# Analyticity and Uniqueness for the Inverse Conductivity Problem

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SOMMARIO. - *Consideriamo il problema inverso di determinare il coefficiente di conduttività* $a = 1 + \mu\chi_D$, $D \subset\subset \Omega$, $\mu =$ *costante, nell'equazione ellittica* $div(a\nabla u) = 0$ *in* $\Omega$, *quando siano assegnati dati al bordo sovradeterminati per una soluzione* $u$ *non banale. Mostriamo che la nonunicità nella determinazione di* $D$ *implica che una porzione* $\Gamma$ *di* $\partial D$ *è soluzione di un particolare problema di frontiera libera. Dimostriamo alcune proprietà di analiticità di tale frontiera libera e, in conseguenza, otteniamo alcuni risultati di unicità per il problema inverso della conduttività.*

SUMMARY. - *We treat the inverse problem of the determination of the conductivity coefficient* $a = 1+\mu\chi_D$, $D \subset\subset \Omega$, $\mu =$ *constant, in the elliptic equation* $div(a\nabla u) = 0$ *in* $\Omega$, *when overdetermined boundary data for one nontrivial solution* $u$ *are assigned . We show that nonuniqueness in the determination of the domain* $D$ *would imply that a part* $\Gamma$ *of* $\partial D$ *is a solution of a particular free boundary problem. We prove analyticity properties of such a free boundary and, consequently, we derive uniqueness results for the inverse conductivity problem.*

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## 1. Introduction

We consider the problem of recovery of a domain $D \subset\subset \Omega$ entering the conductivity coefficient $a(x) = 1+\mu\chi_D(x)$ of the elliptic equation

$$\mathrm{div}(a\nabla u) = 0 \qquad \text{in}\, \Omega \tag{1.1}$$

with the boundary condition

$$u = g \qquad \text{on}\, \partial\Omega\,. \tag{1.2}$$

The additional data are

$$\partial_\nu u = h \qquad \text{on}\, \Gamma_0 \subset \partial\Omega\,. \tag{1.3}$$

where $\Gamma_0$ is an open nonempty part of $\partial\Omega$. Here $\Omega$ is a known bounded domain in $\mathbb{R}^n$, $n \geq 2$, with smooth (for instance $C^2$ ) connected boundary $\partial\Omega$, $\nu$ denotes the exterior unit normal, and $\mu$ is a constant parameter, $\mu > -1$, $\mu \neq 0$. We shall consider both cases when $\mu$ is assumed to be known and when $\mu$ is part of the unknowns.

The question which we are interested in here is the one of uniqueness, that is whether the data $g$, $h$ appearing in (1.2), (1.3) are sufficient to uniquely determine the domain $D$. This problem has attracted a lot of attention in recent years, and most of the results obtained since now can be grouped into two main categories.

(I) *Global uniqueness theorems within "restricted" classes of domains.*

(II) *Local uniqueness theorems in "large" classes of domains.*

A typical example of the results of the group (I) concerns the classes of convex polygons, when the space dimension $n = 2$, or of convex polyhedrons when $n = 3$. Friedman and Isakov [F-I] proved uniqueness results in these classes under additional assumptions relating the diameter of the polygons, or polyhedrons, to their distance to the boundary $\partial\Omega$. Barcelò, Fabes and Seo [B-F-S] were able to remove such additional assumptions, at the cost of prescribing boundary data on $u$ of a very special type. Also other classes of domains have been investigated, unions of disks, cylinders, see [I-P].

In this paper, among other results, we shall prove results of the same flavour of those in [B-F-S] , but with different choices of boundary data which appear to be more practically feasible. Moreover, also the parameter $\mu$ will be considered as an unknown and we prove its unique determination by the boundary data (1.2), (1.3). See Theorems 5.1, 5.3 below.

Concerning group (II), it has been recently proven, [A-I-P], that, when $n = 2$, local uniqueness holds in the class of $C^{1+\lambda}$ simply connected domains. Previous results in this direction concerned classes of planar domains with analytic boundary, [Ch], [B-F-I], [P]. All the above results were based on arguments of linearization, let us recall that a study of the linearized problem is due to Lorenzi and Pagani [L-P]. See also [Is3] for results for a slightly different, but related, problem with interior sources.

The main issue that we want to address in this paper is whether there exist domains $D$ which can be uniquely determined in some "large" class of domains, that is domains on which no, or little, geometrical assumptions are made.

Consider, for instance, the planar case $n = 2$, and let us denote by $\mathcal{D}$ the class of Jordan domains $D \subset\subset \Omega$. We shall find, Theorem 4.1, a subclass $\tilde{\mathcal{D}} \subset \mathcal{D}$, which is dense in the Hausdorff metric, such that any $D \in \tilde{\mathcal{D}}$ is uniquely determined in $\mathcal{D}$ by the boundary data (1.2), (1.3). In fact $\tilde{\mathcal{D}}$ will be chosen as the family of domains $D \in \mathcal{D}$ such that $\partial D$ is not analytic at any of its points.

Before proceeding into the discussion of this result, it is necessary to recall some well-known facts about the direct problem (1.1), (1.2).

