Homoclinic Solutions for Second Order Systems with Expansive Time Dependence

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Sommario. - *Si dimostra l'esistenza di almeno una soluzione omoclina per sistemi Lagrangiani della forma* $-\ddot{u}+u=\alpha(t)\nabla G(u)$ *in* $\mathbf{R}^N$ *dove* $G\in\mathcal{C}^2(\mathbf{R}^N,\mathbf{R})$ *è superquadratica e* $\alpha\in\mathcal{C}^1(\mathbf{R},\mathbf{R})$ *soddisfa la condizione* $\lim_{|t|\to\infty}\dot{\alpha}(t)=0$. *Il metodo è variazionale: le soluzioni omocline del sistema risultano essere punti critici di un opportuno funzionale d'azione. Si dimostra l'esistenza di almeno un punto critico non banale usando l'analisi dei problemi "all'infinito" e argomenti di confronto sui livelli.*

Summary. - *We prove the existence of homoclinic solutions for second order Lagrangian systems of the type* $-\ddot{u}+u=\alpha(t)\nabla G(u)$ *in* $\mathbf{R}^N$ *where* $G\in\mathcal{C}^2(\mathbf{R}^N,\mathbf{R})$ *is superquadratic and* $\alpha\in\mathcal{C}^1(\mathbf{R},\mathbf{R})$ *satisfies the condition* $\lim_{|t|\to\infty}\dot{\alpha}(t)=0$. *The method is variational: solutions being found as critical points of a suitable action functional. We prove the existence of at least one nontrivial critical point using the analysis of problems "at infinity" and level comparison arguments.*

1. Introduction and variational setting

In this paper we study the existence of at least one homoclinic solution for a class of second order Lagrangian systems. In particular we will investigate the problem

$$\begin{cases} -\ddot{u}(t)+u(t)=\alpha(t)\nabla G(u(t)) & \text{in } \mathbf{R}^N \\ u(t)\to 0,\ \dot{u}(t)\to 0 \quad \text{as} & |t|\to\infty \end{cases} \tag{P}$$

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where $u : \mathbf{R} \to \mathbf{R}^N$, $N \geq 1$, and α and G satisfy the following conditions:

(H_1) $G \in \mathcal{C}^2(\mathbf{R}^N; \mathbf{R})$, $\alpha \in \mathcal{C}^1(\mathbf{R}; \mathbf{R})$ and $\alpha(t) > 0$ for all $t \in \mathbf{R}$,

(H_2) there exists $\theta > 2$ such that $0 < \theta G(x) \leq \nabla G(x) \cdot x$ for all $x \in \mathbf{R}^N \setminus \{0\}$,

(H_3) there results $\overline{\alpha} := \limsup_{|t|\to\infty} \alpha(t) > \underline{\alpha} := \liminf_{|t|\to\infty} \alpha(t) > 0$ and $\lim_{|t|\to\infty} \dot{\alpha}(t) = 0$.

Moreover we will need an abstract assumption, see condition (A) below.

Solutions of problem (P) are obtained as critical points of the action functional

$$f(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}} \alpha(t) G(u(t)) dt$$

defined on the Sobolev space $H := H^1(\mathbf{R}; \mathbf{R}^N)$ endowed with the usual norm

$$\|u\| := (\int_{\mathbf{R}} (|\dot{u}(t)|^2 + |u(t)|^2) dt)^{\frac{1}{2}}.$$

It is known, see e.g. [CZR], that if (H_1) and (H_2) hold then $f \in \mathcal{C}^2(H; \mathbf{R})$ and satisfies the geometric assumptions of the Mountain Pass Theorem. In particular, setting $\Gamma := \{\gamma \in \mathcal{C}([0,1]; H) \,:\, \gamma(0) = 0,\ f(\gamma(1)) < 0\}$, there results:

$$c := \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} f(\gamma(s)) \in \mathbf{R}^+.$$

Using variational methods this kind of problem has been widely investigated in recent years assuming various types of time dependence on the potential. We refere e.g. to [AB], [ACZ], [B] [CM], [CZES], [CZR], [R], [S1,2] for the periodic case, to [BB], [CZMN], [STT] for the almost periodic one and to [MNT] in the case of recurrent time dependence. Furthermore we mention [M] and [ACM] for the asymptotically periodic case and [MN] for perturbations of periodic potentials.

For this kind of problem the Palais Smale condition does not hold. In fact we lose compactness of those Palais Smale (PS for short) sequences which carry "mass" at infinity, i.e. PS sequences

(u_n) for which there exists a sequence (τ_n) in $\mathbf{R}$ such that $|\tau_n| \to \infty$ and $\liminf_{n\to\infty} |u_n(\tau_n)| > 0$. It is an important feature of the problem that such sequences can be characterized by solutions of suitable problems at "infinity" associated to (P). Indeed one can show that the sequence of translates $(u_n(\cdot + \tau_n))$ weakly converges in H to a nontrivial solution of one of these problems. In the case of recurrent time dependence (where with recurrent we include periodic, almost periodic and perturbations of periodic potentials) the corresponding problems at infinity have the same structure as the original problem. Using this fact it is possible to select the sequence (τ_n) introduced above so that the sequence $(u_n(\cdot + \tau_n))$ weakly converges to a nontrivial solution of the original problem. For example, in the periodic case the sequence (τ_n) can be made up of multiples of a period of α while in the almost periodic and recurrent cases the construction is more delicate but still possible.

On the other hand in our case, as a consequence of (H_3), the problems at infinity are a continuous family of autonomous problems, and precisely they are

$$\begin{cases} -\ddot{u} + u = \beta \nabla G(u) & \text{in } \mathbf{R}^N \\ u(t) \to 0, \ \dot{u}(t) \to 0 & \text{as } \ |t| \to \infty, \end{cases} \tag{P_β}$$

with β constant between $\underline{\alpha} := \liminf_{|t|\to\infty} \alpha(t)$ and $\overline{\alpha} := \limsup_{|t|\to\infty}$ $\alpha(t)$. Clearly the argument used in the recurrent cases can not be applied. In this case we can overcome the lack of compactness using some level comparison considerations. Indeed, we assume that the action functionals f_β corrisponding to the problems at infinity (P_β) satisfy the following abstract condition:

(A) for every $\beta > 0$ the mountain pass level c_β corresponding to f_β is the smallest nonzero critical value.

