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Periodic Problems for Degenerate Differential Equations

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Summary. - *Extensions of the sum of operators' method of P. Grisvard are used for showing that the degenerate differential equation* $\frac{d}{dt}(Mu(t)) + Lu(t) = f(t)$, $0 \le t \le 1$, *admits one* 1*- periodic solution* u, *according to* $Mu(0) = Mu(1)$.
The parabolic case and the hyperbolic one are discussed as well. Some examples of application to ordinary differential equations and to partial differential equations are given.

Introduction

Let X be a complex Hilbert space and let A be the infinitesimal generator of the C_0 semigroup $\exp(tA)$ in X. In the paper [13], J. Prüss has given a very elegant proof that $1 \in \rho(e^A)$ if and only if $\{2\pi in\} \subset \rho(A)$ and $\sup\{\|(2\pi in - A)^{-1}\|_{L(X)},\ n \in \mathbb{Z}\} < +\infty$, see [13], Theorem 2, p. 850. He also showed that this is equivalent to the property that for all $f \in C([0,1];X)$, the space of X-valued continuous functions on $[0,1]$, the equation

$$\frac{du(t)}{dt} = u'(t) = Au(t) + f(t), \quad 0 \le t \le 1,$$

has precisely one 1-periodic mild solution u, according which $u \in C([0,1];X)$ and

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s)ds, \quad 0 \le t \le 1,$$

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for a certain element u_0 of X. See [13], Theorem 1, p. 849.

The main purpose of this paper consists in extending some results from [13] to the possibly degenerate equation of the type

$$\frac{d}{dt}(Mu(t)) = -Lu(t) + f(t), \quad 0 \leq t \leq 1, \tag{1}$$

where L, M are two closed linear operators from the complex Banach space X into itself, the domain $D(L)$ of L is continuously embedded in $D(M)$ and L has a bounded inverse. We shall associate to (1) the periodicity condition

$$(Mu)(0) = (Mu)(1). \tag{2}$$

One could transform (1), (2) into the multivalued equation

$$v'(t) + Av(t) \ni f(t), \quad 0 \leq t \leq 1, \tag{3}$$

together with

$$v(0) = v(1), \tag{4}$$

where this time $A = LM^{-1}$, $D(A) = M(D(L))$, as done in Favini and Yagi [7] for the parabolic case, and in Yagi [16] for the hyperbolic one, related to the Cauchy problem, and the problem should be reduced to establish that $1 \in \rho(\exp(-A))$. Now, except in a particular (although very interesting) situation, that shall be discussed in section 2, it seems a very hard task to demonstrate that for general multivalued linear operators A an identity like $\sigma(e^{-A}) \setminus \{0\} = \exp(\sigma(-A))$ holds. Moreover, if the data are not sufficiently smooth, "weak" solutions to (3), (4) in some sense corresponding to the non degenerate situation, may be missing at all. The variation of constants formula sometimes available for the solution v of (3), (4) necessarily requires additional regularity to $f(t)$. In other words, in order to $v(0)$ and $v(1)$ to exist, we are compelled to seek for either strict solutions u: $u \in C([0,1]; D(L))$, $Mu \in C^1([0,1]; X)$, or classical solutions $u \in C((0,1]; D(L))$, $Mu \in C^1((0,1]; X)$, with $Mu \in C([0,1]; X)$. This justifies our choice to use a different approach mainly, related to the operational method of P. Grisvard [10], in treating the problem above.

The contents of the paper are as follows. Section 1 is devoted to the parabolic case and then the main result (Theorem 1.1) establishes precisely the time regularity of $f(t)$ guaranteeing that a strict 1-periodic solution u to (1), (2) exists. Many concrete examples of operators M, L verifying our hypotheses can be found in Favini and Yagi [7]. Section 2 is devoted to the best situation, the nearest one to the regular case, where $-A$ generates an analytic semigroup. Then classical solutions to (1), (2) are investigated without requiring any periodicity to $f(t)$. Section 3 concerns the highly degenerate case. Some extensions of well-known results, to be found, a.e., in Haraux [11], are obtained. In particular, we discuss the interesting problem where $z = 0$ is an isolated singularity for the resolvent $(zL + M)^{-1}$. In this peculiar case we shall indicate how to treat the problem in presence of the resonance phenomenon too.

The resonance problem in other situations shall be treated elsewhere.

1. The parabolic case

To begin with, we shortly recall the abstract result in Favini and Yagi [7, 8, 9] concerning the unique solvability and maximal regularity of solutions u of the equation

$$BMu + Lu = h, \tag{E}$$

where h is a given element of the complex Banach space E and

(i) *B is a closed linear operator from E into itself satisfying*

$$\|(B - z)^{-1}\|_{L(E)} \leq C(1 + |\mathfrak{Re}\, z|)^{-1}$$

for all complex numbers z such that $\mathfrak{Re}\, z \leq a_0$, where $a_0 > 0$.

(ii) *M, L are closed linear operators from E into E, $D(L) \subseteq D(M)$, L has a bounded inverse, and*

$$\|M(zM + L)^{-1}\|_{L(E)} \leq C(1 + |z|)^{-\beta}$$

for all $z \in \Sigma_\alpha$, where

$$\Sigma_\alpha = \{z \in \mathbb{C};\ \mathfrak{Re}\, z \geq -c(1 + |\mathfrak{Im}\, z|)^\alpha\}\ ,$$

c being a positive constant and $0 < \beta \leq \alpha \leq 1$.

(iii) *Letting* $T = ML^{-1}$, *then* $B^{-1}T = TB^{-1}$, *or equivalently,* $TBu = BTu$ *for all* $u \in D(B)$.

Denote by $(E, D(B))_{\theta,\infty}$, $0 < \theta < 1$, the real interpolation space between E and $D(B)$, as described in Da Prato and Grisvard [3]. Then one has (Favini and Yagi [7], [9])

PROPOSITION 1.1. *Let us assume* **(i)–(iii)** *and* $2\alpha + \beta > 2$. *If*

$$f \in (E, D(B))_{\theta,\infty} ,$$

with $\frac{2-\alpha-\beta}{\alpha} < \theta < 1$, *then* (E) *has a unique solution* u. *Moreover,*

$$Lu, BMu \in (E, D(B))_{\omega,\infty} ,$$

where $\omega = \alpha\theta + \alpha + \beta - 2$.

The next result will show how Proposition 1.1 allows to solve the periodic problem (1), (2) in the non resonance case. To this end we introduce the notation as follows.

If X is a complex Banach space with norm $\| \ \|_X$, let us define

$$\begin{aligned} E_1 &= C([0,1];X), \text{ endowed with the supremum-norm} \\ E_2 &= C_\pi([0,1];X) = \{u \in E_1;\ u(0) = u(1)\} , \\ & \qquad \|u\|_{E_2} = \|u\|_{E_1},\ u \in E_2, \\ E_3 &= L^p(0,1;X), \\ & \qquad \|u\|_p = (\textstyle\int_0^1 \|u(t)\|_X^p dt)^{1/p},\ 1 < p < \infty. \end{aligned}$$

The operators B_i, $i = 1, 2, 3$, are given by means of

$$\begin{aligned} D(B_1) &= \{u \in C^1([0,1];X) \ ; \quad u(0) = u(1)\} , \\ B_1 u &= du/dt , \quad u \in D(B_1) \\ D(B_2) &= \{u \in C^1([0,1];X) \ ; \quad u(0) = u(1) ,\ u'(0) = u'(1)\} , \\ B_2 u &= du/dt , \quad u \in D(B_2) \\ D(B_3) &= \{u \in W^{1,p}(0,1;X) \ ; \quad u(0) = u(1)\} , \\ B_3 u &= du/dt , \quad u \in D(B_3) \end{aligned}$$

where $C^k([0,1];X)$, $k=1,2,\dots$, denotes the space of all k-times continuously differentiable X-valued functions on $[0,1]$ and $W^{1,p}(0,1;X)$ is the usual Sobolev space, see Barbu [1], p. 18.