Given any measurable set $D$ , and given, for instance, a function $g \in H^{1/2}(\partial\Omega)$, there exists a unique generalized solution $u \in H^1(\Omega)$, which is separately harmonic in the interior $\overset{\circ}{D}$ of $D$ and in $\Omega\setminus\overline{D}$. Moreover $u$ is continuous across $\partial D$, in fact, by the DeGiorgi-Nash-Moser theorem, $u \in C^{\lambda}_{\text{loc}}(\Omega)$, with a Hölder exponent, $\lambda$, $0 < \lambda < 1$, which depends on $n$ and $\mu$ only. See, for instance [G-T] or [L-U]. Higher regularity of $u$ near the two sides of $\partial D$ is achieved, provided corresponding regularity of $\partial D$ is assumed. For instance, if we assume $\partial D \in C^{1+\lambda}$ for some $\lambda$, $0 < \lambda < 1$, then setting

$$u^e = u|_{\Omega\setminus\overline{D}}\,, \quad u^i = u|D\,, \tag{1.4}$$

we have $u^e \in C^{1+\beta}_{\text{loc}}(\Omega \setminus D)$, $u^i \in C^{1+\beta}_{\text{loc}}(\overline{D})$, for some $\beta$, $0 < \beta < 1$, see [D-E-F]. Moreover equation (1.1) can be rewritten as

$$\Delta u^e = 0 \quad \text{in}\, \Omega \setminus \overline{D}\,, \tag{1.5a}$$

$$\Delta u^i = 0 \quad \text{in}\, D \tag{1.5b}$$

$$u^e = u^i \quad \text{on } \partial D\,, \tag{1.5c}$$

$$\partial_\nu u^e = (1+\mu)\partial_\nu u^i \quad \text{on } \partial D\,. \tag{1.5d}$$

Equations (1.5c), (1.5d) are then called the transmission conditions for $u$. Let us also recall that when $\partial D$ is merely Lipschitz, then the transmission conditions continue to hold almost everywhere on $\partial D$, in this case the normal derivatives in (1.5d) can be interpreted in the sense of nontangential limits in $L^2(\partial D)$, see [E-F-V] . Continuing now our previous discussion of the inverse problem, the key step in our argument will be based on the fact that the study of the uniqueness problem leads to the following question (see Theorem 4.1 and its Proof).

*Is it possible that there exists a portion* $\Gamma \subset \partial D$ *and a neighbourhood* $V$ *of* $\Gamma$ *in which* $u^e$ *can be continued harmonically across* $\Gamma$*?*

If this is the case, then by the transmission conditions (1.5c), (1.5d) we are led to the following free boundary problem.

*Given* $u^e$ *harmonic in* $V$*, find a curve (or surface when* $n > 2$*)* $\Gamma \subset V$ *and* $u^i$ *harmonic on one side of* $\Gamma$ *such that*

$$\begin{aligned} u^i &= u^e \quad \text{on } \Gamma\,, \\ (1+\mu)\partial_\nu u^i &= \partial_\nu u^e \quad \text{on } \Gamma\,. \end{aligned}$$

The fact that $u^e$ is harmonic throughout $V$ will imply that, if such a free boundary $\Gamma$ exists and $n = 2$, then it is a piecewise analytic curve, with possible isolated cusps and corners, see Theorem 2.1 . The transmission conditions (1.5c), (1.5d) will require an appropriate reinterpretation when $\partial D$ is merely assumed to be a Jordan curve, this is the content of Lemma 2.2. In the higher dimensional case, $n \geq 3$, the situation is much more complicated, see Theorem 3.1, Example 3.2 and Lemma 3.3. A complete study of the analogous $n$-dimensional free boundary problem remains, in many respects, open. None the less, given the class $\mathcal{D}$ of domains $D \subset\subset \Omega$ such that $\partial D \in C^{1+\lambda}$ for some $\lambda > 0$ and $\partial D$ is connected, we are able to find a dense subclass $\tilde{\mathcal{D}} \subset \mathcal{D}$ of domains which can be uniquely determined in $\mathcal{D}$ by the boundary data (1.2), (1.3). See Theorem 4.1.

In the following Section 2 we treat the two-dimensional free boundary problem.

Section 3 contains the study of the free boundary problem in higher dimensions.

In Section 4 we apply the results of the previous sections to the study of global uniqueness within "large" classes of domains.

Section 5 contains the global uniqueness results in the "restricted" classes of polygons and polyhedrons.

## 2. Analyticity of the free boundary, two-dimensional case

In this section we consider $D$ to be a planar domain. Let us fix a point $z^1 \in \partial D$ and a simply connected neighbourhood $V$ of $z^1$. Assume that $D \cap V$ is a Jordan domain and denote by $z = z(t)$ a conformal map from the unit disk $B_1(0) = \{t \in \mathbb{C} : |t| < 1\}$ onto $D \cap V$ and such that $z(1) = z^1$. We recall that being $D \cap V$ a Jordan domain, such a map is one-to-one and continuous up to $\partial B_1(0)$. We denote $\Gamma = \partial D \cap V$. Let $u \in H^1(V)$ be a generalized solution to the equation

$$\operatorname{div}((1 + \mu\chi_D)\nabla u) = 0 \quad \text{in } V\,. \tag{2.1}$$

Accordingly to (1.4), we set

$$u^e = u|_{V \setminus \overline{D}}\,, u^i = u|_{D\cap V}\,.$$

THEOREM 2.1. *Suppose that $u^e$ can be harmonically continued from $V \setminus \overline{D}$ onto $V$. Let $N \geq 1$ be the integer such that $u^e - u^e(z^1)$ vanishes at $z^1$ of order $N$. Then there exists a neighbourhood $W$ of the point $t = 1$ such that the map $z = z(t)$ can be represented as follows*

$$z(t) = \omega((t-1)^{k/N} f(t)) \quad \textit{for every } t \in W \cap \overline{B_1(0)}\,. \tag{2.2}$$

*Here $k$ is a positive integer, $f$ is an analytic function satisfying $f(1) \neq 0$, and $\omega$ is a conformal map from a neighbourhood of $0$ onto a neighbourhood of $z^1$.*

COROLLARY 2.2. *Under the same assumptions as above, there exists a neighbourhood of $z^1$ in which $\Gamma$ is the union of two regular analytic curves having one common endpoint at $z^1$ and whose tangents at $z^1$ form an angle which is a rational multiple of $\pi$.*
*Furthermore, if we assume $\nabla u^e(z^1) \neq 0$ and that, near $z^1$, $\Gamma$ is a Lipschitz curve, then, in fact, $\Gamma$ is a regular analytic curve.*

The following Lemma will be needed in the proof of Theorem 2.1.