When (A) holds we can prove that there exists a level $c^* > 0$ (determined by the critical levels at infinity c_β) such that the PS condition holds at levels strictly less than c^* at least for those PS sequences (u_n) which verify the additional property $\|u_n - u_{n-1}\| \to 0$ (namely $\overline{PS}$ sequences).

Condition (A) allows us to characterize the level c^* and moreover to ensure the existence of a $\overline{PS}$ sequence at level strictly less than it and therefore the existence of at least one nontrivial solution of (P). The main result that we will prove is the following:

THEOREM 1.1. *If $(H_1)-(H_3)$ and (A) hold then problem (P) admits at least one nontrivial classical solution.*

Finally we point out that our arguments are somewhat related with those in [EL], [L] and [DN] where the autonomous problem at infinity is only one, that is (P_{α_∞}) where $\alpha_\infty = \lim_{|t|\to\infty} \alpha(t)$. In these papers existence results are obtained using level comparison arguments, and variuos kind of hypotheses on the *global* behaviour of α are made to ensure the existence of a PS sequence at the "right" level.

A few comments about assumption (A) are in order. First note that the mountain pass level c_β is always critical for f_β (see e.g. [AB], [C] and [RT]), so that condition (A) only requires that c_β be the smallest critical level for f_β. Moreover (A) is always satisfied in the scalar case $N = 1$ while in the general case $N > 1$ it is well known that (A) holds if G satisfies the condition

(H_4) for all $x \in \mathbf{R}^N \setminus \{0\}$ there results $\quad \nabla G(x) \cdot x < \nabla^2 G(x) x \cdot x.$

See e.g. [RT] for a proof of this.

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2. Preliminary properties

In this section we will collect some preliminary results that will be used in the sequel.

Note that by (H_1) and (H_3) we have that there exist two constants $\overline{a} > \underline{a} > 0$ such that

$$\underline{a} \le \alpha(t) \le \overline{a}, \quad \forall t \in \mathbf{R}. \tag{2.1}$$

Then the following results can be proved (see e.g. [R])

LEMMA 2.1. *There results*

$$\inf_{u\in\mathcal{K}} \|u\| > 0 \quad \textit{and} \quad c^o := \inf_{u\in\mathcal{K}} f(u) > 0.$$

where $\mathcal{K} := \{u \in H \mid u \not\equiv 0,\ \nabla f(u) = 0\}$.

Now we begin to study the PS sequences for the functional f.

LEMMA 2.2. *Let (u_n) be a PS sequence for f at level $b \in \mathbf{R}$. Then (u_n) is bounded in H and $b \geq 0$. Moreover if $f(u_n) \to 0$ then $u_n \to 0$ in H.*

Proof. By (H_2) we have

$$f(u_n) - \frac{1}{\theta}\nabla f(u_n) \cdot u_n \geq (\frac{1}{2} - \frac{1}{\theta})\|u_n\|^2.$$

Therefore (u_n) is bounded in H and we also have that $b \geq 0$ and that if $b = 0$ then $u_n \to 0$. $\diamondsuit$

Furthermore ∇f is weakly continuous in H, i.e. if $u_n \rightharpoonup u_0$ weakly in H then $\nabla f(u_n) \rightharpoonup \nabla f(u_0)$ weakly in H.

The next results can be proved as in [STT].

LEMMA 2.3. *Let (v_n) be a sequence in H such that $v_n \rightharpoonup v_0$ weakly in H. Then as $n \to \infty$ we have*

(*i*) $\int_{\mathbf{R}} |G(v_n - v_0) - G(v_n) + G(v_0)|dt \to 0,$

(*ii*) $\sup_{\varphi \in H}[\int_{\mathbf{R}} |\nabla G(v_n - v_0) - \nabla G(v_n) + \nabla G(v_0)||\varphi|dt] \to 0,$

and, in particular

(*iii*) $|f(v_n - v_0) - f(v_n) + f(v_0)| \to 0,$

(*iv*) $\|\nabla f(v_n - v_0) - \nabla f(v_n) + \nabla f(v_0)\| \to 0.$

In particular it follows

PROPOSITION 2.1. *Let (u_n) be a PS sequence for f at level $b \in \mathbf{R}$. Then there exist a subsequence of (u_n), still denoted u_n, and $u_0 \in H$ such that:*

(*i*) $u_n \rightharpoonup u_0$ *weakly in* H,

(*ii*) $\nabla f(u_0) = 0$ *and* $f(u_0) \leq b$, *(iii)* $(u_n - u_0)$ *is a PS sequence for* f *at level* $b - f(u_0)$.

Since f satisfies the geometric assumptions of the Mountain Pass Theorem we can find a PS sequence for f at level $c > 0$. From Proposition 2.1, this sequence admits a weak limit point $u_0 \in H$ which is a critical point for f. Then we obtain a nontrivial solution of problem (P) if we can prove that $u_0 \not\equiv 0$. Therefore in the next sections we will study in detail the PS sequences weakly convergent to zero.

3. Problems at infinity and compactness properties

In this section we will investigate the problems at infinity associated to (P). First of all one can prove, like in [CZR], the following result concerning a nonvanishing property of the PS sequences.

LEMMA 3.1. *Let* (u_n) *be a PS sequence for* f *at a positive level. Then*

$$\liminf_{n\to\infty} \|u_n\|_\infty > 0$$

where $\|\cdot\|_\infty$ *is the usual norm in* $L^\infty(\mathbf{R};\mathbf{R}^N)$.

In particular for every PS sequence (u_n) at a positive level weakly convergent to zero we can find a sequence (τ_n) in $\mathbf{R}$ with $|\tau_n| \to \infty$ such that $\liminf_{n\to\infty} |u_n(\tau_n)| > 0$. Then if we consider the sequence $v_n = u_n(\cdot + \tau_n)$ we have that, up to a subsequence, this sequence weakly converges to a nonzero limit point. In the sequel we will show using (H_3) that such limit point is a critical point for a functional f_β associated to a suitable problem at infinity.