It is an easy matter to recognize that B_k, $k=1,2,3$, satisfies

$$\|(B_k - z)^{-1}\|_{L(E)} \le |\,\mathfrak{Re}\, z|^{-1}, \mathfrak{Re}\, z < 0;$$

where E is any E_k.

Moreover, $-B_2$ and $-B_3$ generate two contraction semigroups in E_2 and in E_3, respectively. See Da Prato [2], Theorem 2.2, p. 195. This does not hold for B_1 for its domain is not everywhere dense in E_1.

Therefore, for given $k>0$, $B=B_j+k$, $j=1,2,3$, satisfies assumption **(i)**. On the other hand, equation (1) is equivalently written

$$\left(\frac{d}{dt}+k\right)(Mu(t)) + (L-kM)u(t) = f(t)\,, \quad 0 \le t \le 1. \qquad (1.1)$$

It is also to be observed that if the closed linear operators L, M from X into itself satisfy **(ii)** with X instead of E and we identify L, M to the induced operators in E_j, $j=1,2,3$, in an obvious way, then **(ii)** holds for these operators. Since

$$\|M(zM+L-kM)^{-1}\|_{L(E)} \le C(1+|z|)^{-\beta}\ ,$$

for all $z\in\mathbb{C}$, $\mathfrak{Re}\, z \ge k - c(1+|\,\mathfrak{Im}\, z|)^{\alpha}$, let us fix $k=c/2$. Then $\mathfrak{Re}\, z \ge -\frac{c}{2}(1+|\,\mathfrak{Im}\, z|)^{\alpha}$ yields $\mathfrak{Re}\, z \ge k - c(1+|\,\mathfrak{Im}\, z|)^{\alpha}$, so that we are allowed to apply Proposition 1.1 to $B=B_j+k$.

It remains to characterize the interpolation spaces $(E,D(B))_{\theta,\infty}$, $0<\theta<1$, in the different cases. Our first result to this regard reads:

LEMMA 1.2. *We have*

$$\begin{aligned}(E_2,D(B_2))_{\theta,\infty} &= \{f\in C^{\theta}([0,1];X); f(0)=f(1)\}\\ &= C^{\theta}_{\pi}([0,1];X)\ ,\quad 0<\theta<1.\end{aligned}$$

Here $C^{\theta}([0,1];X)$ denotes the space of all X-valued Hölder continuous functions on $[0,1]$, with exponent θ.

Proof. Since $-B_2 - k$, $k > 0$, generates a strongly continuous semigroup of negative type in E_2, we know by Triebel [14], Theorem 1.13.5, pp. 86–87, that

$$\begin{aligned}(E_2, D(B_2))_{\theta,\infty} &= (E_2, D(B_2+k))_{\theta,\infty} \\ &= \Big\{ u \in C_\pi([0,1];X) \ ; \\ &\qquad \sup_{t>0} \sup_{0\le s\le 1} \frac{\|e^{-kt}u(s-t) - u(s)\|_X}{t^\theta} < \infty \Big\}.\end{aligned}$$

Now,

$$\begin{aligned} t^{-\theta}\|e^{-kt}u(s-t) - u(s)\|_X &\le \\ &\le e^{-kt}t^{-\theta}\|u(s-t)-u(s)\|_X + \frac{1-e^{-kt}}{t^\theta}\|u(s)\|_X \\ &\le \sup_{t>0}\sup_{0\le s\le 1}\frac{\|u(s-t)-u(s)\|_X}{t^\theta} + \\ &\quad + \sup_{t>0}\frac{1-e^{-kt}}{t^\theta}\sup_{0\le s\le 1}\|u(s)\|_X \\ &\le C\|u\|_{C^\theta} \end{aligned}$$

implies that $C^\theta_\pi([0,1];X) \subseteq (E_2, D(B_2))_{\theta,\infty}$.

Conversely,

$$\begin{aligned} t^{-\theta}\|u(s-t)-u(s)\|_X &\le \\ &\le \frac{1-e^{-kt}}{t^\theta}\|u(s-t)\|_X + \frac{\|e^{-kt}u(s-t)-u(s)\|_X}{t^\theta} \end{aligned}$$

implies

$$\begin{aligned} \sup_{t>0}\sup_{0\le s\le 1}\frac{\|u(s-t)-u(s)\|_X}{t^\theta} &\le \\ &\le \sup_{t>0}\frac{1-e^{-kt}}{t^\theta}\sup_{0\le s\le 1}\|u(s)\|_X + \\ &\quad + \sup_{t>0}\sup_{0\le s\le 1}\frac{\|e^{-kt}u(s-t)-u(s)\|_X}{t^\theta} \\ &\le C\|u\|_{(E_2,D(B_2))_{\theta,\infty}}. \end{aligned}$$

This concludes the proof. □

We are then allowed to establish the following statement.

THEOREM 1.1. *Let us suppose that L, M satisfy* **(ii)** *in the space X. If $2\alpha+\beta>2$ and $f\in C_\pi^\theta([0,1];X)$, with $\frac{2-\alpha-\beta}{\alpha}<\theta<1$, then (1), (2) has a unique strict solution u such that $Lu\in C_\pi^\omega([0,1];X)$, where $\omega=\alpha\theta+\alpha+\beta-2$.*

One could suppose that if we start with a larger space like E_1 above, then the periodicity condition on $f(t)$ is no longer necessary. On the contrary, this does not hold, as we show in the following theorem.

THEOREM 1.2. $(E_1,D(B_1))_{\theta,\infty}=C_\pi^\theta([0,1];X)$, $0<\theta<1$.

Proof. We recall that $D(B_1)$ coincides with the space $C_\pi^1([0,1];X)$ of all 1-periodic X-valued continuously differentiable functions on $[0,1]$. Define

$$\begin{aligned} F &= \tilde{C}_0([0,1];X)=\{f\in E_1\,;\ f(0)=f(1)=0\}\,,\\ G &= \tilde{C}_0^1([0,1];X)=\{f\in C^1([0,1];X)\,;\ f(0)=f(1)=0\}\,. \end{aligned}$$

Then we know by Lunardi [12], Corollary 1.2.19, p. 19 and Proposition 0.2.2, p. 5, (see also p. 108) that

$$\begin{aligned} (F,G)_{\theta,\infty}=(E_1,G)_{\theta,\infty} &= \tilde{C}_0^\theta([0,1];X)\\ &= \{f\in C^\theta([0,1];X);f(0)=f(1)=0\}. \end{aligned}$$

On the other hand, the mapping T given by

$$Tf=(g,f(0))\,,\ g(t)=f(t)-f(0)\,,\ 0\le t\le 1,$$

is an isomorphism from $C_\pi([0,1];X)$ onto $\tilde{C}_0([0,1];X)\times X$ and from $C_\pi^1([0,1];X)$ onto $G\times X$. Hence, by interpolation, T is an isomorphism from $(C_\pi([0,1];X),C_\pi^1([0,1];X))_{\theta,\infty}$ onto $\tilde{C}_0^\theta([0,1];X)\times X$. Therefore, since it is well known that for general Banach spaces X_0, X_1, with continuous embedding $X_1\subseteq X_0$, $(X_0,X_1)_{\theta,\infty}$ is embedded into the closure of X_1 in X_0, we deduce that

$$\begin{aligned} C_\pi^\theta([0,1];X) &= (C_\pi([0,1];X),C_\pi^1([0,1];X))_{\theta,\infty}\\ &= (C([0,1];X),C_\pi^1([0,1];X))_{\theta,\infty}. \end{aligned}$$

This finishes the proof. □

THEOREM 1.3. *Let L, M satisfy assumption* (ii) *in X with $2\alpha+\beta > 2$. Let $f \in W_\pi^{\theta,p}(0,1;X)$, $\max\{1/p, (2-\alpha-\beta)/\alpha\} < \theta < 1$, where*

$$W_\pi^{\theta,p}(0,1;X) = \Big\{ u \in L^p(0,1;X) \ ; \quad u(0) = u(1)\,, \\ \int_0^1 \int_0^1 \|u(t)-u(s)\|_X^p |t-s|^{-(1+\theta p)}\, dt\, ds < \infty \Big\}.$$

Then problem (1),(2) has a unique strict solution u such that $\frac{d}{dt}(Mu)$ belongs to $W_\pi^{\omega,p}(0,1;X)$, where $\omega = \alpha\theta + \alpha + \beta - 2$.