LEMMA 2.3. *There exist harmonic conjugates $v^e$, $v^i$ to $u^e$, $u^i$ respectively. They satisfy $v^e \in C^\lambda(V \setminus D)$, $v^i \in C^\lambda(\overline{D} \cap V)$ for some $\lambda$, $0 < \lambda < 1$ and, moreover, we have*

$$(1+\mu)v^i = v^e \quad on\, \Gamma\,. \tag{2.3}$$

*Proof.* We construct a so-called stream function associated to $u$, see [B-S] and [A-M] . Consider $w \in H^1(V)$ to be a solution of the first order system

$$\nabla w = (1+\mu\chi_D)(\nabla u)^\perp\,,$$

where $(\cdot)^\perp$ means a counterclockwise rotation of $90^0$. The simple connectedness of $V$ and equation (2.1) provide us with the compatibility conditions of solvability, thus such a $w$ exists and it is unique up to an additive constant. Moreover it satisfies in the weak sense

$$\mathrm{div}((1+\mu\chi_D)^{-1}\nabla w) = 0 \quad \text{in } V\,.$$

In particular, by the already quoted DeGiorgi-Nash-Moser theorem, we have $w \in C^\lambda_{\mathrm{loc}}(V)$ with $0 < \lambda < 1$. Now, let us set

$$v^e = w \quad \text{in} \quad V \setminus D\,, \quad v^i = \frac{1}{1+\mu} w \quad \text{in} \quad \overline{D} \cap V\,,$$

then $v^e$, $v^i$ are harmonic conjugates to $u^e$, $u^i$ respectively, they are Hölder continuous, and they satisfy (2.3). The proof is complete.

*Proof of Theorem 2.1.* Let us introduce the complex analytic functions

$$U^e = u^e + iv^e \quad \text{in } V \setminus D\,, \quad U^i = u^i + iv^i \quad \text{in } \overline{D} \cap V\,.$$

Recalling that $u$ is continuous in $V$ we have $u^i = u^e$ on $\Gamma$, and using (2.3), we obtain

$$(2+\mu)U^e + \mu\overline{U^e} = 2(1+\mu)U^i \quad \text{on } \Gamma\,. \tag{2.4}$$

Let $\gamma$ be an open arc in $\partial B_1(0)$ which is mapped by $z(t)$ into $\Gamma$ and such that $1 \in \gamma$, hence

$$(2+\mu)U^e(z(t)) + \mu\overline{U^e(z(t))} = 2(1+\mu)U^i(z(t)) \quad \text{for every}\, t \in \gamma\,. \tag{2.5}$$

Since $u^e$ can be harmonically continued in $V$, the same holds for $v^e$, and therefore $U^e$ can be analytically continued in $V$. Hence $U^e(z(\cdot))|_\gamma$ can be analytically continued in $B_1(0)$ to a function $F_1$ which is continuous in $B_1(0) \cup \gamma$. We obtain

$$\overline{U^e(z(t))} = \Phi(t) \quad \text{for every } t \in \gamma\,, \tag{2.6}$$

where $\Phi$ is an analytic function in $B_1(0)$ continuous in $B_1(0) \cup \gamma$. By taking the conjugates on both sides of (2.6), setting $\Psi(\bar{t}) = \overline{\Phi(t)}$, and using the identity $\bar{t} = 1/t$ for every $t \in \gamma$, we have $U^e(z(t)) = \Psi(1/t)$ for every $t \in \gamma$, where $\Psi$ is an analytic function in $B_1(0)$ continuous in $B_1(0) \cup \gamma$. Therefore, setting $F_2(t) = \Psi(1/t)$, we arrive at

$$U^e(z(t)) = F_2(t) \quad \text{for every} \quad t \in \gamma\,, \tag{2.7}$$

where $F_2$ is an analytic function in the exterior $G$ of the closed unit disk and continuous in $G \cup \gamma$. Hence by the continuation principle, $U^e(z(\cdot))$ can be analytically continued to $B_1(0) \cup \gamma \cup G$ and, in particular, to a neighbourhood $B_r(1)$ of $1 \in \gamma$. Therefore, we may find a positive integer $k$ and an analytic function $\phi$ in $B_r(1)$, with $\phi(1) \neq 0$, such that

$$U^e(z(t)) = U^e(z^1) + (t-1)^k\phi(t) \quad \text{for every} \quad t \in B_r(1)\,,$$

moreover, since $U^e$ is analytic near $z^1$, we also have

$$U^e(z) = U^e(z^1) + (z-z^1)^N g(z) \quad \text{for every} \quad z \text{ near } z^1\,,$$

where $g$ is an analytic function near $z^1$, with $g(z^1) \neq 0$. We may find a conformal map $z = \omega(\zeta)$ from a neighbourhood of 0 onto a neighbourhood of $z^1$, such that $\omega(0) = z^1$ and also

$$(z-z^1)^N g(z) = \zeta^N\,, \text{when } z = \omega(\zeta)\,, \text{for every}\, \zeta \text{ near } 0\,.$$