More precisely, for every $\beta \in \mathcal{C}^1(\mathbf{R};\mathbf{R})$ with $\overline{a} \geq \beta(t) \geq \underline{a} > 0$ we define the functional $f_\beta : H \to \mathbf{R}$ by setting

$$f_\beta(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}} \beta(t) G(u)\, dt.$$

In the sequel will prove that from (H_3) it follows that the problems at infinity are the ones associated to the functionals f_β with β constant between $\underline{\alpha} := \liminf_{|t|\to\infty} \alpha(t)$ and $\overline{\alpha} := \limsup_{|t|\to\infty} \alpha(t)$.

From now on we will denote by $T_\tau : H \to H$ the translation of parameter $\tau \in \mathbf{R}$ that to every $u \in H$ associates $T_\tau u \in H$ defined by $T_\tau u = u(\cdot + \tau)$. Then we have, as one immediately checks,

$$f_\beta(T_\tau u) = f_{T_{-\tau}\beta}(u),$$

and

$$\nabla f_\beta(T_\tau u)v = \nabla f_{T_{-\tau}\beta}(u)T_{-\tau}v,$$

for all $u,\ v \in H$, $\tau \in \mathbf{R}$ and $\beta \in \mathcal{C}^1(\mathbf{R};\mathbf{R})$.
Note in particular that if β is a positive constant then the functional f_β is invariant under traslations, i.e. $f_\beta(T_\tau u) = f_\beta(u)$ for all $u \in H$ and $\tau \in \mathbf{R}$.

We begin to study the effect of translations on the functional f. First of all we have the following compactness property of the family of traslates of α.

LEMMA 3.2. *Let (τ_n) be a sequence in $\mathbf{R}$ such that $|\tau_n| \to \infty$. Then the sequence $(T_{\tau_n}\alpha)$ admits a convergent subsequence in $L^\infty_{loc}(\mathbf{R};\mathbf{R})$ to a constant $\beta \in [\underline{\alpha}, \overline{\alpha}]$.*

Proof. For every $t \in \mathbf{R}$ let $\alpha_n(t) = T_{\tau_n}\alpha(t)$. Then (α_n) is an equibounded and equicontinuous sequence in $\mathcal{C}^1(\mathbf{R};\mathbf{R})$. Indeed, by (H_1) and (H_3) there exist $M > 0$ and $\overline{a} > \underline{a} > 0$ such that

$$\underline{a} \le \alpha_n(t) \le \overline{a}, \quad \text{and} \quad |\dot{\alpha}_n(t)| \le M \quad \forall t \in \mathbf{R},\ \forall n \ge 1.$$

Then, by the Ascoli-Arzelà Theorem, (α_n) admits a subsequence uniformly convergent on every compact subsets of $\mathbf{R}$. If we consider an exhaustive family of compact subset in $\mathbf{R}$, with a diagonal procedure we obtain that there exists a subsequence of (α_n) which converges in L^∞_{loc} to a function β. Since $\alpha \in \mathcal{C}^1(\mathbf{R};\mathbf{R})$ and $\dot{\alpha}(\xi) \to 0$ as $|\xi| \to \infty$ there results $\alpha_n \in \mathcal{C}^1(\mathbf{R};\mathbf{R})$ and $\dot{\alpha}_n \to 0$ as $n \to \infty$ in L^∞_{loc}, because $|\tau_n| \to \infty$. This implies that $\beta \in \mathcal{C}^1(\mathbf{R};\mathbf{R})$ and $\dot{\beta} \equiv 0$. Therefore β is a constant. Moreover from (H_3) it follows that $\beta \in [\underline{\alpha}, \overline{\alpha}]$. ♢

Now, about the dependence of f_β on β we have:

LEMMA 3.3. *Let β be a positive constant. If (τ_n) is a sequence in $\mathbf{R}$ such that $T_{\tau_n}\alpha \to \beta$ in $L^\infty_{loc}(\mathbf{R};\mathbf{R})$ as $n \to \infty$ then for any compact subset B of H there results:*

$$\sup_{u \in B} |f_{T_{\tau_n}\alpha}(u) - f_\beta(u)| \to 0 \quad \textit{and} \quad \sup_{u \in B} \|\nabla f_{T_{\tau_n}\alpha}(u) - \nabla f_\beta(u)\| \to 0$$

as $n \to \infty$.

Proof. From (H_2) we know that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $G(x) < \varepsilon|x|^2$ and $|\nabla G(x)| < \varepsilon|x|$ for $|x| < \delta$. Then, since B is compact in H we can fix $R_\delta > 0$ such that

$$|u(t)| < \delta \quad \forall t \in \mathbf{R} \quad \text{with } |t| > R_\delta, \quad \forall u \in B.$$

Then we have

$$\begin{aligned}
|f_{T_{\tau_n}\alpha}(u) - f_\beta(u)| &= |\int_{\mathbf{R}} (\beta - T_{\tau_n}\alpha(t))G(u)dt| \\
&\le \int_{|t|\le R_\delta} |\beta - T_{\tau_n}\alpha(t)|G(u)dt + \int_{|t|>R_\delta} |\beta - T_{\tau_n}\alpha(t)|G(u)dt \\
&\le \sup_{|t|\le R_\delta} |\beta - T_{\tau_n}\alpha(t)| \int_{|t|\le R_\delta} G(u)dt + (\beta + \overline{a}) \int_{|t|>R_\delta} G(u)dt \\
&\le C_1 \sup_{|t|\le R_\delta} |\beta - T_{\tau_n}\alpha(t)| + (\beta + \overline{a})\varepsilon \int_{|t|>R_\delta} |u(t)|^2 dt \\
&\le C_1 \sup_{|t|\le R_\delta} |\beta - T_{\tau_n}\alpha(t)| + C_2\varepsilon
\end{aligned}$$

where the constants C_1 and C_2 are indipendent on the choice of $u \in B$. Since $T_{\tau_n}\alpha \to \beta$ in L^∞_{loc} the first term in the last inequality tends to zero as $n \to \infty$ while the second one is less than an arbitrary ε. This shows that

$$\sup_{u\in B} |f_{T_{\tau_n}\alpha}(u) - f_\beta(u)| \to 0.$$

With a similar argument we obtain the second part of the thesis. $\diamondsuit$

Finally, we have the following result that combines the previous ones.