Proof. Since $-\frac{d}{dt}$, with periodic boundary conditions, generates a bounded strongly continuous semigroup of operators in $L^p(0,1;X) = E_3$, (see Da Prato [2], Theorem 2.2, p. 195), with domain $D(B_3)$, the space $(E_3, D(B_3))_{\theta,\infty}$ coincides with the subspace of E_3 consisting of all u such that $u(0) = u(1)$ and

$$\int_0^1 \int_0^1 \frac{\|u(t)-u(s)\|_X^p}{|t-s|^{1+\theta p}} ds\, dt < \infty.$$

See Da Prato ang Grisvard [3], p. 383, too.

Then the affirmation follows directly from Proposition 1.1. □

Theorem 1.1 can be extended to multivalued problems of the type (3), (4) in virtue of the following abstract generalization to Theorem 3.2 in Favini and Yagi [7], p. 363.

THEOREM 1.4. *Let us assume* (i) *and suppose A to be a multivalued linear operator from E into itself such that*

$$\|(z+A)^{-1}\|_{L(E)} \le C(1+|z|)^{-\beta}, \ \forall z \in \Sigma_\alpha, \\ B^{-1}A^{-1} = A^{-1}B^{-1}.$$

If $0 < \beta \le \alpha \le 1$, $2\alpha + \beta > 2$, then for all $f \in (E, D(B))_{\theta,\infty}$, $(2-\alpha-\beta)/\alpha < \theta < 1$, the equation $Bu + Au \ni f$ has a unique solution u with $Bu \in (E, D(B))_{\omega,\infty}$, where $\omega = \alpha\theta + \alpha + \beta - 2$.

Proof. Since $0 \in \rho(A)$, $Au \ni v$ has a unique solution $u = A^{-1}v$. Therefore, if we denote A^{-1} by T, our equation $Bu + Au \ni f$ reads equivalently $BTv + v = f$.

Let $\Gamma : z = a - \frac{c}{2}(1+|y|)^\alpha + iy$, $-\infty < y < +\infty$, and define

$$v = \frac{1}{2\pi i}\int_\Gamma z^{-1}(zT+I)^{-1}B(B-z)^{-1}f\ dz,$$

where $f \in V_\theta = (E, D(B))_{\theta,\infty}$. The above integral is well defined, in view of the estimate

$$|z|^\theta \|B(B-z)^{-1}f\|_E \le C(1+|z|)^{(1-\alpha)(1+\theta)}\|f\|_{V_\theta}.$$

Therefore we have

$$\|v\|_E \le C\int_\Gamma |z|^{-(1+\theta)}(1+|z|)^{1-\beta}(1+|y|)^{(1-\alpha)(1+\theta)}\ |dz|\ \|f\|_{V_\theta}\ ,$$

where the last integral converges in view of the assumption on the parameter θ.

One also proves, referring to the monograph by Favini and Yagi [9] for a detailed check, that $BTv + v = f$ and $BTv \in V_\omega$. On the other hand, these conclusions say nothing else that $f - Bu \in Au$ and $Bu \in V_\omega$, as declared. □

THEOREM 1.5. *Let A be a multivalued linear operator from the complex Banach space X into itself satisfying*

$$\|(z+A)^{-1}\|_{L(X)} \le C(1+|z|)^{-\beta}\ , \quad \forall z \in \Sigma_\alpha.$$

If $0 < \beta \le \alpha \le 1$, $2\alpha + \beta > 2$, then for all $f \in C^\theta_\pi([0,1];X)$, with $\frac{2-\alpha-\beta}{\alpha} < \theta < 1$, the periodic problem

$$\frac{du(t)}{dt} + Au(t) \ni f(t)\ , \quad 0 \le t \le 1,$$
$$u(0) = u(1),$$

has a unique strict solution u with regularity $\frac{du}{dt} \in C^\omega_\pi([0,1];X)$, $\omega = \alpha\theta + \alpha + \beta - 2$.

Proof. It only needs to define the operator B by $D(B) = D(B_2)$, $B = B_2 + \epsilon I$, $\epsilon > 0$, $E = E_2 = C_\pi([0,1]; X)$; instead of A, we consider $A - \epsilon I$, where A is the operator in $C_\pi([0,1]; X)$ with domain $C_\pi([0,1]; D(A))$ induced by the given operator A in a natural way: $D(A)$ is endowed with the graph norm in the product space $X \times X$, so that it becomes a Banach space.

If ϵ is suitably small, Theorem 1.4 applies immediately. □

EXAMPLE 1.1. Let us consider the periodic boundary value problem

$$\frac{\partial(m(x)u)}{\partial t} - \Delta u = f(t,x) \quad \text{in } [0,1] \times \Omega, \tag{1.2}$$

$$u = 0 \quad \text{in } [0,1] \times \partial\Omega, \tag{1.3}$$

$$m(x)u(0,x) = m(x)u(1,x) \quad \text{in } \Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $m(x) \geq 0$ is a given measurable bounded function on Ω, f is a function on $[0,1] \times \Omega$, and $u(t,x)$ is the unknown. The initial value problem $m(x)u(0,x) = v_0$ relative to (1.2), (1.3) has been studied in Favini and Yagi [7], [8] both in the spaces $H^{-1}(\Omega)$, $L^2(\Omega)$ and in $L^p(\Omega)$, $p > 1$. Operators L and M are defined correspondingly in obvious way.

If we take $X = H^{-1}(\Omega)$, then **(ii)** holds with $\alpha = \beta = 1$, but if $X = L^2(\Omega)$, then $\alpha = 1$, $\beta = 1/2$. See Favini and Yagi [7], p. 378. Hence all our results apply immediately to (1.2)–(1.4), too.

EXAMPLE 1.2. Consider, for $t \in [0,1]$ and $x \in [0,\pi]$, the problem

$$\frac{\partial}{\partial t}(\frac{\partial^2}{\partial x^2} + 1)u(t,x) = -a\frac{\partial^2}{\partial x^2}u(t,x) - ku(t,x) + f(t,x), \tag{1.5}$$

$$u(t,0) = u(t,\pi) = \frac{\partial^2}{\partial x^2}u(t,0) = \frac{\partial^2}{\partial x^2}u(t,\pi) = 0, \tag{1.6}$$

$$(\frac{\partial^2}{\partial x^2} + 1)u(0,x) = (\frac{\partial^2}{\partial x^2} + 1)u(1,x), \tag{1.7}$$

where a is a positive constant and k is a real number to be precised later.

If we take $X = C_0([0,\pi]) = \{u \in C([0,\pi]) : u(0) = u(\pi)\}$, K is the realization of $\frac{\partial^2}{\partial x^2}$ with domain

$$D(K) = \{u \in C^2([0,\pi]);\ u(0) = u(\pi) = \frac{\partial^2}{\partial x^2}u(0) = \frac{\partial^2}{\partial x^2}u(\pi) = 0\}\ ;$$

then we take $M = K + I$, $L = aM + (k-a)I$. It is well known that the elements of $\sigma(M)$ are all simple eigenvalues $1 - s^2$, $s \in \mathbb{N}$, so that $zM + L$ has a bounded inverse provided that $M + \frac{k-a}{z+a}I$ has a bounded inverse, and this holds precisely if $|z+a| > \frac{|k-a|}{3}$.

This yields that if $\frac{|k-a|}{3} < a$, that is, $-2a < k < 4a$, then Theorem 1.1 applies with $\alpha = \beta = 1$; therefore for all

$$f \in C^\theta_\pi([0,1]; C_0([0,\pi])), \quad 0 < \theta < 1,$$

problem (1.5)–(1.7) has precisely one strict solution u with regularity

$$\partial^2 u/\partial x^2 \in C^\theta_\pi([0,1]; C_0([0,\pi])).$$

Extensions of the method to higher dimension are allowed by using the general theory elliptic operators in spaces of continuous functions to be found in Lunardi [12].