Denoting $\zeta(t) = \omega^{-1}(z(t))$, we obtain, for a sufficiently small $r$,

$$(\zeta(t))^N = (t-1)^k \phi(t) \quad \text{for every} \quad t \in B_r(1)\,,$$

hence

$$\zeta(t) = (t-1)^{k/N} f(t) \quad \text{for every} \quad t \in B_r(1) \cap \overline{B_1(0)}\,, \tag{2.8}$$

where $f$ denotes an analytic branch of $\phi^{1/N}$. Applying $\omega$ to both sides of (2.8) we obtain (2.2). The proof is complete.

REMARK. Analiticity results for related free boundary problems, like the obstacle problem and the dam problem, are well known, see for instance [F]. In particular, the dam problem can be formulated like ours, however, a quite special geometry of the domain is prescribed and the choice of $u^e$ is very particular ($u^e = x_2$).

*Proof of Corollary 2.2.* The first part of the statement is a straightforward consequence of (2.2). Consider now the case $\nabla u^e(z^1) \neq 0$, that is, $N = 1$. Being $z(t)$ one-to-one in the closed unit disk, we have that only two values of $k$ are admissible in (2.2), $k = 1$ and $k = 2$. If $k = 2$ then $\Gamma$ has an algebraic singular point at $z_1$, and it is locally a Hölder, but not Lipschitz, continuous curve. Thus we are left with the case $k = 1$, from (2.2) we obtain $z'(1) \neq 0$ and the proof is complete.

## 3. Analyticity of the free boundary, $n$-dimensional case

Let us consider now $D$ to be a domain in $\mathbb{R}^n$. Let us fix a point $z^1 \in \partial D$ and a ball $B$ centered at $z_1$, we denote $\Gamma = \partial D \cap B$. Let $u \in H^1(B)$ be a generalized solution to the equation

$$\operatorname{div}((1 + \mu\chi_D)\nabla u) = 0 \quad B\,. \tag{3.1}$$

As before, we set $u^e = u|_{B\setminus D}\,, u^i = u|_{\overline{D\cap B}}$.

THEOREM 3.1. *Suppose that $u^e$ can be harmonically continued onto $B$. If in addition*

$$\Gamma \in C^{1+\lambda}\,, \tag{3.2}$$

*and*

$$\partial_\nu u^e(z^1) \neq 0\,, \tag{3.3}$$

*then $\Gamma$ is a regular analytic hypersurface near $z^1$.*

The following example shows that hypothesis (3.3) is necessary in Theorem 3.1. The same example appears, for a different purpose, in the paper of Kinderlehrer and Nirenberg [K-N].

EXAMPLE 3.2. For any $x \in \mathbb{R}^n$, set $x = (x_1, x_2, y)$ with $x_1, x_2 \in \mathbb{R}$, $y \in \mathbb{R}^{n-2}$, and let $f : \mathbb{R}^{n-2} \to \mathbb{R}$ be any $C^{1+\lambda}$ function. Consider $D = \{x \in \mathbb{R}^n : x_2 > f(y)\}$ and $u(x) = x_1$. We have

$$\Delta u = \operatorname{div}((1 + \mu\chi_D)\nabla u) = 0 \quad \text{in}\, \mathbb{R}^n\,,$$

and also

$$\partial_\nu u^e = \partial_\nu u = 0 \quad \text{everywhere on } \partial D\,,$$

but the regularity of $\partial D$ is the same as the one of $f$.

The above example suggests the following susbtitute for Theorem 3.1 when condition (3.3) does not hold.

LEMMA 3.3. *Suppose that $u^e$ can be harmonically continued onto $B$ and assume (3.2). If in addition*

$$|\nabla u^e(z^1)| \neq 0\,, \tag{3.4}$$

*and also*

$$\partial_\nu u^e = 0 \quad \textit{everywhere on} \quad \Gamma\,, \tag{3.5}$$

*then there exists a neighbourhood $V$ of $z^1$ such that $\Gamma \cap V$ can be represented as a $(n-2)$-parameter family of regular analytic curves.*

*Proof of Theorem 3.1.* Let us denote by $h$ the harmonic continuation of $u^e$ onto $B$, and denote $u = \frac{1+\mu}{\mu}(h - u^i)$ in $\overline{D} \cap B$, we have

$$\Delta u = 0 \quad \text{in} \quad D \cap B\,, \tag{3.5a}$$

$$u = 0 \quad \text{on} \quad \Gamma\,, \tag{3.5b}$$

$$\partial_\nu u = \partial_\nu h \quad \text{on} \quad \Gamma\,. \tag{3.5c}$$

Thus by (3.3), in a neighbourhood of $z^1$ we have $\nabla u \neq 0$ and also

$$\nu = \pm \frac{\nabla u}{|\nabla u|}\,,$$

hence (3.5c) can be rewritten as

$$|\nabla u|^2 - \nabla h \cdot \nabla u = 0 \quad \text{on} \quad \Gamma\,. \tag{3.5c'}$$