Lemma 3.4. *Let (u_n) be a PS sequence for f at level $b > 0$ and let $\beta \in [\underline{\alpha}, \overline{\alpha}]$, $(\tau_n) \subset \mathbf{R}$, $u \in H$ be such that $T_{\tau_n}\alpha \to \beta$ in L^∞_{loc} and*

$$T_{\tau_n}u_n \rightharpoonup u \text{ weakly in } H.$$

Then $\nabla f_\beta(u) = 0$ and $(u_n - T_{-\tau_n}u)$ is a PS sequence for f at level $b - f_\beta(u)$.

Proof. First we prove that $\nabla f_\beta(u) = 0$. Since $T_{\tau_n}u_n \rightharpoonup u$ and ∇f_β is weakly continuous there results $\nabla f_\beta(T_{\tau_n}u_n) \rightharpoonup \nabla f_\beta(u)$. Then, since $\nabla f(u_n) \to 0$, for every $\varphi \in \mathcal{C}_0^\infty$ we obtain

$$\begin{aligned}\nabla f_\beta(u)\cdot\varphi &= \nabla f_\beta(T_{\tau_n}u_n)\cdot\varphi + o(1)\\ &= \nabla f_\beta(T_{\tau_n}u_n)\cdot\varphi - \nabla f(u_n)\cdot T_{-\tau_n}\varphi + o(1)\\ &= \nabla f_\beta(T_{\tau_n}u_n)\cdot\varphi - \nabla f_{T_{\tau_n}\alpha}(T_{\tau_n}u_n)\cdot\varphi + o(1)\\ &= \int_{supp\varphi}(T_{\tau_n}\alpha-\beta)\nabla G(T_{\tau_n}u_n)\cdot\varphi dt + o(1) = o(1)\end{aligned}$$

because $T_{\tau_n}\alpha \to \beta$ in L^∞_{loc} and $(T_{\tau_n}u_n)$ is bounded. Then $\nabla f_\beta(u)\cdot\varphi = 0$ for every $\varphi \in \mathcal{C}_0^\infty$ and therefore $\nabla f_\beta(u) = 0$.
Now we prove that $(u_n - T_{-\tau_n}u)$ is a PS sequence for f at level $b - f_\beta(u)$. Since $T_{\tau_n}u_n \rightharpoonup u$ and $T_{\tau_n}\alpha(t)$ is uniformly bounded in $\mathbf{R}$, by Lemma 2.3 (i) we have

$$\begin{aligned}f(u_n - T_{-\tau_n}u) - f(u_n) &= f_{T_{\tau_n}\alpha}(T_{\tau_n}u_n - u) - f_{T_{\tau_n}\alpha}(T_{\tau_n}u_n)\\ &= -f_{T_{\tau_n}\alpha}(u) + o(1).\end{aligned}$$

Then, using Lemma 3.3, we obtain

$$\begin{aligned}f(u_n - T_{-\tau_n}u) - f(u_n) + f_\beta(u) &= f_\beta(u) - f_{T_{\tau_n}\alpha}(u) + o(1)\\ &= o(1)\end{aligned}$$

and since $f(u_n) \to b$ we obtain $f(u_n - T_{-\tau_n}u) \to b - f_\beta(u)$.
Finally we prove that $\nabla f(u_n - T_{-\tau_n}u) \to 0$. For every $\varphi \in H$ since $T_{\tau_n}u_n \rightharpoonup u$ and $T_{\tau_n}\alpha$ is uniformly bounded, like above by Lemma 2.3 (ii) and Lemma 3.3 we have

$$\begin{aligned}\nabla f(u_n - T_{-\tau_n}u)\cdot\varphi &= \nabla f_{T_{\tau_n}\alpha}(T_{\tau_n}u_n - u)\cdot T_{\tau_n}\varphi\\ &= \nabla f_{T_{\tau_n}\alpha}(T_{\tau_n}u_n)\cdot T_{\tau_n}\varphi - \nabla f_{T_{\tau_n}\alpha}(u)\cdot T_{\tau_n}\varphi + o(1)\|\varphi\|\\ &= \nabla f(u_n)\cdot\varphi - \nabla f_\beta(u)\cdot T_{\tau_n}\varphi + o(1)\|\varphi\|\\ &= o(1)\|\varphi\|,\end{aligned}$$

since $\nabla f(u_n) \to 0$ and $\nabla f_\beta(u) = 0$. Therefore we obtain that $(u_n - T_{\tau_n}u)$ is a PS sequence for f at level $b - f_\beta(u)$. ◊

As a direct consequence of Lemmas 3.1, 3.2 and 3.4 we see that if (u_n) is a PS sequence for f at a positive level weakly convergent to zero then there exists a sequence (τ_n) in $\mathbf{R}$ such that the sequence $(T_{\tau_n}u_n)$ up to a subsequence weakly converges to a critical point of a functional at infinity f_β with $\beta \in [\underline{\alpha}, \overline{\alpha}]$. We have thus proved the following result

Proposition 3.1. *Let (u_n) be a PS sequence for f at level $b > 0$ with $u_n \rightharpoonup 0$. Then there exist a constant $\beta \in [\underline{\alpha}, \overline{\alpha}]$, a function $u \in H \setminus \{0\}$, a subsequence of (u_n), still denoted by u_n, and a sequence (τ_n) in $\mathbf{R}$ with $|\tau_n| \to \infty$ such that*

(i) $T_{\tau_n}\alpha \to \beta$ *in* $L^\infty_{loc}(\mathbf{R};\mathbf{R})$,

(ii) $T_{\tau_n}u_n \rightharpoonup u$ *weakly in* H,

(iii) $\nabla f_\beta(u) = 0$,

(iv) $(u_n - T_{-\tau_n}u)$ *is a PS sequence for f at level* $b - f_\beta(u)$.