2. The case $\alpha = \beta = 1$

In this section we shall prove that if the linear operators L, M satisfy assumption **(ii)** in the complex Banach space X with $\alpha = \beta = 1$, then for all $f \in C^\theta([0,1]; X)$, not necessarily 1-periodic, the problem

$$\frac{d}{dt}(Mu(t)) = -Lu(t) + f(t), \quad 0 < t < 1, \tag{5}$$

has precisely one classical solution

$$u \in C((0,1); D(L))\ , \quad Mu \in C([0,1]; X) \cap C^1((0,1); X)$$

satisfying (2). Indeed, we prove the following statement.

THEOREM 2.1. *Let X be a reflexive Banach space and let L, M be two closed linear operators from X into itself such that*

$$\|M(zM+L)^{-1}\|_{L(X)} \leq C(1+|z|)^{-1}, \ \Re z \geq 0.$$

Then for all $f \in C^\theta([0,1];X)$, $0<\theta<1$, the periodic problem (5), (2) *has a unique classical solution.*

Proof. The assumptions above imply, according to Favini [6], that X splits into the direct sum representation $X = N(T) \oplus R(T)^a$, where $N(T)$ denotes the null space of the operator $T = ML^{-1} \in L(X)$, $R(T)$ is the range of T and $R(T)^a$ is its closure in X.

Moreover, if we use S to denote the restriction of T to $R(T)^a$, then $-S^{-1}$ generates an analytic semigroup in $N(T)^a = Y$. If P is the projection operator onto $N(T)$, letting $Lu = v$, problem (5), (2) reads

$$\frac{d}{dt}(S(I-P)v(t)) + (I-P)v(t) = (I-P)f(t), \quad 0<t<1, \tag{2.1}$$

$$S(I-P)v(0) = S(I-P)v(1), \tag{2.2}$$

$$Pv(t) = Pf(t), \quad 0<t<1. \tag{2.3}$$

Since $f \in C^\theta([0,1];X)$, any classical solution $(I-P)v(t)$ to (2.1) is expressed by

$$\begin{aligned}(I-P)v(t) \;&=\; S^{-1}\exp(-tS^{-1})x + \\ &\quad + S^{-1}\int_0^t \exp(-(t-s)S^{-1})(I-P)f(s)ds,\end{aligned}$$

where $x \in Y$. Hence (2.2) holds if and only if

$$(I-\exp(-S^{-1}))x = \int_0^1 \exp(-(1-s)S^{-1})(I-P)f(s)ds.$$

The key step consists in proving that $1 \in \rho(\exp(-S^{-1}))$.

Since $-S^{-1}$ generates an analytic semigroup, by Davies [4] it is known that this reduces to verify that $2\pi in \notin \sigma(-S^{-1})$ for any integer n.

Now, in view of the assumptions, for all $n \in \mathbb{Z}$,

$$2\pi i n T v + v = f \in X$$

has the unique solution $(2\pi i n T + I)^{-1} f$ if and only if

$$2\pi i n S(I-P)v + (I-P)v = (I-P)f$$

has the unique solution

$$(2\pi i n S + I)^{-1}(I-P)f = S^{-1}(2\pi i n + S^{-1})^{-1}(I-P)f\,.$$

Hence $2\pi i n \in \rho(-S^{-1})$. This completes the proof. □

THEOREM 2.2. *If X is a reflexive Banach space and A is a multi-valued linear operator in X satisfying*

$$\|(z+A)^{-1}\|_{L(X)} \le C(1+|z|)^{-1}, \quad \mathfrak{Re}\, z \ge 0,$$

Then for all $f \in C^\theta([0,1];X)$, $0<\theta<1$, the periodic problem

$$\frac{d}{dt}u(t) + Au(t) \ni f(t), \quad 0<t<1, \tag{6}$$

$$u(0) = u(1), \tag{7}$$

has a unique classical solution.

Proof. If $A^{-1} = T$, then problem (6), (7) transforms into the system

$$\frac{d}{dt}(Tv(t)) + v(t) = f(t), \quad 0<t<1, \quad Tv(0) = Tv(1).$$

In its turn, this is equivalent to (2.1)–(2.3), that is solved noting that $1 \in \rho(\exp(-S^{-1}))$, where S is restriction of T to $R(T)^a$.

We remark that this in fact implies that $1 \in \rho(e^{-A})$, for

$$e^{-A} = \frac{1}{2\pi i}\int_\gamma e^{-z}(z-A)^{-1}dz,$$

where γ is the contour in the complex plane parametrized by $z = c(1+|y|)+iy$, $-\infty < y < \infty$, so that

$$\begin{aligned} e^{-A} &= \frac{1}{2\pi i}\int_\gamma e^{-z}T(zT-I)^{-1}dz \\ &= \frac{1}{2\pi i}\int_\gamma e^{-z}T(zT-I)^{-1}(I-P)dz \\ &= \frac{1}{2\pi i}\int_\gamma e^{-z}S(zS-I)^{-1}(I-P)dz \\ &= \frac{1}{2\pi i}\int_\gamma e^{-z}(z-S^{-1})^{-1}(I-P)dz \\ &= \exp(-S^{-1})(I-P). \end{aligned}$$

Therefore

$$I - e^{-A} = I - \exp(-S^{-1})(I-P) = P + (I - \exp(-S^{-1}))(I-P).$$

Hence, since $I - \exp(-S^{-1})$ has a bounded inverse (in $L(Y)$), we obtain that $I - e^{-A}$ has the inverse $(I-e^{-A})^{-1}$ given by

$$(I-e^{-A})^{-1}y = Py + (I-\exp(-S^{-1}))^{-1}(I-P)y, \ y \in X.$$

It follows that problem (6), (7) has the unique classical solution

$$u(t) = e^{-tA}(I-e^{-A})^{-1}\int_0^1 e^{-(1-s)A}f(s)ds + \int_0^t e^{-(t-s)A}f(s)\,ds\ ,$$

for $0 < t < 1$.

Notice that since $\int_0^1 e^{-(1-s)A}f(s)ds \in D(A)$, if we put

$$w = (I-e^{-A})^{-1}\int_0^1 e^{-(1-s)A}f(s)ds,$$

then $w = e^{-A}w + \int_0^1 e^{-(1-s)A}f(s)ds$ yields that $w \in D(A)$. Therefore, in view of Favini and Yagi [7], p. 364, we conclude that

$$u(0) = (I-e^{-A})^{-1}\int_0^1 e^{-(1-s)A}f(s)ds = u(1).$$

This concludes the proof. □

3. The highly degenerate (hyperbolic) case

In this section we shall show that the operational method described in Paragraph 1 can be adapted to treat problem (1), (2) in the strongly degenerate case where operators L, M satisfy much weaker assumption $\|L(zM+L)^{-1}\|_{L(X)} \leq p(|z|)$, with $p(z)$ a polynomial and $\mathfrak{Re}\, z \geq -\delta$, δ a certain positive constant.

More precisely, we shall use the following result from Favini [6], pp. 434–438.

PROPOSITION 3.1. *Let E be a complex Banach space and the closed linear operators B, L, M from E into itself satisfy commutativity assumption* **(iii)** *and*

(iv) $\|(B-z)^{-1}\|_{L(E)} \leq C(1+|z|)^p$ *for all complex numbers z such that $\mathfrak{Re}\, z \leq a_0$, where $a_0 > 0$, $p \geq -1$.*

(v) $D(L) \subseteq D(M)$*, L has a bounded inverse and*

$$\|L(zM+L)^{-1}\|_{L(E)} \leq C(1+|z|)^m$$

for all $z \in \mathbb{C}$, $\mathfrak{Re}\, z \geq -\delta$, δ being a positive constant, and $m \geq 0$.

If $n = \min\{s \in \mathbb{N};\ s > m+p+1\}$, then equation (E) has a unique solution u for all $h \in D(B^n)$.