Notice that, if we had the additional assumption $u \in C^2(\overline{D} \cap B)$, then the proof would follow by a direct application of Theorem 2 in [K-N] to problem (3.5a)(3.5c'). Under the present regularity assumption (3.2), it seems necessary to reformulate the arguments in [K-N], based on an appropriate hodograph method, in order to take advantage of the divergence structure of (3.5a). We follow the track of Theorem 4.1.4 in [Is1], where similar arguments were used in connection to the inverse problem of potential theory. We can assume

$$\partial_\nu h(z^1) = \partial_\nu u^e(z^1) < 0\,,$$

and also that $z^1 = 0$ and $\nu(0)$ coincides with the negative direction of the $x_1$-axis . We also set $x = (x_1, x')$ with $x' \in \mathbb{R}^{n-1}$. Let us define the hodograph type map

$$x \to y(x), \quad y_1(x) = u(x), \quad y'(x) = x'\,, \tag{3.6}$$

since, by (3.2), we have $u^i \in C^{1+\lambda}(\overline{D} \cap B)$ for some $\lambda$, $0 < \lambda < 1$, hence also $u$ and the map $y = y(x)$ are of class $C^{1+\lambda}(\overline{D} \cap B)$ and, by the Whitney theorem $y(x)$ can be continued to a $C^{1+\lambda}$ map on $B$. Its Jacobian at 0 is, by (3.5c), $\partial_{x_1} u(0) = -\partial_\nu h(0) > 0$. Thus, possibly shrinking the radius of $B$ , the hodograph map from $B$ onto a neighbourhood $V$ of 0 is invertible, with inverse

$$y \to x(y), \quad x_1(y) = w(y), \quad x'(y) = y'\,, \tag{3.7}$$

by (3.5b), and since $\partial_{x_1} u(0) > 0$, we have that the hodograph map transforms $D \cap B$ and $\Gamma$ onto $V^+ = \{y \in V : y_1 > 0\}$ and $\Sigma = \{y \in V : y_1 = 0\}$ respectively. From the results of [Is1], Lemma 4.1.5, we have that (3.5a) induces the following divergence form equation for $w$

$$\partial y_1 \left( \frac{1}{(\partial_{y_1} w)^2} (1 + |\nabla_{y'} w|^2) \right) - 2\mathrm{div}_{y'} \left( \frac{1}{\partial_{y_1} w} \nabla_{y'} w \right) = 0 \quad \text{in } V^*\,. \tag{3.8}$$

This equation can be seen to be elliptic for $w$ provided

$$\partial_{y_1} w > 0\,.$$

This condition is satisfied at $y = 0$, hence everywhere in $V$, provided the radius of $B$ is chosen small enough. Thus (3.8) is a quasilinear uniformly elliptic equation with analytic dependence on its arguments. From (3.5c') one can obtain also a boundary condition for $w$ on $\Sigma$. By differentiation of the identity $u(w(y), y') = y_1$ we have, for $x = x(y)$,

$$\partial_{x_1} u(x) = \frac{1}{\partial_{y_1} w(y)}, \quad \nabla_{x'} u(x) = -\frac{1}{\partial_{y_1} w(y)} \nabla_{y'} w(y)\,,$$

and therefore we obtain

$$1 + |\nabla_{y'} w|^2 - ((\partial_{x_1} h)(w, y') - (\nabla_{x'} h)(w, y') \cdot \nabla_{y'} w)\, \partial_{y_1} w = 0 \text{ on } \Sigma\,. \tag{3.9}$$

This is a regular nonlinear oblique derivative condition, with analytic dependence on its arguments. Applying to (3.8), (3.9) the standard method of difference quotients and the estimates at the boundary of Agmon, Douglis and Nirenberg [A-D-N], Theorem 9.1, we obtain $w \in C^{2+\lambda}(V^+ \cup \Sigma)$. Then we can invoke the analyticity theorem of Morrey, see [M], Theorem 6.7.6', to obtain that $w$ can be continued to a real analytic function defined on a neighbourhood of the origin $y = 0$. Finally, $\Gamma$ is the graph $\{x_1 = w(0, x'\})$ of the analytic function $w(0, \cdot)$. The proof is complete.

Remark. It would be interesting to know whether Theorem 3.1 continues to hold when (3.2) is replaced by $\Gamma \in Lip$. Related regularity results are due to Caffarelli, see for instance [C], but unfortunately, it seems that our problem cannot be directly reduced to his setting.

*Proof of Lemma 3.3.* Possibly shrinking the radius of $B$, we may assume $|\nabla u^e| \neq 0$ everywhere on $B$. The condition (3.5) says that $\nabla u^e(x)$ is tangent to $\Gamma$ at any $x \in \Gamma$. Therefore, for any $x \in \Gamma$, the line of steepest descent of $u^e$ passing through $x$ remains in $\Gamma$. Hence, $\Gamma$ is composed by lines of steepest descent of $u^e$ which are regular analytic curves. The proof is complete.

## 4. Uniqueness in "large" classes of domains

We consider two classes of domains, $\mathcal{D}$, $\tilde{\mathcal{D}}$, which we define in a different fashion depending on the space dimension. When $n = 2$, we set

$$\mathcal{D} = \{D \subset\subset \Omega : D \text{ is a Jordan domain}\,, \} \tag{4.1}$$

$$\tilde{\mathcal{D}} = \{D \in \mathcal{D} : \text{for any } z \in \partial D, \partial D \text{ is not analytic in any neighbourhood of } z\} , \tag{4.2}$$

and, when $n > 2$, we define

$$\mathcal{D} = \Big\{D \subset\subset \Omega : D \text{ is a domain}, \partial D \in C^{1+\lambda} \text{ for some } \lambda > 0 \text{ and } \partial D \text{ is connected}\} , \tag{4.3}$$

$$\tilde{\mathcal{D}} = \{D \in \mathcal{D} : \text{for any } z \in \partial D, \partial D \text{ is not a } (n-2)\text{–parameter family of regular analytic curves in any neighbourhood of } z\} . \tag{4.4}$$

We also impose that the empty set $\emptyset$ belongs to both classes $\mathcal{D}$ and $\tilde{\mathcal{D}}$.