From condition (A) we have that for every $\beta > 0$ the mountain pass level

$$c_\beta := \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} f(\gamma(s))$$

where $\Gamma_\beta := \{\gamma \in \mathcal{C}([0,1];H) \ : \ \gamma(0) = 0, \ f_\beta(\gamma(1)) < 0\}$, is the smallest nontrivial critical level for f_β. Moreover note that for every $\beta \in [\underline{\alpha}, \overline{\alpha}]$ there results $f_\beta(u) \geq f_{\overline{\alpha}}(u)$ for all $u \in H$ so that $\Gamma_\beta \subset \Gamma_{\overline{\alpha}}$. Then we obtain

$$c_{\overline{\alpha}} \leq c_\beta, \quad \forall \beta \in [\underline{\alpha}, \overline{\alpha}] \tag{3.1}$$

and in particular, by (A), we have

$$c_{\overline{\alpha}} = \min_{\beta\in[\underline{\alpha},\overline{\alpha}]} \min_{u\in\mathcal{K}_\beta} f_\beta(u). \tag{3.2}$$

Therefore, from the previous proposition and Lemma 2.2 it follows that every PS sequence for f weakly convergent to zero at level $b < c_{\overline{\alpha}}$ must converge strongly to zero. Indeed if (u_n) is a PS sequence at level $b < c_{\overline{\alpha}}$ weakly convergent to zero but not strongly (i.e. $b > 0$) then, by Proposition 3.1 (iv), we obtain a PS sequence at level strictly less than zero, since $b - f_\beta(u) < 0$ for every nontrivial critical point $u \in H$ of f_β and for every $\beta \in [\underline{\alpha}, \overline{\alpha}]$. This contradicts Lemma 2.2. In particular, by Proposition 2.1, this implies that the PS condition holds at levels strictly less than $c_{\overline{\alpha}}$.

Corollary 3.1. *Every PS sequence for f at level strictly less than $c_{\overline{\alpha}}$ is precompact in H.*

Proof. Let (u_n) be a PS sequence for f at level $b < c_{\overline{\alpha}}$. Then, by Proposition 2.1, up to a subsequence $u_n \rightharpoonup u_0$ weakly in H and $(u_n - u_0)$ is a PS sequence for f at level $b - f(u_0)$ which convergs weakly to zero. But $b - f(u_0) < c_{\overline{\alpha}}$ and therefore, by the previous remark, it must be $u_n \to u_0$ strongly in H. $\diamondsuit$

Considering c^o defined in Lemma 2.1 and using the previous result we obtain

COROLLARY 3.2. *Let (u_n) be a PS sequence for f at level $b \in (0, c^o + c_{\overline{\alpha}})$. Then the following alternative holds:*

(*a*) *either (u_n) weakly converges to zero in H; or*

(*b*) *(u_n) is precompact in H.*

Proof. Let us suppose that (a) does not hold. Then there exists a subsequence of (u_n), still denoted by u_n, weakly convergent in H to a nonzero limit point $u_0 \in H$.

By Proposition 2.1 we have that $\nabla f(u_0) = 0$ and $(u_n - u_0)$ is a PS sequence for f at level $b - f(u_0) \geq 0$ weakly convergent to zero.

Now, since $u_0 \not\equiv 0$, we have $f(u_0) \geq c^o$ and so $b - f(u_0) \leq b - c^o < c_{\overline{\alpha}}$ by assumption. Therefore, by Corollary 3.1, we have that $(u_n - u_0)$ is precompact and since $u_n \rightharpoonup u_0$ we obtain that $u_n \to u_0$ strongly in H. Therefore (b) holds. $\diamondsuit$

In the next section we will show that alternative (a) in the previous corollary never occurs for those PS sequences (u_n) at level $b \in (0, c^*)$, where

$$c^* := \min(c_{\underline{\alpha}}, c^o + c_{\overline{\alpha}}),$$

that satisfy the condition

$$\|u_n - u_{n-1}\| \to 0,$$

namely $\overline{PS}$ sequences. Therefore it will follow that every $\overline{PS}$ sequence for f at level $b \in (0, c^*)$ admits a subsequence that converges in H to a nonzero limit point, that is a nontrivial solution of the problem (P).

4. The $\overline{PS}$ sequences

We begin to describe the PS sequences at a positive level weakly convergent to zero. The next result is a standard characterization of PS sequences and it can be found in all papers on homoclinic solutions. In our case it takes the following form:

PROPOSITION 4.1. *Let (u_n) be a PS sequence for f at level $b > 0$ such that $u_n \rightharpoonup 0$ weakly in H. Then there exist $q \in \mathbf{N}$, q constants $\beta_i \in [\underline{\alpha}, \overline{\alpha}]$, q functions $v_i \in H \setminus \{0\}$, a subsequence of (u_n), still denoted by u_n, and q sequences $(\theta_n^i) \subset \mathbf{R}$, with $1 \le i \le q$ such that*

(i) $\nabla f_{\beta_i}(v_i) = 0$ *for all* $i = 1, ..., q$,

(ii) $u_n - \sum_{i=1}^q T_{\theta_n^i} v_i \to 0$ *strongly in* H *as* $n \to \infty$,

(iii) $b = \sum_{i=1}^q f_{\beta_i}(v_i)$,

(iv) $|\theta_n^i| \to \infty$, $|\theta_n^i - \theta_n^j| \to \infty$ *for all* $i \ne j = 1, ..., q$ *as* $n \to \infty$.