Though we can discuss the case $L^p(0,1;X)$, $1 < p < \infty$, as in Theorem 1.3, for sake of brevity we confine our discussion to solutions of (1), (2) in the space of continuous functions on the interval $[0,1]$. In other words, we shall take $E = C([0,1];X)$.

We first have the theorem as follows.

THEOREM 3.1. *Let X be a complex Banach space and let L, M be closed linear operators from X into itself such that $D(L) \subseteq D(M)$, L has a bounded inverse and for some constants $m \geq 0$, $\delta > 0$,*

$$\|L(zM+L)^{-1}\|_{L(X)} \leq C(1+|z|)^m, \ \forall z \in \mathbb{C}, \ \mathfrak{Re}\, z \geq -\delta. \qquad (3.1)$$

If n is the smallest integer greater than $m+1$, then for all $f \in C^n([0,1];X)$ with $f^{(i)}(0) = f^{(i)}(1)$, $i = 0, 1, \ldots, n-1$, problem (1), (2) has a unique strict solution u.

Proof. By using notation as in Section 1, take $k = 2\delta$, $B = B_1 + k$. Identifying L, M with the operators induced by them in $E = C([0,1];X)$, problem (1), (2) reads equivalently $BMu+(L-kM)u = f$.

On the other hand, it is readily checked that assumption (3.1) guarantees that

$$\|zM(zM + L - kM)^{-1}\|_{L(E)} \le C(1 + |z|)^m, \quad \mathfrak{Re}\, z \ge -k/2.$$

Then Proposition 3.1 gives directly the desired result. □

COROLLARY 3.1. *Let X be a complex Hilbert space with inner product $\langle\ ,\ \rangle$. If L, M are two self-adjoint operators in X, $M \ge 0$, $L > 0$, $D(L^{1/2}) \subseteq D(M)$, then for all $f \in C^3([0,1];X)$, $f^{(i)}(0) = f^{(i)}(1)$, $i = 0,1,2$, problem (1), (2) has one and only one strict solution.*

Proof. If $zMu + Lu = f \in X$, then $z\|M^{1/2}u\|^2 + \|L^{1/2}u\|^2 = \langle f, u\rangle$, that is

$$\begin{aligned} \mathfrak{Re}\, z\|M^{1/2}u\|^2 + \|L^{1/2}u\|^2 &= \mathfrak{Re}\langle f, u\rangle\ , \\ |\,\mathfrak{Im}\, z|\|M^{1/2}u\|^2 &= |\,\mathfrak{Im}\langle f, u\rangle|\ , \end{aligned}$$

where we used $\|u\|$ to denote the norm of $u \in X$.

Summing up both the members of the preceding equalities we obtain

$$\begin{aligned} (\mathfrak{Re}\, z + |\,\mathfrak{Im}\, z|)\|M^{1/2}u\|^2 + \|L^{1/2}u\|^2 &= \mathfrak{Re}\langle f, u\rangle + |\,\mathfrak{Im}\langle f, u\rangle| \\ &\le C\|f\|\|L^{1/2}u\|. \end{aligned}$$

Since M is L-bounded (according to Kato's notion), with L-bound equal to zero, it is known that $zM + L$ is closed and its adjoint $(zM + L)^*$ coincides with $\overline{z}M + L$. We refer to Weidmann [15], p. 109.

Therefore, for all z in the region $\{\mathfrak{Re}\, z + |\,\mathfrak{Im}\, z| \ge 0\} \cup \{|z| \le \epsilon\}$, for suitable positive ϵ, there exists the inverse $(zM + L)^{-1} \in L(X)$ with $\|M(zM + L)^{-1}\|_{L(X)} \le \text{Const}$.

Then Theorem 3.1 is applicable with $m = 1$. □

Corollary 3.2. *Let X be a complex Hilbert space with inner product $\langle\ ,\ \rangle$ and let L, M be two closed linear operators in X such that $D(L) \subseteq D(M)$, $0 \in \rho(L)$, M (respectively, M^*) is L-bounded (respectively, L^*-bounded) with L-bound (respectively, L^*-bound) equal to 0. Moreover, suppose that*

$$\mathfrak{Re}\langle Mu, Lu\rangle \geq 0\ , \qquad \text{for all } u \in D(L),$$
$$\mathfrak{Re}\langle M^*f, L^*f\rangle \geq 0\ , \qquad \text{for all } f \in D(L^*).$$

Then for all $\epsilon > 0$ and every $f \in C^3([0,1];X)$, $f^{(i)}(0) = f^{(i)}(1)$, $i = 0,1,2$, the problem

$$\frac{d}{dt}(Mu(t)) = -(L+\epsilon M)u(t) + f(t)\,, \qquad 0 \leq t \leq 1,$$
$$Mu(0) = Mu(1),$$

has a unique strict solution.

Proof. Let $(zM + L + \epsilon M)u = f \in X$. Then

$$(\mathfrak{Re}\, z + \epsilon)\|Mu\|^2 + \mathfrak{Re}\langle Lu, Mu\rangle = \mathfrak{Re}\langle f, Mu\rangle$$

implies that $\|Mu\| \leq (\mathfrak{Re}\, z + \epsilon)^{-1}\|f\|$, where $\mathfrak{Re}\, z > -\epsilon$.

On the other hand, setting $(z+\epsilon)u = v$ gives $Mv + (z+\epsilon)^{-1}Lv = f$, so that

$$\langle Mv, Lv\rangle + \frac{\mathfrak{Re}\, z + \epsilon - i\,\mathfrak{Im}\, z}{|z+\epsilon|^2}\|Lv\|^2 = \langle f, Lv\rangle.$$

Therefore, by taking real parts, in view of the assumptions, we have

$$\frac{\mathfrak{Re}\, z + \epsilon}{|z+\epsilon|^2}\|Lv\|^2 \leq \|f\|\|Lv\|\,,\ \mathfrak{Re}\, z > -\epsilon.$$

This implies that $zM + L + \epsilon M$ is one-to-one for these z's and has a closed range. Applying the same trick to the adjoint equation, since the assumptions assure that $(zM + L)^* = \overline{z}M^* + L^*$ as well, we deduce that $(\overline{z} + \epsilon)M^* + L^*$ is one-to-one, that is, the range of $zM + L + \epsilon M$ is everywhere dense in X.

This concludes the proof that $zM + L + \epsilon M$ has a bounded inverse for all z with $\mathfrak{Re}\, z > -\epsilon$. Moreover, condition (3.1) is verified with $m = 1$ if we take a suitable δ. □

REMARK 3.1. We observe that Corollary 3.2 can be viewed as a linear degenerate version, that ensures regular solutions as well, of Corollary 9 in Haraux [11], p. 175, where existence of weak and strong 1-periodic solutions to the multivalued equation $v' + Av + \epsilon v \ni f(t)$, $\epsilon > 0$, is established, A being a maximal monotone operator in a Hilbert space.

The case $\epsilon = 0$, the resonant one, needs more attention and shall be considered elsewhere.

Propositions 1.1 and 3.1 have a large range of application. In particular, they allow to handle periodic problems of different type, as it is shown in the next result, where we confine us to prove only the affirmation corresponding to Proposition 3.1.

COROLLARY 3.3. *Let the closed linear operators L, M satisfy assumption (3.1) in the complex Hilbert space X. Then the elliptic problem*

$$\frac{d^2}{dt^2}(Mu(t)) - Lu(t) = f(t), \quad 0 \le t \le 1, \tag{3.2}$$

$$(Mu)(0) = (Mu)(1), \quad \frac{d}{dt}(Mu)(0) = \frac{d}{dt}(Mu)(1), \tag{3.3}$$

has a unique solution u, with $Lu \in L^2(0,1;X)$, $Mu \in W^{2,2}(0,1;X)$, for all $f \in W^{2(m+1),2}(0,1;X)$, $\frac{d^i f}{dt^i}(0) = \frac{d^i f}{dt^i}(1)$, $i = 0, 1, ..., 2m+1$.