Observe that it is a rather straightforward matter to prove that $\tilde{\mathcal{D}}$ is dense in $\mathcal{D}$, with respect to the Hausdorff metric.

THEOREM 4.1. *Let $D \in \tilde{\mathcal{D}}$, and let $\mu$ be a given constant, $-1 < \mu$, $\mu \neq 0$. Let $u$ be a non constant solution of (1.1). For any open, nonempty, subset $\Gamma_0$ of $\partial\Omega$, the data $g$, $h$ in (1.2), (1.3) uniquely determine $D$ in $\mathcal{D}$.*

REMARK. It might seem that this uniqueness result can be of little practical use since the condition $D \in \tilde{\mathcal{D}}$ is unstable under small perturbations. However, we wish to stress that the principal aim here is to show the existence of domains $D$ in the larger class $\mathcal{D}$ which are uniquely determined by one data pair $g$, $h$.

*Proof of Theorem 4.1.* Let us denote $D_1 = D$ and $u_1 = u$. We assume, by contradiction, that there exists $D_2 \in \mathcal{D}$, $D_2 \neq D_1$ such that the solution $u_2$ of (1.1), (1.2) when $D$ is replaced with $D_2$, satisfies also (1.3). Let $G$ be the connected component of $\Omega \setminus (\overline{D_1 \cup D_2})$ such that $\partial\Omega \subset \partial G$.

We show that our assumption by contradiction implies that $\Gamma = (\partial D_1 \setminus \overline{D_2}) \cap \partial G$ has nonempty interior in $\partial D_1$. Were it not so, we would have $\partial G \subset \partial\Omega \cup \partial D_2$, and, being $\partial D_2$ connected, $D_1 \subset D_2$. By [A], Theorem 1.1 , and by the definition of the class $\mathcal{D}$, this last condition would imply $D_1 = D_2$.

Since $u_1$, $u_2$ are both harmonic in $G$ and have the same Cauchy data on $\Gamma_0$, we have $u_1 = u_2$ in $G$. Therefore, using for both $u_1$, $u_2$, the notation (1.4) where $D$ is replaced with $D_1$, $D_2$, respectively, we have that, on $\Gamma$, $u_1^e$ can be harmonically continued inside $D_1$ by the function $u_2^e$. Now, $u_2^e$ is nonconstant, because, otherwise also $g$ and $u_1$ would be constant, thus the set of critical points $\{x \in \Gamma : \nabla u_1^e = 0\} = \{x \in \Gamma : \nabla u_2^e = 0\}$ has empty interior in $\Gamma$. Let us now distinguish the cases, $n = 2$, $n > 2$. When $n = 2$, we may apply Theorem 2.1 to $u_1$, and find a regular analytic arc contained in $\Gamma$. When $n > 2$, either we may find $z^1 \in \Gamma$ such that $\partial_\nu u_1^e(z^1) \neq 0$ or there exists an open subset of $\Gamma$ on which $\partial_\nu u_1^e = 0$. Thus, by Theorem 3.1 and Lemma 3.3, we obtain that $\Gamma$ contains a $(n-2)$-parameter family of regular analytic curves. Thus, for any dimension, $n \geq 2$, we have $D_1 \notin \tilde{\mathcal{D}}$, which contradicts our hypothesis. The proof is complete.

## 5. Uniqueness in "restricted" classes of domains

In this section we shall consider the unique determination of the domain $D$ within the class $\mathcal{P}$ of, possibly empty, convex polygons or polyhedrons $D \subset\subset \Omega$, when $n = 2$, $n = 3$, respectively. We shall need some additional information on the boundary data, for instance, when $n = 2$, we assume that, roughly speaking, the Dirichlet data $g$ has only one maximum on $\partial\Omega$. On the other hand, we will be able to identify also the parameter $\mu$. For this purpose, let us denote by $I = (-1, 0) \cup (0, \infty)$ the range of values of $\mu$. Notice that the case $\mu = 0$ is excluded, because in equation (1.1), this case can also be expressed by setting $D = \emptyset$.

Let us consider first the case $n = 2$ and let $\partial\Omega$ be decomposed into two arcs $\gamma$, $\delta$. We shall assume that the Dirichlet data $g$ in (1.2) is nonconstant and satisfies

$$g \quad \text{is monotone on } \gamma \text{ and on } \delta, \text{ separately}. \tag{5.1}$$

THEOREM 5.1. *Let $n = 2$, let $D \in \mathcal{P}$, and let $\mu \in I$. Let $u$ be the solution of (1.1), (1.2) and let $g$ satisfy (5.1). For any open, nonempty, subset $\Gamma_0$ of $\partial\Omega$, the data $g$, $h$ in (1.2), (1.3) uniquely determine $D$ in $\mathcal{P}$ and $\mu$ in $I$.*

We shall make use of the following Lemma.

LEMMA 5.2. *Let $g$ satisfy (5.1) and let $u$ be the solution to (1.1), (1.2). The gradient of $u$ never vanishes in $\Omega \setminus \partial D$.*

*Proof.* See Theorem 2.7 in [A-M].