Proof. Since (u_n) is a PS sequence at a positive level weakly convergent to zero, by Proposition 3.1 we know that there exist a constant $\beta_1 \in [\underline{\alpha}, \overline{\alpha}]$, a function $v_1 \in H \setminus \{0\}$, a subsequence of (u_n), still denoted by u_n, and a sequence (τ_n) in $\mathbf{R}$ with $|\tau_n| \to \infty$ such that:

. $T_{\tau_n} u_n \rightharpoonup v_1$ weakly in H,

. $T_{\tau_n} \alpha \to \beta_1$ in L^∞_{loc},

. $\nabla f_{\beta_1}(v_1) = 0$,

. $(u_n - T_{-\tau_n} v_1)$ is a PS sequence for f at level $b - f_{\beta_1}(v_1)$.

Setting $\theta_n^1 = -\tau_n$, two cases may occur:
(I) if $b - f_{\beta_1}(v_1) = 0$ then by Lemma 2.2 $\|u_n - T_{\theta_n^1} v_1\| \to 0$ and the proposition is proved with $q = 1$,
(II) if $b - f_{\beta_1}(v_1) > 0$, setting $b_1 = b - f_{\beta_1}(v_1)$ and $u_n^1 = u_n - T_{\theta_n^1} v_1$, (u_n^1) is a PS sequence for f at level $b_1 > 0$ with $u_n^1 \rightharpoonup 0$ and we can apply Proposition 3.1 to the sequence (u_n^1).

To show that this procedure ends, it is enough to prove that for some $q \geq 1$ there results

$$b - f_{\beta_1}(v_1) - f_{\beta_2}(v_2) - \ldots - f_{\beta_q}(v_q) = 0. \tag{4.1}$$

Indeed, since $\nabla f_{\beta_i}(v_i) = 0$ and $\beta_i \in [\underline{\alpha}, \overline{\alpha}]$ for all $i = 1, \ldots, q$, by (3.2) we have

$$f_{\beta_i}(v_i) \geq c_{\beta_i} \geq c_{\overline{\alpha}}.$$

So that after at most $[\frac{b}{c_{\overline{\alpha}}}]$ steps we obtain (4.1). This completes the proof. $\diamondsuit$

Starting from the previous proposition one can prove like in [STT] (or otherwise like in [MNT] or [CZMN]) the following result:

PROPOSITION 4.2. *Let (u_n) be a $\overline{PS}$ sequence for f at level $b > 0$ with $u_n \rightharpoonup 0$. Then there exists a sequence (τ_n) in* **R** *such that:*

(i) $\liminf_{n\to\infty} |u_n(\tau_n)| > 0$,

(ii) $\lim_{n\to\infty} |\tau_n| = \infty$,

(iii) $\lim_{n\to\infty} |\tau_n - \tau_{n-1}| = 0$.

In the previous section we have seen (Proposition 3.1) that for every PS sequence (u_n) for f at level $b > 0$ weakly convergent to zero there exist a sequence (τ_n) in **R** and a constant $\beta \in [\underline{\alpha}, \overline{\alpha}]$ such that, up to a subsequence, $T_{\tau_n}\alpha \to \beta$ in L^∞_{loc} and $(T_{\tau_n}u_n)$ weakly converges in H to a nonzero critical point of f_β.

This property has been used in all recent papers as a starting point for the analysis of PS sequences. In the case of a recurrent time dependence of the potential, it has been used in combination with Proposition 4.2 in order to show that a suitable subsequence of $(T_{\tau_n}u_n)$ converges to a particular nonzero limit point. The fact that this limit point is the required solution is a consequence of the recurrence hypoteses: in these cases the original problem *coincides* with one of the problems at infinity.

In our case these arguments cannot work due to the fact that the problems at infinity are all different from the original one. However the analysis of $\overline{PS}$ sequences discloses remarkable properties. Indeed in the next proposition we will prove, combining Propositions 3.1

and 4.2, that for *every* $\beta \in [\underline{\alpha}, \overline{\alpha}]$ and for every $\overline{PS}$ sequence (u_n) there exists a sequence (τ_n) in $\mathbf{R}$ such that, up to a subsequence, the sequence $(T_{\tau_n} u_n)$ weakly tends in H to a nonzero critical point of f_β. This is the fundamental property which will allow us to conclude the proof.

PROPOSITION 4.3. *Let (u_n) be a $\overline{PS}$ sequence for f at level $b > 0$ with $u_n \rightharpoonup 0$. Then for every $\beta \in [\underline{\alpha}, \overline{\alpha}]$ there exist a subsequence of (u_n), still denoted by u_n, a function $u_0 \in H \setminus \{0\}$ and a sequence (τ_n) in $\mathbf{R}$ such that:*

(a) $T_{\tau_n} u_n \rightharpoonup u_0$ *weakly in* H,

(b) $\nabla f_\beta(u_0) = 0$,

(c) $(u_n - T_{-\tau_n} u_0)$ *is a PS sequence for f at level $b - f_\beta(u_0)$.*

Proof. Let (τ_n) be the sequence in $\mathbf{R}$ associated to (u_n) via Proposition 4.2 and let (σ_k) be a sequence in $\mathbf{R}$ such that $|\sigma_k| \to \infty$ and $\alpha(\sigma_k) \to \beta$ as $n \to \infty$ (such sequence exists by (H_3)). By (ii) and (iii) in Proposition 4.2 we obtain that there exists a subsequence of (τ_{n_k}) of (τ_n), such that $|\tau_{n_k} - \sigma_k| \to 0$ as $k \to \infty$. Then, if we consider the sequence $(T_{\tau_{n_k}} \alpha)$, there results

$$T_{\tau_{n_k}} \alpha \to \beta \quad \text{in } L^\infty_{loc}(\mathbf{R}, \mathbf{R}).$$

Indeed, for all $R > 0$ and for all $t \in [-R, R]$ we have

$$\begin{aligned} |T_{\tau_{n_k}} \alpha(t) - \beta| &\le |T_{\tau_{n_k}} \alpha(t) - T_{\sigma_k} \alpha(t)| + |T_{\sigma_k} \alpha(t) - \beta| \\ &= |\alpha(t + \tau_{n_k}) - \alpha(t + \sigma_k)| + |\alpha(t + \sigma_k) - \beta|. \end{aligned} \tag{4.2}$$