Proof. Define an operator B in $E = L^2(0,1;X)$, by

$$\begin{aligned} D(B) &= \{v \in W^{2,2}(0,1;X)\,; v(0) = v(1), \frac{dv}{dt}(0) = \frac{dv}{dt}(1)\}\ , \\ Bv &= -\frac{d^2 v}{dt^2} = -v" \ , \quad v \in D(B). \end{aligned}$$

Then B is a positive self-adjoint operator in E, as readily seen (Haraux [11], p. 188).

Hence we can conclude applying Proposition 3.1, with $p = -1$, solving (3.2), (3.3). □

Before establishing the next statement, we recall that if $z = 0$ is a polar singularity for the resolvent $(z + ML^{-1})^{-1}$ of order $m + 1$, $m = 0, 1, \dots$, in the Hilbert space X, that is,

$$\|L(zL + M)^{-1}\|_{L(X)} \le \frac{C}{|z|^{m+1}}, \quad 0 < |z| \le \epsilon,$$

for suitable positive ϵ, then

$$\|L(zM + L)^{-1}\|_{L(X)} \le C|z|^{-m}, \quad \forall z, |z| \ge 1/\epsilon.$$

Moreover, the representation $X = N(T^{m+1}) \oplus R(T^{m+1})$ holds, where $R(T^{m+1})$ is a closed subspace of X and $T = ML^{-1}$. We refer to Yosida [17], p. 229.

Since we always assume that $D(L) \subseteq D(M)$, we know that $zT + I$ has a bounded inverse for any z, $|z| < \frac{1}{\|T\|_{L(X)}}$.

Hence, if $\epsilon > \|T\|_{L(X)}$, then Theorem 3.1 applies directly. In the particular case $m = 0$ we are even allowed to use Theorem 1.1, as it has been done in Example 1.2.

But in general ϵ is small and thus resonance phenomena may arise. To deepen in some detail this case, we want to give a more direct approach to (1), (2), introducing the projection operator P onto $N(T^{m+1})$, and the restrictions T_1 and T_2 to $N(T^{m+1})$ and $N(T^{m+1})$ respectively. The change of variable $Lu = v$, together with the well known commutativity of the involved operators T_1 and T_2 with P, leads that problem (1), (2) splits into the two systems

$$\frac{d}{dt}(T_1 P v(t)) = -Pv(t) + Pf(t), \quad 0 \le t \le 1, \tag{3.4}$$

$$T_1 P v(0) = T_1 P v(1), \tag{3.5}$$

and

$$\frac{d}{dt}(T_2(I - P)v(t)) = -(I - P)v(t) + (I - P)f(t), \quad 0 \le t \le 1, \tag{3.6}$$

$$T_2(I - P)v(0) = T_2 P v(1), \tag{3.7}$$

where (3.4), (3.5) is a problem in $N(T^{m+1})$ and (3.6), (3.7) is considered in $R(T^{m+1})$.

Since $R(T^{m+1}) = R(T^r)$ for all $r \in N$, $r \geq m+1$, T_2 has a bounded inverse in $R(T^{m+1})$ and (3.6), (3.7) is equivalent to

$$\frac{d}{dt}((I-P)v)(t)) = -T_2^{-1}(I-P)v(t) + T_2^{-1}(I-P)f(t), \qquad (3.8)$$

$$0 \leq t \leq 1,$$

$$(I-P)v(0) = Pv(1). \qquad (3.9)$$

Moreover, since $T_1^{m+1} = 0$, to solve (3.4) we necessarily need to request regularity like $f \in C^m([0,1], X)$, so that

$$Pv(t) = Pf(t) - T_1Pf^{(1)}t) + T_1^2Pf^{(2)}(t) - \ldots + (-1)^m T_1^m Pf^{(m)}(t).$$

Hence, (notice that if $m = 0$, then $TP = 0$), for $i = 0, 1$,

$$\begin{aligned} T_1Pv(i) &= T_1Pf(i) - T_1^2Pf^{(1)}(i) + \ldots + (-1)^{m-1}T_1^m f^{(m-1)}(i) \\ &= P[Tf(i) - T^2 f^{(1)}(i) + \ldots + (-1)^{m-1}T^m f^{(m-1)}(i)]. \end{aligned}$$

Let us consider now the regular system (3.8), (3.9).

In view of Haraux [11], Corollary 9, p. 157, we know that it has a 1-periodic solution $(I-P)v(t)$ if and only if for each $j \in \mathbb{Z}$, $|j| \leq \frac{1}{2\pi}\|T_2^{-1}\|_{L(R(T^{m+1}))}$, the integral

$$\int_0^1 T_2^{-1}(I-P)f(s)e^{-2\pi jis}ds \in R(T_2^{-1} + 2\pi jiI),$$

that is,

$$\int_0^1 (I-P)f(s)e^{-2\pi jis}ds = (I + 2\pi jiT_2)(I-P)x_j, \qquad (3.10)$$

for certain elements $x_j \in X$, $0 < |j| \leq a_0 = \frac{1}{2\pi}\|T_2^{-1}\|_{L(R(T^{m-1}))}$. Now, by Yosida [14], p. 228, (3.10) is equivalently written

$$\int_0^1 (I-P)f(s)e^{-2\pi jis}ds = (I-P)(I + 2\pi jiT)x_j,$$

or else

$$\int_0^1 f(s)e^{-2\pi jis}ds - (I + 2\pi jiI)x_j \in N(T^{m+1}),\ 0 < |j| \le a_0. \quad (3.11)$$

Combining these results, we can establish the following one.

THEOREM 3.2. *Let $z = 0$ be a polar singularity of order $m+1$ for $(z-T)^{-1}$, with $T = ML^{-1}$, and M, L closed linear operators from the Hilbert space X into itself, $D(L) \subseteq D(M)$, $L^{-1} \in L(X)$.*
Let $a_0 = \frac{1}{2\pi}\|T_2^{-1}\|_{L(R(T^{m+1}))}$. If $f \in C^m([0,1];X)$ satisfies the compatibility relations (3.11) *and (if $m > 0$)*

$$\sum_{i=1}^{m}(-1)^{i-1}T^i\left(\frac{d^{i-1}}{dt^{i-1}}f(0) - \frac{d^{i-1}}{dt^{i-1}}f(1)\right) \in R(T^{m+1}) \ ,$$

then problem (1), (2) *has at least one strict solution.*

Henceforth we shall illustrate use of the theorems above by means of some examples from ordinary and partial differential equations.

EXAMPLE 3.1. Let $f, g, h \in C([0,1])$. Then the algebraic differential problem

$$\begin{aligned} \frac{dv}{dt}(t) &= -u(t) + f(t), \\ 0 &= -v(t) + g(t), \\ \frac{dv}{dt}(t) + \frac{1}{2\pi i}\frac{dw}{dt}(t) &= -w(t) + h(t), \quad 0 \le t \le 1, \end{aligned}$$

$$v(0) = v(1)\ , \quad w(0) = w(1)\ ,$$

has obviously a solution if and only if $g \in C^1([0,1])$, $g(0) = g(1)$ and

$$\int_0^1 e^{2\pi is}(h(s) - g'(s))ds = \int_0^1 e^{2\pi is}(h(s) + 2\pi i g(s))ds = 0.$$

The solutions are then given by

$$\begin{aligned} u(t) &= f(t) - g'(t), \\ v(t) &= g(t), \\ w(t) &= e^{-2\pi is}h_0 + 2\pi i\int_0^t e^{-2\pi i(1-s)}(h(s) - g'(s))ds, \end{aligned}$$

where h_0 is an arbitrary complex number.

Here $M(u,v,w)=(v,0,v+\frac{1}{2\pi i}w)$, $L=I$, so that $T=M$, $z=0$ is a polar singularity of order 2 for the resolvent $(z-M)^{-1}$, $N(T^2)$ is the space generated by the vectors $(1,0,0)$ and $(0,i,2\pi i)$, $R(T^2)$ is the space generated by $(0,0,1)$. Hence, $P(u,v,w)=(u,v,-2\pi iv)$, $(I-P)(u,v,w)=(0,0,w+2\pi iv)$.