Remark. Let us remark that the arguments in [A-M] are based on an adaptation of the classical index calculus to the gradients of weak solutions of divergence form elliptic equations. In fact in [A-M], it is proved that, in a generalized sense, $\nabla u$ has zero index at every point of $\Omega$, and that this generalized notion of index coincides with the classical one at points where $\nabla u$ is smooth.

Let us also notice here that the approach in [A-M] would allow us to state Lemma 5.2 when different types of boundary data are prescribed on $u$. For instance, we could replace the Dirichlet condition (1.2), with the Neumann data

$$\partial_\nu u = h \quad \text{on } \partial\Omega\,, \tag{5.2}$$

the monotonicity conditions (5.1) should be replaced by the conditions

$$h \geq 0 \quad \text{on} \quad \gamma, \quad h \leq 0 \quad \text{on} \quad \delta\,. \tag{5.3}$$

Also boundary conditions of mixed type could be considered, like, for instance, the following one. Suppose that $\partial\Omega$ is decomposed into 4 consecutive arcs $\alpha$, $\beta$, $\gamma$, $\delta$, and prescribe, instead of (1.2), the boundary condition

$$u = 1 \text{ on } \alpha, \quad u = -1 \text{ on } \gamma, \quad \partial_\nu u = 0 \text{ on } \beta \cup \delta\,. \tag{5.4}$$

Consequently, Theorem 5.1 continues to hold when the boundary condition (1.2), (5.1) is replaced by (5.2), (5.3) or by (5.4) . Of course, it will be necessary to assume, in place of (1.3), that on $\Gamma_0$, the Cauchy data $\nabla u|_{\Gamma_0}$ are known.

*Proof of Theorem 5.1.* First we prove the unique determination of $D$. Assume the opposite, then there are convex polygons $D_1$, $D_2 \in \mathcal{P}$, $D_1 \neq D_2$, numbers $\mu_1$, $\mu_2 \in I$ and corresponding solutions $u_1$, $u_2$ satisfying the same boundary data (1.2), (1.3). Since $D_1 \neq D_2$ and since they are convex, there is one vertex $z^1$ of one of them (say, of $D_1$) which is outside of the second one. Again by the convexity, we have that $\Omega \setminus (\overline{D_1 \cup D_2})$ is a connected open set, on such a set $u_1$, $u_2$ are both harmonic and hence must coincide because they have equal Cauchy data on $\Gamma_0$. The function $u_2$ is harmonic in a neighbourhood of $z^1$, and by Lemma 5.2, we have $\nabla u_2(z^1) \neq 0$. Applying Corollary 2.2 with $u = u_1$ we obtain that $\partial D_1$ is a regular analytic curve near

$z^1$. This is a contradiction because $z^1$ is a vertex of $D_1$. Hence we have $D_1 = D_2$. Let us suppose now $\mu_1 \leq \mu_2$, then the conductivity coefficients in (1.1) are $a_1 = 1 + \mu_1\chi_{D_1} \leq a_2 = 1 + \mu_2\chi_{D_1}$. By the monotonicity argument used in [A], Theorem 1.1, see also [Is2] Section 1.1, we obtain that since $u_1$, $u_2$ have the same Cauchy data on $\Gamma_0$, then we must have $a_1 = a_2$. The proof is complete.

Let us consider now the case $n = 3$. We will prescribe that the Dirichlet data in (1.2) have the form

$$g = \chi_E\,, \tag{5.5}$$

where $E$ is a given open subset of $\partial\Omega$. Notice that in this case, $g \notin H^{1/2}(\partial\Omega)$ and the solution u has to be meant as the $H^1_{\mathrm{loc}}(\Omega)$ limit of solutions to (1.1) satisfying (1.2) with $g$ replaced with smooth approximations. Let us just observe that, by our smoothness assumptions on $\partial\Omega$, we have that $u$ is continuous up to $\partial\Omega$ where $g$ is continuous, that is on $\partial\Omega \setminus \partial E$. Here $\partial E$ denotes the boundary of $E$ relative to $\partial\Omega$.

Our uniqueness theorem below will be based on arguments of symmetry, rather than of analyticity, see [F-I] and [B-F-S]. For this purpose we shall need the following definition. Given an oriented line $\alpha$ in $\mathbb{R}^3$ and an angle $\theta \in (0, 2\pi)$, we denote by $R(\alpha, \theta)$ the rotation around $\alpha$ of angle $\theta$.

Definition. Given a positive integer $K$, we say that the pair of sets $(\Omega, E)$ satisfies the $S_K$ condition, if there exists a point $P$, $K$ distinct lines $\alpha_1, \ldots, \alpha_K$ passing through $P$, and angles $\theta_1, \ldots, \theta_K$ such that we have

$$R(\alpha_k, \theta_k)\Omega = \Omega,\ R(\alpha_k, \theta_k)E = E, \text{ for every } k = 1, \ldots, K\,. \tag{5.6}$$

Theorem 5.3. *Let $n = 3$, let $D \in \mathcal{P}$, and let $\mu \in I$. Let $u$ be the solution of (1.1), (1.2) and let $g$ be given by (5.5). Suppose that the pair $(\Omega, E)$ does not satisfy the condition $S_4$. For any open, nonempty, subset $\Gamma_0$ of $\partial\Omega$, the data $g$, $h$ in (1.2), (1.3) uniquely determine $D$ in $\mathcal{P}$ and $\mu$ in $I$.*

Example. Let us show a simple procedure which exibits, for a given domain $\Omega$, subsets $E \subset \partial\Omega$ such that the pair $(\Omega, E)$ does not satisfy the condition $S_2$, and hence also $S_4$. This construction is due to Tuljak, [T].