The first term in the last expression of (4.2) converges uniformly to zero by the uniform continuity of α. Concerning the second one note that

$$\begin{aligned} |\alpha(t + \sigma_k) - \beta| &\le |\alpha(t + \sigma_k) - \alpha(\sigma_k)| + |\alpha(\sigma_k) - \beta| \\ &= |\dot{\alpha}(\xi_k)| R + |\alpha(\sigma_k) - \beta| \\ &\le |\dot{\alpha}(\xi_k)| R + |\alpha(\sigma_k) - \beta|, \end{aligned}$$

for some ξ_k between σ_k and $t + \sigma_k$. As $|\sigma_k| \to \infty$, we obtain $|\xi_k| \to \infty$ and so by (H_3) we have that $\dot{\alpha}(\xi_k) \to 0$. Moreover by construction

$\alpha(\sigma_k) \to \beta$. Then we have that also the second term in (4.2) tends to zero uniformly in $[-R, R]$.
Consider now the sequence $(T_{\tau_{n_k}} u_{n_k})$. Since $(T_{\tau_{n_k}} u_{n_k})$ is bounded in H, it admits a subsequence, still denoted by $T_{\tau_{n_k}} u_{n_k}$, weakly convergent in H to some u_0. From (i) in Proposition 4.2 we have $\liminf_{k\to\infty} |T_{\tau_{n_k}} u_{n_k}(0)| > 0$ hence $u_0 \not\equiv 0$ and moreover, as $T_{\tau_{n_k}} \alpha \to \beta$ in L^∞_{loc} and $T_{\tau_{n_k}} u_{n_k} \rightharpoonup u_0$ weakly in H, by Lemma 3.4 we obtain that $\nabla f_\beta(u_0) = 0$ and that $(u_n - T_{\tau_n} u_0)$ is a PS sequence for f at level $b - f_\beta(u_0)$. $\diamondsuit$

In particular if $b < c_{\underline{\alpha}}$ and if we choose $\beta = \underline{\alpha}$ in the previous proposition we obtain:

COROLLARY 4.1. *Every $\overline{PS}$ sequence for f at level $b \in (0, c_{\underline{\alpha}})$ admits a subsequence weakly convergent to a nonzero limit point.*

Proof. Let (u_n) be a $\overline{PS}$ sequence at level $b \in (0, c_{\underline{\alpha}})$ and, arguing by contradiction, suppose that every subsequence of (u_n) weakly converges to zero. Then $u_n \rightharpoonup 0$ and, using the previous proposition with $\beta = \underline{\alpha}$, we obtain that there exist $u_0 \in H \setminus \{0\}$, a sequence (τ_n) in $\mathbf{R}$ and a subsequence of (u_n), still denoted by u_n, such that setting $v_n = u_n - T_{-\tau_n} u_0$ there results $\nabla f_{\underline{\alpha}}(u_0) = 0$ and (v_n) is a PS sequence for f at level $b - f_{\underline{\alpha}}(u_0)$.

As $\nabla f_{\underline{\alpha}}(u_0) = 0$ and $u_0 \not\equiv 0$ we have, by condition (A), $f_{\underline{\alpha}}(u_0) \geq c_{\underline{\alpha}}$. Moreover, by assumption $b < c_{\underline{\alpha}}$. Therefore $b - f_{\underline{\alpha}}(u_0) < 0$ and then, since (v_n) is a PS sequence at level $b - f_{\underline{\alpha}}(u_0)$, we get a contradiction by Lemma 2.2. $\diamondsuit$

In the previous section we have proved that the PS condition holds at levels in $(0, c_{\overline{\alpha}})$. Now, as a direct consequence of the previous corollary and Corollary 3.2, we have that the $\overline{PS}$ condition holds at levels in $(0, c^*)$ where

$$c^* := \min(c_{\underline{\alpha}}, c^o + c_{\overline{\alpha}}).$$

More precisely the following result holds (compare with Corollary 3.1, noting that $c^* > c_{\overline{\alpha}}$):

COROLLARY 4.2. *Every $\overline{PS}$ sequence for f at level $b \in (0, c^*)$ is precompact in H.*

Therefore we obtain a nontrivial critical point for f if we can find a $\overline{PS}$ sequence with level in $(0, c^*)$. To this aim we recall that since f satisfies the geometric assumptions of the Mountain Pass Theorem we can find a PS sequence at the mountain pass level c. Moreover by a result given in [CZES] for every ε small enough there exists a $\overline{PS}$ sequence for f at some level $b \in [c-\varepsilon, c+\varepsilon]$. So it is enough to prove that the mountain pass level c is strictly less than c^*. To prove this we need the following result due to P. Caldiroli (see [Cal], Theorem 1.1 and Lemma 2.2).

THEOREM 4.1. *For every $\beta > 0$ there exist $u_\beta \in H$ and $\gamma_\beta \in \Gamma_\beta$ such that:*

(i) $\nabla f_\beta(u_\beta) = 0$ *and* $f_\beta(u_\beta) = c_\beta$,

(ii) $u_\beta \in \gamma_\beta([0,1])$ *and* $\max_{s\in[0,1]} f_\beta(\gamma_\beta(s)) = f_\beta(u_\beta) = c_\beta$.

Then we have

LEMMA 4.1. *For every $0 < \beta_1 < \beta_2$ there results $c_{\beta_1} > c_{\beta_2}$. Moreover there results $c \le c_{\overline{\alpha}}$ and in particular $c < c_{\underline{\alpha}}$.*

Proof. First we prove that $c_{\beta_1} > c_{\beta_2}$. Let $\gamma_{\beta_1} \in \Gamma_{\beta_1}$ be the optimal path for f_{β_1} given in the previous theorem. Now, since $\beta_1 < \beta_2$, there results

$$f_{\beta_1}(u) > f_{\beta_2}(u), \quad \forall u \in H \setminus \{0\}. \tag{4.3}$$

In particular it follows that $\gamma_{\beta_1} \in \Gamma_{\beta_2}$. Then let $s_0 \in [0,1]$ be such that

$$\max_{s\in[0,1]} f_{\beta_2}(\gamma_{\beta_1}(s)) = f_{\beta_2}(\gamma_{\beta_1}(s_0)) \ge c_{\beta_2}.$$