Thus the conditions in Theorem 3.1 read, since $j=-1$ only,

$$T(f(0)-f(1),g(0)-g(1),h(0)-h(1))= \\ =(g(0)-g(1),0,g(0)-g(1)+\frac{1}{2\pi i}(h(0)-h(1))\in R(T^2)$$

and there exists $(\overline{u},\overline{v},\overline{w})$ such that

$$\Big(\int_0^1 f(s)e^{2\pi is}ds+\overline{u}-2\pi i\overline{v}\ , \\ \int_0^1 g(s)e^{2\pi is}ds+\overline{v}\ ,\ \int_0^1 h(s)e^{2\pi is}ds-2\pi i\overline{v}\Big)\in N(T^2).$$

The first one reduces to $g(0)-g(1)$, and the second one becomes

$$\int_0^1 f(s)e^{2\pi is}ds+\overline{u}-2\pi i\overline{v}\quad \text{arbitrary,} \\ -2\pi i\left(\int_0^1 g(s)e^{2\pi is}ds+\overline{v}\right)=\int_0^1 h(s)e^{2\pi is}ds-2\pi i\overline{v},$$

that is precisely the aforementioned compatibility relation.

Sometimes we can substitute assumptions on the operator $T=ML^{-1}$ by corresponding hypotheses on M only. This happens, for instance, when M and L have a common domain and commute according to $(z-M)^{-1}L^{-1}=L^{-1}(z-M)^{-1}$ for all $z\in\rho(M)$. The following two examples clarify the typical situation.

Example 3.2. Let K be a densely defined closed linear operator from the Hilbert space X into itself, $k_0\in\mathbb{R}$, such that $z=0$ is a polar singularity of order 1 for the resolvent $(z+k_0+K)^{-1}$, and the spectrum of K consists of a countable set of (real) eigenvalues, $-k_0$ being the greatest one among them.

Let $\delta\in\rho(-K)\cap\mathbb{R}$, and consider the periodic problem

$$\frac{d}{dt}((k_0+K)u(t)) = Ku(t) + \delta u(t) + f(t), \quad 0 \le t \le 1, \tag{3.12}$$

$$(k_0+K)u(0) = (k_0+K)u(1), \tag{3.13}$$

where $f \in C([0,1];X)$. Let $M = k_0 + K$, so that, denoting by P the projection operator onto $N(M)$, problem (3.12), (3.13) reduces to

$$\frac{d}{dt}(\tilde{M}(I-P)u(t)) = \tag{3.14}$$

$$= \tilde{M}(I-P)u(t) + (\delta - k_0)(I-P)u(t) + (I-P)f(t),$$

$$0 \le t \le 1,$$

$$(I-P)u(0) = (I-P)u(1), \tag{3.15}$$

$$Pu(t) = \frac{1}{k_0 - \delta}Pf(t), \quad 0 \le t \le 1,$$

$\tilde{M}$ being the restriction of M to $R(M)$. Now (3.14) is equivalent to

$$\frac{d}{dt}((I-P)u(t)) = ((\delta - k_0)\tilde{M}^{-1} + I)(I-P)u(t) + \tilde{M}^{-1}(I-P)f(t)\,,$$

and thus we are allowed to apply Haraux's result [11], Corollary 9, p. 158, according which (3.14), (3.15) has a 1-periodic strict solution if and only if $\forall m \in \mathbb{Z}$, $|m| \le a_0 = (2\pi)^{-1}\|I + (\delta - k_0)\tilde{M}^{-1}\|_{L(R(M))}$,

$$\int_0^1 (I-P)f(s)e^{-2\pi ims}ds \in R((2\pi im - 1)\tilde{M} - (\delta - k_0))\ ;$$

that is,

$$\begin{aligned}\int_0^1 f(s)e^{-2\pi ims}ds &\in R((2\pi im - 1)M - (\delta - k_0)) \\ &= R(M - \frac{\delta - k_0}{2\pi im - 1}).\end{aligned}$$

If $m = 0$, this reads

$$\int_0^1 f(s)ds \in R(K+\delta) = X.$$

On the other hand, if $m \neq 0$, since $\delta - k_0 \neq 0$ in view of the assumptions, then $R(M - \frac{\delta - k_0}{2\pi i m - 1}) = X$ again. Hence problem (3.12), (3.13) admits one 1-periodic solution.

In particular, the present argument applies to $X = L^2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $K = \Delta$, with $D(K) = H_0^1(\Omega) \cap H^2(\Omega)$.

EXAMPLE 3.3. Let K be a densely defined closed linear operator from the Hilbert space X into itself, satisfying the same properties as in Example 3.2, that is, $z = 0$ is a polar singularity of order 1 for the resolvent $(z + k_0 + K)^{-1}$, where $k_0 \in \mathbb{R}$, and the spectrum of K consists of a countable set of (real) eigenvalues, $-k_0$ being the greatest one among them.

Let $\delta \in \mathbb{C}$ and consider the equation

$$\frac{d}{dt}((k_0 + K)u(t)) = iKu(t) + \delta u(t) + f(t), \quad 0 \leq t \leq 1, \tag{3.16}$$

with periodic condition (3.13). By using the same notation as in Example 3.2, system (3.16), (3.13) splits into

$$\frac{d}{dt}(\tilde{M}(I-P)u(t)) = \tag{3.17}$$

$$= i\tilde{M}(I-P)u(t) + (\delta - ik_0)(I-P)u(t) + (I-P)f(t),$$

$$0 \leq t \leq 1,$$

$$(I-P)u(0) = (I-P)u(1), \tag{3.18}$$

$$(ik_0 - \delta)Pu(t) = Pf(t), \qquad 0 \leq t \leq 1.$$

Let us denote by $-k_j$, $j = 0, 1, \dots$, the eigenvalues of K and assume $\delta \neq ik_j$, $j = 0, 1, \dots$. This guarantees, between other things, that last equation has a unique solution $Pu(t)$. Obviously, we maintain that $f(t)$ is a continuous X-valued function.

Application of Haraux's result [11], Corollary 9, p. 157, leads that (3.17), (3.18) has a 1-periodic strict solution if and only if $\forall m \in \mathbb{Z}$,

$|m| \leq a_0 = \frac{1}{2\pi}\|I - (i\delta + k_0)\tilde{M}^{-1}\|_{L(R(M))}$,

$$\int_0^1 (I-P)f(s)e^{-2\pi ims}ds \in R((2\pi im - 1)\tilde{M} + (i\delta + k_0)),$$

that is,

$$\int_0^1 f(s)e^{-2\pi ims}ds \in R((2\pi im - 1)M + (i\delta + k_0)).$$

On the other hand, the last range coincides with the range $R(K + \frac{2\pi imk_0 + i\delta}{2\pi im - 1})$. Hence, if $\mathfrak{Im}\,\delta \neq -k_j$, then it is all of X. Analogously, this is the case if $\mathfrak{Im}\,\delta = -k_j$ and

$$-\mathfrak{Re}\,\delta \neq 2\pi m(k_0 + k_j), \quad j = 1, 2, \ldots$$

Instead of, if $\mathfrak{Im}\,\delta = -k_j$ and $-\mathfrak{Re}\,\delta = 2\pi m(k_0 + k_j)$, $j = 1, 2, \ldots$, we have resonance situation.

EXAMPLE 3.4. Let H be a complex Banach space with norm $\|\cdot\|_H$ and inner product $\langle\ ,\ \rangle$. Let A, B, C be closed linear operators from H into itself such that

$$A = A^* > 0,\ D(A)^c = H,\ C = C^* \geq 0,\ D(C) = H, \tag{3.19}$$

$$D(A^{1/2}) \subseteq D(B) \cap D(B^*), \tag{3.20}$$

$$\mathfrak{Re}\langle Bu, u\rangle \geq c_0\|u\|^2,\ \mathfrak{Re}\langle B^*v, v\rangle \geq c_0\|v\|^2, \tag{3.21}$$
$$u \in D(B), v \in D(B^*),$$

where c_0 is a positive constant.