Suppose, without loss of generality, that the origin is contained in $\Omega$, and let $t \in \mathbb{R}$ be such that $(0,0,t) \in \Omega$. Set $E = \{x \in \partial\Omega : x_3 > t\}$. Notice that $\partial E$ is contained in a plane orthogonal to the $x_3$-axis, hence E may have rotational symmetries only around lines parallel to the $x_3$-axis, otherwise $\partial E$ would be contained in the intersection line of two incident planes. Therefore (5.6) may be satisfied with $K = 1$ at most.

*Proof of Theorem 5.3.* As in Theorem 5.1, let us assume by contradiction that there are convex polyhedrons $D_1$, $D_2 \in \mathcal{P}$, $D_1 \neq D_2$, numbers $\mu_1$, $\mu_2 \in I$ and corresponding solutions $u_1$, $u_2$ satisfying the same boundary data (1.2), (1.3). As before, we may assume that there is one vertex $z^1$ of $D_1$ which is outside $D_2$. Again we obtain that, in $\Omega \setminus (\overline{D_1 \cup D_2})$, $u_1 = u_2$. In particular we have that, near $z^1$, $u_1^e = u_2$ and this last function in harmonic in a full neighbourhood of $z^1$. Let $\alpha_1, \ldots, \alpha_K$, $K \geq 3$ be the lines containing the edges of $D_1$ ending at $z^1$. We may then apply a result in [F-I], Lemma 4.1, and obtain that there exist angles $\theta_1, \ldots, \theta_K$ such that we have

$$\begin{aligned} u_2 = u_2 \circ R(\alpha_k, \theta_k) \text{ in a neighbourhood } B \text{ of } z^1\,, \\ \text{for every } k = 1, \ldots, K\,. \end{aligned} \tag{5.7}$$

Notice that, by composition of the rotations $R(\alpha_k, \theta_k)$, $u_2$ has also rotation symmetries around the rotated axes $R(\alpha_j, \theta_j)\alpha_k$ for every $j, k = 1, \ldots, K$, $j \neq k$. Being the edges of $D_1$ at $z^1$ at least 3, we obtain that (5.7) holds for at least $K = 4$ distinct axes. Let us fix $k = 1, \ldots, K$. By the convexity of $D_2$, we may choose a half line $\alpha_k^+ \subset \alpha_k$ starting from $z^1$, which does not intersect $\overline{D_2}$. Let $P_k \in \partial\Omega \cap \alpha_k$ be the first point of $\partial\Omega$ encountered when moving from $z^1$ along $\alpha_k^+$. Let $T_k$ be a, sufficiently small, tubular neighbourhood of the segment $\overline{z^1 P_k}$ and let $T_k'$ be the connected component of $T_k \cap \overline{\Omega} \cap R(\alpha_k, \theta_k)\overline{\Omega}$ which contains $\overline{z^1 P_k}$. By (5.7) and by harmonic continuation, we obtain

$$u_2 = u_2 \circ R(\alpha_k, \theta_k) \quad \text{in } T_k'\,.$$

Let us show that $\partial\Omega \cap T_k = R(\alpha_k, \theta_k)(\partial\Omega \cap T_k)$. Were it not so, we could find a point $Q \in T_k' \setminus R(\alpha_k, \theta_k)(\partial\Omega \cap T_k))$ and a neighbourhood $V$ of $Q$ in which $u_2$ could be harmonically continued by the function $u_2 \circ R(\alpha_k, \theta_k)$, thus $u_2$ is continuous near $Q$, hence it identically equals 0 or 1 on $\partial\Omega \cap V$. This is impossible, because by the strong maximum principle and by (1.2), (5.5), the interior values of $u_2$, and hence those of $(u_2 \circ R(\alpha_k, \theta_k))|_V$, are strictly between 0 and 1. Therefore, $\partial\Omega \cap T_k = R(\alpha_k, \theta_k)(\partial\Omega \cap T_k)$ and, in a neighbourhood, relative to $\overline{\Omega}$, of such a set, we have $u_2 = u_2 \circ R(\alpha_k, \theta_k)$. We can repeat the above argument, combining the maximum principle and harmonic continuation, all around $\partial\Omega$. We obtain

$$\partial\Omega = R(\alpha_k, \theta_k)\partial\Omega\,,$$

$$u_2 = u_2 \circ R(\alpha_k, \theta_k) \quad \text{in a neighbourhood, relative to } \overline{\Omega}, \text{ of } \partial\Omega\,.$$

Hence, by (1.2) and (5.5), (5.6) holds with $K \geq 4$, that is $(\Omega, E)$ satisfies the $S_4$ condition, contrary to our hypothesis. Therefore we have $D_1 = D_2$. Next, we obtain $\mu_1 = \mu_2$ by the same argument as in Theorem 5.1. The proof is complete.

REMARK. It can be noticed that, in view of the arguments used in Theorems 5.1, 5.3, it is possible to prove with minor adaptations, the following variation of Theorem 4.1 in which also the parameter $\mu$ is unknown, at the cost of searching the unknown domain $D$ in the restricted class $\tilde{\mathcal{D}}$ rather than in $\mathcal{D}$.

THEOREM 4.1'. *Let $D \in \tilde{\mathcal{D}}$, and let $\mu \in I$. Let $u$ be a non constant solution of (1.1). For any open, nonempty, subset $\Gamma_0$ of $\partial\Omega$, the data $g$, $h$ in (1.2), (1.3) uniquely determine $D$ in $\tilde{\mathcal{D}}$.*

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