From (ii) in the previous theorem and (4.3) we have

$$c_{\beta_1} \ge f_{\beta_1}(\gamma_{\beta_1}(s_0)) > f_{\beta_2}(\gamma_{\beta_1}(s_0)) \ge c_{\beta_2}$$

since $\gamma_{\beta_1}(s_0) \not\equiv 0$. Then we have $c_{\beta_1} > c_{\beta_2}$.
Now, to prove that $c \le c_{\overline{\alpha}}$, for every $\varepsilon > 0$ let $\gamma \in \Gamma_{\overline{\alpha}}$ be such that

$$\sup_{s\in[0,1]} f_{\overline{\alpha}}(\gamma(s)) \le c_{\overline{\alpha}} + \varepsilon.$$

By (H_3) there exists a sequence (τ_n) in $\mathbf{R}$ such that $T_{\tau_n}\alpha \to \overline{\alpha}$ in L^∞_{loc}. From Lemma 3.3, since $\gamma([0,1])$ is compact in H, we obtain

$$f(T_{-\tau_n}\gamma(s)) = f_{T_{\tau_n}\alpha}(\gamma(s)) \to f_{\overline{\alpha}}(\gamma(s)), \tag{4.4}$$

uniformly with respect to $s \in [0,1]$. Then for n big enough we have $T_{-\tau_n}\gamma \in \Gamma$ and as $f_{\overline{\alpha}}(\gamma(s)) \le c_{\overline{\alpha}} + \varepsilon$ for all $s \in [0,1]$ by (4.4) we have $c \le c_{\overline{\alpha}} + \varepsilon$. Since $\varepsilon > 0$ was arbitrarily fixed, we obtain $c \le c_{\overline{\alpha}}$. $\diamondsuit$

We can now prove the existence of at least one solution of problem (P).

Proof of Theorem 1.1. From Corollary 4.2 we obtain a non-trivial critical point for f if we can find a $\overline{PS}$ sequence for f at level $b \in (0, c^*)$. But this is certainly the case since by Lemma 4.1 the mountain pass level c satisfies $c \le c_{\overline{\alpha}} < c_{\underline{\alpha}}$. So that $c < c^* = \min(c^o + c_{\overline{\alpha}}, c_{\underline{\alpha}})$. $\diamondsuit$

References

[ACM] Abenda A., Caldiroli P. and Montecchiari P., *Multibump solutions for Duffing-like systems*, Preprint SISSA (1994).

[AB] Ambrosetti A. and Bertotti M. L., *Homoclinics for second order conservative systems*, Partial differential equation and related subjects, Ed. M. Miranda, Pitman Research Notes in Math. Ser. (1992).

[ACZ] Ambrosetti A. and Coti Zelati V., *Multiple homoclinic orbits for a class of conservative systems*, Rend. Sem. Mat. Padova **89** (1993), 177-194.

[BB] Bertotti M. L. and Bolotin S., *A variational approach for homoclinics in almost periodic Hamiltonian systems*, Preprint (1995).

[B] Bolotin S., *Existence of homoclinic motions*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **6** (1980), 98-103.

[C] Caldiroli P., *A new proof of the existence of homoclinic orbits for a class of autonomous second oreder Hamiltonian systems in* $\mathbf{R}^N$, to appear in Math. Nachr.

[CM] Caldiroli P. and Montecchiari P., *Homoclinics orbits for second order Hamiltonian systems with potential changing sign*, Comm. Appl. Nonlinear Analysis **1** (1994), 97-129.

[CZES] Coti Zelati V., Ekeland I. and Séré E., *A variational approach to homoclinic orbits in Hamiltonian systems*, Math. Ann. **288** (1990), 133-160.

[CZMN] Coti Zelati V., Montecchiari P. and Nolasco M., *Multibump homoclinic solutions for a class of second order, almost periodic Hamiltonian systems*, to appear in Nodea.

[CZR] Coti Zelati V. and Rabinowitz P. H., *Homoclinic orbits for second order Hamiltonian systems possesing superquadratic potentials*, J. Amer. Math. Soc. **4** (1991), 693-727.

[DN] Ding W. J. and Ni W. M., *On the existence of positive entire solutions of a semilinear ellipitic equation*, Arch. Rat. Mech. Anal. **91** (1986), 283-308.

[EL] Esteban M. J. and Lions P. L., *Existence and nonexistence results for semilinear elliptic problems in unbounded domains*, Arch. Rat. Mech. Anal. **93** (1981), 1-14.

[L] Lions P. L, *On positive solutions of semilinear elliptic equations in unbounded domains*, Arch. Rat. Mech. Anal. **93** (1981), 1-14.

[M] Montecchiari P., *Existence and multiplicity of homoclinic solutions for a class of asymptotically periodic second order Hamiltonian systems*, Ann. Mat. Pura Appl. (IV) **168** (1995), 317-354.

[MN] Montecchiari P. and Nolasco M., *Multibump solutions for pertubations of periodic second order Hamiltonian systems*, to appear in Nonlinear Analysis.

[MNT] Montecchiari P., Nolasco M. and Terracini S., *Multiplicity of homoclinics for time recurrent second order systems*, to appear in Calculus of Variation.

[R] Rabinowitz P. H., *Homoclinic orbits for a class of Hamiltonian systems*, Proc. Roy. Soc. Edinburgh **114 A** (1990), 33-38.

[RT] Rabinowitz P. H. and Tanaka K., *Some results on connecting orbits for a class of Hamiltonian systems*, Math. Z. **206** (1991), 473-479.

[S1] Séré E., *Existence of infinitely many homoclinic orbits in Hamiltonian systems*, Math. Zeit. **209** (1991), 27-42.

[S2] Séré E., *Looking for the Bernoulli shift*, Ann. IHP Anal. Non-linéaire **10** (1993), 561-590.

[STT] Serra E., Tarallo M. and Terracini S., *On the existence of homoclinic solutions for almost periodic second order systems*, to appear in Ann. IHP Anal. Nonlinéaire.

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