (D(A^*), L^2([0,2\pi];d\gamma))_\theta.$$

A is then an operator of type $(0,2)$, such that A^{is} is bounded for all $s \in \mathbb{R}$ and for every $0 < \theta < 1, (D(A), L^2([0,2\pi];d\gamma))_\theta \neq (D(A^*), L^2([0,2\pi];d\gamma))_\theta$. This finishes the construction.

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