Given $f \in C([0,1]; H)$, the second order differential equation

$$\frac{d}{dt}\left(C\left(\frac{du}{dt}\right)\right) + B\frac{du}{dt} + Au = f(t), \quad 0 \leq t \leq 1,$$

is formally written $\frac{d}{dt}(Mw(t)) + Lw(t) = F(t)$, $0 \leq t \leq 1$, where $w = (u, v)$, $F(t) = (0, f(t))$, $L(u, v) = (-v, Au + Bv)$, $(u, v) \in D(L) = D(A) \times D(A^{1/2})$, $M(x, y) = (x, Cy)$, $(x, y) \in D(M) =$

$D(A^{1/2}) \times H = X$. The space X is a Hilbert space with respect to the inner product $\langle\ ,\ \rangle_X$ given by

$$\langle (x,y),(x_1,y_1)\rangle_X = \langle A^{1/2}x, A^{1/2}x_1\rangle + \langle y, y_1\rangle.$$

In view of assumptions (3.19), (3.20), it is seen that L is a closed linear operator from X into itself. Moreover, given $\mathfrak{Re}\, z > 0$, if $zMw + Lw = h = (f,g) \in X$, $w = (u,v)$, multiplying the equation by w with respect to the inner product in X and taking real parts, we obtain

$$\mathfrak{Re}\, z\{\|A^{1/2}u\|^2 + \|C^{1/2}v\|^2\} + \mathfrak{Re}\langle Bv, v\rangle \le \\ \le \mathfrak{Re}\{\langle A^{1/2}f, A^{1/2}u\rangle + \langle g, v\rangle\}\ ,$$

that is, in virtue of (3.21),

$$\min\{\mathfrak{Re}\, z, c_0\}\|w\|_X \le k\|h\|_X \tag{3.22}$$

On the other hand, the adjoint operator L^* of L is given by $L^*(u,v) = (v, -Au + B^*v)$, so that the estimate

$$\min\{\mathfrak{Re}\, z, c_0\}\|w\|_X \le k\|(\overline{z}M^* + L^*)w\|_X \tag{3.23}$$

holds as well. Combining (2.22) and (2.23) we conclude that $zM + L$ has a bounded inverse for $\mathfrak{Re}\, z > 0$ and $\|(zM+L)^{-1}\|_{L(X)} \le k(\min\{\mathfrak{Re}\, z, c_0\})^{-1}$.

Hence, for all $\mathfrak{Re}\, z \ge -\frac{\epsilon}{2}$ we have $\|L(zM + L + \epsilon M)^{-1}\|_{L(X)} \le k(1+|z|)$ ensuring that Theorem 3.1 applies, with $m = 1$, provided that $L + \epsilon M$, with $\epsilon > 0$, takes the place of L.

Summarizing up, by translating the result to the original problem, we have proved that under assumptions (2.19)–(2.21), for all $f \in C^3([0,1]; H)$, $f^{(i)}(0) = f^{(i)}(1)$, $i = 0,1,2$, the periodic problem

$$\frac{d}{dt}(C(\frac{du}{dt})) + (B + 2\epsilon C)\frac{du}{dt} + (A + \epsilon B + \epsilon^2 C)u = f(t), \quad 0 \le t \le 1\ ,$$

$$u(0) = u(1), \quad C\frac{du}{dt}(0) = C\frac{du}{dt}(1)$$

has a unique strict solution $u \in C([0,]; D(A)) \cap C^1([0,1]; D(B))$, $C\frac{du}{dt} \in C^1([0,1]; H)$. Notice that the conclusion is true in the particular case of the operator $C = 0$.

We shall indicate two concrete examples of partial differential equations to which our result is applicable.

The first one concerns a Poisson wave equation with damping in a bounded region $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$. Indeed, it is the equation of hyperbolic-parabolic type

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(m(x)\frac{\partial u}{\partial t}\right) + (n(x) + 2\epsilon m(x))\frac{\partial u}{\partial t} + \\
&\qquad\qquad - (\Delta - \epsilon n(x) - \epsilon^2 m(x))u = f(t,x), \\
&\qquad\qquad\qquad\qquad \text{in } [0,1] \times \Omega, \\
&u = 0, \qquad\qquad\qquad\qquad \text{in } [0,1] \times \partial\Omega, \\
&u(0,x) = u((1,x), \qquad\qquad \text{in } \Omega, \\
&m(x)\frac{\partial u}{\partial t}(0,x) = m(x)\frac{\partial u}{\partial t}(1,x), \quad \text{in } \Omega,
\end{aligned}
$$

where $m(x)$ is a continuous nonnegative function on $\overline{\Omega}$, $n(x)$ is continuous, real valued on Ω, $n(x) \geq n_0 > 0$ for all $x \in \Omega$. Here we take $H = L^2(\Omega)$, $A = -\Delta$, $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, so that $D(A^{1/2}) = H_0^1(\Omega)$, C is multiplication by $m(x)$ and B is multiplication by $n(x)$, with the maximal domain.

Obviously, all what remains to do is to add some conditions guaranteeing that (3.20) holds. Since for all $q > 1$

$$\int_\Omega n(x)^2|u(x)|^2 dx \leq \left(\int_\Omega |u(x)|^{2q/(q-1)} dx\right)^{(q-1)/q} \left(\int_\Omega n(x)^{2q} dx\right)^{1/q}$$

we see that if

$$\int_\Omega n(x)^{2q} dx < \infty, \quad q > 1, \tag{3.24}$$

then the Sobolev imbeddings

$$
\begin{aligned}
&H^1(\Omega) \subset L^p(\Omega), \qquad 2 \leq p \leq \frac{2n}{n-2}, \text{ if } n > 2, \\
&H^1(\Omega) \subset L^p(\Omega), \qquad 2 \leq p < \infty, \text{ if } n = 2,
\end{aligned}
$$

imply that if $n > 2$ and (3.24) is verified with $2q \geq n$, then $H_0^1(\Omega) \subset D(B)$. On the other hand, if $n = 2$, then (3.24) with $q > 1$ suffices to have the continuous imbedding $H_0^1(\Omega) \subset D(B)$ as well.

The case $n = 1$ can be improved a bit. In this case $\Omega = (a, b)$, where $a < b$ are any real numbers. We can estimate the integral $\int_\Omega n(x)^2 |u(x)|^2 dx$ by

$$k_1 \left\{ \int_a^{(a+b)/2} n(x)^2 (x-a) \left(\int_a^x |u'(y)|^2 dy \right) dx + \right.$$
$$\left. + \int_{(a+b)/2}^b n(x)^2 (b-x) \left(\int_x^b |u'(y)|^2 dy \right) dx \right\} \leq$$
$$\leq k_2 \int_a^b |u'(x)|^2 dx, \quad u \in H_0^1(a,b),$$

provided that the integrals

$$\int_a^{(a+b)/2} n(x)^2 (x-a) dx \qquad \text{and} \qquad \int_{(a+b)/2}^b n(x)^2 (b-x) dx$$

converge. For instance, this is the case when $n(x) = (x-a)^{-\sigma}(b-x)^{-\tau}$, where $\sigma, \tau < 1$.

The second example below shows that our assumptions allow us to consider differential operators in the role of B, too. For sake of simplicity, we confine to the one dimensional situation $\Omega = (0,1)$, $H = L^2(0,1)$. The operator A is $-\frac{d^2}{dx^2}$, with $D(A) = H_0^1(0,1) \cap H^2(0,1)$, the operator C is the multiplication by $m(x) \geq 0$ as above and B is given by

$$D(B) = \{u \in H^1(0,1)\,;\ u(0) = 0\}, \quad Bu = \frac{du}{dx} + \epsilon_0 u,$$

where ϵ_0 is an arbitrary positive number. Since

$$D(B^*) = \{u \in H^1(0,1)\,;\ u(1) = 0\}, \quad B^* u = -\frac{du}{dx} + \epsilon_0 u,$$

all the hypotheses (3.19)–(3.21) are clearly satisfied.

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